1. (20 points) Consider the curve

\[ f(x) = \frac{x^4}{16} + \frac{1}{2x^2}. \]

(a) Calculate the arc length function \( s(t) \) starting at \( x = 1 \), that computes the length of the curve from \((1, f(1))\) to \((t, f(t))\).

(b) Calculate the arc length from \( x = 2 \) to \( x = 4 \).

Solution: (a) Since

\[ f'(x) = \frac{x^3}{4} - \frac{1}{x^3}, \]

the arc length function is given by

\[ s(t) = \int_1^t \sqrt{1 + \left(\frac{x^3}{4} - \frac{1}{x^3}\right)^2} \, dx \]

\[ = \int_1^t \sqrt{\left(\frac{x^3}{4} + \frac{1}{x^3}\right)^2} \, dx \]

\[ = \int_1^t \frac{x^3}{4} + \frac{1}{x^3} \, dx \]

\[ = \left. \frac{t^4}{16} - \frac{1}{2t^2} + \frac{7}{16} \right|_1^t \]

(b) By the definition of the arc length function, \( s(4) \) is the arc length from \( t = 1 \) to \( t = 4 \) and \( s(2) \) is the arc length from \( t = 1 \) to \( t = 4 \), so the arc length from \( t = 2 \) to \( t = 4 \) is

\[ s(4) - s(2) = 15 + \frac{3}{32}. \]

2. (20 points)

(a) Find the area of the surface of revolution obtained by rotating the curve \( y = x^2 \), for \( 0 \leq x \leq 2 \), about the \( y \)-axis.
(b) Find the area of the surface of revolution obtained by rotating the curve \( x = 1 + |y| \), for \(-1 \leq y \leq 1\), about the \( y \)-axis.

**Solution:** (a) We compute

\[
A = \int_0^2 2\pi x \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx = 2\pi \int_0^2 x \sqrt{1 + 4x^2} \, dx.
\]

Setting \( u = 1 + 4x^2 \), we have \( du = 8xdx \), so that \( xdx = du/8 \). Also when \( x = 0 \), we have \( u = 1 \), and when \( x = 2 \), \( u = 17 \). This gives

\[
A = \frac{\pi}{4} \int_1^{17} \sqrt{u} \, du = \frac{\pi}{4} \left[ \frac{2}{3} u^{3/2} \right]_1^{17} = \frac{\pi}{6} \left( 17^{3/2} - 1 \right).
\]

**Solution:** (b) Recall that \(|y| = \begin{cases} y & \text{if } y \geq 0, \\ -y & \text{if } y < 0 \end{cases}\) and therefore \( \frac{dx}{dy} = \begin{cases} 1 & \text{if } y \geq 0, \\ -1 & \text{if } y < 0 \end{cases}\). We now compute

\[
A = \int_{-1}^1 2\pi (1 + |y|) \sqrt{1 + \left(\frac{dx}{dy}\right)^2} \, dy
\]

\[
= 2\pi \int_{-1}^0 (1 - y) \sqrt{1 + (-1)^2} \, dy + 2\pi \int_{0}^1 (1 + y) \sqrt{1 + (1)^2} \, dy
\]

\[
= 2\sqrt{2}\pi \left[ y - \frac{y^2}{2} \right]_{-1}^0 + 2\sqrt{2}\pi \left[ y + \frac{y^2}{2} \right]_0^1
\]

\[
= 2\sqrt{2}\pi \left( - \left( 1 - \frac{(-1)^2}{2} \right) \right) + 2\sqrt{2}\pi \left( 1 + \frac{1^2}{2} \right) = 6\sqrt{2}\pi.
\]

3. **(20 points)** Determine whether the following series converge or diverge. Justify your answers, making sure to name the convergence test(s) that you are using.

(a)

\[
\sum_{n=1}^{\infty} \frac{3^n + 1}{2^n - 1} = 4 + \frac{10}{3} + \frac{28}{7} + \frac{82}{15} + \cdots
\]

(b)

\[
\sum_{n=2}^{\infty} \frac{1}{n \ln(n)^3} = \frac{1}{2 \ln(2)^3} + \frac{1}{3 \ln(3)^3} + \frac{1}{4 \ln(4)^3} + \cdots
\]

**Solution:** (a) We will show that the terms in the series do not tend to zero and \( n \) tends to infinity. We have

\[
\frac{3^n + 1}{2^n - 1} > \frac{3^n}{2^n} = \left( \frac{3}{2} \right)^n
\]
so
\[
\lim_{n \to \infty} \frac{3^n + 1}{2^n - 1} > \lim_{n \to \infty} \left(\frac{3}{2}\right)^n = \infty.
\]
hence the series diverges by the Divergence Test.

(b) We will use the Integral Test and show that the improper integral
\[
\int_2^\infty \frac{dx}{x \ln(x)^3}
\]
converges. We will use the substitution
\[
u = \ln(x) \quad du = \frac{dx}{x},
\]
which gives
\[
\int_2^\infty \frac{dx}{x \ln(x)^3} = \left. \frac{1}{2u^2} \right|_\ln 2 = \frac{1}{2(\ln 2)^2}.
\]
This the integral; converges, so the series does.

