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## MATHEMATICS TEST

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MATHEMATICS TEST

Time—170 minutes

66 Questions

**Directions:** Each of the questions or incomplete statements below is followed by five suggested answers or completions. In each case, select the one that is the best of the choices offered and then mark the corresponding space on the answer sheet.

Computation and scratchwork may be done in this examination book.

**Note:** In this examination:

- (1) All logarithms are to the base  $e$  unless otherwise specified.
- (2) The set of all  $x$  such that  $a \leq x \leq b$  is denoted by  $[a, b]$ .

1. If  $f(g(x)) = 5$  and  $f(x) = x + 3$  for all real  $x$ , then  $g(x) =$

- (A)  $x - 3$                       (B)  $3 - x$                       (C)  $\frac{5}{x + 3}$                       (D) 2                      (E)  $\frac{2}{5}$
- 

2.  $\lim_{x \rightarrow 0} \frac{\tan x}{\cos x} =$

- (A)  $-\infty$                       (B)  $-1$                       (C) 0                      (D) 1                      (E)  $\infty$
- 

3.  $\int_0^{\log 4} e^{2x} dx =$

- (A)  $\frac{15}{2}$                       (B) 8                      (C)  $\frac{17}{2}$                       (D)  $\frac{\log 16}{2} - 1$                       (E)  $\log 4$
- 

4. Let  $A - B$  denote  $\{x \in A: x \notin B\}$ . If  $(A - B) \cup B = A$ , which of the following must be true?

- (A)  $B$  is empty.  
 (B)  $A \subseteq B$   
 (C)  $B \subseteq A$   
 (D)  $(B - A) \cup A = B$   
 (E) None of the above
- 

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5. If  $f(x) = |x| + 3x^2$  for all real  $x$ , then  $f'(-1)$  is
- (A)  $-7$                       (B)  $-5$                       (C)  $5$                       (D)  $7$                       (E) nonexistent
- 

6. For what value of  $b$  is the value of  $\int_b^{b+1} (x^2 + x) dx$  a minimum?
- (A)  $0$                       (B)  $-1$                       (C)  $-2$                       (D)  $-3$                       (E)  $-4$
- 

7. In how many of the eight standard octants of  $xyz$ -space does the graph of  $z = e^{x+y}$  appear?
- (A) One                      (B) Two                      (C) Three                      (D) Four                      (E) Five
- 

8. Suppose that the function  $f$  is defined on an interval by the formula  $f(x) = \sqrt{\tan^2 x - 1}$ . If  $f$  is continuous, which of the following intervals could be its domain?

- (A)  $\left(\frac{3\pi}{4}, \pi\right)$
- (B)  $\left(\frac{\pi}{4}, \frac{\pi}{2}\right)$
- (C)  $\left(\frac{\pi}{4}, \frac{3\pi}{4}\right)$
- (D)  $\left(-\frac{\pi}{4}, 0\right)$
- (E)  $\left(-\frac{3\pi}{4}, -\frac{\pi}{4}\right)$
- 

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9.  $\int_0^1 \frac{x}{2-x^2} dx =$

- (A)  $-\frac{1}{2}$       (B)  $\frac{5}{3}$       (C)  $\frac{\log 2 - e}{2}$       (D)  $-\frac{\log 2}{2}$       (E)
- 

10. If  $f''(x) = f'(x)$  for all real  $x$ , and if  $f(0) = 0$  and  $f'(0) = -1$ , then  $f(x) =$

- (A)  $1 - e^x$       (B)  $e^x - 1$       (C)  $e^{-x} - 1$       (D)  $e^{-x}$       (E)
- 

11. If  $\phi(x, y, z) = x^2 + 2xy + xz^{\frac{3}{2}}$ , which of the following partial derivatives are identically zero?

