Function Fields of Class Number One

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Outline

1 Motivation
   - Origin of the Problem
   - Previous Work

2 Our Result
   - Main Result
   - Basic Ideas

3 Summary

4 Further questions

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Classification of imaginary quadratic fields was completed by Heegner and Stark in 1969. Goldfeld and Gross-Zagier showed in 1983 that imaginary quadratic fields of given class number can be effectively classified.
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In the case of (global) function fields, there are no archimedean places at ‘infinity’, so there is no canonical ring of integers and its class group. The usual substitute, which we will use below, is the divisor class group of degree zero.

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Theorem (Leitzel, Madan and Queen): Except for a possible exception of genus 4 function field with field of constants $F_2$, there are only 7 (up to isomorphism) class number one function fields of positive genus.
LIST OF ALL SEVEN CASES

- $g = 1, y^2 + y = x^3 + x + 1$ over $F_2$
- $g = 1, y^2 = x^3 + 2x + 2$ over $F_3$
- $g = 1, y^2 + y = x^3 + \eta$ over $F_4$
- $g = 2, y^2 + y = x^5 + x^3 + 1$ over $F_2$
- $g = 2, y^2 + y = (x^3 + x^2 + 1)(x^3 + x + 1)^{-1}$ over $F_2$
- $g = 3, y^4 + xy^3 + (x^2 + x)y^2 + (x^3 + 1)y + (x^4 + x + 1) = 0$ over $F_2$
- $g = 3, y^4 + (x^3 + x + 1)y + (x^4 + x + 1) = 0$ over $F_2$
In 1975, J. Leitzel, M. Madan and C. Queen claimed to prove that there is no genus 4 function field with field of constants $F_2$ having class number one.

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Theorem: There is only one function field (up to isomorphism) of genus 4 having class number one.
Why we have such a nice bound on genus and size of constant field (what makes it much easier than in number field)?
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FACTS WE NEED

- **Class Number Formula:** $K$ is a function field then $h = L(1)$ where $h$ is the class number of $K$, $L(t)$ is the $L-$ function of $F$.

- **Hasse-Weil Theorem:** Let $K$ be a function field over $F_q$ and $L(t) = \prod_{1}^{2g} (t - \alpha_i)$ is its $L-$ function then $\alpha_i$’s have absolute value $\sqrt{q}$.

- **Weil Bound:** $|K(F_q) - q - 1| \leq 2g\sqrt{q}$ where $K(F_q)$ denotes the number of rational points on $K$ over $F_q$. 
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• Every hyperelliptic function field has a degree 1 or 2 prime divisor.
• Canonical class of non-hyperelliptic curve embeds it into $P^3$.
• Every genus 4 canonical curve is a complete intersection of a degree 2 surface and a degree 3 surface.
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Key: Using Hasse Weil Theorem and Weil Bound we get 
\[(\sqrt{q} + 1)^2g \geq h \geq (\sqrt{q} - 1)^2g.\]
Lemma 1 (M. Madan, C. Queen): Let $K$ is a function field with $g = 4$, $h = 1$ then the coefficient field is $F_2$, $N_1 = N_2 = N_3 = 0$, $N_4 = 1$, $L(t) = 1 - 3t + 2t^2 + t^4 + 8t^6 - 24t^7 + 16t^8$, where $t = 2^s$ and $N_i$ denotes the number of degree $i$ primes in $K$. 
Lemma 2 (Manin): Let function field $K$ over $F_2$ with $L$–function given above, then the Cartier operator acts on the canonical class by the matrix

$$C = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{pmatrix}$$ (1)
Lemma 3: $K$ is a function field over $F_2$, $\Omega_K$ is the module of differentials of $K$ and $S$ represents the symmetric algebra of $\Omega_K$ over $K$ then there is a derivation $D : S \to S$ such that $\forall W \in S, DW = (CW)^2$ where $C$ is the Cartier operator on $\Omega_K$. 
Let $x \in K$ but no in $K^2$, then $S = K[dx]$, where $d : K \to \Omega_K$ is the universal derivation. We define $D : S \to S$, by $D(\sum_n f_n(dx)^n) = \sum_n (df_n)(dx)^n$. And $\forall y \in K, \exists f, g \in K$, such that $f^2 + g^2x = y$, we get $D(ydx) = D(f^2 + g^2x)dx = (gdx)^2 = (C(ydx))^2$. 
By $N_1 = N_2 = N_3 = 0$ we know our function field $K$ is not hyperelliptic and the canonical class \( \{ W_1, W_2, W_3, W_4 \} \) gives the canonical embedding of $K$ into $P^3$ and $W_i, W_j$ has no common zero.

Now by induction on the quadratic forms we can deduce that up to non-degenerate linear transformation our degree 2 surface is equivalent to one of the 24 cases.
Using all above argument and by SAGE we calculate all rational points with degree $\leq 4$ on these 24 surfaces and we get a unique one:

$$F_2 : W_1 W_2 + W_1 W_3 + W_1 W_4 + W_2 W_4 + W_1^2 + W_3^2 + W_4^2 = 0,$$

Then apply Lemma 2 and 3 we get:

$$F_3 : W_2^2 W_3 + W_1 W_4^2 + W_3^3 + W_3 W_4 + W_1^2 W_2 + W_4^3 + W_1^2 W_3 + W_3 W_4^2 = 0$$
There is one and only one function field of genus 4 and class number 1:

\[
y^5 + y^3 + y^2(x^3 + x^2 + x) + y(x^7 + x^5 + x^4 + x^3 + x)/(x^4 + x + 1) = \\
(x^{13} + x^{12} + x^8 + x^6 + x^2 + x + 1)/(x^4 + x + 1)^2, \text{ over } F_2.
\]
Given an abelian finite group, whether it’s a class group for some function field?

There are some alternative definitions of class group of function field and the similar class number one problem can be asked again.

How about the function field of dimension $\geq 2$?

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REFERENCE


