

THE ASYMPTOTIC NUMBER OF SIMPLE SINGULAR VECTOR TUPLES OF A CUBICAL TENSOR

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ABSTRACT. S. Ekhad and D. Zeilberger recently proved that the multivariate generating function for the number of simple singular vector tuples of a generic $m_1 \times \cdots \times m_d$ tensor has an elegant rational form involving elementary symmetric functions, and provided a partial conjecture for the asymptotic behavior of the cubical case $m_1 = \cdots = m_d$. We prove this conjecture and further identify completely the dominant asymptotic term, including the multiplicative constant. Finally, we use the method of differential approximants to conjecture that the *subdominant connective constant* effect observed by Ekhad and Zeilberger for a particular case in fact occurs more generally.

1. INTRODUCTION

In this note, we confirm a conjecture of Ekhad and Zeilberger [3] regarding the number of simple singular vector tuples of a generic $m_1 \times \cdots \times m_d$ complex tensor. We refer the reader to the work of Friedland and Ottaviani [4] for the definitions of these terms, as they are not important to the content of this article. Therein, the authors prove the following theorem.

Theorem 1.1 (Friedland and Ottaviani [4, Theorem 1]). *The number of simple singular vector tuples of a generic $m_1 \times \cdots \times m_d$ complex tensor is equal to the coefficient of $t_1^{m_1-1} \cdots t_d^{m_d-1}$ in the expression*

$$\prod_{i=1}^d \frac{\widehat{t}_i^{m_i} - t_i^{m_i}}{\widehat{t}_i - t_i}, \quad \text{where } \widehat{t}_i = \left(\sum_{j=1}^d t_j \right) - t_i.$$

Denoting by $a_d(m_1, \dots, m_d)$ the quantity described in Theorem 1.1, Ekhad and Zeilberger [3] derived a rational generating function for this multi-indexed sequence.

Theorem 1.2 (Ekhad and Zeilberger [3, Proposition 1]). *Let $e_i(x_1, \dots, x_d)$ be the i th elementary symmetric function*

$$e_i(x_1, \dots, x_d) = \sum_{1 \leq r_1 < \cdots < r_i \leq d} x_{r_1} \cdots x_{r_i}.$$

Then, the multivariate generating function $A_d(x_1, \dots, x_d)$ for the sequence $a_d(m_1, \dots, m_d)$ is

$$A_d(x_1, \dots, x_d) = \sum_{m_1, \dots, m_d \geq 0} a_d(m_1, \dots, m_d) x_1^{m_1} \cdots x_d^{m_d} = \left(1 - \sum_{i=2}^d (i-1) e_i(x_1, \dots, x_d) \right)^{-1} \prod_{i=1}^d \frac{x_i}{1-x_i}.$$

Date: October 15, 2017.

We are primarily interested in the number of simple singular vector tuples of tensors for which $m_1 = \cdots = m_d$, known as *cubical tensors*. Denote

$$C_d(n) = a_d(\underbrace{n, \dots, n}_{d \text{ times}}),$$

and observe that Theorem 1.2 implies that the generating function $F_d(x)$ of the sequence $\{C_d(n)\}_{n \geq 0}$ is the diagonal of $A_d(x_1, \dots, x_d)$; that is,

$$F_d(x) = \sum_{n \geq 0} C_d(n) x^n = \sum_{n \geq 0} [x_1^n \cdots x_d^n] A_d(x_1, \dots, x_d) x^n.$$

Here $[x_1^n \cdots x_d^n] A(x)$ denotes the coefficient of $x_1^n \cdots x_d^n$ in $A(x)$.

A univariate generating function $A(x)$ is said to be D-finite if it is the solution of a nontrivial linear differential equation with polynomial coefficients (in x), and a sequence $a(n)$ is said to be P-recursive if it satisfies a recurrence relation of the form

$$p_0(n)a(n) + p_1(n)a(n-1) + \cdots + p_k(n)a(n-k) = 0$$

where each $p_i(n)$ is a polynomial and $p_0(n) \neq 0$. These two notions are in fact equivalent—a generating function $A(x)$ is D-finite if and only if the coefficients of its power series expansion are P-recursive.

