1. (11 points)

(a) Given 12 people, how many ways can we divide them into 3 teams with 4 players each, called the Cardinals, the Bluejays, and the Vultures?

(b) What is the number of ways if we don’t distinguish the three teams, and just divide the 12 people into 3 groups of 4 people each?

Answer:

(a) The answer is given by the multinomial coefficient
\[
\binom{12}{4, 4, 4} = \frac{12!}{(4!)^3}
\]

(b) We would divide the answer in part (a) by 3!, since the order of teams doesn’t matter:
\[
\frac{1}{3!} \cdot \binom{12}{4, 4, 4} = \frac{12!}{3! \cdot (4!)^3}
\]

2. (12 points) Suppose the probability that a tourist has visited Europe is 2/3. If a tourist has visited Europe, then the probability that the tourist went to Paris is 1/2. Lastly, the probability that a tourist visited Paris but didn’t go to the Louvre is 1/4. What is the probability that a tourist has visited the Louvre? (The Louvre is a famous museum that has the Mona Lisa and other well known paintings.)

Answer:

Let \( E \) be the event that a tourist visited Europe, \( F \) the event that a tourist visited Paris and \( G \) the event that a tourist visited the Louvre. Since everyone who has visited the Louvre has been to Paris, and everyone who has been to Paris has been to Europe, we have \( G \subset F \subset E \), thus \( G = E \cap F \cap G \). It follows that
\[
P(G) = P(EFG) = P(E)P(F|E)P(G|EF) = \frac{2}{3} \cdot \frac{1}{2} \cdot \left(1 - \frac{1}{4}\right) = \frac{1}{4}.
\]
3. **(11 points)** Suppose the devil forces you to flip a biased coin with probability $3/4$ of tails. If it does land on tails, then your probability of going to heaven is only $1/6$. But if it lands on heads, your probability of going to heaven is $2/3$. Later you find yourself at the pearly gates which lead to heaven, but you’ve forgotten all details of earthly life except for the material in Math 201 (and the rules of the devil’s game). What is the probability that the coin landed on heads?

**Answer:**

Let $V$ be the event that the you go to heaven, and let $H$ be the event that the coin lands on heads. We are looking for $P(H|V)$. Bayes’ formula tells us that

$$P(H|V) = \frac{P(V|H)P(H)}{P(V)}$$

We know:

$$P(H) = \frac{1}{4} \quad P(H^c) = \frac{3}{4}$$

$$P(V|H) = \frac{2}{3} \quad P(V|H^c) = \frac{1}{6}$$

and we can compute

$$P(V) = P(V|H)P(H) + P(V|H^c)P(H^c) = \frac{2}{3} \cdot \frac{1}{4} + \frac{1}{6} \cdot \frac{3}{4} = \frac{7}{24}$$

So finally,

$$P(H|V) = \frac{P(V|H)P(H)}{P(V)} = \frac{(2/3)(1/4)}{(7/24)} = \frac{4}{7}$$

4. **(11 points)** For events $A, B$, prove that

$$1 - P(A \cap B) = P(A^c \cup B^c)$$

Be sure to give a symbolic argument and not just a Venn diagram.

**Answer:**

Recall from De Morgan’s laws, that

$$(A \cap B)^c = A^c \cup B^c$$

Since $1 - P(E) = P(E^c)$, and setting $E = A \cap B$, we get

$$1 - P(A \cap B) = P((A \cap B)^c) = P(A^c \cup B^c)$$
5. **(11 points)** A perfect nail produced in a factory has length 1.5 inches and diameter .25 inches. The length of a nail is called “bad” if it is more than .05 inches larger or smaller than 1.5 inches, and “good” otherwise. The diameter is bad if it is more than .02 inches larger or smaller than .25 inches, and good otherwise. After examining a sample of 1000 nails, an inspector reports that the probability that a randomly selected nail has a bad length is .2, the probability that the diameter is bad is .25, and the probability that both the length and diameter are good is .8. After reviewing the report, you conclude that the inspector must have made a mistake. Why?

**Answer:**

Many ways to answer this. The easiest: only .75 of the nails had a good diameter. Call \(D\) the event of having a good diameter, and \(L\) the event of having a good length. So \(P(D) = .75\). But \(DL \subseteq D\), so \(P(DL) \leq .75\). But \(.8 > .75\).

6. **(11 points)** There are 8 balls in an urn, of which 2 are black, 2 are red, 2 are blue, and 2 are yellow. You draw 3 balls at random from the urn without replacement. What is the probability that two of them are the same color?

**Answer:**

There are \(\binom{8}{3} = 56\) ways to draw 3 balls from the urn. The number of ways to draw 3 balls and have two of them be the same color is equal to the number of ways to pick a color for the repeated ball (4 possibilities) and then pick a third ball (6 possibilities). So the probability of two balls being the same color is \(\frac{(4)(6)}{56} = \frac{3}{7}\).

7. **(11 points)** A certain website has the following restriction on passwords: a password must start with three letters, either upper or lower case, followed by a two digit number (i.e. a number bigger than 9 and less than 100). How many different passwords are possible?

**Answer:**

Since there are 26 letters and the first three characters of a password can be either upper or lower case, there are 52 choices for each letter. By the principle of counting, the number of choices for the three letters is \(52^3\). Regardless of the choices of the letters, there will be 90 choices for the number-portion of the password. Using the generalised basic principle of counting we obtain that there are \(52^3 \times 90\) different passwords.

Alternatively, we could argue that regardless of the choice of the first 3 letters, we will have 9 choices for the first digit and 10 choices for the last one, so there are \(52^3 \times 9 \times 10\) different
passwords.

8. (11 points)

(a) There are six runners in a race, numbered 1, 2, 3, 4, 5, and 6. You observe that runner 6 finishes before runner 5, but do not know anything else about the order in which the runners finish. How many possible orders of finishing are there for the six runners?

(b) In addition to observing runner 6 finish before runner 5, suppose that someone tells you that runner 2 finished before runner 1, and runner 1 finished before runner 3. How many possible orders of finishing are there given this new information?

Answer:

There are 6! orderings of the 6 runners with no restrictions. There are 2! ways of swapping runners 6 and 5 for any given ordering, so there are 6!/2 ways answers to (a). Likewise, there are 3! ways of rearranging runners 2, 1, and 3. So there are

$$\frac{6!}{2!3!} = 60$$

allowed orderings of the runners in part (b).

9. (11 points) Suppose we have an irregular 6 sided die such that when thrown, any even number is twice as likely to appear as any odd number. What is the probability that we get a 3 when we throw this die?

Answer:

Let a be the probability of throwing a 1. Then we must have $P(\{1\}) = P(\{3\}) = P(\{5\}) = a$ and $P(\{2\}) = P(\{4\}) = P(\{6\}) = 2a$. Since the sets $\{1\}$, $\{2\}$, $\{3\}$, $\{4\}$, $\{5\}$, $\{6\}$ are mutually disjoint, we have $P(\bigcup_{i=1}^{6} \{i\}) = \sum_{i=1}^{6} P(\{i\}) = 9a$. Since $\bigcup_{i=1}^{6} \{i\} = \{1, 2, 3, 4, 5, 6\}$ is the full sample space, we must have $P(\bigcup_{i=1}^{6} \{i\}) = 1$. Thus $9a = 1$ and $P(\{3\}) = a = \frac{1}{9}$. 