I.  $\frac{\partial^2 \phi}{\partial y^2}$

II.  $\frac{\partial^2 \phi}{\partial x \partial y}$

III.  $\frac{\partial^2 \phi}{\partial z \partial y}$

- (A) III only  
(B) I and II only  
(C) I and III only  
(D) II and III only  
(E) I, II, and III
- 

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12.  $\lim_{x \rightarrow 0} \frac{\sin 2x}{(1+x)\log(1+x)} =$

- (A)  $-2$                       (B)  $-\frac{1}{2}$                       (C)  $0$                       (D)  $\frac{1}{2}$
- 

13.  $\lim_{n \rightarrow \infty} \int_1^n \frac{1}{x^n} dx =$

- (A)  $0$                       (B)  $1$                       (C)  $e$                       (D)  $\pi$                       (E)  $\frac{1}{2}$
- 

14. At a 15 percent annual inflation rate, the value of the dollar would decrease by approximately one-half every 5 years. At this inflation rate, in approximately how many years would the dollar be worth  $\frac{1}{1,000,000}$  of value?

- (A) 25                      (B) 50                      (C) 75                      (D) 100
- 

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15. Let  $f(x) = \int_1^x \frac{1}{1+t^2} dt$  for all real  $x$ . An equation of the line tangent to the graph of  $f$  at the point  $(2, f(2))$  is
- (A)  $y - 1 = \frac{1}{5}(x - 2)$       (B)  $y - \text{Arctan } 2 = \frac{1}{5}(x - 2)$       (C)  $y - 1 = (\text{Arctan } 2)(x - 2)$   
(D)  $y - \text{Arctan } 2 + \frac{\pi}{4} = \frac{1}{5}(x - 2)$       (E)  $y - \frac{\pi}{2} = (\text{Arctan } 2)(x - 2)$
- 

16. Let  $f(x) = e^{g(x)}h(x)$  and  $h'(x) = -g'(x)h(x)$  for all real  $x$ . Which of the following must be true?
- (A)  $f$  is a constant function.  
(B)  $f$  is a linear nonconstant function.  
(C)  $g$  is a constant function.  
(D)  $g$  is a linear nonconstant function.  
(E) None of the above
- 

17.  $1 - \sin^2\left(\text{Arccos } \frac{\pi}{12}\right) =$
- (A)  $\sqrt{\frac{1 - \cos \frac{\pi}{24}}{2}}$       (B)  $\sqrt{\frac{1 - \cos \frac{\pi}{6}}{2}}$       (C)  $\sqrt{\frac{1 + \cos \frac{\pi}{24}}{2}}$       (D)  $\frac{\pi}{6}$       (E)  $\frac{\pi}{12}$
- 

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18. If  $f(x) = \sum_{n=0}^{\infty} (-1)^n x^{2n}$  for all  $x \in (0, 1)$ , then  $f'(x) =$
- (A)  $\sin x$       (B)  $\cos x$       (C)  $\frac{1}{1+x^2}$       (D)  $\frac{-2x}{(1+x^2)^2}$       (E)
- 

19. Which of the following is the general solution of the differential equation

$$\frac{d^3y}{dt^3} - 3\frac{d^2y}{dt^2} + 3\frac{dy}{dt} - y = 0?$$

- (A)  $c_1e^t + c_2te^t + c_3t^2e^t$   
(B)  $c_1e^{-t} + c_2te^{-t} + c_3t^2e^{-t}$   
(C)  $c_1e^t - c_2e^{-t} + c_3te^{t^2}$   
(D)  $c_1e^t + c_2e^{2t} + c_3e^{3t}$   
(E)  $c_1e^{2t} + c_2te^{-2t}$
- 

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20. Which of the following double integrals represents the volume of the solid bounded above by the graph of  $z = 6 - x^2 - 2y^2$  and bounded below by the graph of  $z = -2 + x^2 + 2y^2$ ?

