MATH 201

PRACTICE FINAL EXAM PART B

Dec 9, 2012

NAME (please print legibly): ____________________________________________
Student ID Number: __________________________________________________

• No notes or electronic devices other than watches are allowed on this exam.

• Please show all your work. You may use back pages if necessary. You may not receive full credit for a correct answer if there is no work shown. Attempt all questions.

• Please circle your Instructor: Pakianathan (MWF 12-12:50), Petridis (MW 3:25-4:40)

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<th>Part-A</th>
<th>QUESTION</th>
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Part-A
Part-B

- In the actual final you will be given a copy of the table of p.201 in the book.
- You will also be given the list below with three mass functions and three density functions.
- You will not be able to do all the questions until after the last day of class.

Useful Formulas

Probability mass functions of discrete random variables

- Geometric with parameter $p$: $\Pr(X = k) = (1 - p)^{k-1}p$.

- Negative binomial with parameters $p$ and $r$: $\Pr(X = k) = \binom{k-1}{r-1} p^r (1 - p)^{k-r}$.

- Hypergeometric with parameters $N$, $m$ and $n$: $\Pr(X = k) = \binom{m}{k} \binom{N-m}{n-k} \binom{N}{n}$.

Probability density functions of continuous random variables

- Exponential with rate $\lambda$: $f(t) = \lambda e^{-\lambda t}$ for $t \geq 0$ and zero otherwise.

- Normal with mean $\mu$ and variance $\sigma^2$: $f(x) = \frac{1}{\sqrt{2\pi} \sigma} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$ for $x \in \mathbb{R}$.

- Gamma with parameters $\alpha$ and $\lambda$:

  $$f(t) = \frac{\lambda e^{-\lambda t} (\lambda t)^{\alpha-1}}{\Gamma(\alpha)} \text{ for } t \geq 0$$

  and zero otherwise. $\Gamma(\alpha) = \int_0^\infty e^{-y} y^{\alpha-1} dy$ for all $\alpha > 0$. 

3
1. (11 pts) Answer the following questions in the space provided. $X$ and $Y$ will always be random variables. Each of the following 4 questions is different and the variables in each part are not related to the variables in the other parts.

1. (3 pts) If $X$ is a normal random variable with mean 2 and variance 3 and $Y$ is a normal random variable with mean 5 and variance 12, and $X$ and $Y$ are independent, then $X+Y$ is a __________________________ random variable with mean __________ and variance __________.

2. (3pts) If $X \sim \chi^2_n$ is chi-squared with parameter $n$, then its mean is: ________________ .
   It is a Gamma random variable with $\alpha = $ ________________ and $\lambda = $ ________________.

3. (3 pts) If $X_1, \ldots, X_n$ are independent identically distributed random variables with mean $\mu$ and variance $\sigma^2$, then the formula for the sample mean is $\bar{X} := $ __________________________
   and $\bar{X}$ has mean equal to ________________ and has variance equal to ________________.

4. (2pts) If $X+Y = 10$, then the value of the correlation of $X$ and $Y$ is ________________.
   Your answer should be a number.
2. (8 pts) You have two sons: a 17 year old and an 18 year old. Statistical evidence shows that the height of an 18 year old male resident of Rochester can be approximated by a normal random variable with mean 6 feet and variance 1/4 foot. Using the table in p. 201 of the book answer the following questions. [In the exam you will be given a copy of the table.]

(a) (4pts) Find the probability that your younger son will be taller than 5 feet when he turns 18. Your answer should be given to two decimal places.

