SCALING PROPERTIES OF A MOVING POLYMER

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ABSTRACT. We set up an SPDE model for a moving, weakly self-avoiding polymer with intrinsic length J taking values in $(0, \infty)$. Our main result states that the effective radius of the polymer is approximately $J^{5/3}$; evidently for large J the polymer undergoes stretching. This contrasts with the equilibrium situation without the time variable, where many earlier results show that the effective radius is approximately J.

For such a moving polymer taking values in $\mathbf{R}^2$, we offer a conjecture that the effective radius is approximately $J^{5/4}$.

1. INTRODUCTION

Because of their widespread presence in the physical world, polymers have been intensively studied in chemistry, statistical mechanics, probability, and other fields. See Doi and Edwards [8] for a wide-ranging treatment from the physical point of view, and den Hollander [7], Giacomin [10], and Bauerschmidt et. al. [1] for rigorous mathematical results. König and van der Hofstad [17] discuss the one-dimensional case.

From the mathematical point of view, the study of polymers is hampered by the many complicated factors influencing their shapes. The simplest model for a polymer is a random walk where the time parameter of the walk represents the distance along the polymer. In this model we assume that new segments are attached to the end of the polymer with a random orientation. Perhaps the most important modification of this model is to penalize self-intersection; clearly two segments of the polymer cannot occupy the same position at the same time. If self intersection is prohibited, then we are led to study self-avoiding random walks. There is a large literature on this subject, see Madras and Slade [14]. One feature of interest is the macroscopic extension of the polymer, and there are various ways to quantify this notion. We give

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our own definition later, which we call the effective radius or simply the radius. The extension is often measured by the variance of the end-to-end distance, $E[|S_n|^2]$, where S_n is the location of the polymer at n units from its beginning S_0. From now on we will also assume that $S_0 = 0$.

A famous problem is to show that when $(S_n)_{n \in \mathbf{N}_0}$ is the simple random walk on $\mathbf{Z}^d$ with self-avoiding paths, we have $E[|S_n|^2] \approx Cn^{2\nu}$ where

(1.1) $$\nu = \begin{cases} 1 & \text{if } d = 1 \\ 3/4 & \text{if } d = 2 \\ 0.588\ldots & \text{if } d = 3 \\ 1/2 & \text{if } d \geq 4 \end{cases}$$

and there should be a logarithmic correction for $d = 4$. The case of $d = 1$ is obvious and for $d \geq 5$ the result has been verified by Hara and Slade [13, 12]. The other cases are open except for partial results in the $d = 4$ case, see page 400 of [1] and also [2] and the references therein. The same results should hold for weakly self-avoiding random walks, that is, when the probability of a path of length n is penalized by an exponential term involving the number of self-intersections.

The case $d = 1$ is the simplest, but it still presents some challenging problems. For example, consider weakly self-avoiding one-dimensional simple random walks $(S_n)_{n \in \mathbf{N}_0}$. With $S_0 = 0$, one could try to characterize the limiting speed,

$$\lim_{n \to \infty} \frac{1}{n} \left(E\left[S_n^2 \right] \right)^{1/2}$$

and there is a fairly complete answer, see Greven and den Hollander [11]. Here the key observation was that the occupation measure for simple random walk obeys a Markov property similar to the Ray-Knight theorem for Brownian motion. There has also been work on the continuous-time situation, see van der Hofstad, den Hollander, and König [16].

One limitation of the above models is that they do not take time into account. Of course a real polymer changes its shape over time. On page 5 of [7] den Hollander comments:

> "We will not (!) consider models where the length or the configuration of the polymer changes with time (e.g. due to growing or shrinking, or to a Metropolis dynamics associated with the Hamiltonian for an appropriate choice of allowed transitions). These non-equilibrium

situations are very interesting and challenging indeed,
but so far the available mathematics is very thin."

The goal of this paper is to make a contribution in this direction, for
the continuous case and in one dimension. We would also like to point
out that in our situation, it is not clear that the occupation measure
has a Markov property as in the time-independent case.

1.1. **Setup.** In this section we motivate and define our weakly self-
avoiding polymer model, and define the radius. The Rouse model is
commonly used to study moving polymers without self-avoidance; the
polymer is modeled as a sequence of balls connected by springs, with
friction due to an ambient fluid. Doi and Edwards [8], in equation
(4.9) on page 92 explain how to take a limit and obtain the following
stochastic partial differential equation (SPDE), which is also called the
Edwards-Wilkinson model in the context of surface growth (see Funaki
[9] for a rigorous derivation),

(1.2)
$$\partial_t u = \partial_x^2 u + \dot{W}(t, x),$$
$$u(0, x) = u_0(x),$$

where $(\dot{W}(t, x))_{t \geq 0, x \in [0, J]}$ is a two-parameter white noise. We assume
that the intrinsic length of the polymer is J, by which we mean that
$x \in [0, J]$. Since the ends of the polymer are not fixed, we impose
Neumann boundary conditions

(1.3)
$$\partial_x u(t, 0) = \partial_x u(t, J) = 0.$$

We also assume that u_0 is continuous on $[0, J]$.

As is well-known, we do not expect solutions $(u(t, x))_{t \geq 0, x \in [0, J]}$ to be
differentiable in either variable, so we must regard (1.2) as shorthand
for an integral equation, usually called the mild form:

(1.4)
$$u(t, x) = G_t(u_0)(x) + \int_0^t \int_0^J G_{t-s}(x, y) W(dy ds),$$

where

$$G_t(f)(x) = \int_0^J G_t(x, y) f(y) dy$$

and $G_t(x, y) = G_t^J(x, y)$ is the Neumann heat kernel on $x, y \in [0, J]$
which solves

$$\partial_t G_t(x, y) = \partial_x^2 G_t(x, y),$$
$$\partial_x G_t(0, y) = \partial_x G_t(J, y) = 0,$$
$$G_0(x, y) = \delta(x - y).$$

Writing

$$G_t^{\mathbf{R}}(x) = \frac{1}{\sqrt{4\pi t}} \exp\left(-\frac{x^2}{4t}\right),$$

for the heat kernel on $\mathbf{R}$, we have

$$(1.5) \quad G_t(x,y) = \sum_{k\in\mathbf{Z}} G_t^{\mathbf{R}}(x-y-2kJ) + \sum_{k\in\mathbf{Z}} G_t^{\mathbf{R}}(x+y-(2k+1)J).$$

It is well-known that for $t > 0$ the process $x \to u(t,x)$ is a Brownian motion plus a smooth function, see Exercise 3.10 in Chapter III.4 of Walsh [19] and also Proposition 1 in Mueller and Tribe [15]. Therefore we may define an occupation measure and a local time as follows, where $m(\cdot)$ is Lebesgue measure,

$$L_t(A) = m\{x \in [0,J] : u(t,x) \in A\}$$

$$(1.6)$$

$$\ell_t(y) = \frac{L_t(dy)}{dy}.$$

Now we define a weakly self-avoiding process. For continuous processes, the usual way of doing this is to weight the original probability measure by the exponential of the integral of local time squared, see [7] Section 3.1. One might think that instead of the fixed-time occupation measure $L_t(A)$ defined above, we should consider the overall occupation measure $m\{(t,x) \in [0,T] \times [0,J] : u(t,x) \in A\}$. However, at different times there is no reason that two parts of the polymer cannot be in the same position, so we study L_t as defined above. Then $\ell_t(y)$ represents the density of values of x for which $u(t,x) \in dy$, for t fixed.