(A)  $4 \int_{x=0}^{x=2} \int_{y=0}^{y=\sqrt{2}} (8 - 2x^2 - 4y^2) dy dx$

(B)  $\int_{x=-2}^{x=2} \int_{y=-\sqrt{(4-x^2)/2}}^{y=\sqrt{(4-x^2)/2}} (8 - 2x^2 - 4y^2) dy dx$

(C)  $4 \int_{y=0}^{y=\sqrt{2}} \int_{x=-\sqrt{4-2y^2}}^{x=\sqrt{4-2y^2}} dx dy$

(D)  $\int_{y=-\sqrt{2}}^{y=\sqrt{2}} \int_{x=-2}^{x=2} (8 - 2x^2 - 4y^2) dx dy$

(E)  $2 \int_{y=0}^{y=\sqrt{2}} \int_{x=0}^{x=\sqrt{4-2y^2}} (8 - 2x^2 - 4y^2) dx dy$

---

21. Let  $a$  be a number in the interval  $[0, 1]$  and let  $f$  be a function defined on  $[0, 1]$  by

$$f(x) = \begin{cases} a^2 & \text{if } 0 \leq x \leq a, \\ ax & \text{otherwise.} \end{cases}$$

Then the value of  $a$  for which  $\int_0^1 f(x) dx = 1$  is

- (A)  $\frac{1}{4}$                       (B)  $\frac{1}{3}$                       (C)  $\frac{1}{2}$                       (D) 1                      (E) nonexistent
- 

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22. If  $b$  and  $c$  are elements in a group  $G$ , and if  $b^5 = c^3 = e$ , where  $e$  is the unit element of  $G$ , then the inverse of  $b^2cb^4c^2$  must be
- (A)  $b^3c^2bc$                       (B)  $b^4c^2b^2c$                       (C)  $c^2b^4cb^2$                       (D)  $cb^2c^2b^4$                       (E)  $cbc^2$
- 
23. Let  $f$  be a real-valued function continuous on the closed interval  $[0, 1]$  and differentiable on the open interval  $(0, 1)$  with  $f(0) = 1$  and  $f(1) = 0$ . Which of the following must be true?
- I. There exists  $x \in (0, 1)$  such that  $f(x) = x$ .  
II. There exists  $x \in (0, 1)$  such that  $f'(x) = -1$ .  
III.  $f(x) > 0$  for all  $x \in [0, 1)$ .
- (A) I only                      (B) II only                      (C) I and II only                      (D) II and III only                      (E) I, II, and III
- 
24. If  $A$  and  $B$  are events in a probability space such that  $0 < P(A) = P(B) = P(A \cap B) < 1$ , which of the following CANNOT be true?
- (A)  $A$  and  $B$  are independent.                      (B)  $A$  is a proper subset of  $B$ .                      (C)  $A \neq B$   
(D)  $A \cap B = A \cup B$                       (E)  $P(A)P(B) < P(A \cap B)$
- 

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25. Let  $f$  be a real-valued function with domain  $[0, 1]$ . If there is some  $K > 0$  such that  $f(x) - f(y) \leq K|x - y|$  for all  $x$  and  $y$  in  $[0, 1]$ , which of the following must be true?
- (A)  $f$  is discontinuous at each point of  $(0, 1)$ .
  - (B)  $f$  is not continuous on  $(0, 1)$ , but is discontinuous at only countably many points of  $(0, 1)$ .
  - (C)  $f$  is continuous on  $(0, 1)$ , but is differentiable at only countably many points of  $(0, 1)$ .
  - (D)  $f$  is continuous on  $(0, 1)$ , but may not be differentiable on  $(0, 1)$ .
  - (E)  $f$  is differentiable on  $(0, 1)$ .
- 

26. Let  $\mathbf{i} = (1, 0, 0)$ ,  $\mathbf{j} = (0, 1, 0)$ , and  $\mathbf{k} = (0, 0, 1)$ . The vectors  $\mathbf{v}_1$  and  $\mathbf{v}_2$  are orthogonal if  $\mathbf{v}_1 = \mathbf{i} + \mathbf{j} - \mathbf{k}$  and  $\mathbf{v}_2 =$
- (A)  $\mathbf{i} + \mathbf{j} - \mathbf{k}$                       (B)  $\mathbf{i} - \mathbf{j} + \mathbf{k}$                       (C)  $\mathbf{i} + \mathbf{k}$                       (D)  $\mathbf{j} - \mathbf{k}$                       (E)  $\mathbf{i} - \mathbf{k}$
- 

