MATH 282

FINAL EXAM

May 07, 2008

NAME (please print legibly):  

Your University ID Number:  

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1. (12 points)

Please state the following definition of theorem.

(a) The Argument Principle

(b) Picard's theorem

(c) Uniform convergence

(d) Open set

(e) Taylor series of $f(z)$ about $z_0$

(f) $u$ is a harmonic conjugate of $v$
3. (6 points)

Let

\[ f(z) = \frac{e^{\frac{1}{z-2}}}{z(z+1)^4}. \]

(a) Find and classify the singularities of \( f \)
- essential sing. at \( z=2 \)
- pole order 1 at \( z=0 \)
- pole order 4 at \( z=-1 \)

(b) Find

\[ \oint_{|z|=\frac{1}{2}} f(z) \, dz = 2\pi i \, \text{Res}(f) \]

\[ = 2\pi i \, \lim_{z \to 0} e^{\frac{z-2}{z} \ln \left( \frac{z}{z+1} \right)^4} \]

\[ = \frac{2\pi i}{e^2}. \]
4. (5 points)

Verify

\[ \int_{0}^{\infty} \frac{1}{\sqrt{x(x+4)}} \, dx = \frac{\pi}{2}. \]

Let \( I = \int_{0}^{\infty} \frac{dx}{\sqrt{x(x+4)}} \). \( \Gamma = C_e + \delta_1 + C_\rho + \delta_2 \) branch with \( \theta \in (0, \pi) \).

\[ \lim_{\rho \to \infty} \int_{C_p} \frac{1}{\sqrt{x(x+4)}} \, dz = 2\pi i \text{ Res} (z) \]

\[ \lim_{\rho \to \infty} \int_{C_p} \frac{1}{\sqrt{x(x+4)}} \, dz = 2\pi i \lim_{z \to -4} \frac{1}{\sqrt{z}} \]

\[ = 2\pi i \lim_{z \to -4} \frac{1}{\sqrt{z+4}} \]

\[ = \pi \]

\[ \Rightarrow \quad \Pi = \lim_{\rho \to \infty} \int_{C_p} \frac{1}{\sqrt{x(x+4)}} \, dz = 2I = \boxed{\frac{\pi}{2}} \]

\[ |f(z)| = \frac{1}{\sqrt{z+4}} \leq \frac{1}{\sqrt{4}} = \frac{1}{2} \]

On \( C_\rho^a \): \( |f| \leq \frac{1}{\sqrt{\rho^a}} \rightarrow 1 \int_{C_{\rho^a}} |f| \, dz \leq \frac{2\pi}{\sqrt{\rho^a}} \)

\[ \rho \to 0, \quad \Rightarrow 0 \]

\[ |f| \leq \frac{1}{\sqrt{1-e^{-4}}} \]

\[ \Rightarrow \quad \int_{C_e} |f| \, dz \leq \frac{2\pi \sqrt{e}}{e^{-4}} \longrightarrow 0. \]
5. (5 points)

Please verify

\[
p.v. \int_{-\infty}^{\infty} \frac{e^{2ix}}{x} \, dx = i\pi.
\]

\[
= \lim_{\varepsilon \to 0^+} \left( \int_{-\varepsilon}^{0} + \int_{0}^{\varepsilon} \right) \frac{e^{2ix}}{x} \, dx
\]

\[
= \lim_{\varepsilon \to 0^+} \int_{-\varepsilon}^{0} \frac{e^{2ix}}{x} \, dx - \lim_{\varepsilon \to 0^+} \int_{0}^{\varepsilon} \frac{e^{2ix}}{x} \, dx
\]

No residues

\[
= 0 - \lim_{\varepsilon \to 0^+} \int_{\varepsilon}^{0} \frac{2iz}{z} \, dz
\]

\[
= -\pi \lim_{\varepsilon \to 0^+} \left( \frac{2iz}{z} \right) = 0
\]

\[
= i\pi \lim_{\varepsilon \to 0^+} \left( \frac{e^{2iz}}{z} \right) = i\pi
\]
6. (5 points)

Please show

\[ \int_0^{2\pi} \frac{1}{2 - \cos(\theta)} \, d\theta = \frac{2\pi}{\sqrt{3}}. \]

Note: \( \text{Res}(2 - \sqrt{3}) = -\frac{1}{2\sqrt{3}}. \)

\[ \begin{align*}
\theta &= e^{i\theta} \\
\frac{d\theta}{dz} &= dz \\\n\cos \theta &= \frac{z + \frac{1}{z}}{2} \\
\end{align*} \]
7. (3 points)

Consider

\[ f(z) = \frac{1}{z(z - 1)}. \]