The theory of D-finite functions (see, e.g., Zeilberger [13], Christol [2], and Lipshitz [7]) guarantees that each of the functions $F_d(x)$ is D-finite, as they are diagonals of rational functions. Unfortunately, current implementations of constructive approaches to finding $F_d(x)$ cannot handle even $d = 5$.

Ekhad and Zeilberger [3] provide the recurrence relation for $C_3(n)$ and use this to find that the asymptotic behavior of the sequence is

$$C_3(n) \sim \frac{2}{\pi\sqrt{3}} 8^n n^{-1}.$$

The exponential growth rate 8 (sometimes called the *connective constant*) and the polynomial exponent -1 are derived rigorously from the recurrence relation for $C_3(n)$, while the multiplicative constant $2/(\pi\sqrt{3})$ is estimated through the calculation of many initial terms. After calculating 160 initial terms of $C_4(n)$, Ekhad and Zeilberger further conjecture that

$$C_4(n) \sim \alpha_4 81^n n^{-3/2},$$

for an unknown constant α_4 . Combining these with other numerical calculations, Ekhad and Zeilberger ultimately conjecture that

$$C_d(n) \sim \alpha_d ((d-1)^d)^n n^{(1-d)/2}.$$

In Section 2 we confirm this conjecture, and more, by using multivariate asymptotic methods to prove the following theorem.

Theorem 1.3. For $d \geq 3$, the number $C_d(n)$ of simple singular vector tuples of a d -dimensional $n \times \cdots \times n$ cubical tensor is asymptotically

$$C_d(n) = \frac{(d-1)^{d-1}}{(2\pi)^{(d-1)/2} d^{(d-2)/2} (d-2)^{(3d-1)/2}} ((d-1)^d)^n n^{(1-d)/2} \left(1 + O\left(\frac{1}{n}\right)\right).$$

In Section 3 we discuss the intractability of the computational problem of determining $F_d(x)$ exactly for small d , and we apply the method of differential approximants to explore the phenomenon of subdominant connective constants.

2. THE ASYMPTOTIC BEHAVIOR OF $C_d(n)$

Only recently have the techniques of analytic combinatorics been reliably extended to the multivariate case. In this section we appeal primarily to two articles of Raichev and Wilson [12, 11] and one of Pemantle and Wilson [10]. We start by repeating the necessary definitions and theorems from these articles.

For a d -dimensional complex vector \mathbf{x} , define

$$G_d(\mathbf{x}) = \prod_{i=1}^d x_i,$$

$$H_d(\mathbf{x}) = \left(\prod_{i=1}^d (1 - x_i) \right) \left(1 - \sum_{i=2}^d (i-1) e_i(\mathbf{x}) \right),$$

so that $A_d(\mathbf{x}) = G_d(\mathbf{x})/H_d(\mathbf{x})$ is the generating function whose main diagonal asymptotic behavior we wish to compute. Going forward, we will drop the subscript when the context is clear.

Let \mathcal{V} be the variety defined by $H(\mathbf{x}) = 0$. For complex \mathbf{x} , define the polydisk $\mathbf{D}(\mathbf{x})$ and the torus $\mathbf{T}(\mathbf{x})$ by

$$\mathbf{D}(\mathbf{x}) = \{ \mathbf{x}' : |x'_i| \leq |x_i| \text{ for all } i \},$$

$$\mathbf{T}(\mathbf{x}) = \{ \mathbf{x}' : |x'_i| = |x_i| \text{ for all } i \}.$$

A point $\mathbf{x} \in \mathcal{V}$ is said to be *minimal* if all of its coordinates are non-zero and $\mathcal{V} \cap \mathbf{D}(\mathbf{x}) \subset \mathbf{T}(\mathbf{x})$. Further, \mathbf{x} is *strictly minimal* if it is the only point of \mathcal{V} in $\mathbf{T}(\mathbf{x})$.