27. If the curve in the  $yz$ -plane with equation  $z = f(y)$  is rotated around the  $y$ -axis, an equation of the resulting surface of revolution is
- (A)  $x^2 + z^2 = [f(y)]^2$
  - (B)  $x^2 + z^2 = f(y)$
  - (C)  $x^2 + z^2 = |f(y)|$
  - (D)  $y^2 + z^2 = |f(y)|$
  - (E)  $y^2 + z^2 = [f(x)]^2$
- 

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28. Let  $A$  and  $B$  be subspaces of a vector space  $V$ . Which of the following must be subspaces of  $V$ ?

- I.  $A + B = \{a + b : a \in A \text{ and } b \in B\}$
- II.  $A \cup B$
- III.  $A \cap B$
- IV.  $\{x \in V : x \notin A\}$

- (A) I and II only
  - (B) I and III only
  - (C) III and IV only
  - (D) I, II, and III only
  - (E) I, II, III, and IV
- 

29.  $\lim_{n \rightarrow \infty} \sum_{k=1}^n \left( \frac{1}{k} - \frac{1}{2k} \right) =$

- (A) 0                      (B) 1                      (C) 2                      (D) 4                      (E)  $\frac{1}{2}$
- 

30. If  $f(x_1, \dots, x_n) = \sum_{1 \leq i < j \leq n} x_i x_j$ , then  $\frac{\partial f}{\partial x_n} =$

- (A)  $n!$                       (B)  $\sum_{1 \leq i < j < n} x_i x_j$                       (C)  $\sum_{1 \leq i < j < n} (x_i + x_j)$                       (D)  $\sum_{j=1}^n x_j$                       (E)  $\sum_{j=1}^{n-1} x_j$
- 

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31. If  $f(x) = \begin{cases} \sqrt{1-x^2} & \text{for } 0 \leq x \leq 1 \\ x-1 & \text{for } 1 < x \leq 2, \end{cases}$

then  $\int_0^2 f(x) dx$  is

(A)  $\frac{\pi}{2}$

(B)  $\frac{\sqrt{2}}{2}$

(C)  $\frac{1}{2} + \frac{\pi}{4}$

(D)  $\frac{1}{2} + \frac{\pi}{2}$

(E) undefined

---

32. Let  $R$  denote the field of real numbers,  $Q$  the field of rational numbers, and  $Z$  the ring of integers. Which following subsets  $F_i$  of  $R$ ,  $1 \leq i \leq 4$ , are subfields of  $R$ ?

$$F_1 = \{a/b : a, b \in Z \text{ and } b \text{ is odd}\}$$

$$F_2 = \{a + b\sqrt{2} : a, b \in Z\}$$

$$F_3 = \{a + b\sqrt{2} : a, b \in Q\}$$

$$F_4 = \{a + b\sqrt[4]{2} : a, b \in Q\}$$

(A) No  $F_i$  is a subfield of  $R$ .

(B)  $F_3$  only

(C)  $F_2$  and  $F_3$  only

(D)  $F_1, F_2,$  and  $F_3$  only

(E)  $F_1, F_2, F_3,$  and  $F_4$

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33. If  $n$  apples, no two of the same weight, are lined up at random on a table, what is the probability that they lined up in order of increasing weight from left to right?

