MATH 282

FINAL EXAM

May 4, 2005

Print your name in LEGIBLE CAPITAL LETTERS:

SOLUTION

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TOTAL 100
1. (10 pts) Define the 3 types of isolated singularities.

   If \( f \) has an isolated singularity at \( z_0 \), then \( \sum_{k=-\infty}^{\infty} a_k (z-z_0)^k \) on some punctured disk \( 0 < |z-z_0| < r \).

Case 1: If \( a_k = 0 \) for all \( k < 0 \), \( f \) has a removable singularity at \( z_0 \).

Case 2: If there is \( m > 0 \) such that \( a_{-m} \neq 0 \) and \( a_k = 0 \) for all \( k < -m \), then \( f \) has a pole of order \( m \) at \( z_0 \).

Case 3: If \( a_k \neq 0 \) for infinitely many \( k < 0 \), then \( f \) has an essential singularity at \( z_0 \).

2. (10 pts) Define the residue of an isolated singularity.

   If \( \sum_{k=-\infty}^{\infty} a_k (z-z_0)^k \) on \( 0 < |z-z_0| < r \), then \( \text{Res}(f, z_0) = a_{-1} \).
3. (10 points) Give a formula for the residue of a pole of order $m$.

In this case: \[ \text{Res}(f; z_0) = \frac{1}{(m-1)!} \lim_{z \to z_0} \frac{d^{m-1}}{dz^{m-1}} [(z-z_0)^m f(z)] \]

Recall this comes from: \( f(z) = \frac{q_m}{(z-z_0)^m} + \cdots + \frac{q_1}{z-z_0} + q_0 + a(z-z_0) \)

\( \Rightarrow (z-z_0)^m f(z) = q_m + \cdots + q_1 (z-z_0)^{m-1} + q_0 (z-z_0)^m + \cdots \)

Then \( \frac{d^{m-1}}{dz^{m-1}} [(z-z_0)^m f(z)] = (m-1)! \frac{d^{m-1}}{dz^{m-1}} f(z) + q_m (z-z_0)^m + \cdots \)

\( \Rightarrow \quad q_{m-1} = \frac{1}{(m-1)!} \lim_{z \to z_0} d^{m-1} \frac{d^{m-1}}{dz^{m-1}} [(z-z_0)^m f(z)] \)

4. (10 points) State the Cauchy residue theorem for computing a loop integral

\[ \int_{\Gamma} f(z) \, dz \]

where \( \Gamma \) is a positively oriented simple closed curve.

If \( f \) is analytic on and inside \( \Gamma \), except for finitely many isolated singularities \( z_1, \ldots, z_n \) inside \( \Gamma \), then \( \int_{\Gamma} f(z) \, dz = 2\pi i \sum_{k=1}^{n} \text{Res}(f; z_k) \)
Compute the following integrals. Show all your work.

Be sure to put the relevant singularities in the box provided. Do the same for the relevant residues and also for the value of the integral.

5. (10 points) \[ \oint_{|z|=1} (z + 1)e^{\frac{1}{z}}\,dz = 2\pi i \text{Res}(0) = 2\pi i \left( \frac{3}{2} \right) \]

\[
(2+1)\cdot \frac{1}{2^1} \\
= (2+1)\left(1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots \right) \\
= 2 + 2\left(1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots \right) \\
\text{converges on } \{2170\}. \\
\Rightarrow \quad q - 1 = 1 + \frac{1}{2^1} = \left( \frac{3}{2} \right)
\]

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<th>the relevant singularities are</th>
<th>( z = 0 )</th>
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<td>the relevant residues are</td>
<td>( \text{Res}(0) = \frac{3}{2} )</td>
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<tr>
<td>the integral</td>
<td>( 3\pi i )</td>
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6. (10 points) \[ \int_{|z|=2} \frac{e^z}{z^2(z+1)} \, dz \]

\[ \text{the relevant singularities are} \]
\[ z=0, -1 \]

\[ \text{the relevant residues are} \]
\[ \text{Res}(0) = 0, \quad \text{Res}(-1) = \frac{e^{-1}}{-1} = \frac{1}{e} \]

\[ \text{the integral} = \frac{2\pi i}{e} \]

[Diagram showing contour integration with poles at 0 and -1]
7. (10 pts) \[ \int_{0}^{2\pi} \frac{d\theta}{1 + \cos^2 \theta} \]

\[ \int = \frac{\sqrt{2}}{2} \int_{\theta = 1}^{\infty} \frac{d\theta}{i \theta (1 + \frac{1}{4} (\theta^2 + \frac{1}{2}))} = \frac{4}{i} \int_{\theta = 1}^{\infty} \frac{d\theta}{\theta^4 + 6\theta^2 + 1} \]

\[ z^4 + 6z^2 + 1 = 0 \]

\[ \Rightarrow z^2 = \frac{1}{2} (-6 \pm \sqrt{36 - 4}) = -3 \pm 2\sqrt{2} \]

\[ \sqrt{x + 3 + 2\sqrt{2}} \text{ inside } |z| < 1 \]

\[ \sqrt{x - 3 - 2\sqrt{2}} \text{ outside } |z| > 1 \]

\[ \text{Relevant singularities are: } z_1 = \sqrt{3 - 2\sqrt{2}} \text{ i } \quad z_2 = -\sqrt{3 - 2\sqrt{2}} \text{ i } \]