For many practical examples, the primary obstacle in computing the asymptotic expansion of the diagonal (or more generally, the asymptotic expansion in any direction) is detecting which points of \mathcal{V} contribute to the asymptotic behavior. We will use a variety of direct calculations to show that for each d , the asymptotic behavior of the sequence $C_d(n)$ is governed by a single strictly minimal point in the positive orthant \mathbb{R}_+^d .

Theorem 1.3 will then be proved by applying a theorem of Raichev and Wilson [12]. The theorem is stated below; we have simplified it to apply only to asymptotic behavior along the main diagonal $F_{n\mathbf{1}}$ of a d -variate generating function $F(\mathbf{x}) = G(\mathbf{x})/H(\mathbf{x})$.¹

¹More specifically, we have substituted $\alpha = 1$ and $p = 1$.

Definitions of new terms are given after the statement of the theorem, and we use the short-hand $\partial_i H(\mathbf{y})$ to denote the partial derivative of $H(\mathbf{x})$ with respect to x_i , evaluated at $\mathbf{x} = \mathbf{y}$. We also use $\widehat{\mathbf{x}}_i$ to denote \mathbf{x} with the i th coordinate deleted and $\widehat{\mathbf{x}}_{i,j}$ to denote \mathbf{x} with the i th and j th coordinates deleted.

Theorem 2.1 (Raichev and Wilson [12, Theorem 3.2]). *Let $d \geq 2$. If $\mathbf{c} \in \mathcal{V}$ is strictly minimal, smooth with $c_d \partial_d H(\mathbf{c}) \neq 0$, critical and isolated, and nondegenerate, then for all $N \in \mathbb{N}$,*

$$F_{n\mathbf{1}} = c^{-n\mathbf{1}} \left[\left((2\pi n)^{d-1} \det \tilde{g}''(0) \right)^{-1/2} \sum_{k < N} n^{-k} L_k(\tilde{u}_0, \tilde{g}) + O\left(n^{-(d-1)/2-N}\right) \right]$$

as $n \rightarrow \infty$.

The quantities $\det \tilde{g}''(0)$, L_k , \tilde{u}_0 , and \tilde{g} will be defined later, as needed. For now it suffices to remark that they can all be computed and hence Theorem 2.1 permits the computation of the asymptotic behavior of the main diagonal to arbitrary precision.

There are a number of hypotheses that must be verified to apply Theorem 2.1. We have already stated what it means for a point in \mathcal{V} to be strictly minimal. Further, $\mathbf{c} \in \mathcal{V}$ is *smooth* if $\partial_i H(\mathbf{c}) \neq 0$ for some i , $\mathbf{c} \in \mathcal{V}$ is *critical* if it's smooth and

$$c_1 \partial_1 H(\mathbf{c}) = c_2 \partial_2 H(\mathbf{c}) = \cdots = c_d \partial_d H(\mathbf{c}),$$

\mathbf{c} is *isolated* if there is a neighborhood around \mathbf{c} in which it is the only critical point, and \mathbf{c} is *nondegenerate* if $\det \tilde{g}''(0) \neq 0$.

We claim that

$$\mathbf{c} = \left(\underbrace{\frac{1}{d-1}, \dots, \frac{1}{d-1}}_{d \text{ times}} \right),$$

satisfies the hypotheses of Theorem 2.1 and therefore is the sole contributing point to the asymptotic behavior of $C_d(n)$. The verification of this claim relies on tedious computation using several properties of the symmetric functions $e_i(\mathbf{x})$; these will be stated as they are required. To simplify notation, denote

$$P(\mathbf{x}) = \prod_{i=1}^d (1 - x_i),$$

$$S(\mathbf{x}) = 1 - \sum_{i=2}^d (i-1) e_i(\mathbf{x}).$$

Proposition 2.2. *The point c lies in the variety \mathcal{V} .*

Proof. It suffices to show that $S(\mathbf{c}) = 0$. Observe that

$$e_i(k\mathbf{1}) = \binom{d}{i} k^i,$$

and therefore

$$S(\mathbf{c}) = 1 - \sum_{i=2}^d (i-1)e_i(\mathbf{c}) = 1 - \sum_{i=2}^d (i-1) \binom{d}{i} \left(\frac{1}{d-1}\right)^i = 0,$$

the final equality being verified by the computer algebra system Maple (which itself employs an algorithm of Zeilberger [14]). \square

