Probability 201– Solutions for Midterm 1 of Fall 2004

Question 1

(a) The answer is 13. Here are the two most natural ways to compute the number of permissible committees. Let $A$ and $B$ be the faculty who don’t like each other.

(I) The number of permissible committees equals the number of all possible committees minus the number of committees where both $A$ and $B$ are members. The number of all possible committees is $\binom{8}{6} = \binom{8}{2} = 28$ and the number of committees with both $A$ and $B$ as members is $\binom{6}{4} = \binom{6}{2} = 15$, as in addition to $A$ and $B$ one must chose 4 committee members from the remaining 6 faculty. Subtracting gives $28-15=13$.

(II) The number of permissible committees equals the number of committees where neither $A$ nor $B$ are members plus the number of committees where only $A$ is a member plus the number of committees where only $B$ is a member. The first summand is $\binom{6}{6} = 1$ as one has to chose 6 committee members from the 6 faculty that are not $A$ nor $B$; the second summand is $\binom{6}{5} = \binom{6}{1} = 6$ as in addition to having $A$ the committee must have 5 members from the 6 faculty who are neither $A$ nor $B$; similarly there are 6 committees that have only $B$ as a member. Adding everything together gives $1+6+6=13$.

Example 4b in p.6 is similar.

(b) Not in the material for Midterm 1.

Question 2

(a) $A$ is Anna and $B$ is Bob.

(i) The answer is $1/4$. We consider ordered seating arrangements, which are permutations of the people present at the party. There are $8!$ such permutations (the first person can sit in 8 places, the second in 7 and so on). To calculate the probability that $A$ and $B$ sit next to each other we only need to count the number of seating arrangements in which they sit next to each other and divide by $8!$. Here are two ways to answer the question.

(I) Think of $A$ and $B$ as a single entity that occupies two consecutive seats. This means that we have 7 entities we would like to sit in a row ($A&B$ and the remaining 6 people). There are $7!$ ways to do this (the first entity has 7 places to sit, the second 6 and so on). Once the order of the entities has been decided one needs to decide in which order $A$ and $B$ sit in the two consecutive seats. There are 2 (or 2! if you prefer) ways to sit them and hence by
the fundamental principle of counting $2 \cdot 7!$ ways for $A$ and $B$ to sit next to each other. The probability they sit next to each other is therefore $2 \cdot 7!/8! = 2/8 = 1/4$.

(II) The number of ways that $A$ and $B$ can sit down next to each other equals the number of ways they can sit next to each other when $A$ sits at the end points of the row plus the number of ways they can sit next to each other when $A$ does not sit at the end points of the row. The first summand is $2 \cdot 6!$, as there are 2 possible positions for $A$ and $B$ to sit (either $AB$ at the beginning or $BA$ at the end) and $6!$ ways for the remaining 6 guests to sit in a row. The second summand is $6 \cdot 2 \cdot 6!$ as there are 6 places for $A$ to sit, given her place there are 2 places for $B$ to sit and $6!$ ways for the remaining 6 guests to sit in a row. Adding everything gives $(2 + 6 \cdot 2) \cdot 6! = 14 \cdot 6! = 2 \cdot 7!$ and so the probability they sit next to each other is $2 \cdot 7!/8! = 2/8 = 1/4$.

(b) The answer this time is $2/7$. Here are two ways to compute the probability.

(I) In a round table $A$ must have two neighbours. Because of the overall symmetry each of the other guests is equally likely to be her left or her right neighbour. So the probability that $B$ is her left (or right) neighbour is $1/7$. So

$$P(B \text{ sits next to } A) = P(B \text{ sits to the right of } A) + P(B \text{ sits to the left of } A) = 1/7 + 1/7 = 2/7.$$ 

(II) We work with cyclical orderings of 8 guests. There are $7!$ such orderings as where the first guest sits does not matter, the second guest can sit in any of the remaining 7 seats, the third in any of the remaining 6 seats and so on. So to get the probability that $A$ sits next to $B$ we only need to compute the number of cyclic seating arrangements in which $A$ and $B$ sit next to each other and divide by $7!$. Once $A$ has been sat there are 2 possible seats for $B$ and $6!$ ways for the remaining guests to sit (note here that the remaining guests essentially sit in a row as $A$ and $B$ define the end points). Thus the probability is $2 \cdot 6!/7! = 2/7$.

(b) The answer is $2/47$. The key is to observe that the only way you end up with 4 of a kind is to end up with 4 aces. Let us quickly recall some facts: there are 52 cards, 4 of which are aces and 48 of which are other cards; there are 5 cards in your hand, of which 3 are aces and 2 are other cards; and there are 47 cards remaining, of which 1 is an ace and 46 are other cards.

To end up with 4 aces you need to select a pair of cards that includes the lone ace. There are 46 such pairs. The total number of pairs you can select is $\binom{47}{2}$ and so the probability you end up with 4 aces is $46/\binom{47}{2} = 2/47$. 

