1. (18 points)
Suppose that $X$ is a random variable such that

$$X = \begin{cases} 
5 & \text{with probability } \frac{1}{4} \\
2 & \text{with probability } \frac{1}{2} \\
-1 & \text{with probability } \frac{1}{4}
\end{cases}$$

(a) Find $EX$.
(b) Find $E[X^2]$.
(c) Find $\text{Var}(X)$.

Solution:
(a) Since the expectation is a sum of the outcomes, weighted by the probabilities, we get

$$EX = 5 \cdot \frac{1}{4} + 2 \cdot \frac{1}{2} - 1 \cdot \frac{1}{4} = 2$$

(b) $E[X^2] = 5^2 \cdot \frac{1}{4} + 2^2 \cdot \frac{1}{2} + (-1)^2 \cdot \frac{1}{4} = \frac{25}{4} + \frac{4}{2} + \frac{1}{4} = \frac{17}{2}$

(c) Using parts (a) and (b), we get

$$\text{Var}(X) = E[X^2] - (EX)^2 = \frac{17}{2} - 2^2 = \frac{9}{2}$$

2. (20 points) Suppose that interest rates are not constant. Today is the beginning of the year, and you know that

(i) You can buy or sell an annuity paying $20,000 at the end of each year, ending after 10 payments, for $V_1 = $180,000 (today’s value).

(ii) You can buy or sell an annuity paying $10,000 at the end of each 6-month period, ending after 20 payments, for $V_2 = $190,000 (today’s value).

What is the price (today) of an annuity paying $30,000 at the end of each odd-numbered 6-month period, that is, at the end of the 1st, 3rd, 5th, and so on, and ending after 10 years? Justify your answer.

Solution: We can use the first two annuities to replicate the third one. To be specific, we can buy 3 shares of the second annuity, and short sell 3/2 shares of the first annuity. The price would be

$$3 \cdot V_2 - \frac{3}{2} \cdot V_1 = 3 \times $190,000 - \frac{3}{2} \times $180,000$$

$$= $570,000 - $270,000$$

$$= $300,000$$
3. (20 points) Suppose that $t$ is the current time, and you enter a contract which costs $V(t)$ at the current time. But at a future time $T$, you must exchange a share of stock $A$ for two shares of stock $B$. Call the prices of the two stocks at time $t$ as $S^A_t$ and $S^B_t$ respectively. Find $V(t)$ in terms of information that would be known at time $t$,

(a) assuming that neither stock pays income.
(b) assuming that stock $A$ pays no income, but stock $B$ pays income $I^B$ per share at time $T_1$, assuming $t < T_1 < T$.

Solution:
(a) We use the replicating portfolio method.

<table>
<thead>
<tr>
<th>Portfolio A</th>
<th>Asset</th>
<th>Value at $t$</th>
<th>Value at $T$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 long share contract</td>
<td>$V(t)$</td>
<td>$2S^B_T - S^A_t$</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Portfolio B</th>
<th>Asset</th>
<th>Value at $t$</th>
<th>Value at $T$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2 shares stock B</td>
<td>$2S^B_t$</td>
<td>$2S^B_T$</td>
<td></td>
</tr>
<tr>
<td>-1 shares stock A</td>
<td>$-S^A_t$</td>
<td>$-S^A_T$</td>
<td></td>
</tr>
<tr>
<td>total</td>
<td>$2S^B_t - S^A_t$</td>
<td>$2S^B_T - S^A_T$</td>
<td></td>
</tr>
</tbody>
</table>

By the replicating portfolio principle, we find

$$V(t) = 2S^B_t - S^A_t$$

(b) We again use the replicating portfolio method. At time $T_1$, we use a negative ZCB payment to cancel out the income from stock $B$.

<table>
<thead>
<tr>
<th>Portfolio A</th>
<th>Asset</th>
<th>Value at $t$</th>
<th>Value at $T$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 long share contract</td>
<td>$V(t)$</td>
<td>$2S^B_T - S^A_t$</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Portfolio B</th>
<th>Asset</th>
<th>Value at $t$</th>
<th>Value at $T_1$</th>
<th>Value at $T$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2 shares stock B</td>
<td>$2S^B_t$</td>
<td>$2S^B_{T_1} + 2I^B$</td>
<td>$2S^B_T$</td>
<td></td>
</tr>
<tr>
<td>-2$I^B$ shares ZCB, maturity $T_1$</td>
<td>$-2I^B Z(t,T_1)$</td>
<td>$-2I^B$</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>-1 shares stock A</td>
<td>$-S^A_t$</td>
<td>$-S^A_{T_1}$</td>
<td>$-S^A_T$</td>
<td></td>
</tr>
<tr>
<td>total</td>
<td>$2S^B_t - S^A_t$</td>
<td>$2S^B_{T_1} - S^A_{T_1}$</td>
<td>$2S^B_T - S^A_T$</td>
<td></td>
</tr>
</tbody>
</table>