Proposition 2.3. *The point \mathbf{c} is strictly minimal in \mathcal{V} .*

Proof. The variety \mathcal{V} can be written as the union

$$\mathcal{V} = \{\mathbf{x} : x_i = 1 \text{ for some } i\} \cup \{\mathbf{x} : S(\mathbf{x}) = 0\}.$$

This union is not disjoint.

Suppose that \mathbf{c} were not minimal. Then, there would exist a minimal point $\mathbf{y} \in \mathcal{V} \cap D(\mathbf{c})$ different from \mathbf{c} . Since $|y_i| \leq 1/(d-1)$ for all i , we must have $S(\mathbf{y}) = 0$.

Consider the variety \mathcal{V}' defined by $S(\mathbf{y}) = 0$. We say that a polynomial P is *aperiodic* if the set of integer combinations of the exponent vectors of its monomials is all of \mathbb{Z}^d . For example, $x_1 + x_1^2 x_2$ is aperiodic because the \mathbb{Z} -span of $\{(1,0), (2,1)\}$ is \mathbb{Z}^2 , while $x_1^2 + x_2^2$ is not aperiodic.

Proposition 3.17 from Pemantle and Wilson [10] states that if $H = 1 - P$ for an aperiodic polynomial P , then every minimal point of the variety defined by $H = 0$ is strictly minimal and lies in the positive orthant.² Applying this to S and \mathcal{V}' , we conclude that $0 < y_i \leq 1/(d-1)$ for all i .

It follows that

$$\sum_{i=2}^d (i-1)e_i(\mathbf{y}) \leq \sum_{i=2}^d (i-1) \binom{d}{i} \left(\frac{1}{d-1}\right)^i = 1,$$

with equality only when $\mathbf{y} = \mathbf{c}$. Therefore \mathbf{c} is minimal, and again by Proposition 3.17 from [10], \mathbf{c} is in fact strictly minimal. \square

Proposition 2.4. *The point \mathbf{c} is smooth.*

Proof. Observe first that

$$\frac{\partial}{\partial x_j} e_i(\mathbf{x}) = e_{i-1}(\widehat{\mathbf{x}}_j).$$

²The proof of Proposition 3.17 in [10] does not rely on their Assumption 3.6.

Therefore,

$$\begin{aligned}
\partial_d H(\mathbf{c}) &= (\partial_d P(\mathbf{c}))S(\mathbf{c}) + P(\mathbf{c})(\partial_d S(\mathbf{c})) \\
&= - \left(\prod_{i=1}^{d-1} (1 - c_i) \right) S(\mathbf{c}) + P(\mathbf{c}) \left(- \sum_{i=2}^d (i-1) e_{i-1}(\widehat{\mathbf{c}}_d) \right) \\
&= - \left(\frac{d-2}{d-1} \right)^d \left(\frac{d}{d-1} \right)^{d-2} \\
&= - \frac{(d-2)^d d^{d-2}}{(d-1)^{2d-2}} \neq 0.
\end{aligned}$$

Thus \mathbf{c} is a smooth point. □

Proposition 2.5. *The point \mathbf{c} is critical.*

Proof. To prove the criticality of \mathbf{c} we must verify that $c_j \partial_j H(\mathbf{c}) = c_k \partial_k H(\mathbf{c})$ for all j, k . As both \mathbf{c} and H are symmetric, this is trivially true. □