If $P_{T,J}$ denotes the original probability measure of $(u(t,x))_{t\in[0,T],x\in[0,J]}$, we define the probability $Q_{T,J,\beta}$ as follows. For clarity, we will let $E^{P_{T,J}}, E^{Q_{T,J,\beta}}$ denote the expectations with respect to $P_{T,J}$ and $Q_{T,J,\beta}$ respectively. We write E for $E^{P_{T,J}}$. Let

$$(1.7) \qquad \mathcal{E}_{T,J,\beta} = \exp\left(-\beta \int_0^T \int_{-\infty}^\infty \ell_t(y)^2 dydt\right),$$

$$Z_{T,J,\beta} = E[\mathcal{E}_{T,J,\beta}] = E^{P_{T,J}}[\mathcal{E}_{T,J,\beta}].$$

where β is a parameter representing inverse temperature, following the usual convention in statistical mechanics. Then we define

$$(1.8) \qquad Q_{T,J,\beta}(A) = \frac{1}{Z_{T,J,\beta}} E\left[\mathcal{E}_{T,J,\beta}\mathbf{1}_A\right].$$

For ease of notation, we will usually drop the subscripts except for T and write

$$P_T = P_{T,J}, \qquad Q_T = Q_{T,J,\beta}, \qquad \mathcal{E}_T = \mathcal{E}_{T,J,\beta}, \qquad Z_T = Z_{T,J,\beta}.$$

Finally, we define the radius of the polymer u. The most common definition of the radius of a polymer $(p(x))_{x\in[0,J]}$ involves the end-to-end distance $|p(J) - p(0)|$, but we find it more convenient to study the standard deviation of the distance from the center of mass. Define

$$(1.9) \qquad \bar{u}(t) = \frac{1}{J} \int_0^J u(t, x)dx$$

and define the radius of $(u(t, x))_{t\in[0,T], x\in[0,J]}$ to be

$$R(T, J) = \left[\frac{1}{TJ} \int_0^T \int_0^J \left(u(t, x) - \bar{u}(t) \right)^2 dx dt \right]^{1/2}.$$

1.2. **Statement of the main result.** For any $\beta, J > 0$ we define

$$(1.10) \qquad h(\beta, J) = \begin{cases} 1, & \text{for } 0 \le \beta J^{7/2} \le e, \\ \log(\beta J^{7/2}), & \text{for } \beta J^{7/2} > e. \end{cases}$$

Here is our main result.

Theorem 1.1. *There are constants $\varepsilon_0, K_0, K_1 > 0$ not depending on β, J such that the following hold.*
  (i) For all $J > 0$ and $\beta \ge eJ^{-7/2}$ we have

$$\lim_{T\to\infty} Q_T \left[\varepsilon_0 h(\beta, J)^{-1} \beta^{1/3} J^{5/3} \le R(T, J) \le K_0 h(\beta, J)^{1/2} \beta^{1/3} J^{5/3} \right] = 1.$$

  (ii) For all $J \ge 1$ and $0 < \beta < eJ^{-7/2}$ we have

$$\lim_{T\to\infty} Q_T \left[\varepsilon_0 \beta^{1/3} J^{5/3} \le R(T, J) \le K_1 J^{5/3} \right] = 1.$$

Remark 1.2. *There is a barrier to sharpening the second inequality in (ii) with respect to dependence on β, see Remark 4.3 and the explanation after (4.6). So there is a gap in the upper bound of (ii) regarding β. But we would like to point out that most results for weakly self-avoiding polymers do not give end-to-end distance depending on β (or its analogue) either.*

Remark 1.3. *We give an intuitive justification for Theorem 1.1 in the Appendix.*

1.3. **Outline of the proof.** The following strategy for bounding the right side of (1.8) was already used in Bolthausen [4].
  In view of statements (i) and (ii) of Theorem 1.1, we define

$$(1.11) \qquad K_2 = \begin{cases} \beta^{1/3} h(\beta, J)^{1/2} K_0 & \text{if } \beta \ge eJ^{-7/2} \\ K_0 & \text{if } \beta < eJ^{-7/2} \end{cases}$$

and choose $K_0 > 0$ later. Define the following events.

(1.12)
$$A_T^{(1)} = \{R(T, J) < \varepsilon_0 h(\beta, J)^{-1}\beta^{1/3}J^{5/3}\},$$
$$A_T^{(2)} = \{R(T, J) > K_2\beta^{1/3}J^{5/3}\}.$$

It suffices to show that for $i = 1, 2$ we have

$$\lim_{T\to\infty} Q_T\big(A_T^{(i)}\big) = 0.$$

From (1.8) we see that it is enough to find:

(1) a lower bound on Z_T, derived in Section 2,
(2) and an upper bound on $E^{P_T}\big[\mathcal{E}_T \mathbf{1}_{A_T^{(i)}}\big]$ for $i = 1, 2$, obtained in Sections 3 and 4, respectively.

Finally, the upper bounds divided by the lower bound should vanish as $T \to \infty$.

As mentioned above, Greven and den Hollander [11] give a precise result for the length of the growing polymer in the case without an extra time parameter. Their argument depends on special properties of the local time or occupation measure, which are not available in our case. Bolthausen's argument [4] starts from first principles and gives a less precise result, but does not depend on these special properties, so we carry over some of his ideas.

1.4. **A conjecture about two dimensions.** We build on the physical reasoning of Flory (see Madras and Slade [14], subsection 2.2) and offer a conjecture about the case in which the polymer takes values in $\mathbf{R}^2$. The reasoning is given in the Appendix.

Here we assume that $\mathbf{u} = (u_1, u_2)$ is a vector-valued solution to (1.2), where $\mathbf{u}_0$ is also vector valued, and the Neumann boundary conditions (1.3) still hold. Also, we assume that $\dot{\mathbf{W}} = (\dot{W}_1, \dot{W}_2)$ is a vector of independent white noises.

Since we believe that $x \to \mathbf{u}(t, x)$ behaves like a two-dimensional Brownian motion, the local time will not exist. Instead, we should use either

(1) Varadhan's renormalized self-intersection local time [18], or
(2) A mollified version of local time such as

$$\ell_t^\phi(y) = \int_0^J \phi(\mathbf{u}(t, x) - y)dx$$

where $y \in \mathbf{R}^2$ and $\phi : \mathbf{R}^2 \to [0, \infty)$ is compactly supported in a neighborhood of 0.

Replacing ℓ by one of these alternatives, we define $R(T, J)$ as before. We do not state our conjecture as precisely as Theorem 1.1, nor do we speculate about the dependence of $R(T, J)$ on β.

Conjecture 1.1. *With high probability,*

$$R(T, J) \approx J^{5/4}.$$

2. Lower Bound on the Partition Function

In this section we derive a lower bound on the partition function Z_T which was defined in (1.7). This bound is given in the following proposition. Recall that h was defined in (1.10).

Proposition 2.1. *For any $\beta > 0$ and $J > 0$ we have*

$$\liminf_{T \to \infty} \frac{1}{T} \log Z_T \geq -C J^{1/3} h(\beta, J) \beta^{2/3},$$

where $C > 0$ is a constant independent from J and β.

The proof of Proposition 2.1 is delayed to the end of this section as we will need additional ingredients in order to derive this result.

We first define the radius R_u and then derive the scaling properties of $R_u(T, J)$, $\mathcal{E}_{T,J,\beta}$ and $Z_{T,J,\beta}$ in J. We therefore introduce some additional definitions and notation, which will be used later on. Let

$$(2.1) \qquad \theta_u(t, J) := \left[\frac{1}{J} \int_0^J \left(u(t, x) - \bar{u}(t) \right)^2 dx \right]^{1/2}, \quad 0 \leq t \leq T.$$

We also write $\theta_\varphi(J)$ when φ is a function which may not depend on t; the definition is the same as for θ_u.

Define

$$(2.2) \qquad R_\varphi(T, J) = \left(\frac{1}{T} \int_0^T \theta_\varphi(t, J)^2 dt \right)^{1/2}.$$

Let $(w(t, x))_{t \geq 0, x \in D}$ be a double-indexed stochastic process, where $D \subset \mathbb{R}$ is a compact set. In analogy to the local time of u which was defined in (1.6), we define $\ell_t^w = (\ell_t^w(y))_{y \in \mathbf{R}}$ as the local time of $w(t, \cdot)$, whenever it exists. Moreover we define $\mathcal{E}_{T,J,\beta}^w$ $(Z_{T,J,\beta}^w)$ as exponential of the squared local time of w (partition function), which corresponds to $\mathcal{E}_{T,J,\beta}$ $(Z_{T,J,\beta})$ in (1.7). Recall that $Q_{T,J,\beta}$ was defined in (1.8). In similar way we define $Q_{T,J,\beta}^w$ when we refer to the process w.

Finally, recall that u satisfies (1.2) on $x \in [0, J]$, with boundary conditions (1.3). In the following lemma we establish the scaling properties of $R_u(T, J)$, $\mathcal{E}_{T,J,\beta}^u$ and $Q_{T,J,\beta}^u(\cdot)$ in J.

Lemma 2.2. *Let $T, J > 0$. Define*

(2.3) $$v(t, x) := J^{-1/2} u(J^2 t, Jx).$$

Then the following holds:

   (i) *v satisfies (1.2) on $x \in [0, 1]$, with a different white noise.*
   (ii) *For any constant $\beta > 0$ we have*

$$\mathcal{E}^u_{T,J,\beta} \overset{\mathcal{D}}{=} \mathcal{E}^v_{TJ^{-2},1,\beta J^{7/2}}, \quad Q^u_{T,J,\beta}(\cdot) = Q^v_{TJ^{-2},1,\beta J^{7/2}}(\cdot).$$

(iii)

$$R_u(T, J) \overset{\mathcal{D}}{=} J^{1/2} R_v(J^{-2}T, 1).$$

The proof of Lemma 2.2 is postponed to Section 5.