(A)  $\frac{1}{2}$

(B)  $\frac{1}{n}$

(C)  $\frac{1}{n!}$

(D)  $\frac{1}{2^n}$

(E)  $\frac{1}{n}$

---

34.  $\frac{d}{dx} \int_0^{x^2} e^{-t^2} dt =$

(A)  $e^{-x^2}$

(B)  $2e^{-x^2}$

(C)  $2e^{-x^4}$

(D)  $x^2 e^{-x^2}$

(E)  $2x e^{-x^2}$

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35. Let  $f$  be a real-valued function defined on the set of integers and satisfying  $f(x) = \frac{1}{2}f(x-1) + \frac{1}{2}f(x+1)$ . Which of the following must be true?
- I. The graph of  $f$  is a subset of a line.
  - II.  $f$  is strictly increasing.
  - III.  $f$  is a constant function.
- (A) None  
(B) I only  
(C) II only  
(D) I and II  
(E) I and III
- 

36. If  $F$  is a function such that, for all positive integers  $x$  and  $y$ ,  $F(x, 1) = x + 1$ ,  $F(1, y) = 2y$ , and  $F(x + 1, y + 1) = F(F(x, y + 1), y)$ , then  $F(2, 2) =$
- (A) 8                      (B) 7                      (C) 6                      (D) 5                      (E) 4
- 

37. If  $\det \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & k \end{pmatrix} = 9$ , then  $\det \begin{pmatrix} 3a & 3b & 3c \\ g-4a & h-4b & k-4c \\ d & e & f \end{pmatrix} =$
- (A)  $-108$                       (B)  $-27$                       (C)  $3$                       (D)  $12$                       (E)  $27$
- 

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38.  $\lim_{n \rightarrow \infty} \frac{3}{n} \sum_{i=1}^n \left[ \left( \frac{3i}{n} \right)^2 - \left( \frac{3i}{n} \right) \right] =$
- (A)  $-\frac{1}{6}$                       (B) 0                      (C) 3                      (D)  $\frac{9}{2}$                       (E)
- 

39. For a real number  $x$ ,  $\log(1 + \sin 2\pi x)$  is not a real number if and only if  $x$  is

- (A) an integer
- (B) nonpositive
- (C) equal to  $\frac{2n-1}{2}$  for some integer  $n$
- (D) equal to  $\frac{4n-1}{4}$  for some integer  $n$
- (E) any real number
- 

40. If  $x$ ,  $y$ , and  $z$  are selected independently and at random from the interval  $[0, 1]$ , then the probability that  $x \geq yz$  is

- (A)  $\frac{3}{4}$                       (B)  $\frac{2}{3}$                       (C)  $\frac{1}{2}$                       (D)  $\frac{1}{3}$                       (F)
- 

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41. If  $A = \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix}$ , then the set of all vectors  $X$  for which  $AX = X$  is

(A)  $\left\{ \begin{pmatrix} a \\ b \end{pmatrix} \mid a = 0 \text{ and } b \text{ is arbitrary} \right\}$

(B)  $\left\{ \begin{pmatrix} a \\ b \end{pmatrix} \mid a \text{ is arbitrary and } b = 0 \right\}$

(C)  $\left\{ \begin{pmatrix} a \\ b \end{pmatrix} \mid a = -b \text{ and } b \text{ is arbitrary} \right\}$

(D)  $\left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}$

(E) the empty set

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42. What is the greatest value of  $b$  for which any real-valued function  $f$  that satisfies the following properties must also satisfy  $f(1) < 5$ ?

- (i)  $f$  is infinitely differentiable on the real numbers;
- (ii)  $f(0) = 1$ ,  $f'(0) = 1$ , and  $f''(0) = 2$ ; and
- (iii)  $|f'''(x)| < b$  for all  $x$  in  $[0, 1]$ .

