AN IMPROVED UNIQUENESS RESULT FOR A SYSTEM OF SDE RELATED TO THE STOCHASTIC WAVE EQUATION

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Abstract. We improve on the strong uniqueness results of [GLM+17], which deal with the following system of SDE.

$$\begin{aligned} dX_t &= Y_t dt \\ dY_t &= |X_t|^\alpha dB_t \end{aligned}$$

and $X_0 = x_0, Y_0 = y_0$. For $(x_0, y_0) \neq (0,0)$, we show that short-time uniqueness holds for $\alpha > -1/2$.

1. Introduction

The purpose of this note is to improve a uniqueness result of [GLM+17]. First we state our result, and then we recall some motivation. Let X_t, Y_t solve the following system of stochastic differential equations (SDE) for $\alpha \in \mathbf{R}$.

$$\begin{aligned} dX_t &= Y_t dt \\ dY_t &= |X_t|^\alpha dB_t, \end{aligned} \tag{1.1}$$

with initial data $X_0 = x_0, Y_0 = y_0$. Here B_t is a standard one-dimensional Brownian motion. For the standard theory of SDE such as (1.1), see Chapter V of [Pro05].

We recall the results of Theorems 1.1 and 1.2 from [GLM+17], which are stated together as follows.

Theorem 1 (Gomez, Lee, Mueller, Neuman, and Salins). *If $\alpha > 1/2$ and $(x_0, y_0) \neq (0,0)$, then (1.1) has a unique solution in the strong sense, up to the time τ at which the solution (X_t, Y_t) first takes the value $(0,0)$ or blows up. Moreover the unique strong solution never reaches the origin.*

2010 *Mathematics Subject Classification.* Primary, 60H10.

Key words and phrases. Stochastic differential equations, uniqueness.

CM: Supported by a Simons grant.

In our main result we prove that the lower bound on α could be extended to $\alpha > -1/2$.

Theorem 2. *If $\alpha > -1/2$ and $(x_0, y_0) \neq (0,0)$, then (1.1) has a unique solution in the strong sense, up to the time τ at which the solution (X_t, Y_t) first takes the value $(0,0)$ or blows up. Moreover the unique strong solution never reaches the origin.*

Remark 1. *The point $(x_0, y_0) = (0,0)$ plays a special role. As proved in Theorem 3 of* [GLM+17]*, if $0 < \alpha < 1$ then with this initial condition both strong and weak uniqueness fail.*

Now we give some motivation for (1.1). Uniqueness questions for SDE such as $dX_t = a(X_t)dt + b(X_t)dB_t$ have been studied for a long time. Existence and uniqueness hold for Lipschitz coefficients a, b, see Section V.3 of [Pro05]. The coefficient $a(x)$ can be badly behaved, but the best result for $b(x)$, due to Yamada and Watanabe [YW71], is that $b(x)$ should be Hölder continuous of order at least 1/2. However, Yamada and Watanabe's method is essentially one dimensional, and does not carry over to systems except in special cases such as radial symmetry.

For stochastic PDE, existence and uniqueness hold for most equations in the case of Lipschitz continuous coefficients. A case of special interest is the SPDE for the superprocess,

$$\partial_t u(t,x) = \Delta u(t,x) + |u(t,x)|^{1/2}\dot{W}(t,x), \quad x \in \mathbf{R},\, t \geq 0.$$

with appropriate initial data, usually nonnegative. Here $\dot{W}(t,x)$ is two-parameter white noise. For such initial data, weak uniqueness among nonnegative solutions is known [Per02], and strong uniqueness among nonnegative solutions is an important unsolved problem. If the exponent 1/2 is replaced by $\gamma > 0$, then we know that strong uniqueness holds among solutions taking values in $\mathbf{R}$ if $\gamma > 3/4$ [MP11], and strong uniqueness fails for $\gamma < 3/4$ [MMP14]. The strong uniqueness results for $\gamma > 3/4$ also hold if $|u|^\gamma$ is replaced by a function of u which is Hölder continuous with index γ.