\[ \text{Res} \left( \frac{4z}{i(z - z_1)(z - z_2)(z^2 + 3 + 2\sqrt{2})} ; z_1 \right) = \lim_{z \to z_1} \frac{4z}{i(z - z_1)(z^2 + 3 + 2\sqrt{2})} = \frac{4z_1}{i(z_1 - z_2)(z_1^2 + 3 + 2\sqrt{2})} \]

\[ z_2 = -z_1 \Rightarrow \frac{2}{i(z_1^2 + 3 + 2\sqrt{2})} = \frac{1}{2\sqrt{2}i} \]

\[ \text{the relevant residues are both are } \frac{1}{2\sqrt{2}i} \]

\[ \text{the integral = } \sqrt{2\pi i} \]

| the relevant singularities are | \[ \pm \sqrt{3 - 2\sqrt{2}} \text{ i } \] |
| the relevant residues are | both are \[ \frac{1}{2\sqrt{2}i} \] |
| the integral = | \[ \sqrt{2\pi i} \] |
8. (10 pts) \[ \int_{0}^{\infty} \frac{dx}{(x^2 + 1)(x^2 + 9)} = \frac{1}{2} \nu \int_{0}^{\infty} \frac{dx}{(x^2 + 1)(x^2 + 9)} \quad \text{since func. is even} \]

\[ f(\zeta) = \frac{1}{(\zeta^2 + 1)(\zeta^2 + 9)} = \frac{1}{2} \int_{C_{P}^{+}} f(z) \, dz \]

\[ = \frac{1}{2} \left[ \text{Res}(f; i) + \text{Res}(f; 3i) \right] \]

\[ = \pi i \left[ \frac{1}{2} \text{Res}(f; 3i) + \frac{1}{2} \text{Res}(f; i) \right] \]

\[ = \pi i \left[ \frac{1}{2} \frac{1}{i^2} + \frac{1}{2} \frac{1}{(3i)^2} \right] \]

\[ = \pi i \left[ \frac{1}{16i} - \frac{1}{48i} \right] \]

\[ = \pi i \left( \frac{1}{24i} \right) = \frac{\pi}{24} \]

Since \( \text{deg}(x^2 + 1)(x^2 + 9) = 4 > 0 + 2 \Rightarrow \text{The integral goes to zero as } \rho \rightarrow 0 \)

On \( C_{P}^{+} \), \( |f(z)| \leq \frac{1}{\rho^2} \frac{1}{(1 + \frac{1}{2})(1 + \frac{9}{2})} \leq \frac{2}{(21)^{\frac{1}{2}}} \) when \( \rho \) is sufficiently large

\[ \text{On } C_{P}^{+} \quad |f(z)| \leq \frac{2}{\rho^2} \Rightarrow \left| \int_{C_{P}^{+}} f(z) \, dz \right| \leq \frac{2}{\rho^2} \pi \rho = \frac{2\pi}{\rho^{3}} \rightarrow 0, \quad \rho \rightarrow 0 \]

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| the relevant singularities are | \( z = i \), \( z = 3i \) |
| the relevant residues are | \( \text{Res}(i) = \frac{1}{16i} \), \( \text{Res}(3i) = \frac{1}{48i} \) |
| the integral = | \( \frac{\pi}{24} \) |
9. (10 pts) \[ \int_{-\infty}^{\infty} \frac{dx}{x^2 + 4x + 5} \]

Let \( f(z) = \frac{1}{z^2 + 4z + 5} \)

\[ \Gamma = \delta + C_p^+ \]

\[ \Gamma \to 0 \]

\[ \text{Let } \Gamma \to 0 \quad \text{by thin Annulus since} \]

\[ \text{deg}(x^2 + 4x + 5) = 2 + 0 = \text{deg}(1) + 2 \]

\[ \|f\|_{C_p^+} = \frac{1}{|2|} \left| \frac{1}{\frac{\pi}{2} + \frac{\pi}{2} + \frac{\pi}{2}} \right| = \frac{2}{|2|} \quad \text{for } 21 \text{ stub. large} \]

So on \( \Gamma \to 0 \quad \|f\|_{C_p^+} \leq \frac{2}{|2|} \Rightarrow \left| \int_{C_p^+} fdz \right| \leq \frac{2}{|2|} \left| \Gamma \right| \rho = \frac{2\pi}{|2|} \Rightarrow 0 \)

| the relevant singularities are | \[ z = -2 + i \] |
| the relevant residues are | \[ \frac{1}{2i} \] |
| the integral = | \[ \pi \] |
10. (10 pts) \[ \int_0^\infty \frac{\cos(2x)}{x^2 + 1} \, dx = \frac{1}{2} \text{Re} \left( \int_0^\infty \frac{e^{2ix}}{x^2 + 1} \, dx \right) \]

by Jordan's Lemma

\[ \text{let } f = \frac{e^{2ix}}{x^2 + 1} \]

\[ \text{let } f = \frac{e^{2ix}}{x^2 + 1} \]

\[ = 2\pi i \text{ Res} (f, i) \]

\[ = 2\pi i \ln \frac{e^{2i}}{2i} \]

\[ = 2\pi i \left( \frac{e^2}{2i} \right) = \frac{\pi}{e^2} \]

\[ \Gamma = C_p^+ + \gamma \]

So, \[ \int_0^\infty \frac{\cos(2x)}{x^2 + 1} \, dx = \frac{1}{2} \text{Re} \left( \frac{\pi}{e^2} \right) = \frac{\pi}{2e^2} \]

the relevant singularities are \[ z = i \]

the relevant residues are \[ \frac{1}{2i e^2} \]

the integral = \[ \frac{\pi}{2e^2} \]