Math 282: Complex Analysis

Final
May 6, 2015

NAME (please print legibly): ________________________________
Your University ID Number: ________________________________

• The presence of calculators, cell phones, iPods and other electronic devices at this exam is strictly forbidden.

• Show your work and justify your answers. You may not receive full credit for a correct answer if insufficient work is shown or insufficient justification is given.

• Clearly circle or label your final answers, on those questions for which it is appropriate.

• Scratch paper is provided following the last page.
1. (10 points) Evaluate

\[(1 - i)^i.\]
2. (10 points) Find and plot all solutions to the equation $z^4 = -1$. 


3. (10 points) Show that if

\[(\bar{z})^2 = z^2\]

then \(z\) is either real or pure imaginary.
4. **(10 points)** Find the values of $z$ for which the following series converge.

(a) 

$$
\sum_{n=1}^{\infty} \frac{z^n}{n(n+1)}
$$

(b) 

$$
\sum_{n=1}^{\infty} \frac{(-1)^{n-1} z^{2n-1}}{(2n-1)!}
$$

(c) 

$$
\sum_{n=1}^{\infty} n!(z - 3i)^n
$$
5. (10 points) Find the Laurent series of the function

\[ f(z) = \frac{1}{z - 3} \]

in each of the following domains.

(a) \(|z - 3| > 0\).

(b) \(|z| > 3\).
6. (10 points) Evaluate
\[ \int_{\Gamma} \frac{1}{(z - i)^2(z + 4 - i)} \, dz, \]
where \( \Gamma \) is the circle \( |z| = 4 \) transversed once clockwise. Explain your method.
7. (12 points)

(a) Sketch the following two contours and integrate the function \( f(z) = \bar{z} \) along them. [Note: you may do this problem for \( a = 0 \) for 50% partial credit.]

(i) The circular arc \( z = a + e^{i\theta} \), where \( a \) is any complex number and \( 0 \leq \theta \leq \pi/2 \), directed counterclockwise.
(ii) The line segment from $a + 1$ to $a + i$, with the same $a$ as in part (a).

(b) Does path independence hold for the function $f(z)$? Why or why not?
8. (14 points) Let $\Gamma$ be the circle of radius 2 centered at $z = 1$, transversed once counterclockwise. Find the following integrals by any method you wish. Explain your answer.

(a) $\int_{\Gamma} \frac{dz}{z - 1}$

(b) $\int_{\Gamma} (z - 1)^2 dz$

(c) $\int_{\Gamma} \frac{dz}{(z - 1)^2}$
(d) \[ \int_{\Gamma} \frac{dz}{z - 2} \]

(e) \[ \int_{\Gamma} \frac{dz}{(z - 2)^2} \]

(f) \[ \int_{\Gamma} \frac{dz}{z - 5i} \]

(g) \[ \int_{\Gamma} \frac{dz}{(z - 5i)^2} \]
9. (14 points) Indicate whether the following statements are true or false. You do not need to justify your answer.

(a) Let \( \Gamma = \{ z = e^{i\theta} \mid 0 \leq \theta \leq \pi \} \) be a contour inside a domain \( D \), and let \( f(z) \) be any analytic function on this domain. Then
\[
\int_{\Gamma} f(z)\,dz = 0
\]
True False

(b) If \( f(z) \) is any analytic function on a simply connected domain \( D \) and \( \Gamma_1, \Gamma_2 \) are contours inside \( D \) with the same start point and the same end point, then
\[
\int_{\Gamma_1} f(z)\,dz = \int_{\Gamma_2} f(z)\,dz
\]
True False

(c) If \( u(x, y) \) is a harmonic function, then there exists an analytic function whose imaginary part is \( u(x, y) \).
True False

(continued next page)
(d) If \( f(z) = u(x, y) + iv(x, y) \) is analytic then

\[
\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}.
\]

True  False

(e) If a function is not analytic in a domain \( D \), then the integral of \( f \) along a contour \( \Gamma \) inside \( D \) depends on the parametrization of \( \Gamma \).

True  False

(f) \( f(z) = e^z \) is a multi-valued function.

True  False

(h) \( \text{Log } z \), the principal branch of \( \text{log } z \), is a multi-valued function.

True  False
10. (10 points) Find the maximum value of \(|f(z)|\) in \(|z| \leq 1\) for the function \(f(z) = z^2 - 3z\). Explain your reasoning.
11. (10 points) Find all singularities of the following function. State the type of singularity and if relevant specify its order.

\[
\frac{\sin z - z}{z(z + i)^2(e^{\frac{1}{\pi i}} - 1)}
\]
12. **(10 points)** Find a harmonic function $\phi(x, y)$ defined on $\mathbb{C} \setminus \{(x, y) | x < 0 \text{ and } y > 0\}$ with the following boundary conditions:

\[
\begin{align*}
\phi(0, y) &= 3 \quad \text{for } y > 0, \\
\phi(x, 0) &= 9 \quad \text{for } x < 0.
\end{align*}
\]

\[ \int_{0}^{2\pi} \frac{d\theta}{(5 - 3\sin \theta)}. \]

\[ \int_{0}^{\infty} \frac{\cos mx}{(x^2 + n^2)^2} \, dx, \quad \text{for } m \text{ and } n \text{ any non-zero integers}. \]
15. (10 points) Compute the following integral. Explain your method. Justify any limit arguments. Sketch any contour you use.

\[ \int_0^\infty \frac{\sin nx}{nx} \, dx, \quad \text{for any non-zero integer } n. \]
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