By the replicating portfolio principle, we find

$$V(t) = 2S^B_t - S^A_t - 2I^B Z(t,T_1)$$

4. (21 points) Recall that we discussed forward currency contracts, assuming a fixed dollar interest rate $r_d$ and a euro interest rate $r_e$. Both rates are for continuous compounding. As usual, $t$ is the current time and $T$ is some future time. We discussed a forward contract on one euro, with maturity $T$. 
Assuming that the contract has price 0 at the current time \( t \), we found that the forward price (in dollars) for one euro is

\[
F(t, T) = X_t e^{(r_d - r_e)(T-t)}.
\]

Suppose that instead of constant interest rates \( r_d, r_e \) we have zero coupon bond prices \( Z_d(t, T) \), \( Z_e(t, T) \). Find a new formula for \( F(t, T) \) and justify it using the portfolio replication principle.

**Solution:** We use the replication principle. Note that at the current time \( t \), one euro ZCB costs \( Z_e(t, T) \) euros, or \( Z_e(t, T)X_t \) dollars.

<table>
<thead>
<tr>
<th>Portfolio A</th>
<th>Value at ( t )</th>
<th>Value at ( T )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 long share contract</td>
<td>0</td>
<td>1 euro -( K ) dollars</td>
</tr>
<tr>
<td>( K ) shares ZCB in dollars, maturity ( T )</td>
<td>( KZ_d(t, T) ) dollars</td>
<td>( K ) dollars</td>
</tr>
<tr>
<td>total</td>
<td>( KZ_d(t, T) ) dollars</td>
<td>1 euro</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Portfolio B</th>
<th>Value at ( t )</th>
<th>Value at ( T )</th>
</tr>
</thead>
<tbody>
<tr>
<td>One euro ZCB</td>
<td>( Z_e(t, T)X_t ) dollars</td>
<td>1 euro</td>
</tr>
</tbody>
</table>

By the replication principle,

\[
KZ_d(t, T) = Z_e(t, T)X_t
\]

and so

\[
F(t, T) = K = \frac{Z_e(t, T)}{Z_d(t, T)}X_t.
\]

5. (21 points) In problem 4, we wrote \( F(t, T) \) for the forward price (in dollars) at the current time \( t \), of one euro at time \( T \). Suppose we are thinking of a swap, and we have times \( t < T_0 < T_1 < \cdots < T_n \). Consider a swap at rate \( K \), where at each time \( T_1, \ldots, T_n \), you pay \( K \) dollars and receive 1 euro. Note that the time \( T_0 \) is not included.

In the following, express your answers in terms of \( K \), \( Z_e(t, T) \) and \( Z_d(t, T) \). You can also use \( F(t, T) \) without giving a formula for it, to avoid carrying over errors from problem 4. Remember that your answers must only use information known at the current time \( t \). You may express your answers in terms of a sum.

(a) The dollar leg is the stream of dollar payments you make. Find the value \( V^d(t) \) of the dollar leg at time \( t \).

(b) Find the value \( V^e(t) \) of the Euro leg at time \( t \), that is, the stream of euros that you receive. The value must be given in dollars.

(c) If the contract is worth 0 at time \( t \), find the swap rate \( K \).

**Solution:**

(a) Using zero coupon bond prices, we find

\[
V^d(t) = \sum_{i=1}^{n} KZ_d(t, T_i)
\]
(b) One euro received at time $T$ can be priced at time $t$ using the forward rate $F(t, T)$. These prices must also be brought into present value by multiplication by $Z(t, T_i)$. So we get

$$V^e(t) = \sum_{i=1}^{n} F(t, T_i) Z_d(t, T_i)$$

We could also use a euro ZCB, and then convert to dollars. That would give

$$V^e(t) = X_t \sum_{i=1}^{n} Z_e(t, T_i)$$

which is the same answer if we use our solution for $F(t, T)$.

(c) If the contract is worth 0 at time $t$, the value of the two legs must be equal at time $t$. We conclude that

$$\sum_{i=1}^{n} K Z_d(t, T_i) = \sum_{i=1}^{n} F(t, T_i) Z_d(t, T_i)$$

and so

$$K = \frac{\sum_{i=1}^{n} F(t, T_i) Z_d(t, T_i)}{\sum_{i=1}^{n} Z_d(t, T_i)}$$

Using the second method from part (b) would give us

$$K = X_t \frac{\sum_{i=1}^{n} Z_e(t, T_i)}{\sum_{i=1}^{n} Z_d(t, T_i)}$$

These two answers are the same, if we expand $F(t, T)$. 