Much less is known about the stochastic wave equation

$$\partial_t^2 u = \Delta u + |u|^\alpha \dot{W}(t,x) \tag{1.2}$$

and analogous existence and uniqueness results are currently out of reach. Thus we are led to study SDE analogues of (1.2) such as

$$\ddot{u}(t) = |u(t)|^\alpha \dot{B}_t.$$

If we write $X_t = u(t)$ and $Y_t = \dot{u}(t)$, we arrive at (1.1).

2. Proof of Theorem 2

First, recall that from Yamada and Watanabe [YW71], we know that the existence of a weak solution together with strong uniqueness implies existence and uniqueness in the strong sense.

Step 1: Construction of a weak solution. When $0 < \alpha \leq 1/2$, the construction of a weak solution and the proof that it almost surely never hits the origin is similar to the proof of Theorem 1.2 in [GLM$^+$17], hence it is omitted.

Assume now that $-1/2 < \alpha \leq 0$, and fix the initial point $(x_0, y_0) \neq (0,0)$.

We use the following transformation which was used in the proof of Theorem 1.2 in [GLM$^+$17]. Define

$$h(x) := \frac{1}{2\alpha+1}|x|^{2\alpha+1}\text{sgn}(x), \quad h^{-1}(x) := (2\alpha+1)^{\frac{1}{2\alpha+1}}|x|^{\frac{1}{2\alpha+1}}\text{sgn}(x). \tag{2.1}$$

Observe that

$$dh(x) = |x|^{2\alpha}dx, \quad dh^{-1}(x) = (2\alpha+1)^{\frac{-2\alpha}{2\alpha+1}}|x|^{-\frac{2\alpha}{2\alpha+1}}.$$

Note that $h(x)$ is continuous and increasing in $\mathbf{R}$ even for $-1/2 < \alpha \leq 0$, and therefore the inverse function $h^{-1}(x)$ is well defined. However, for $-1/2 < \alpha < 0$, $dh(x)$ is infinite at the origin so the transformation in Theorem 1.2 of [GLM$^+$17] does not apply directly (see (3.4)–(3.6) therein). Since $-1/2 < \alpha \leq 0$, it follows that $dh^{-1}(x)$ is continuous in $\mathbf{R}$.

Let

$$\tilde{V}_t = h(x_0) + y_0 t + \int_0^t \tilde{B}_s ds, \quad \tilde{Y}_t = y_0 + \tilde{B}_t, \tag{2.2}$$

where $\{\tilde{B}_t\}_{t\geq 0}$ is a standard Brownian motion. We define the following time change

$$T(t) = \int_0^t (2\alpha+1)^{-\frac{2\alpha}{2\alpha+1}}|\tilde{V}_s|^{-\frac{2\alpha}{2\alpha+1}}ds. \tag{2.3}$$

Note that

$$P(T(t) < \infty, \text{ for all } 0 \leq t < \infty) = 1, \tag{2.4}$$

since $-\frac{2\alpha}{2\alpha+1} \geq 0$ for $-1/2 < \alpha \leq 0$, and $\tilde{V}_s$ has continuous trajectories.

We further define the inverse time change,

$$T^{-1}(t) = \inf\{s \geq 0 : T(s) > t\}. \tag{2.5}$$

From Remark 5.2 in [GLM$^+$17] we get that $|\tilde{V}_t| \vee |\tilde{Y}_t| \to \infty$ as $t \to \infty$, while both $\tilde{V}_t$ and $\tilde{Y}_t$ are recurrent process, hence it follows that