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Question 3

(a) The first percentage is about 41 and the second is about 56.

Let’s set some notation. $F$ is the event that a member of the class is female, $M$ is the event that a member of the class is male and $G$ the event that a member of the class is a graduate.

The question gives us the following information: $P(M) = 0.62$, which implies $P(F) = 1 - P(M) = 0.38$; $P(G|F) = 0.48$ and $P(G|M) = 0.37$.

By formula (3.1) on p. 65 we have

$$P(G) = P(G|F) \cdot P(F) + P(G|M) \cdot P(M)$$

$$= 0.48 \cdot 0.38 + 0.37 \cdot 0.62 = 0.4118.$$ 

So the percentage of graduates is about 41.

We are also asked to compute $P(M|G)$.

$$P(M|G) = \frac{P(MG)}{P(G)}$$

$$= \frac{P(G|M) \cdot P(M)}{P(G)}$$

$$= \frac{0.37 \cdot 0.62}{0.4118} = \frac{0.2294}{0.4118} \simeq 0.557.$$ 

So the percentage of graduates that is male is about 56.

[Aside: A quick way to get an estimate for the probability $P(G)$ is to approximate both 0.38 and 0.37 by 0.375. Then $P(G)$ is about 0.375 \cdot (0.48 + 0.62) = 0.375 \cdot 1.1 = 0.4125 \simeq 0.41$. A quick way to estimate the probability $P(M|G)$ is to note from (a slight variation of the) above that $P(G) \simeq 0.37 \cdot 1.1$. We also have $P(MG) = 0.37 \cdot 0.62$ so $P(M|G) \simeq 0.62/1.1 = 6.2 \cdot 1/11 \simeq 6.2 \cdot 0.09 = 0.558 \simeq 0.56$.]

(b) There have to be 12 white round blocks.

Let $B$ be the event that a block is black, $W$ the event that a block is white, $S$ the event that a block is square and $R$ the event that a block is round. We are given the following information: $|BS| = 5, |WS| = 6$ and $|RB| = 10$. Let $x = |WR|$ be the quantity we are asked to determine.

Let’s make some quick deductions from the data. The total number of blocks is $5 + 6 + 10 + x = 21 + x$, $|B| = |BS| + |BR| = 5 + 10 = 15$ and $|S| = |SB| + |SW| = 5 + 6 = 11$.

We want to find $x$ so that $B$ and $S$ are independent events. We will then be done by Proposition 4.1 on p.81 as $W = B^c$ and $R = W^c$. 

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It is therefore enough to find the value of $x$ that ensures that $P(BS) = P(B) \cdot P(S)$, which is equivalent to

$$\frac{|BS|}{\text{total number of blocks}} = \frac{|B|}{\text{total number of blocks}} \cdot \frac{|S|}{\text{total number of blocks}}.$$ 

After some rearranging we get $(21 + x)|BS| = |B| \cdot |S|$, which gives $(21 + x)^5 = 15 \cdot 11$ and finally $x = 33 - 21 = 12$.

**Question 4**

(a) Not in the material for Midterm 1.

(b) The probability $P$ is $1.9 \cdot 0.9^9$, which is about 0.74.

This is because

$$P(\text{package passes federal requirements}) = P(\text{package has no defective disks}) + P(\text{package has exactly one defective disk}).$$

A package has no defective disks if all its 10 disks function. Each disk functions with probability $1 - 0.1 = 0.9$ and so because of the independence the probability a package has no defective disks is $0.9^{10}$.

A package has exactly one defective disk with probability $10 \cdot 0.1 \cdot 0.9^9$ as there are 10 choices for the defective disk, the probability this particular disk is defective is 0.1 and the probability the remaining 9 disks function is $0.9^9$. So

$$P(\text{package passes federal requirements}) = 0.9^{10} + 10 \cdot 0.1 \cdot 0.9^9 = 0.9^9(0.9 + 1) = 1.9 \cdot 0.9^9 \simeq 0.74.$$  

[Aside: A quick way to get an estimate for the quantity $1.9 \cdot 0.9^9$ is to write $0.9^9 = (1 - 0.1)^9$ and use the first four terms of the binomial theorem: $(1 - 0.1)^9 \simeq 1 - \binom{9}{1}0.1 + \binom{9}{2}0.1^2 - \binom{9}{3}0.1^3 = 1 - 0.9 + 0.36 - 0.072 \simeq 0.39$. Then $1.9 \cdot 0.9^9 \simeq 1.9 \cdot 0.39 \simeq 0.74$.]

(c) The answer is $3 \cdot (1 - P) \cdot P^2$.

There are 3 distinct sequences of three packages, two of which meet federal requirements and one that doesn’t. Each of these package sequences appears with probability $(1 - P) \cdot P \cdot P$ because of the independence.

Example 4f in p.82 is similar to both parts (b) and (c).

(d) Not in the material for Midterm 1.