Proposition 2.6. *The point \mathbf{c} is isolated.*

Proof. Let $\epsilon > 0$ be small and let B be the ϵ -neighborhood around \mathbf{c} . Suppose $\mathbf{y} \in B$ is critical. It must then be true that

$$y_1 \partial_1 H(\mathbf{y}) = y_d \partial_d H(\mathbf{y}).$$

Using the calculation performed in Proposition 2.4 along with the fact that $S(\mathbf{y}) = 0$, it follows that

$$y_1 \sum_{i=2}^d (i-1) e_{i-1}(\widehat{\mathbf{y}}_1) = y_d \sum_{i=2}^d (i-1) e_{i-1}(\widehat{\mathbf{y}}_d).$$

Using the identity

$$e_i(x_1, \dots, x_d) = x_1 e_{i-1}(x_2, \dots, x_d) + e_i(x_2, \dots, x_d)$$

(with the convention that $e_d(x_2, \dots, x_d) = 0$), the equality becomes

$$y_1 \sum_{i=2}^d (i-1) (y_d e_{i-1}(\widehat{\mathbf{y}}_{1,d}) + e_i(\widehat{\mathbf{y}}_{1,d})) = y_d \sum_{i=2}^d (i-1) (y_1 e_{i-1}(\widehat{\mathbf{y}}_{1,d}) + e_i(\widehat{\mathbf{y}}_{1,d})).$$

By canceling like terms, we see

$$y_1 \sum_{i=2}^d (i-1) e_i(\widehat{\mathbf{y}}_{1,d}) = y_d \sum_{i=2}^d (i-1) e_i(\widehat{\mathbf{y}}_{1,d}).$$

Since the function

$$U(\mathbf{x}) = \sum_{i=2}^d (i-1) e_i(\widehat{\mathbf{x}}_{1,d})$$

is nonzero and continuous at $\mathbf{x} = \mathbf{c}$, ϵ can be chosen small enough to ensure that $U(\mathbf{y}) \neq 0$. Dividing both sides by $U(\mathbf{y})$ yields

$$y_1 = y_d,$$

and symmetry implies that \mathbf{y} has the form $y\mathbf{1}$ for some $y \in \mathbb{C}$.

Noting that

$$H(y\mathbf{1}) = 1 - \sum_{i=2}^d (i-1) \binom{d}{i} y^i = (y+1)^{d-1} ((1-d)y+1),$$

we find that $\mathbf{y} = \mathbf{c}$. Therefore, \mathbf{c} is isolated. \square

The last hypothesis to check is that \mathbf{c} is nondegenerate. This amounts to checking that $\det \tilde{g}''(0) \neq 0$. In non-symmetric cases, the definition of $\tilde{g}''(0)$ is quite cumbersome. Thankfully, Proposition 4.2 of [12] proves that in cases where H and \mathbf{c} are symmetric,

$$\det \tilde{g}''(0) = dq^{d-1},$$

where

$$q = 1 + \frac{c_1}{\partial_d H(\mathbf{c})} (\partial_{dd} H(\mathbf{c}) - \partial_{1d} H(\mathbf{c})).$$

Proposition 2.7. *The point \mathbf{c} is nondegenerate.*

Proof. We showed in Proposition 2.4 that

$$\partial_d H(\mathbf{x}) = - \left(\prod_{i=1}^{d-1} (1-x_i) \right) S(\mathbf{x}) + P(\mathbf{x}) \left(- \sum_{i=2}^d (i-1) e_{i-1}(\widehat{\mathbf{x}}_d) \right)$$

and

$$\partial_d H(\mathbf{c}) = - \frac{(d-2)^d d^{d-2}}{(d-1)^{2d-2}}.$$

Additionally, noting that the first term in the left summand and the second term in the right summand are independent of x_d , we have

$$\begin{aligned} \partial_{dd} H(\mathbf{x}) &= - \left(\prod_{i=1}^{d-1} (1-x_i) \right) \partial_d S(\mathbf{x}) + (\partial_d P(\mathbf{x})) \left(- \sum_{i=2}^d (i-1) e_{i-1}(\widehat{\mathbf{x}}_d) \right) \\ &= 2 \left(\prod_{i=1}^{d-1} (1-x_i) \right) \left(\sum_{i=2}^d (i-1) e_{i-1}(\widehat{\mathbf{x}}_d) \right), \end{aligned}$$

proving

$$\partial_{dd} H(\mathbf{c}) = 2 \left(\frac{d-2}{d-1} \right)^{d-1} \left(\frac{d}{d-1} \right)^{d-2} = \frac{2d^{d-2}(d-2)^{d-1}}{(d-1)^{2d-3}}.$$