$\lim_{t\to\infty} T(t) = \infty$ a.s. and therefore

$$P(T^{-1}(t) < \infty, \text{ for all } 0 \le t < \infty) = 1. \tag{2.6}$$

Define

$$X_t = h^{-1}\big(\tilde{V}_{T^{-1}(t)}\big), \quad t \ge 0. \tag{2.7}$$

First, we explicitly compute $T^{-1}(t)$:

$$\begin{aligned}\frac{d}{dt}T^{-1}(t) &= \frac{1}{\frac{d}{ds}T(s)|_{s=T^{-1}(t)}} = (2\alpha+1)^{\frac{2\alpha}{2\alpha+1}}|\tilde{V}_{T^{-1}(t)}|^{\frac{2\alpha}{2\alpha+1}} \\ &= (2\alpha+1)^{\frac{2\alpha}{2\alpha+1}}|h(X_t)|^{\frac{2\alpha}{2\alpha+1}} = |h^{-1}(h(X_t))|^{2\alpha} = |X_t|^{2\alpha}.\end{aligned}$$

It follows that

$$T^{-1}(t) = \int_0^t |X_s|^{2\alpha} ds. \tag{2.8}$$

From (2.2) and (2.8) we get that

$$\begin{aligned} d\tilde{V}_{T^{-1}(t)} &= (y_0 + \tilde{B}_{T^{-1}(t)})dT^{-1}(t) \\ &= (y_0 + \tilde{B}_{T^{-1}(t)})|X_t|^{2\alpha}dt.\end{aligned}$$

On the other hand, from (2.7) we get,

$$\begin{aligned} d\tilde{V}_{T^{-1}(t)} &= dh(X_t) \\ &= |X_t|^{2\alpha}dX_t,\end{aligned}$$

and therefore we have

$$dX_t = (y_0 + \tilde{B}_{T^{-1}(t)})dt. \tag{2.9}$$

From (2.4) we have $\lim_{t\to\infty} T^{-1}(t) = \infty$, a.s., hence using (2.8) we can define

$$Y_t = y_0 + \tilde{B}_{T^{-1}(t)}. \tag{2.10}$$

From the Dambis-Dubins-Schwarz theorem (see Revuz and Yor [RY99], page 181, Theorem 1.6) we get that $\{Y_t\}_{t\ge 0}$ satisfies

$$Y_t = y_0 + \int_0^t |X_s|^{\alpha} dB_s^{(1)}, \tag{2.11}$$

where $B_t^{(1)}$ is another standard Brownian motion.

From (2.2) and (2.9)–(2.11) it follows that

$$(X_t, Y_t) = (h^{-1}(\tilde{V}_{T^{-1}(t)}), \tilde{Y}_{T^{-1}(t)}), \tag{2.12}$$

is a weak solution to (1.1).

In was proved in Section 3 of [GLM+17] that $(\tilde{V}_t, \tilde{Y}_t)$ never equals $(0,0)$, that is,

$$P\big((\tilde{V}_t, \tilde{Y}_t) \neq (0,0) \text{ for } t > 0\big) = 1.$$

Together with (2.12) and (2.6) it follows that

$$P\big((X_t, Y_t) \neq (0,0) \text{ for } t > 0\big) = 1.$$

Step 2: Proof of strong uniqueness. Let $(X_t^i, Y_t^i) : i = 1, 2$ be two solutions of (1.1) starting from $(x_0, y_0) \neq 0$, moreover let τ_n for a natural number n be the first time t at which either

$$|(X_t^1, Y_t^1)|_{\ell^\infty} \wedge |(X_t^2, Y_t^2)|_{\ell^\infty} \leq 2^{-n}$$

or

$$|(X_t^1, Y_t^1)|_{\ell^\infty} \vee |(X_t^2, Y_t^2)|_{\ell^\infty} \geq 2^{n},$$

where $|(x,y)|_{\ell^\infty} = |x| \vee |y|$ is the ℓ^∞ norm.