(A) 1

(B) 2

(C) 6

(D) 12

(E) 24

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43. Let  $n$  be an integer greater than 1. Which of the following conditions guarantee that the equation

$$x^n = \sum_{i=0}^{n-1} a_i x^i \text{ has at least one root in the interval } (0, 1) ?$$

I.  $a_0 > 0$  and  $\sum_{i=0}^{n-1} a_i < 1$

II.  $a_0 > 0$  and  $\sum_{i=0}^{n-1} a_i > 1$

III.  $a_0 < 0$  and  $\sum_{i=0}^{n-1} a_i > 1$

- (A) None
- (B) I only
- (C) II only
- (D) III only
- (E) I and III

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44. If  $x$  is a real number and  $P$  is a polynomial function, then  $\lim_{h \rightarrow 0} \frac{P(x + 3h) + P(x - 3h) - 2P(x)}{h^2} =$

- (A) 0                      (B)  $6P'(x)$                       (C)  $3P''(x)$                       (D)  $9P''(x)$                       (E)  $\infty$
- 

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45. Consider the system of equations

$$ax^2 + by^3 = c$$

$$dx^2 + ey^3 = f$$

where  $a$ ,  $b$ ,  $c$ ,  $d$ ,  $e$ , and  $f$  are real constants and  $ae \neq bd$ . The maximum possible number of real solutions  $(x, y)$  of the system is

- (A) none                      (B) one                      (C) two                      (D) three                      (E) five
- 

46. If  $x^3 - x + 1 = a_0 + a_1(x - 2) + a_2(x - 2)^2 + a_3(x - 2)^3$  for all real numbers  $x$ , then  $(a_0, a_1, a_2, a_3)$  is

(A)  $\left(1, \frac{1}{2}, 0, -\frac{1}{8}\right)$

(B)  $(1, -1, 0, 1)$

(C)  $(7, 6, 10, 1)$

(D)  $(7, 11, 12, 6)$

(E)  $(7, 11, 6, 1)$

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47. Let  $C$  be the ellipse with center  $(0, 0)$ , major axis of length  $2a$ , and minor axis of length  $2b$ . The value of  $\oint_C x \, dy - y \, dx$  is

(A)  $\pi\sqrt{a^2 + b^2}$

(B)  $2\pi\sqrt{a^2 + b^2}$

(C)  $2\pi ab$

(D)  $\pi ab$

(E)  $\frac{\pi ab}{2}$

---

48. Let  $G_n$  denote the cyclic group of order  $n$ . Which of the following direct products is NOT cyclic?

(A)  $G_{17} \times G_{11}$

(B)  $G_{17} \times G_{11} \times G_5$

(C)  $G_{17} \times G_{33}$

(D)  $G_{22} \times G_{33}$

(E)  $G_{49} \times G_{121}$

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49. Let  $X$  be a random variable with probability density function

$$f(x) = \begin{cases} \frac{3}{4}(1 - x^2) & \text{if } -1 \leq x \leq 1, \\ 0 & \text{otherwise.} \end{cases}$$

What is the standard deviation of  $X$ ?

- (A) 0                      (B)  $\frac{1}{5}$                       (C)  $\frac{\sqrt{30}}{15}$                       (D)  $\frac{1}{\sqrt{5}}$                       (E)
- 

50. The set of all points  $(x, y, z)$  in Euclidean 3-space such that

$$\begin{vmatrix} 1 & x & y & z \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{vmatrix} = 0$$

is

- (A) a plane containing the points  $(1, 0, 0)$ ,  $(0, 1, 0)$ , and  $(0, 0, 1)$   
(B) a sphere with center at the origin and radius 1  
(C) a surface containing the point  $(1, 1, 1)$   
(D) a vector space with basis  $\{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$   
(E) none of the above
- 

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51. An automorphism  $\phi$  of a field  $F$  is a one-to-one mapping of  $F$  onto itself such that  $\phi(a + b) = \phi(a) + \phi(b)$  and  $\phi(ab) = \phi(a)\phi(b)$  for all  $a, b \in F$ . If  $F$  is the field of rational numbers, then the number of distinct automorphisms of  $F$  is

- (A) 0                      (B) 1                      (C) 2                      (D) 4                      (E) infinite
- 

52. Let  $T$  be the transformation of the  $xy$ -plane that reflects each vector through the  $x$ -axis and then doubles the vector's length.