From Lemma 2.2(ii) it follows that we can prove Proposition 2.1 for $J = 1$ and then use the scaling properties of Z_T to generalize the result for any $J > 0$. Therefore in the remainder of the section we assume that $J = 1$.

Next we state a few useful facts on the Fourier decomposition of the solution to (1.2), which are taken from Chapter III.4 of [19]. Note that the stochastic heat equation in Chapter III.4 of [19] includes also a linear drift term, however this only changes the eigenvalues of the equation and does not affect the eigenfunctions.

Let $(\varphi_n(\cdot), -\lambda_n)_{n \geq 0}$ be the sequence of orthonormal Neumann eigenfunctions and eigenvalues of the Laplacian on $[0, 1]$, with $(\lambda_n)_{n \geq 0}$ in increasing order. Then

(2.4) $$\varphi_0(x) = 1, \qquad \lambda_0 = 0$$

and for $n \geq 1$,

(2.5) $$\varphi_n(x) = \sqrt{2} \cos(n\pi x), \qquad \lambda_n = n^2 \pi^2.$$

Recall that the mild form of (1.2) was defined in (1.4). Define

(2.6) $$N(t, x) = \int_0^t \int_0^J G_{t-s}(x, y) W(dyds), \quad t \geq 0, \ x \in [0, J].$$

It follows that for $J = 1$,

(2.7) $$N(t, x) = \sum_{n=0}^{\infty} X_t^{(n)} \varphi_n(x),$$

where

(2.8)
$$dX_t^{(n)} = -\lambda_n X_t^{(n)} dt + dB_t^{(n)},$$
$$X_0^{(n)} = 0$$

and $(B_\cdot^{(n)})_{n\geq 0}$ is a collection of independent Brownian motions.

As in Section 3 of [4] we define a measure $\hat{P}_T^{(a)} = \hat{P}_{T,1,\beta}^{(a)}$ that adds a drift to the process. In our situation, it would be logical to simply add a constant drift to the white noise. However, adding such drift would just shift the solution to (1.2) to the right (or left). So we would like to add a drift which increases or decreases with x.

In order to do that, we take the noise $\dot{W}$ and add a drift $a\varphi_1(\cdot)$. In what what follows we fix $T > 0$. Recall that

$$(2.9) \qquad \int_0^1 \varphi_1^2(x)dx = 1.$$

Then the Cameron-Martin formula for spacetime white noise (see Dawson [6], Theorem 5.1) gives

(2.10)
$$\frac{d\hat{P}_T^{(a)}}{dP_T} = \exp\left(\int_0^T\int_0^1 a\varphi_1(x)W(dxdt) - \frac{1}{2}\int_0^T\int_0^1 a^2\varphi_1^2(x)dxdt\right)$$
$$= \exp\left(\int_0^T\int_0^1 a\varphi_1(x)W(dxdt)\right)\Big/\hat{\zeta}(T,a),$$

where

$$(2.11) \qquad \hat{\zeta}(T,a) = \exp\left(\frac{Ta^2}{2}\right).$$

We write $\hat{E}$ for the expectation $E^{\hat{P}_T^{(a)}}$ with respect to $\hat{P}_T^{(a)}$. Let $\beta > 0$, and recall that $\ell_t(\cdot)$ is the local time of the process $x \to u(t,x)$. Now using Jensen's inequality, we get

$$\log Z_T = \log\hat{E}\left[\exp\left(-\beta\int_0^T\int_{-\infty}^\infty \ell_t(y)^2 dydt - \log\frac{d\hat{P}_T^{(a)}}{dP_T}\right)\right]$$

(2.12)
$$\geq -\beta\hat{E}\left[\int_0^T\int_{-\infty}^\infty \ell_t(y)^2 dydt\right] - \hat{E}\left[\log\frac{d\hat{P}_T^{(a)}}{dP_T}\right]$$

$$=: -\beta I_1(T) - I_2(T).$$

In what follows we derive the asymptotic behaviour of $I_i(T)$ for $i = 1, 2$ when $T \to \infty$, which will help us to prove Proposition 2.1.

Proposition 2.3. *There exist positive constants $C_1, C_2 > 0$ not depending on J, T, a such that:*

(i) *for any $0 < a \leq 1$ we have*

$$\lim_{T\to\infty}\frac{I_1(T)}{T} = \frac{C_1}{a},$$

(ii) *for any $a > 1$ we have*

$$\lim_{T \to \infty} \frac{I_1(T)}{T} = C_2 \frac{\log a}{a}.$$

Proof. From (1.4) it follows that for any continuous initial condition u_0, as $t \to \infty$, $G_t(u_0)(x)$ converges to a constant uniformly in x. Since adding a constant does not change the radius of the solution, we can ignore the influence of the initial data and start with $u(0, \cdot)$ which is a pinned stationary distribution with respect to $\hat{P}$ (see e.g. equation (2.1) in [15]). Let $t_0 \geq 0$ and $x_0 \in [0, 1]$. For $t_0 \leq t$ and $x \in [0, 1]$ define

$$u_{t_0,x_0}(t, x) := \int_{t_0}^{t} \int_{0}^{1} G_{t-r}(x, y) \widetilde{W}(dydr)$$

$$(2.13) \qquad + \int_{-\infty}^{t_0} \int_{0}^{1} \left[G_{t-r}(x, y) - G_{t_0-r}(x_0, y) \right] \widetilde{W}(dydr)$$

$$+ a \int_{-\infty}^{0} \int_{0}^{1} \left[G_{t-r}(x, y) - G_{t_0-r}(x_0, y) \right] \varphi_1(y) dydr,$$

where $\widetilde{W}$ is a space-time white noise.

We first study the solution u to (2.13) under a $\hat{P}_T^{(a)}$, which adds drift $a\varphi_1(y)$ to the noise $\widetilde{W}(dyds)$ for $s \in [0, T]$.

Therefore, we replace $\widetilde{W}(dyds)$ with $\hat{W}(dyds) + a\varphi_1(y)dyds$ for $s \in [0, T]$, $y \in [0, 1]$. So from (1.4) we get that u_{t_0,x_0} satisfies the following equation under $\hat{P}_T^{(a)}$, for $t_0 \leq t \leq T$ and $x \in [0, 1]$,

$$u_{t_0,x_0}(t, x) := \int_{t_0}^{t} \int_{0}^{1} G_{t-r}(x, y) \hat{W}(dydr)$$

$$+ a \int_{t_0}^{t} \int_{0}^{1} G_{t-r}(x, y) \varphi_1(y) dydr$$

$$(2.14) \qquad + \int_{-\infty}^{t_0} \int_{0}^{1} \left[G_{t-r}(x, y) - G_{t_0-r}(x_0, y) \right] \hat{W}(dydr)$$

$$+ a \int_{-\infty}^{t_0} \int_{0}^{1} \left[G_{t-r}(x, y) - G_{t_0-r}(x_0, y) \right] \varphi_1(y) dydr$$

$$=: (A) + a \cdot (B) + (C) + a \cdot (D).$$

Recall that (φ_n, λ_n) were defined in (2.5). We will use the $\mathbf{L}^2$-eigenfunction expansion of the heat kernel

$$(2.15) \qquad G_t(x, y) = \sum_{n=0}^{\infty} e^{-\lambda_n t} \varphi_n(x) \varphi_n(y).$$

Since λ_n is the eigenvalue corresponding to φ_n, we have for every $n \geq 1$,
(2.16)
$$\int_0^1 \varphi_n(y) G_{t-s}(x,y) dy = e^{-\lambda_n(t-s)} \varphi_n(x) = \exp\left(-\pi^2 n^2 (t-s)\right) \varphi_n(y).$$

In the following lemma we verify that the integrals (C) and (D) converge. Note that (C) and (D) depend on (t, t_0, x, x_0) so we often abbreviate $(C)(t, t_0, x, x_0)$ and $(D)(t, t_0, x, x_0)$.

Lemma 2.4. *For any $T \geq 0$ the following holds:*

  (i)
$$\sup_{t_0 \leq t \leq T} \sup_{x, x_0 \in [0,1]} E[(C)(t, t_0, x, x_0)^2] < \infty,$$

  (ii)
$$\sup_{t_0 \leq t \leq T} \sup_{x, x_0 \in [0,1]} |(D)(t, t_0, x, x_0)| < \infty.$$

The proof of Lemma 2.4 is postponed to Section 6.