(b) (4 pts) A friend of yours, who is a statistician, asks how tall your older son is and you reply that he is taller than exactly 85% of 18 year old male Rochesterians. What should her guess for his height be? Your answer should be given in feet (not inches) and be accurate to one decimal place.
3. (8 pts) Last year you bought a light bulb that was advertised to last three years on average. Assuming that the lifetime of the bulb is an exponential random variable, what is the probability that it lasts at least another year? Explain your reasoning carefully, clearly stating any properties of the exponential random variable that you use. Your answer should be a number, though you don’t have to evaluate it.
4. (15 pts) The pair of random variables \((X,Y)\) has joint distribution

\[
f(x,y) = \begin{cases} 
2e^{-\frac{x}{2} - 4y} , & 0 \leq x, y < \infty \\
0, & \text{otherwise}
\end{cases}.
\]

(a) (3pts) Find the marginal distribution of \(X\). Show all your work.

\[f_X(x) = \]

(b) (3pts) Find the marginal distribution of \(Y\). Show all your work.

\[f_Y(y) = \]
(c) (2pts) Are \( X \) and \( Y \) independent? Briefly justify your answer.

(d) (7pts) Find the probability density function of the random variable \( Z = X + Y \). Show all your work.

\[ f_Z(z) = \]
5. (10 pts) The Green Line bus is scheduled to leave from the bus stop near your house at 8 am every weekday. In practice this means that it arrives at a time uniformly distributed in the time interval [7:58 am, 8:02 am]. You don’t want to either miss the bus or wait too long, so you try to be there just before it arrives. In practice this means that you arrive at a time uniformly distributed in the time interval [7:56 am, 8:00 am]. What is the probability that you arrive at the bus stop before the bus?
6. (16 pts) The pair of random variables \((X, Y)\) has joint distribution

\[
f(x, y) = \begin{cases} \frac{1}{2}e^{-x}, & 0 \leq x < \infty, -x < y < x \\ 0, & \text{otherwise} \end{cases}
\]

(a) (5 pts) Find the expected value of the random variable \(XY\). Show all your work.

\[
\mathbb{E}[XY] =
\]

(b) (4 pts) Find an expression for the marginal distribution of \(Y\). Show all your work.

\[
f_Y(y) =
\]
(c) (5 pts) Find the expected value of $Y$. Show all your work.

\[ \mathbb{E}[Y] = \]

(d) (2 pts) Calculate the covariance of $X$ and $Y$. Show all your work.

\[ \text{Cov}(X, Y) = \]
7. (10 pts) In this question it is assumed that the probability a family has a boy is the same as the probability it has a girl. You are told that a certain family has four children, but are given no other information. If \( X \) is the random variable that counts the number of boys and \( Y \) is the random variable that counts the number of girls calculate the covariance of \( X \) and \( Y \) showing all your work and stating the properties you may use.

\[
\text{Cov}(X, Y) =
\]
8. (10 pts) (a) (3 pts) State carefully the Central Limit Theorem for a sequence of independent identically distributed random variables $X_1, X_2, \ldots$ with mean $\mu$ and variance $\sigma^2$.

(b) (7 pts) Use the Central Limit Theorem and the table on p. 201 of the book to estimate the probability that when a coin is flipped $n$ times one gets between $n/2 - \sqrt{n}$ and $n/2 + \sqrt{n}$ heads. Here $n$ is assumed to be a large integer. Your answer should be accurate to two decimal places. Do not perform the "continuity correction" discussed in the book as the effect is small and will make your answer messy in this situation.
9. **(12 pts)** (a) (3 pts) State Markov’s inequality for a nonnegative random variable \(X \geq 0\) with mean \(\mu\) and a positive constant \(a > 0\).

\[
\Pr(X \geq a) \leq
\]

(b) (3pts) State Chebyshev’s inequality for a nonnegative random variable \(X \geq 0\) with mean \(\mu\) and variance \(\sigma^2\) and a positive constant \(a > 0\).

\[
\Pr(|X - \mu| \geq a) \leq
\]

You are told that during rush hour a metro comes in on London’s Victoria line every two minutes. Find an upper bound on the probability that you will have to wait longer than four minutes when:

(c) (3pts) No other information is available.

\[
\Pr(\text{wait longer than four minutes}) \leq
\]

(d) (3pts) It is known that the waiting time is a random variable with variance 1.

\[
\Pr(\text{wait longer than four minutes}) \leq
\]