If  $A$  is the  $2 \times 2$  matrix such that  $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = A\begin{bmatrix} x \\ y \end{bmatrix}$  for each vector  $\begin{bmatrix} x \\ y \end{bmatrix}$ , then  $A =$

(A)  $\begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}$

(B)  $\begin{bmatrix} \frac{\sqrt{2}}{2} & 1 \\ 1 & -\frac{\sqrt{2}}{2} \end{bmatrix}$

(C)  $\begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}$

(D)  $\begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}$

(E)  $\begin{bmatrix} 0 & -2 \\ -2 & 0 \end{bmatrix}$

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53. Let  $r > 0$  and let  $C$  be the circle  $|z| = r$  in the complex plane. If  $P$  is a polynomial function, then  $\int_C P(z) dz =$
- (A) 0
  - (B)  $\pi r^2$
  - (C)  $2\pi i$
  - (D)  $2\pi P(0)i$
  - (E)  $P(r)$
- 

54. If  $f$  and  $g$  are real-valued differentiable functions and if  $f'(x) \geq g'(x)$  for all  $x$  in the closed interval  $[0, 1]$ , which of the following must be true?
- (A)  $f(0) \geq g(0)$
  - (B)  $f(1) \geq g(1)$
  - (C)  $f(1) - g(1) \geq f(0) - g(0)$
  - (D)  $f - g$  has no maximum on  $[0, 1]$ .
  - (E)  $\frac{f}{g}$  is a nondecreasing function on  $[0, 1]$ .
- 

55. Let  $p$  and  $q$  be distinct primes. There is a proper subgroup  $J$  of the additive group of integers which contains exactly three elements of the set  $\{p, p + q, pq, p^q, q^p\}$ . Which three elements are in  $J$ ?
- (A)  $pq, p^q, q^p$
  - (B)  $p + q, pq, p^q$
  - (C)  $p, p + q, pq$
  - (D)  $p, p^q, q^p$
  - (E)  $p, pq, p^q$
- 

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56. For a subset  $S$  of a topological space  $X$ , let  $\text{cl}(S)$  denote the closure of  $S$  in  $X$ , and let  $S' = \{x: x \in \text{cl}(S - \{x\})\}$  denote the derived set of  $S$ . If  $A$  and  $B$  are subsets of  $X$ , which of the following statements are true?

- I.  $(A \cup B)' = A' \cup B'$
- II.  $(A \cap B)' = A' \cap B'$
- III. If  $A'$  is empty, then  $A$  is closed in  $X$ .
- IV. If  $A$  is open in  $X$ , then  $A'$  is not empty.

- (A) I and II only
  - (B) I and III only
  - (C) II and IV only
  - (D) I, II, and III only
  - (E) I, II, III, and IV
- 

57. Consider the following procedure for determining whether a given name appears in an alphabetized list of  $n$  names.

Step 1. Choose the name at the middle of the list (if  $n = 2k$ , choose the  $k$ th name); if that is the given name, you are done; if the list is only one name long, you are done. If you are not done, go to Step 2.

Step 2. If the given name comes alphabetically before the name at the middle of the list, apply Step 1 to the first half of the list; otherwise, apply Step 1 to the second half of the list.

If  $n$  is very large, the maximum number of steps required by this procedure is close to

- (A)  $n$
  - (B)  $n^2$
  - (C)  $\log_2 n$
  - (D)  $n \log_2 n$
  - (E)  $n^2 \log_2 n$
- 

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58. Which of the following is an eigenvalue of the matrix

$$\begin{pmatrix} 2 & 1-i \\ 1+i & -2 \end{pmatrix}$$

over the complex numbers?

- (A) 0                      (B) 1                      (C)  $\sqrt{6}$                       (D)  $i$                       (E)  $1+i$
- 

59. Two subgroups  $H$  and  $K$  of a group  $G$  have orders 12 and 30, respectively. Which of the following could NOT be the order of the subgroup of  $G$  generated by  $H$  and  $K$ ?