In the following two lemmas, which will be proved in Section 6, we derive two essential properties of the pinned string.

Assume now that t_0 and x_0 are fixed. For any $t > 0$, let $\hat{g}_{t,x_1,x_2}$ be the density function for $u_{t_0,x_0}(t, x_2) - u_{t_0,x_0}(t, x_1)$ with respect to $\hat{P}_T^{(a)}$. First, we reformulate the expected local time integral in terms of an integral over $\hat{g}_{t,x_1,x_2}(0)$.

Lemma 2.5 (Rephrasing the local time integral). *For any $0 \leq t \leq T$ we have*

(2.17)
$$\hat{E}\left[\int_{-\infty}^{\infty} \ell_t(y)^2 dy\right] = \int_0^1 \int_0^1 \hat{g}_{t,x_1,x_2}(0) dx_2 dx_1.$$

Next we derive the shift invariance property of the pinned string.

Lemma 2.6 (Shift invariance of the pinned string). *Let $0 \leq t_0 \leq T$ and $x_0 \in [0,1]$. Then under the measure $\hat{P}_T^{(a)}$ the random field*

$$U_{t_0,x_0}(t, x_1, x_2) := u_{t_0,x_0}(t, x_1) - u_{t_0,x_0}(t, x_2), \quad t \in [0, T], \ x_1, x_2 \in [0,1],$$

is stationary in t. That is, for any $t_0 < t_1 < t_2 \leq T$ we have

$$\left(U_{t_0,x_0}(t_1, x_1, x_2)\right)_{x_1,x_2 \in [0,1]} \overset{\mathcal{D}}{=} \left(U_{t_0,x_0}(t_2, x_1, x_2)\right)_{x_1,x_2 \in [0,1]}.$$

From Lemmas 2.5 and 2.6, it follows that in order to bound $I_1(T)$, we can restrict our discussion to the case where $t = t_0 = 0$ and $x_0 = 1/2$

in (2.14). Then we have

$$u_{0,1/2}(0,x) = \int_0^\infty \int_0^1 \left[G_r(x,y) - G_r(1/2,y) \right] \hat{W}(dydr)$$

(2.18)
$$+ a \int_0^\infty \int_0^1 \left[G_r(x,y) - G_r(1/2,y) \right] \varphi_1(y) dy dr$$

$$=: (E)(x) + a \cdot (F)(x).$$

Recall that $\{\ell_t(y)\}_{y \in \mathbf{R}}$ is the local time of $u_{0,1/2}(t, \cdot)$. From (2.12) and Lemmas 2.5 and 2.6, we therefore have

$$(2.19) \quad I_1(T) = T\hat{E}\left[\int_{-\infty}^\infty \ell_0(y)^2 dy \right] = T \int_0^1 \int_0^1 \hat{g}_{0,x_1,x_2}(0) dx_2 dx_1.$$

From (2.19) we conclude that the scaling properties of $I_1(T)$ are given by $\hat{g}_{0,x_1,x_2}$.

Notation. In order to simplify the notation we write $\hat{g}_{x_1,x_2}$ instead of $\hat{g}_{0,x_1,x_2}$ for the rest of this section. We also use the notation $u(t,x)$ instead of $u_{0,1/2}(t,x)$ where there is no ambiguity.

In the following lemma we derive some essential bounds on the second moment for the increments of $u_{0,1/2}$, in the drift-less case, that is when $a = 0$ in (2.18). Since $u_{0,1/2}$ is a Gaussian process, this bound also applies to the variance of its increments for any $a > 0$.

Lemma 2.7. *Assume that $a = 0$ in (2.18). Then there exist constants $C_1, C_2 > 0$ such that for all $x_1, x_2 \in [0,1]$ and $t \in [0,T]$,*

$$C_1|x_1 - x_2| \le E\left[(u(t,x_1) - u(t,x_2))^2 \right] \le C_2|x_1 - x_2|.$$

The proof of Lemma 2.7 is given in Section 7.

In order to study $\hat{g}_{x_1,x_2}$, we need analyze the drift term (F) in (2.18). Since $\varphi_1(1/2) = 0$ we get from (2.16) that

$$(F)(x) = \int_0^\infty e^{-\lambda_1 t} \left[\varphi_1(x) - \varphi_1(1/2) \right] dt$$

$$= \int_0^\infty e^{-\lambda_1 t} \varphi_1(x) dt$$

$$= \frac{\sqrt{2}}{\pi^2} \cos(\pi x),$$

where we used (2.5) in the last equality.

Define

$$\mathcal{D}(x_1,x_2) := (F)(x_1) - (F)(x_2), \quad 0 \le x_1, x_2 \le 1.$$

In the following lemma we derive a lower bound on $\mathcal{D}(x_1,x_2)$.

Lemma 2.8. *For all $0 \leq x_1 \leq x_2 \leq 1$ we have*

$$\mathcal{D}(x_1, x_2) \geq \begin{cases} \frac{\sqrt{2}}{8\pi^2}(x_2 - x_1)(x_2 + x_1), & \text{for } x_1 + x_2 \leq 1, \\ \frac{\sqrt{2}}{2\pi^2}(x_2 - x_1)\left(1 - \frac{x_2 + x_1}{2}\right), & \text{for } 1 < x_1 + x_2 \leq 2. \end{cases}$$

The proof of Lemma 2.8 is given in Section 7.

We define g_{x_1, x_2} as the density function for $u(0, x_2) - u(0, x_1)$ in (2.18) when $a = 0$. From Lemmas 2.7 and 2.8 we get that there exist constants $C_1, C_2 > 0$ such that

$$\int_0^1 \int_0^1 \hat{g}_{x_1, x_2}(0) dx_2 dx_1 = 2 \int_0^1 \int_{x_1}^1 \hat{g}_{x_1, x_2}(0) dx_2 dx_1$$

$$= 2 \int_0^1 \int_{x_1}^1 g_{x_1, x_2}(\mathcal{D}(x_1, x_2)) dx_2 dx_1$$

$$\leq C_1 \int_0^1 \int_{x_1}^1 (x_2 - x_1)^{-1/2} \exp\left(-\frac{C_2 a^2 \mathcal{D}(x_1, x_2)^2}{x_2 - x_1}\right) dx_2 dx_1$$

$$(2.20) \quad \leq C_1 \int_0^1 \int_{x_1}^1 \mathbf{1}_{\{x_1 + x_2 \leq 1\}} (x_2 - x_1)^{-1/2}$$

$$\times \exp\left(-C_3 a^2 (x_1 + x_2)^2 (x_2 - x_1)\right) dx_1 dx_2$$

$$+ C_1 \int_0^1 \int_{x_1}^1 \mathbf{1}_{\{x_1 + x_2 > 1\}} (x_2 - x_1)^{-1/2}$$

$$\times \exp\left(-C_4 a^2 \left(1 - \frac{x_1 + x_2}{2}\right)^2 (x_2 - x_1)\right) dx_1 dx_2$$

$$=: C_1 \big(\mathcal{I}_1(a) + \mathcal{I}_2(a)\big).$$

The result then follows from following lemma, which is proved in Section 7.

Lemma 2.9. *There exist constants $C_1, C_2 > 0$ not depending on a, such that for $i = 1, 2$ we have:*

(i) *for all $0 < a \leq 1$,*

$$\mathcal{I}_i(a) \leq \frac{C_1}{a},$$

(ii) *for all $a > 1$,*

$$\mathcal{I}_i(a) \leq C_2 \frac{\log a}{a}.$$

$\square$

Next we analyze $I_2(T)$ from (2.12).