Finally, as in the proof of Theorem 1.1 in [GLM+17], let $Y_t^{i,n} = Y_{t\wedge\tau_n}^i$ and $X_t^{i,n} = \int_0^t Y_s^{i,n} ds$. Notice that $(X^{i,n}, Y^{i,n})$ solve

$$\begin{aligned} dX_t^{i,n} &= Y_t^{i,n} dt \\ dY_t^{i,n} &= |X_t^{i,n}|^\alpha \mathbf{1}_{[0,\tau_n]}(t) dB_t \end{aligned} \tag{2.13}$$

and that $(X_t^{i,n}, Y_t^{i,n}) = (X_t^i, Y_t^i)$ if $t \leq \tau_n$. Define

$$D_t = E\left[\left(X_t^{1,n} - X_t^{2,n}\right)^2\right].$$

Recall that $x \mapsto |x|^\alpha$ is a Lipschitz continuous function except in a neighborhood of $x = 0$. As discussed in Section 2 of [GLM+17], there is a sequence of stopping times

$$\begin{aligned} \sigma_0^i &= 0 \\ \sigma_{k+1}^i &= \inf\{t > \sigma_k^i : X_t^{i,n} = 0\}. \end{aligned}$$

These stopping times form a discrete set and do not accumulate.

In order to prove uniqueness up to time τ_n, it is enough to prove that $X_t^{1,n} = X_t^{2,n}$ for all $t \in [0, \sigma_k^i \wedge \tau_n]$ for any k and any i. We do this in two steps.

First, assume that $x_0 \neq 0$. We will argue that $\sigma_1^1 = \sigma_1^2$ and $X_t^{1,n} = X_t^{2,n}$ for all $t \in [0, \sigma_1^1]$. If $|X_t^{1,n}| \wedge |X_t^{2,n}| > 0$ for all $t \in [0, \tau_n]$, then a minimum is attained and because the coefficients in (2.13) are Lipschitz continuous when $|X_t^{i,n}|$ is bounded away from zero, standard uniqueness arguments show that $X_t^{1,n} = X_t^{2,n}$ for $t \in [0, \tau_n]$. So we assume that there exists $i \in \{1, 2\}$ such that $\sigma_1^i \leq \tau_n$. That is, at least one of the $X_t^{i,n}$ hits zero before τ_n. For $\delta < |x_0|$, let $\rho^\delta = \inf\{t > 0 : |X_t^{1,n}| \wedge |X_t^{2,n}| < \delta\}$. Because the coefficients of (2.13) are Lipschitz continuous when

$\delta < |X^{i,n}_t|$, standard arguments can be used to show that $X^{1,n}_t = X^{2,n}_t$ for all $t \in [0, \rho_\delta]$. By letting $\delta \to 0$ it is clear that $X^{1,n}_t = X^{2,n}_t$ for all $t \in [0, \lim_{\delta\to 0} \rho_\delta)$. From the continuity of $X^{i,n}$, $i = 1, 2$ it follows that $\lim_{\delta\to 0} \rho_\delta = \sigma^1_1 \wedge \sigma^2_1$, the first time that one of the $X^{i,n}_t$ hits zero. Therefore, $X^{1,n}_t = X^{2,n}_t$ for all $t \in [0, \sigma^1_1 \wedge \sigma^2_1)$ and by again by continuity we can conclude that $X^{1,n}_{\sigma^1_1\wedge\sigma^2_1} = X^{2,n}_{\sigma^1_1\wedge\sigma^2_1} = 0$ so that $\sigma^1_1 = \sigma^2_1$.

Second, assume that $x_0 = 0$.

It is enough to prove the uniqueness of the solutions to (2.13) starting at $X^{i,n}_0 = 0$ up to the first time that either one of $|X^{i,n}_t|$'s hits level 2^{-n}. Therefore, we can restrict time t to the interval $[0, \eta]$, where η is the first time $t < \tau_n$ at which

$$|X^{1,n}_t| \vee |X^{2,n}_t| = 2^{-n}.$$

If there is no such time, then let $\eta = \tau_n$. Then using the strong Markov property we can restart the process at η and use the previous step to prove uniqueness up to time σ^1_1.