Furthermore,

$$\begin{aligned}\partial_{1d}H(\mathbf{x}) &= (\partial_{1d}P(\mathbf{x}))S(\mathbf{x}) + (\partial_dP(\mathbf{x}))(\partial_1S(\mathbf{x})) + (\partial_1P(\mathbf{x}))(\partial_dS(\mathbf{x})) + P(\mathbf{x})(\partial_{1d}S(\mathbf{x})) \\ &= \left(\prod_{i=2}^{d-1} (1-x_i) \right) \left(S(\mathbf{x}) + (1-x_1) \left(\sum_{i=2}^d (i-1)e_{i-1}(\widehat{\mathbf{x}}_1) \right) \right. \\ &\quad \left. + (1-x_d) \left(\sum_{i=2}^d (i-1)e_{i-1}(\widehat{\mathbf{x}}_d) \right) \right. \\ &\quad \left. - (1-x_1)(1-x_d) \left(\sum_{i=2}^d (i-1)e_{i-2}(\widehat{\mathbf{x}}_{1,d}) \right) \right),\end{aligned}$$

proving,

$$\partial_{1d}H(\mathbf{c}) = \left(\frac{d-2}{d-1} \right)^{d-1} \left(2 \left(\frac{d}{d-1} \right)^{d-2} - 2 \left(\frac{d-2}{d-1} \right) \left(\frac{d}{d-1} \right)^{d-3} \right) = \frac{4d^{d-3}(d-2)^{d-1}}{(d-1)^{2d-3}}.$$

We can now compute q :

$$q = 1 + \frac{c_1}{\partial_d H(\mathbf{c})} (\partial_{dd}H(\mathbf{c}) - \partial_{1d}H(\mathbf{c})) = \frac{d-2}{d}.$$

Finally,

$$\det \tilde{g}''(0) = dq^{d-1} = d \left(\frac{d-2}{d} \right)^{d-1} = \frac{(d-2)^{d-1}}{d^{d-2}} \neq 0.$$

Hence, \mathbf{c} is nondegenerate. \square

Having verified the hypotheses of Theorem 2.1 for \mathbf{c} , we will now define and compute several of the quantities in its conclusion. To find the first-order asymptotic behavior, we consider the case $N = 1$ in the theorem. Applying the appropriate simplifications to the definitions of L_k , \tilde{u}_0 , and \tilde{g} in [12], we find that

$$L_0(\tilde{u}_0, \tilde{g}) = \frac{G(\mathbf{c})}{-c_d \partial_d H(\mathbf{c})} = \left(\frac{1}{d-1} \right)^d \left(\frac{(d-1)^{2d-1}}{d^{d-2}(d-2)^d} \right) = \frac{(d-1)^{d-1}}{d^{d-2}(d-2)^d}.$$

Assembling all computed quantities into the conclusion of Theorem 2.1 yields

$$C_d(n) = \frac{L_0(\tilde{u}_0, \tilde{g})}{\sqrt{(2\pi)^{d-1} \det \tilde{g}''(0)}} ((d-1)^d)^n n^{(1-d)/2} \left(1 + O\left(\frac{1}{n}\right) \right)$$

and so

$$C_d(n) = \frac{(d-1)^{d-1}}{(2\pi)^{(d-1)/2} d^{(d-2)/2} (d-2)^{(3d-1)/2}} ((d-1)^d)^n n^{(1-d)/2} \left(1 + O\left(\frac{1}{n}\right) \right),$$

proving Theorem 1.3.

The computation of the asymptotic behavior of off-diagonal sequences can also be performed using the same techniques. In this case, however, the loss of symmetry will complicate some of the necessary calculations.