Lemma 2.10. *For any $J, T, a > 0$ we have*

$$I_2(T) = \frac{a^2 T}{2}.$$

Proof. From (2.11) and (2.12) it follows that

$$I_2(T) = \hat{E}\left[\log \frac{d\hat{P}_T^{(a)}}{dP_T}\right]$$

(2.21)
$$= \hat{E}\left[\int_0^T \int_0^J a\varphi_1(x) W(dxdt)\right] - \log \hat{\zeta}(T, a)$$

$$= \hat{E}\left[W(T, a\varphi_1)\right] - \frac{a^2 T}{2},$$

where we write

$$W(T, a\varphi_1) = \int_0^T \int_0^J a\varphi_1(x) W(dxdt)$$

Using (2.10) we get

$$\hat{E}\left[W(T, a\varphi_1)\right] = E\left[W(T, a\varphi_1)\frac{d\hat{P}_T^{(a)}}{dP_T}\right]$$

(2.22)
$$= \frac{a}{\hat{\zeta}(T, a)} E\left[W(T, \varphi_1) \exp\left(aW(T, \varphi_1)\right)\right]$$

$$= \frac{a}{\hat{\zeta}(T, a)} \frac{d}{da} E\left[\exp\left(aW(T, \varphi_1)\right)\right].$$

In the preceding line we have the derivative of a moment generating function. Let

$$X = W(T, \varphi_1),$$

and

$$\psi(a) = E\left[\exp\left(aX\right)\right].$$

Note that $X \sim N(0, \sigma^2)$ and from (2.9) we conclude that

$$\sigma^2 = \int_0^T \int_0^1 \varphi_1(y)^2 dy = T.$$

It follows that

$$\psi(a) = \exp\left(\frac{a^2 T}{2}\right),$$

$$\frac{d}{da}\psi(a) = aT \exp\left(\frac{a^2 T}{2}\right).$$

Using this moment generating function computation in (2.22) and using (2.11), we get

$$(2.23) \qquad \hat{E}\left[W(T, a\varphi_1)\right] = \frac{1}{\hat{\zeta}(T, a)} a^2 T \exp\left(\frac{a^2 T}{2}\right) = a^2 T.$$

Pulling together (2.21) and (2.23), we have

$$I_2(T) = \hat{E}\left[W(T, a\varphi_1)\right] - \log\hat{\zeta}(T, a) = a^2 T - \frac{a^2}{2}T = \frac{1}{2}a^2 T.$$

$\square$

Now we are ready to prove Proposition 2.1.

Proof of Proposition 2.1. We first prove the proposition for $J = 1$, that is for a solution v to (1.2) on $x \in [0, 1]$.

From (2.12), Proposition 2.3(i), and Lemma 2.10 we get for any $0 \leq \beta \leq e$ and $0 \leq a \leq 1$,

$$\liminf_{T\to\infty} \frac{1}{T}\log Z_{T,1,\beta} \geq \lim_{T\to\infty} \frac{1}{T}\left(-\beta I_1(T) - I_2(T)\right)$$
$$= -C\frac{\beta}{a} - \frac{a^2}{2}.$$

Next we choose $a = (\beta/e)^{1/3}$, and get that there exists a constant $\tilde{C} > 0$ such that

$$(2.24) \qquad \liminf_{T\to\infty} \frac{1}{T}\log Z_{T,1,\beta} \geq -\tilde{C}\beta^{2/3}.$$

Again from (2.12), Proposition 2.3(ii) and Lemma 2.10 we get for any $\beta > e$ and $a > 1$,

$$\liminf_{T\to\infty} \frac{1}{T}\log Z_{T,1,\beta} \geq \lim_{T\to\infty} \frac{1}{T}\left(-\beta I_1(T) - I_2(T)\right)$$
$$= -C\frac{\beta \log a}{a} - \frac{a^2}{2}.$$

As before we choose $a = (\beta/e)^{1/3}$, and get that there exists a constant $\hat{C} > 0$ such that

$$(2.25) \qquad \liminf_{T\to\infty} \frac{1}{T}\log Z_{T,1,\beta} \geq -\hat{C}\beta^{2/3}\log\beta.$$

Let $J > 0$ and let u be the solution to (1.2) on $x \in [0, J]$. Now we use Lemma 2.2(ii) and (1.7) to get $Z_{T,J,\beta}^u = Z_{TJ^{-2},1,\beta J^{7/2}}^v$. Together

with (2.24) we have for $0 \leq \beta J^{7/2} \leq e$,

$$
\begin{aligned}
\liminf_{T \to \infty} \frac{1}{T} \log Z^u_{T,J,\beta} = J^{-2} \liminf_{T \to \infty} \frac{1}{TJ^{-2}} \log Z^v_{TJ^{-2},1,\beta J^{7/2}} \\
\geq -J^{-2} \beta^{2/3} J^{7/3} C \\
\geq -\beta^{2/3} J^{1/3} C.
\end{aligned}
$$

Similarly using (2.25) we get for all $\beta J^{7/2} \geq e$,

$$
\begin{aligned}
\liminf_{T \to \infty} \frac{1}{T} \log Z^u_{T,J,\beta} = J^{-2} \liminf_{T \to \infty} \frac{1}{TJ^{-2}} \log Z^v_{TJ^{-2},1,\beta J^{7/2}} \\
\geq -C J^{-2} \beta^{2/3} J^{7/3} \log(\beta J^{2/7}).
\end{aligned}
$$

By the definition of $h(\beta, J)$ from (1.10), the result follows. $\qquad\square$

3. Small distance tail estimate

In this section we derive an upper bound for the tail behaviour of $R(T, J)$, which is given in the following proposition. Recall that Q_T and $A_T^{(1)}$ were defined in (1.8) and (1.12), respectively, and that $A_T^{(1)}$ depends on ε_0.

Proposition 3.1. *We can choose $\varepsilon_0 > 0$ small enough so that*

$$
\lim_{T \to \infty} Q_T\big(A_T^{(1)}\big) = 0.
$$

Proof. We first prove the proposition for $J = 1$, that is, for a solution v to (1.2) on $x \in [0, 1]$.

We first define the event $A_{K,T} = \{R(T, 1) \leq K\}$. Recall that θ_v was defined in (2.1). From (2.2) we get

$$
R(T, 1)^2 = \frac{1}{T} \int_0^T \theta_v(t, 1)^2 dt.
$$

Hence we have on $A_{K,T}$

(3.1) $|\{t \in [0, T] : \theta_v(t, 1)^2 \leq 2K^2\}| \geq T/2,$

where $|\cdot|$ denotes the Lebesgue measure. From (2.1) it follows that if $\theta_v(t, 1)^2 \leq 2K^2$ then

(3.2) $|\{x \in [0, 1] : v(t, x) \in [\bar{v}(t) - 2K, \bar{v}(t) + 2K]\}| \geq 1/2.$

Let $d_t^\pm = \bar{v}(t) \pm 2K$, and note that $d_t^+ - d_t^- = 4K$. Then from (1.6), (3.1) and (3.2) it follows that on $A_{K,T}$ we have

(3.3) $\left|\left\{t \in [0, T] : \int_{d_k^-}^{d_k^+} \ell_t^v(x) dx \geq 1/2\right\}\right| \geq T/2.$

Using Jensen's inequality and (3.3), we get on $A_{T,K}$

$$\int_0^T \int_{-\infty}^\infty \ell_t^v(y)^2 dy dt \geq 4K \int_{-\infty}^\infty \left(\int_{d_K^-}^{d_K^+} \ell_t(y)^2 \frac{dy}{4K} \right) dt$$

(3.4)
$$\geq 4K \int_0^T \left(\int_{d_K^-}^{d_K^+} \ell_t(y) \frac{dy}{4K} \right)^2 dt$$

$$\geq \frac{T}{32K}.$$

Note that on the event $A_T^{(1)}$ in (1.12) we have

(3.5) $$R(T,1) < \varepsilon_0 h(\beta,1)^{-1}\beta^{1/3}.$$

Hence from (3.4) we get that on $A_T^{(1)}$,

$$\int_0^T \int_{-\infty}^\infty \ell_t(x)^2 dx dt \geq \frac{T}{32\varepsilon_0 h(\beta,1)^{-1}\beta^{1/3}}.$$

Recall the definition of $\mathcal{E}_T$ in (1.7). Thus for any $\varepsilon_0 > 0$ we have
(3.6)
$$E\left[\mathcal{E}_T \mathbf{1}\left(R(T,1) < \varepsilon_0 h(\beta,1)^{-1}\beta^{1/3} \right) \right] \leq \exp\left(-\beta \frac{T}{32\varepsilon_0 h(\beta,1)^{-1}\beta^{1/3}} \right).$$

Using Proposition 2.1 with $J = 1$ and (3.6) we get

$$\lim_{T\to\infty} \frac{1}{T} \log Q_T \left(R(T,1) < \varepsilon_0 h(\beta,1)^{-1}\beta^{1/3} \right)$$

$$\leq \lim_{T\to\infty} \frac{1}{T} \log E\left[\mathcal{E}_T \mathbf{1}\left(R(T,1) < \varepsilon_0 h(\beta,1)^{-1}\beta^{1/3} \right) \right]$$

(3.7)
$$- \liminf_{T\to\infty} \frac{1}{T} \log Z_T$$

$$\leq -\beta^{2/3} h(\beta,1) \left(\frac{1}{32\varepsilon_0} - C \right).$$

Hence by choosing ε_0 small enough we get the result for $J = 1$.