Without loss of generality we can assume that $y_0 > 0$. Following the argument starting at the bottom of page 5 of [GLM+17], we first note that

$$X^{i,n}_t = \int_0^t \int_0^s |X^{i,n}_r|^\alpha \mathbf{1}_{[0,\tau_n]}(r) dB_r ds.$$

By the Cauchy-Schwarz inequality and Ito's isometry, we get

$$\begin{aligned}
E\left[\left(X^{1,n}_t - X^{2,n}_t\right)^2\right] &\le tE\int_0^t \left(\int_0^s \left(|X^{1,n}_r|^\alpha - |X^{2,n}_r|^\alpha\right)\mathbf{1}_{[0,\tau_n]}(r)dB_r\right)^2 ds \\
&= tE\int_0^t \int_0^s \left(|X^{1,n}_r|^\alpha - |X^{2,n}_r|^\alpha\right)^2 \mathbf{1}_{[0,\tau_n]}(r)drds \\
&\le tE\int_0^t \int_0^t \left(|X^{1,n}_r|^\alpha - |X^{2,n}_r|^\alpha\right)^2 drds \\
&\le t^2E\int_0^t \left(|X^{1,n}_r|^\alpha - |X^{2,n}_r|^\alpha\right)^2 dr.
\end{aligned}$$

Thus, for the stopping time $\eta > 0$ and any $t \in (0, \eta)$,

$$D_t \le t^2E\int_0^t \left(|X^{1,n}_r|^\alpha - |X^{2,n}_r|^\alpha\right)^2 dr.$$

Now the mean value theorem gives, for $0 < a < b$, that for some $c \in (a, b)$ we have

$$b^\alpha - a^\alpha = \alpha c^{\alpha-1}(b - a) \le |\alpha| a^{\alpha-1}(b - a).$$

Thus for $t \in [0,\eta]$, using the lower bound on $X_t^{i,n}$ from (2.3) in [GLM⁺17] we get

$$D_t \leq |\alpha| 2^{-n(\alpha-1)} t^2 \int_0^t r^{2\alpha-2} D_r dr. \tag{2.14}$$

By assumption, for $i=1,2$, $Y_t^{i,n}$ is almost surely continuous. It follows that

$$\lim_{t\downarrow 0} \frac{X_t^{i,n}}{t} = \lim_{t\downarrow 0} \frac{1}{t} \int_0^t Y_r^{i,n} dr = y_0 \tag{2.15}$$

exists.

From (2.13) we have

$$|X_t^{i,n}| \leq \int_0^t |Y_s^{n,i}| ds \leq 2^n t,$$

hence it follows that

$$\left(X_t^{1,n} - X_t^{2,n}\right)^2 \leq 2^{2(n+1)} t^2.$$

Then from dominated convergence we get

$$\lim_{t\downarrow 0} \frac{D_t}{t^2} = (y_0 - y_0)^2 = 0.$$

Let

$$V_t = \frac{D_t}{t^2}.$$

By the above, $V_0 = 0$ exists as a limit. Using (2.14) we conclude

$$V_t \leq C_n \int_0^t r^{2\alpha} V_r dr, \quad \text{for all } t \in (0,\eta),$$

and by Gronwall's lemma,

$$\begin{aligned} V_t &\leq V_0 \exp\left(\int_0^t C_n r^{2\alpha} dr\right) \\ &\leq V_0 \exp\left(\frac{C_n}{2\alpha+1} t^{2\alpha+1}\right) \\ &= 0. \end{aligned}$$

This shows uniqueness for $\alpha > -1/2$.

Finally, by using the strong Markov property and starting over at time $\sigma_1^1 = \sigma_1^2$, we can extend our uniqueness result up to time $\sigma_2^1 = \sigma_2^2$. By repeating this argument and using the fact that the σ_k^i cannot accumulate, we can prove uniqueness up to time τ_n.

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