What are the maximal disks or annuli centered at 0 where a Taylor series expansion or Laurent series expansion of \( f \) about 0 exists? (Remark: You may sketch the regions or describe them.)

\[ 0 < |z| < 1 \]

\[ |z| < 1 \]

\[ 1 < |z| \]

8. (3 points)

Let \( C \) be the contour defined by the square with vertices \( 2 + 2i, -2 + 2i, -2 - 2i \) and \( 2 - 2i \), positively oriented. Please calculate the integral

\[ \int_C e^{1/z^2} \, dz = 2 \pi i \text{ Res } \left( \frac{1}{z^2} \right) = 0 \]

(Hint: recall that \( e^z = \sum_{k=0}^{\infty} \frac{z^k}{k!} \).)

\[ e^{1/z^2} = 1 + \frac{1}{2} + \frac{1}{2!} + \frac{1}{3!} + \cdots \]

Note: \( a_{-1} \) term is zero

\[ \Rightarrow \text{ Res } e^{1/z^2} = 0 \]
9. (2 points)
Find the real part of \( \frac{4 - 3i}{1 + i} \):
\[
\frac{(4 - 3i)(-1 - i)}{2} = \frac{-4 - 3i + 3 - 4i}{2} = \frac{-7 - 7i}{2}
\]
\[
\Rightarrow \text{Real part} = -\frac{7}{2}
\]

10. (4 points)
Please find and sketch \((1 + i)^i\).
\[
\begin{align*}
\Rightarrow & \quad e^{\frac{\text{Log}(1+i)}{i} - \text{arg}(1+i)} \\
& = e^{\frac{\text{Log}(\sqrt{2})}{i} \cdot \frac{\pi}{2} + \frac{\pi}{2} i} \\
& = e^{\text{Log}(\sqrt{2})} \cdot e^{\frac{\pi}{2} i} \quad \text{for } k \in \mathbb{Z}
\end{align*}
\]
11. (5 points)

Suppose that $f$ is entire and $|f(z)| \leq |z|^3$ for all $|z| > r_0$ for some $r_0$. Prove that $f$ is a polynomial of degree at most 3.

**Cauchy Estimates:** $f$ analytic on the set $|z-z_0| \leq r$ and $|f(z)| \leq M$ on $|z-z_0| = r \Rightarrow |f^{(n)}(z_0)| \leq \frac{n! M}{r^n}, n = 1, 2, ...$

For any $R > r_0$, $|f| \leq |z|^3 = R^3$ on $|z| = R$.

(Note $f$ entire so can choose $R$ as large as we want.)

\[ \Rightarrow |f^{(n)}(z_0)| \leq \frac{n! R^3}{R^n} = \frac{n!}{R^{n-3}}. \]

Since this holds for any $R > r_0$, we must have for $n \geq 3$:

\[ |f^{(n)}(z_0)| \leq \lim_{R \to \infty} \frac{n!}{R^{n-3}} = 0. \]

\[ \Rightarrow f^{(n)}(z_0) = 0 \text{ for } n > 3. \]

Since $f$ entire $\Rightarrow f = \sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!} z^k$ has ROC = $\infty$.

Since $f^{(k)}(z_0) = 0$ for $k > 3$ \[ \Rightarrow f = f(z_0) + f'(z_0) z + f''(z_0) z^2 + f'''(z_0) z^3 \text{ on all } z \in \mathbb{C}. \]
12. (7 points)

(a) Show

\[ \lim_{R \to \infty} \int_{|z|=R} \frac{1}{2^2(z-1)^6} \, dz = 0. \]

\[ |f(z)| = \left| \frac{1}{2^2} \cdot \frac{1}{|z-1|^6} \right| \leq \left| \frac{1}{2^2} \cdot \frac{1}{|z-1|^6} \right| = \left| \frac{1}{2^2} \cdot \frac{1}{|1-\frac{1}{2}|^6} \right| \]

\[ \Rightarrow \frac{1}{|z-1|^6} \leq \frac{1}{(1-\frac{1}{2})^6} \]

\[ \Rightarrow \int_{|z|=R} \frac{dz}{z^2(z-1)^6} \leq \frac{2\pi}{R^7} \cdot \frac{1}{\left| 1-\frac{1}{2} \right|^6} \]

\[ \Rightarrow \int_{R \to \infty} = 0. \]

(b) Use part (a) to show

\[ \int_{|z|=2} \frac{1}{z^2(z-1)^6} \, dz = 0. \]

\[ \int_{|z|=2} \frac{dz}{z^2(z-1)^6} = \pi i \left( \text{Res}(0) + \text{Res}(1) \right) = \frac{dz}{z^2(z-1)^6} \]

\[ \text{Res}(0) = \text{Res}(1) \text{ for } R > 1. \]

So, \( \int_{|z|=R} \frac{dz}{z^2(z-1)^6} \) is constant in \( R \) if \( R > 1 \).