Let $J > 0$ and let u be the solution to (1.2) on $x \in [0, J]$. Define

$$\beta_u = J^{-7/2}\beta_v.$$

Now we use Lemma 2.2(ii) and (iii) together with (3.7) and

(3.8) $$h(\beta_u, J) = h(\beta_v, 1),$$

to get

$$\frac{1}{T}Q_{T,J,\beta_u}\left(R_u(T,J) \leq \varepsilon_0 h(\beta_u,J)^{-1}\beta_u^{1/3}J^{5/3}\right)$$

$$(3.9) \quad = \frac{1}{T}Q_{TJ^{-2},1,\beta_v}\left(J^{1/2}R_v(TJ^{-2},1) \leq \varepsilon_0 h(\beta_v,1)^{-1}J^{-7/6}\beta_v^{1/3}J^{5/3}\right)$$

$$= J^{-2}\frac{1}{TJ^{-2}}Q_{TJ^{-2},1,\beta_v}\left(R_v(TJ^{-2},1) \leq \varepsilon_0 h(\beta_v,1)^{-1}\beta_v^{1/3}\right).$$

It follows that

$$\lim_{T\to\infty}\frac{1}{T}Q_{T,J,\beta_u}\left(R_u(T,J) \leq \varepsilon_0 h(\beta_u,J)^{-1}\beta_u^{1/3}J^{5/3}\right)$$

$$\geq -J^{-2}\beta_v^{2/3}h(\beta_u,J)\left(\frac{1}{32\varepsilon_0}-C\right)$$

$$= -\beta_u^{2/3}J^{1/3}h(\beta_u,J)\left(\frac{1}{32\varepsilon_0}-C\right).$$

Hence by choosing ε_0 small enough we get the result for any $J > 0$. □

4. Large distance tail estimate

In this section we derive an upper bound for tail probability of $R(T,J)$, which is given in the following proposition. Recall that Q_T and $A_T^{(2)}$ were defined in (1.8) and (1.12), respectively, and that K_2 was a constant appearing in the definition of A_2.

Proposition 4.1. *We can choose K_2 so large that*

$$\lim_{T\to\infty}Q_T\left(A_T^{(2)}\right) = 0.$$

Before we start with the proof of Proposition 4.1 we introduce the following large deviation result. Let $\gamma > 0$ and define $(X_t)_{t\geq 0}$ to be an Ornstein-Uhlenbeck process satisfying

$$(4.1) \qquad \begin{aligned} dX_t &= -\gamma X_t dt + dW_t, \\ X_0 &= 0 \end{aligned}$$

and let

$$\mathcal{S}_T := \int_0^T X_t^2 dt.$$

Lemma 3.1 of Bercu and Rouault [3], which relies on Bryc and Dembo [5], states the following.

Lemma 4.2. *$T^{-1}\mathcal{S}_T$ satisfies a large deviation principle with good rate function*

(4.2)
$$I(c) = \begin{cases} \frac{(2\gamma c - 1)^2}{8c} & \text{if } c > 0, \\ +\infty & \text{otherwise.} \end{cases}$$

Remark 4.3. *In fact $I(c) = 0$ if $c = (2\gamma)^{-1}$. As noted in (1.5) of Bercu and Rouault [3],*

$$\lim_{T \to \infty} \frac{\mathcal{S}_T}{T} = \frac{1}{2\gamma}$$

almost surely. So this lemma is only useful to us when $c > (2\gamma)^{-1}$.

Proof of Proposition 4.1. Once again we first prove the proposition for $J = 1$, that is for a solution v to (1.2) on $x \in [0,1]$.

Observe that since $G_t(u_0)(x)$ tends to a constant uniformly in x as $t \to \infty$, we need only deal with $N(\cdot, \cdot)$. So for the purposes of this proof, we will assume that u_0 is identically 0 and thus $u(t,x) = N(t,x)$ for $t \geq 0$, $x \in [0, J]$.

Also recall that θ_u was defined in (2.1) and that we use θ_φ for a function φ which may not depend on t. Recall that $(X_\cdot^{(n)})_{n \geq 0}$ was defined in (2.8). We also define

$$S_T^{(n)} = \int_0^T \left(X_t^{(n)}\right)^2 dt.$$

Recall that N was defined in (2.6), and that (2.7) states

$$N(t,x) = \sum_{n=0}^{\infty} X_t^{(n)} \varphi_n(x).$$

We define $\bar{N}(t) = \int_0^1 N(t,x) dx$, as we defined $\bar{u}$. Since $\varphi_0(x) = 1$ for $x \in [0,1]$ and since the $(\varphi_n)_{n \geq 0}$ are orthogonal, we have $\bar{N}(t) = X_t^{(0)} \varphi_0(x)$, for all $x \in [0,1]$. Therefore

$$N(t,x) - \bar{N}(t,x) = \sum_{n=1}^{\infty} X_t^{(n)} \varphi_n(x).$$

Again using the fact that $(\varphi_n)_{n \geq 1}$ are orthonormal, we get

(4.3)
$$R_N(T,1)^2 = \sum_{n=1}^{\infty} \frac{1}{T} \int_0^T (X_t^{(n)})^2 dt = \sum_{n=1}^{\infty} \frac{1}{T} S_T^{(n)}.$$

We bound the final term in (4.3) using the following lemma . We first observe that for a family of events $(A_T)_{T>0}$ that satisfies a large

deviations principle with rate function $I > 0$ we have

$$\limsup_{T \to \infty} \frac{1}{T} \log P(A_T) \leq -I.$$

Then, for all $\kappa > 1$ there exists T_κ such that for $T > T_\kappa$ we have

$$\frac{1}{T} \log P(A_T) \leq -\frac{I}{\kappa}.$$

and so for $T > T_\kappa$

(4.4) $$P(A_T) \leq \exp\left(-\frac{TI}{\kappa}\right).$$

In our case, we have a family of events $(A_T^{(n)})_{n \geq 1, T \geq 0}$ where

$$A_T^{(n)} := \left\{\frac{1}{T} S_T^{(n)} > Kn^{-2}\right\}.$$

So in principle we would have different times $T_\kappa^{(n)}$ in (4.4). In the proof of the following lemma we use the fact that our processes $X_t^{(n)}$ are related in law and by scaling to a single Ornstein-Uhlenbeck process, so we can choose a single T_κ which works for all n. For the next lemma, we choose $\kappa = 2$.

Lemma 4.4. *There exists $\overline{T} > 0$ such that if $T > \overline{T}$ then for all $n \geq 1$ we have*

$$P\left(\frac{1}{T} S_T^{(n)} > Kn^{-2}\right) \leq \exp\left(-\frac{TnI(K)}{2}\right),$$

where $I(\cdot)$ was defined in (4.2).

Before proving this lemma, we use it to finish the proof of Proposition 4.1 in the case $J = 1$.

Let $\beta > 0$ be a fixed constant. Recall that for every $K_0 > 0$, K_2 was defined in (1.11). Define

(4.5) $$c_0 = \left(\sum_{n=1}^{\infty} \frac{1}{n^2}\right)^{-1} = \frac{6}{\pi^2}.$$

Now taking $c_n = 2^{-3} c_0 K_2^2 n^{-2}$ and $\gamma = \lambda_n = n^2 \pi^2$ in (4.2), we find that $c_n \gamma = 2^{-3} c_0 \pi^2 K_2^2$. Following Remark 4.3, in order to use Lemma 4.2, we need to show that

(4.6) $$2c_n \gamma = 2^{-2} \pi^2 c_0 K_2^2 > 1, \quad \text{for all } n \geq 1.$$

We show that this requirement is satisfied in the following two cases. Case 1: for $\beta \geq e$, $K_2^2 = \beta^{2/3} \log \beta K_0^2 \geq K_0^2$. Note that we need $\beta > e$ here since $\log \beta$ could be arbitrarily small near $\beta = 1$. Choosing

$K_2^2 > \frac{4}{\pi^2 c_0}$ will be sufficient for (4.6). Case 2: when $0 < \beta < e$, $K_2^2 = K_0^2$, and again $K_2^2 > \frac{4}{\pi^2 c_0}$ will be sufficient for (4.6). From (4.2), (4.5) and (4.6) it follows that for K_0 (and therefore K_2) large enough,

$$
\begin{aligned}
I(c_n) &= \frac{(2\gamma c_n - 1)^2}{8 c_n} \\
(4.7) \qquad &= \frac{(2^{-2}\pi^2 c_0 K_2^2 - 1)^2}{c_0 K_2^2 n^{-2}} \\
&\geq \frac{1}{2} n^2 K_2^2, \quad \text{for all } n \geq 1.
\end{aligned}
$$