3. COMPUTATIONAL ASPECTS AND SUBDOMINANT CONNECTIVE CONSTANTS

All known automatic methods for computing diagonals of rational functions, either exactly or asymptotically, suffer from large run-times. Recent advances have improved the situation, though such calculations still remain out of reach for even reasonably sized rational functions in more than a few variables. We comment on two such implementations.

Apagodu and Zeilberger [1] provide an algorithm that produces a linear recurrence with polynomial coefficients (in n) for the diagonal coefficients of a rational function. Applying the algorithm to $C_3(n)$ returns, after a few hours, a recurrence of order 6 with polynomial coefficients of degree at most 7. We did not attempt to apply the algorithm to $C_4(n)$. Ekhad and Zeilberger [3] note that it is much faster to generate terms of the sequence $C_3(n)$ and guess a linear recurrence. More recently, Lairez [6] has provided a Magma implementation to produce the differential equation satisfied by the diagonal of a rational function. It finds the generating function for $C_3(n)$ in a few seconds and the generating function for $C_4(n)$ in about 40 minutes.

On the asymptotic side, recent work of Melczer and Salvy [8] provides an improved algorithm to rigorously compute the asymptotic behavior of diagonals of rational functions. Their implementation provides the correct asymptotic behavior for $C_3(n)$ and $C_4(n)$ in a few seconds, and that of $C_5(n)$ in a few minutes.

Upon the calculation of the linear recurrence satisfied by $C_3(n)$, Ekhad and Zeilberger note that for the correct initial conditions the connective constant (better known in some circles as the exponential growth rate) is 8. However, for most other initial conditions, the resulting sequence would have connective constant 9. Though we are not able to find linear recurrences for $C_d(n)$ for $d \geq 5$, we can provide some evidence that this phenomenon of subdominant connective constants persists for all values of d .

We employ the method of differential approximants, pioneered by Guttmann and Joyce [5] and a favorite tool of statistical mechanists. It allows for experimental estimation of the asymptotic behavior of a sequence given only a finite number of known initial terms. A forthcoming article [9] by the present author will explore the inner workings of the method, its usefulness to enumerative combinatorics, and provide an open-source implementation.

Using the first 100 terms of $C_3(n)$, the method of differential approximants predicts that the generating function $F_3(x)$ has, as expected, a singularity located at

$$x \approx 0.12500000000000000000000000000001 \pm (2 \cdot 10^{-32})$$

corresponding to the known connective constant 8. More interestingly, it also detects a singularity located at

$$x \approx 0.111111111111113 \pm (4 \cdot 10^{-14}).$$

In most cases, this would imply a connective constant 9. Being in that case the dominant singularity, we would expect it to be estimated *more* accurately than the singularities further from the origin, not less. In our experience, this indicates that a sequence has a subdominant connective constant, as is known to be true in this case.

Applying the same process to the first 100 terms of $C_4(n)$ yields estimates for the location of singularities of $F_4(x)$ at

$$x \approx 0.0123456790123456790123456790123456790123456790123457 \pm 2 \cdot 10^{-52}, \text{ and}$$

$$x \approx 0.007999999999999999 \pm (5 \cdot 10^{-17}).$$

The first indicates the known connective constant 81, while the second indicates that this connective constant is subdominant to a connective constant 125.

The first 70 terms of $C_5(n)$ are sufficient to predict the location of singularities of $F_5(x)$ to be

$$x \approx 0.0009765624996 \pm 4 \cdot 10^{-39}, \text{ and}$$

$$x \approx 0.0004164930 \pm (2 \cdot 10^{-10}),$$

implying that the known connective constant 1024 is subdominant to a connective constant 2401.

This evidence leads us to conjecture that the known connective constants $(d-1)^d$ of all $C_d(n)$ are subdominant to the connective constants $(2d-3)^{d-1}$ for generic solutions to the linear recurrence for $C_d(n)$.

ACKNOWLEDGEMENTS

The author would like to thank the referees for their careful reading and feedback, which significantly improved this article.

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