And from part (a), \( \int_{R \to \infty} \frac{dz}{z^2(z-1)^6} = 0 \) implies

\[ \int_{|z|=2} \frac{dz}{z^2(z-1)^6} = 0 \text{ for every } R > 1. \]

In particular,

\[ \int_{|z|=2} \frac{dz}{z^2(z-1)^6} = 0. \]
13. (4 points)

Consider \( f(z) = \frac{1}{z} \). If \( f \) has an anti-derivative in the following domain \( D \), find one. If not, explain why it does not exist.

(a) \( D : |z - 1| < 1. \)

Here \( \text{Log}(z) \) is an anti-deriv. since (princ. branch) \( \text{Log}(z) \) is analytic on \( D \) and \( (\text{Log}(z))' = \frac{1}{z} \) on \( D \).

(b) \( D = \mathbb{C} \setminus \{0\} \). No anti-deriv. on this set. For any \( z \in D \), there is a branch of \( \text{Log}(z) \) with \( (\text{Log}(z))' = \frac{1}{z} \) analytic at that point. So any anti-deriv. of \( \frac{1}{z} \) on any domain must be of the form \( \text{Log}(z) + \text{Constant} \), for some branch of \( \text{Log} \). But no branch can be defined analytic on \( D \), since there is a discontinuity in the argument i.e. we must take a branch cut somewhere to define a single-valued \( \text{Log} \).

14. (5 points)

Show the function \( f(x,y) = (x^2 - y^2 - 3x) + i(2xy - 3y) \) is entire.

\[
\begin{align*}
\nabla_x &= 2x - 3 = \nabla_y \\
\n\nabla_y &= -2y = -\nabla_x \\
\text{entire by Cauchy Riemann equations.}
\end{align*}
\]
15. (5 points)

Let \( f(z) = \sum_{k=0}^{\infty} \frac{3^k}{k!} z^k \). Please compute

\[
\oint_{|z|=1} \frac{f(z)\sin(z)}{z^2} \, dz = 2\pi i \text{ Res}(0) = 2\pi i
\]

\[
\sin(z) = z - \frac{z^3}{3!} + \ldots
\]

\[
\frac{f(z)\sin(z)}{z^2} = \frac{1}{z^2} \left( 1 + 3z + \frac{3^2}{2} z^2 + \frac{3^3}{3!} z^3 + \ldots \right)
\times (z - \frac{z^3}{3!} + \ldots)
\]

\[
= \frac{1}{2^2} (2 + 3z^2 + \ldots)
\]

\[
= \frac{1}{2} + 3 + \ldots
\]

\Rightarrow \text{Res}(0) = 1

16. (5 points)

Sketch where the principal branch of \((3 - z^2)^{\frac{1}{5}}\) is analytic.

\[\text{pr} \left[ (3 - z^2)^{\frac{1}{5}} \right] \text{ not analytic if } 3 - z^2 \text{ is real or negative}.\]

\[3 - (x+iy)^2 = 3 - x^2 - y^2 - 2ixy \rightarrow \text{in } (-\infty, 0] \text{ if: } 2xy = 0 \Rightarrow x = 0 \text{ or } y = 0\]

\(x = 0\): Need \(3 - x^2 y^2 \leq 0\) also

\[\Rightarrow 3 + y^2 \leq 0 \text{ does not happen.}\]

\(y = 0\): \(3 - x^2 y^2 \leq 0 \Rightarrow 3 - x^2 \leq 0\)

\[x^2 \geq 3 \]

\[|x| \geq \sqrt{3}\]
17. (5 points)

Let \( f \) be a function analytic inside and on a bounded domain \( D \). Suppose \( \text{Re}(f) \) is constant on the boundary. Prove \( f \) is constant on \( D \). (Hint: use the maximum-modulus theorem for harmonic functions.)

\[
\text{If } f = u + iv \text{ analytic } \Rightarrow u \text{ is harmonic.}
\]

Let \( \partial D \) be the boundary of \( D \). \( D \) bounded \( \Rightarrow \max_{z \in \partial D} u \) and \( \min_{z \in \partial D} u \) occur on \( \partial D \) by max/min principle.

If \( u \equiv C \) on \( \partial D \), then \( \max_{z \in \partial D} u = C = \min_{z \in \partial D} u \).

\( \text{(const.)} \)

But this implies \( u \equiv C \) on all of \( D \).

Then, \( u_x = u_y = 0 \Rightarrow v_x = v_y = 0 \) by Cauchy-Riemann equations. By this, we know \( v \) is constant on \( D \).

\( \Rightarrow f = u + iv \) is constant on \( D \).