From (4.3) we get

$$
\begin{aligned}
P\left(R_N(T,1) \geq K_2\right) &= P\left(R_N(T,1)^2 \geq K_2^2\right) \\
(4.8) \qquad &\leq \sum_{n=1}^{\infty} P\left(\frac{1}{T} S_T^{(n)} > c_0 K_2^2 n^{-2}\right),
\end{aligned}
$$

where we have used (4.5) in the last inequality. Using this (4.8) together with Lemma 4.4 and (4.7) give us that

$$
\limsup_{T \to \infty} P\left(R_N(T,1) \geq K_2\right) \leq \limsup_{T \to \infty} \sum_{n=1}^{\infty} \exp\left(-T I(c_0 K_2 n^{-2})/2\right) = 0.
$$

Proof of Lemma 4.4. Let $(Y_t)_{t \geq 0}$ satisfy (4.1) with $\gamma = 1$, that is $Y_0 = 0$ and $dY = -Y\,dt + dB$. For $a, b > 0$ let

$$
\begin{aligned}
Z_t^{(a,b)} &= a Y_{bt} \\
B_t^{(a,b)} &= a B_{bt}.
\end{aligned}
$$

Then we find

$$
\begin{aligned}
dZ_t^{(a,b)} &= d(a Y_{bt}) \\
&= -a Y_{bt} d(bt) + a\, dB_{bt} \\
&= -b Z_t^{(a,b)} + dB_t^{(a,b)}
\end{aligned}
$$

and also $Z_0^{(a,b)} = 0$.

Choosing $b = \lambda_n = \pi^2 n^2$ and $a = (\pi n)^{-1}$ and writing $Y^{(n)} = Z^{(a,b)}$, we see that $B^{(n)} := B^{(a,b)}$ is a standard Brownian motion. So for $t \geq 0$

$$
dY_t^{(n)} = -\lambda_n Y^{(n)} dt + dB_t^{(n)}
$$

and $(X_t^{(n)})_{t\geq 0}$ and $(Y_t^{(n)})_{t\geq 0}$ are equal in distribution. Then Lemma 4.2 implies that

$$P\left(\frac{1}{T}S_T^{(n)} > Kn^{-2}\right) = P\left(\frac{1}{T}\int_0^T (X_t^{(n)})^2 dt > Kn^{-2}\right)$$

$$= P\left(\frac{1}{T}\int_0^T (Y_t^{(n)})^2 dt > Kn^{-2}\right)$$

$$= P\left(\frac{1}{T}\int_0^T \left((\pi n)^{-1}Y_{\pi^2 n^2 t}\right)^2 dt > Kn^{-2}\right)$$

$$= P\left(\frac{1}{\pi^2 n^2 T}\int_0^{\pi^2 n^2 T} (Y_s)^2 ds > \pi^2 K\right),$$

where we have used the change of variable $s = \pi^2 n^2 t$.

Then Lemma 4.2 finishes the proof of Lemma 4.4. As we remarked, this also finishes the proof of Lemma 4.1 when $J = 1$. □

Now use scaling to finish the proof of Lemma 4.1 for all $J > 0$. Let u be the solution to (1.2) on $x \in [0, J]$. Recall that v is the solution to (1.2) on $x \in [0, 1]$. Now we use Lemma 2.2(ii) and (iii) together with (3.8) to get along the same lines as (3.9) that

$$Q_{TJ^{-2},1,\beta_v}\left(R_v(TJ^{-2}, 1) \leq K_0 h(\beta_v, 1)^{1/2}\beta_v^{1/3}\right)$$
$$= Q_{T,J,\beta_u}\left(R_u(T, J) \leq K_0 h(\beta_u, J)^{1/2}\beta_u^{1/3}J^{5/3}\right),$$

where $\beta_u = J^{-7/2}\beta_v$. From (1.11), (1.12) and Proposition 4.1(i) with $J = 1$ we get that there exists $K_0 > 0$ such that for any $\beta_v \geq e$ we have,

$$\lim_{T\to\infty} Q_{T,1,\beta_v}\left(R_v(T, 1) \leq K_0 h(\beta_v, 1)^{1/2}\beta_v^{1/3}\right) = 1.$$

It follows that

$$\lim_{T\to\infty} Q_{T,J,\beta_u}\left(R_u(T, J) \leq K_0 \beta_u^{1/3} h(\beta_u, J)^{1/2} J^{5/3}\right) = 1,$$

for $\beta_u = J^{-7/2}\beta_v \geq eJ^{-7/2}$, so we get (i) for all $J > 0$.

From Lemma 4.4 with $J = 1$, we get that there exists $K_0 > 0$ such that for any $0 < \beta_v \leq e$ we have,

$$\lim_{T\to\infty} Q_{T,1,\beta_v}\left(R_v(T, 1) \leq K_0\right) = 1.$$

Therefore, using the same scaling argument, for any $J \geq 1$ and $\beta_u \leq eJ^{-7/2}$ we have

$$\lim_{T\to\infty} Q_{T,J,\beta_u}\left(R_u(T, J) \leq \tilde{K}_0 J^{5/3}\right) = 1,$$

where $\tilde{K}_0 = K_0 J^{-7/6} \leq K_0$. Then we get (i) for all $J \geq 1$.

This finishes the proof of Proposition 4.1. $\qquad\square$

5. Scaling of the Polymer

This section is dedicated to the proof of Lemma 2.2. Throughout this section we assume that $u(t,x)$ satisfies equation (1.2) on $x \in [0, J]$ and that v is defined as in (2.3).

Proof of Lemma 2.2. (i) This is a well known property of the stochastic heat equation, but we give a short derivation for completeness. First, recall that $(G_t^J(x))_{t \geq 0, x \in [0,J]}$ is the Neumann heat kernel on $[0, J]$, and that the following scaling equalities hold.

$$G_{J^{-2}t}^1(J^{-1}x, J^{-1}y) = J \cdot G_t^J(x, y),$$

$$W^{J^2,J}(dy\,ds) := J^{-3/2}W(Jdy, J^2ds) \overset{\mathcal{D}}{=} W(dyds),$$

where $\overset{\mathcal{D}}{=}$ means equality in distribution. The first equality above follows from (1.5) and the scaling properties of the heat kernel on $\mathbf{R}$. Then by the definition (2.3) of $v(t,x)$,

$$
\begin{aligned}
v(t,z) &= J^{-1/2}u(J^2t, Jz) \\
&= J^{-1/2}\int_0^J G_J(J^2t, Jz - y)u_0(y)dy \\
&\quad + J^{-1/2}\int_0^{J^2t}\int_0^J G_J(J^2t - s, Jz - y)W(dyds) \\
&= J^{-3/2}\int_0^J G_1(t, z - J^{-1}y)u_0(y)dy \\
&\quad + J^{-3/2}\int_0^{J^2t}\int_0^J G_1(t - J^{-2}s, z - J^{-1}y)W(dyds) \\
&= \int_0^1 G_1(t, z - w)v_0(w)dw \\
&\quad + \int_0^t\int_0^1 G_1(t - r, z - w)W^{J^2,J}(dwdr),
\end{aligned}
$$

This proves part (i).

(ii) We first derive the relation between ℓ_t^u and ℓ_t^v. From (1.6) and (2.3) we get

$$\ell_t^v(y) = \partial_y \int_0^1 \mathbf{1}_{(-\infty,y]}(v(t,x))dx$$

$$= \partial_y \int_0^1 \mathbf{1}_{(-\infty,y]}(J^{-1/2}u(J^2t, Jx))dx$$

$$= \partial_y \int_0^1 \mathbf{1}_{(-\infty,J^{1/2}y]}(u(J^2t, Jx))dx.$$

Changing variables to $z = Jx$, $w = J^{1/2}y$ we get

$$\ell_t^v(y) = J^{1/2}\partial_w \int_0^J \mathbf{1}_{(-\infty,w]}(u(J^2t, z))J^{-1}dz$$

$$= J^{-1/2}\ell_{J^2t}^u(w).$$

It follows that

$$\ell_t^u(w) = J^{1/2}\ell_{J^{-2}t}^v(y) = J^{1/2}\ell_{J^{-2}t}^v(J^{-1/2}w).$$

Thus,

$$\int_0^T \int_{-\infty}^\infty \ell_t^u(w)^2dwdt = J \int_0^T \int_{-\infty}^\infty \ell_{J^{-2}t}^v(J^{-1/2}w)^2dwdt.$$

Making the change of variable $s = J^{-2}t$ and going back to the original variable $w = J^{1/2}y$, we get

$$(5.1) \quad \begin{aligned}\int_0^T \int_{-\infty}^\infty \ell_t^u(w)^2dwdt &= J \int_0^{J^{-2}T} \int_{-\infty}^\infty \ell_s^v(y)^2 J^{1/2}dy J^2 ds \\ &= J^{7/2} \int_0^{J^{-2}T} \int_{-\infty}^\infty \ell_s^v(w)^2dyds.\end{aligned}$$

Let $\beta > 0$. From (1.7) and (5.1) we get

$$\mathcal{E}_{T,J,\beta_u}^u = \exp\left(-\beta \int_0^T \int_{-\infty}^\infty \ell_t^u(y)^2dydt\right)$$

$$= \exp\left(-\beta J^{7/2} \int_0^{J^{-2}T} \int_{-\infty}^\infty \ell_s^v(w)^2dyds\right)$$

$$= \mathcal{E}_{J^{-2}T,1,\beta J^{7/2}}^v.$$

From the above equation together with (1.8) we get

$$Q_{T,J,\beta}^u(\cdot) = Q_{TJ^{-2},1,\beta J^{7/2}}^v(\cdot),$$

and we have proved (ii).