4. (20 points)

(a) Find the area of one petal of the polar rose \(r = 2 \cos(4\theta)\) pictured below.

\[\text{Solution:}\] We need to find consecutive zeros of \(r = 2 \cos(4\theta)\). These will give the limits of integration, because the petal will close for those \(\theta\) values. If \(0 = 2 \cos(4\theta)\), then \(4\theta = \pi/2, 3\pi/2\), so \(\theta = \pi/8, 3\pi/8\) are the limits of integration.
Area \[= \frac{1}{2} \int_{\pi/8}^{3\pi/8} 4 \cos^2(4\theta) d\theta \]
\[= 2 \int_{\pi/8}^{3\pi/8} \frac{1 + \cos(8\theta)}{2} d\theta \]
\[= \theta + \frac{\sin(8\theta)}{8} \bigg|_{\pi/8}^{3\pi/8} \]
\[= \pi/4 \]

Other correct integrals: \[\frac{1}{8\pi} \int_{0}^{2\pi} 4 \cos^2(4\theta) d\theta, \frac{1}{2} \int_{\pi/8}^{3\pi/8} 4 \cos^2(4\theta) d\theta, \frac{1}{2} \int_{3\pi/8}^{5\pi/8} 4 \cos^2(4\theta) d\theta, \text{ etc.}\]
(b) The parametric curve given by $x = 4t^3 - 3t$, $y = t^2 + 1$ intersects the $y$-axis at 3 different values of $t$. What are the **equations of the tangent lines** to the curve at each of these points?

**Solution:** Solve $x = 4t^3 - 3t = 0$ to get $t = 0, \pm \frac{\sqrt{3}}{2}$. We have

$$\frac{dy}{dx} = \frac{2t}{12t^2 - 3}.$$

At $t = 0$, the tangent is horizontal with $y$-intercept $y = 1$, so we get $y = 1$. At $t = \frac{\sqrt{3}}{2}$, at $(x, y) = (0, 1 + 3/4)$, the tangent has slope $\frac{\sqrt{3}}{6}$. At $t = -\frac{\sqrt{3}}{2}$, also at $(x, y) = (0, 1 + 3/4)$, the tangent has slope $-\frac{\sqrt{3}}{6}$. So the two lines are

$$y = \pm \frac{\sqrt{3}}{6} x + 7/4.$$
5. (20 points) Find the sum of each of the following series.

(a) \[
\sum_{n=2}^{\infty} \frac{2}{n^2 - 1} = \frac{2}{3} + \frac{2}{8} + \frac{2}{15} + \frac{2}{24} + \frac{2}{35} + \cdots
\]

**Hint:** Use partial fractions.

(b) \[
\sum_{n=0}^{\infty} \left(\frac{1}{6 + (-1)^n}\right)^n = 1 + \frac{1}{5} + \frac{1}{7^2} + \frac{1}{5^3} + \frac{1}{7^4} + \cdots
\]

**Hint:** Consider the evenly and oddly indexed terms separately.

**Solution:** (a) Using partial fractions we find that
\[
\frac{2}{n^2 - 1} = \frac{1}{n - 1} - \frac{1}{n + 1}.
\]

Hence we can rewrite the series as
\[
\sum_{n=2}^{\infty} \frac{2}{n^2 - 1} = \sum_{n=2}^{\infty} \left(\frac{1}{n - 1} - \frac{1}{n + 1}\right)
= \left(\frac{1}{1} - \frac{1}{3}\right) + \left(\frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{1}{5}\right) + \left(\frac{1}{4} - \frac{1}{6}\right) + \left(\frac{1}{5} - \frac{1}{7}\right) + \cdots.
\]

All the negative terms are cancelled by subsequent positive ones, leaving only the first two positive terms. Hence the sum is 3/2.

(b) Collecting the evenly and oddly indexed terms into separate series, we get
\[
\sum_{n=0}^{\infty} \left(\frac{1}{6 + (-1)^n}\right)^n = 1 + \frac{1}{5} + \frac{1}{7^2} + \frac{1}{5^3} + \frac{1}{7^4} + \cdots
= \left(1 + \frac{1}{7^2} + \frac{1}{7^4} + \cdots\right) + \left(\frac{1}{5} + \frac{1}{5^3} + \frac{1}{5^5} + \cdots\right)
= \left(1 + \frac{1}{7^2} + \frac{1}{7^4} + \cdots\right) + \frac{1}{5} \left(1 + \frac{1}{5^2} + \frac{1}{5^4} + \cdots\right).
\]

This is the sum of two geometric series, namely
\[
\sum_{m=0}^{\infty} \frac{1}{49^m} + \frac{1}{5} \sum_{m=0}^{\infty} \frac{1}{25^m} = \frac{1}{1 - (1/49)} + \frac{1}{5} \left(\frac{1}{1 - (1/25)}\right)
= \frac{49}{48} \cdot \frac{25}{5 \cdot 24} = \frac{5 \cdot 49 + 2 \cdot 25}{5 \cdot 48} = \frac{295}{240} = \frac{59}{48}.
\]