- (A) 30                      (B) 60                      (C) 120                      (D) 360                      (E) Countable infinity
- 

60. Let  $A$  and  $B$  be subsets of a set  $M$  and let  $S_0 = \{A, B\}$ . For  $i \geq 0$ , define  $S_{i+1}$  inductively to be the collection of subsets  $X$  of  $M$  that are of the form  $C \cup D$ ,  $C \cap D$ , or  $M - C$  (the complement of  $C$  in  $M$ ), where  $C, D \in S_i$ . Let  $S = \bigcup_{i=0}^{\infty} S_i$ . What is the largest possible number of elements of  $S$ ?

- (A) 4  
(B) 8  
(C) 15  
(D) 16  
(E)  $S$  may be infinite.
- 

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61. A city has square city blocks formed by a grid of north-south and east-west streets. One automobile route from City Hall to the main firehouse is to go exactly 5 blocks east and 7 blocks north. How many different routes from City Hall to the main firehouse traverse exactly 12 city blocks?

(A)  $5 \cdot 7$

(B)  $\frac{7!}{5!}$

(C)  $\frac{12!}{7!5!}$

(D)  $2^{12}$

(E)  $7!5!$

---

62. Let  $R$  be the set of real numbers with the topology generated by the basis  $\{(a, b) : a < b, \text{ where } a, b \in R\}$ . If  $X$  is the subset  $[0, 1]$  of  $R$ , which of the following must be true?

- I.  $X$  is compact.
- II.  $X$  is Hausdorff.
- III.  $X$  is connected.

(A) I only

(B) II only

(C) III only

(D) I and II

(E) II and III

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63. Let  $R$  be the circular region of the  $xy$ -plane with center at the origin and radius 2.

Then  $\iint_R e^{-(x^2 + y^2)} dx dy =$

- (A)  $4\pi$
  - (B)  $\pi e^{-4}$
  - (C)  $4\pi e^{-4}$
  - (D)  $\pi(1 - e^{-4})$
  - (E)  $4\pi(e - e^{-4})$
- 

64. Let  $V$  be the real vector space of real-valued functions defined on the real numbers and having derivatives of all orders. If  $D$  is the mapping from  $V$  into  $V$  that maps every function in  $V$  to its derivative, what are all the eigenvectors of  $D$ ?

- (A) All nonzero functions in  $V$
  - (B) All nonzero constant functions in  $V$
  - (C) All nonzero functions of the form  $ke^{\lambda x}$ , where  $k$  and  $\lambda$  are real numbers
  - (D) All nonzero functions of the form  $\sum_{i=0}^k c_i x^i$ , where  $k > 0$  and the  $c_i$ 's are real numbers
  - (E) There are no eigenvectors of  $D$ .
- 

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65. If  $f$  is a function defined by a complex power series expansion in  $z - a$  which converges for  $|z - a| < 1$  and diverges for  $|z - a| > 1$ , which of the following must be true?

- (A)  $f(z)$  is analytic in the open unit disk with center at  $a$ .
  - (B) The power series for  $f(z + a)$  converges for  $|z + a| < 1$ .
  - (C)  $f'(a) = 0$
  - (D)  $\int_C f(z) dz = 0$  for any circle  $C$  in the plane.
  - (E)  $f(z)$  has a pole of order 1 at  $z = a$ .
- 

66. Let  $n$  be any positive integer and  $1 \leq x_1 < x_2 < \dots < x_{n+1} \leq 2n$ , where each  $x_i$  is an integer. Which of the following must be true?

- I. There is an  $x_i$  that is the square of an integer.
- II. There is an  $i$  such that  $x_{i+1} = x_i + 1$ .
- III. There is an  $x_i$  that is prime.

- (A) I only
  - (B) II only
  - (C) I and II
  - (D) I and III
  - (E) II and III
- 

IF YOU FINISH BEFORE TIME IS CALLED, YOU MAY CHECK YOUR WORK ON THIS TEST.