(iii) Recall the definitions of $R_v(T, 1)$ and $R_u(J^2T, J)$ from (2.2). From (1.9) and (2.3) we get

$$
\begin{aligned}
\bar{v}(t) &= \int_0^1 v(t, x)dx \\
&= J^{-1/2} \int_0^1 u(J^2t, Jx)dx \\
&= J^{-3/2} \int_0^J u(J^2t, x)dx \\
&= J^{-1/2}\bar{u}(J^2t, x).
\end{aligned}
$$

(5.2)

Using (2.3), (5.2) and making the change of variables $s = J^2t$, $y = Jx$, we get

$$
\begin{aligned}
R_v(T, 1)^2 &= \frac{1}{T} \int_0^T \int_0^1 \Big(v(t, x) - \bar{v}(t)\Big)^2 dxdt \\
&= \frac{1}{TJ} \int_0^T \int_0^1 \Big(u(J^2t, Jx) - \bar{u}(J^2t)\Big)^2 dxdt \\
&= \frac{1}{TJ^4} \int_0^{TJ^2} \int_0^J \Big(u(s, y) - \bar{u}(s)\Big)^2 dydt \\
&= J^{-1} R_u(J^2T, J)^2.
\end{aligned}
$$

Taking the square root and defining $\tilde{T} = TJ^{-2}$, we have

$$
R_u(\tilde{T}, J) = J^{1/2} R_v(J^{-2}\tilde{T}, 1),
$$

and we get (iii). $\qquad\qquad\square$

6. Properties of the pinned string

This section we prove Lemmas 2.4, 2.5 and 2.6.

Proof of Lemmas 2.4. (i) Recall that (C) was defined in (2.14). Using Ito's isometry, (2.5), (2.15), Fubini's theorem and the fact that $\{\varphi_n\}_{n\geq 0}$

is an orthonormal basis, we get

$$\hat{E}[(C)^2] = \int_{-\infty}^{t_0} \int_0^1 \left[G_{t-r}(x,y) - G_{t_0-r}(x_0,y)\right]^2 dy dr$$

$$= \int_{-\infty}^{t_0} \sum_{n=0}^{\infty} \sum_{j=0}^{\infty} \left[e^{-(t-r)\lambda_n} \varphi_n(x) - e^{-(t_0-r)\lambda_n} \varphi_n(x_0)\right]$$

(6.1)
$$\times \left[e^{-(t-r)\lambda_j} \varphi_j(x) - e^{-(t_0-r)\lambda_j} \varphi_j(x_0)\right] \int_0^1 \varphi_n(y)\varphi_j(y) dy dr$$

$$= \int_{-\infty}^{t_0} \sum_{n=0}^{\infty} \left[e^{-(t-r)\lambda_n} \varphi_n(x) - e^{-(t_0-r)\lambda_n} \varphi_n(x_0)\right]^2 dr$$

$$\leq 2 \int_0^{\infty} \sum_{n=1}^{\infty} \left[e^{-(t-t_0+r)\lambda_n} + e^{-r\lambda_n}\right]^2 dr,$$

where we dropped the $n = 0$ term because $\lambda_0 = 0$ and $\varphi_0 \equiv 1$, so the difference in the brackets is zero. Also, we used the fact that $\|\varphi_n\|_{\infty}^2 \leq 2$ for all $n \geq 1$, which follows from (2.5). It follows that

$$\hat{E}[(C)^2] \leq 2 \sum_{n=0}^{\infty} \int_0^{\infty} \left[2e^{-r\lambda_n}\right]^2 dr \leq 8 \sum_{n=0}^{\infty} (2\lambda_n)^{-1} < \infty,$$

where we used (2.5) in the last inequality.

(ii) From (2.16), we get

$$|(D)| = \left|\int_{-\infty}^{t_0} \int_0^1 \left[G_{t-r}(x,y) - G_{t_0-r}(x_0,y)\right] \varphi_1(y) dy dr\right|$$

$$= \left|\int_0^{\infty} \int_0^1 \left[G_{t-t_0+r}(x,y) - G_r(x_0,y)\right] \varphi_1(y) dy dr\right|$$

$$\leq \int_0^{\infty} \left[e^{-\lambda_1(t-t_0+r)} + e^{-\lambda_1 r}\right] |\varphi_1(x)| dr$$

$$< \infty.$$

$\square$

Proof of Lemma 2.5. Let $\hat{f}_{t,x_1,x_2}$ be the joint probability density of $(u(t,x_1), u(t,x_2))$ under $\hat{P}_T^{(a)}$.

Let $S_1, S_2 \subset \mathbf{R}$. Then from (1.6) we have

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mathbf{1}_{S_1}(y_1)\mathbf{1}_{S_2}(y_2)\ell_t(y_1)\ell_t(y_2)dy_2dy_1$$

$$= \int_0^1 \int_0^1 \mathbf{1}_{S_1}(u(t,x_1))\mathbf{1}_{S_2}(u(t,x_2))dx_2dx_1.$$

Therefore

$$\hat{E}\left[\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mathbf{1}_{S_1}(y_1)\mathbf{1}_{S_2}(y_2)\ell_t(y_1)\ell_t(y_2)dy_2dy_1\right]$$

$$= \int_0^1 \int_0^1 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mathbf{1}_{S_1}(y_1)\mathbf{1}_{S_2}(y_2)\hat{f}_{t,x_1,x_2}(y_1,y_2)dy_2dy_1dx_2dx_1.$$

By using Dynkin's monotone class theorem we can move from indicator functions to simple functions to nonnegative measurable functions to bounded measurable functions $(h(y_1,y_2))_{y_1,y_2\in\mathbf{R}}$, we get

$$\hat{E}\left[\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(y_1,y_2)\ell_t(y_1)\ell_t(y_2)dy_2dy_1\right]$$

$$= \int_0^1 \int_0^1 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(y_1,y_2)\hat{f}_{t,x_1,x_2}(y_1,y_2)dy_2dy_1dx_2dx_1.$$

Finally, replacing $h(y_1,y_2)$ by an approximate identity $a(y_1-y_2)$ and taking a limit, we get

$$\hat{E}\left[\int_{-\infty}^{\infty} \ell_t(y)^2dy\right] = \int_0^1 \int_0^1 \int_{-\infty}^{\infty} \hat{f}_{t,x_1,x_2}(y,y)dydx_2dx_1.$$

Recall that $\hat{g}_{t,x_1,x_2}$ is the probability density function of $u(t,x_2) - u(t,x_1)$ under $\hat{P}_T^{(a)}$. Then by elementary probability,

$$\hat{g}_{t,x_1,x_2}(z) = \int_{-\infty}^{\infty} \hat{f}_{t,x_1,x_2}(y,z+y)dy$$

and we get that

$$\hat{E}\left[\int_{-\infty}^{\infty} \ell_t(y)^2dy\right] = \int_0^1 \int_0^1 \hat{g}_{t,x_1,x_2}(0)dx_2dx_1.$$

$\square$

Proof of Lemma 2.6. Let $0 \le t_0 \le T$ and $x_0 \in [0,1]$. From (2.14) we have for any $t_0 \le t \le T$ and $x_1, x_2 \in [0,1]$,

$$u_{t_0,x_0}(t,x_1) - u_{t_0,x_0}(t,x_2)$$

$$= \int_{t_0}^t \int_0^1 \big(G_{t-r}(x_1,y) - G_{t-r}(x_2,y)\big)\hat{W}(dydr)$$

$$+ a \int_{t_0}^t \int_0^1 \big(G_{t-r}(x_1,y) - G_{t-r}(x_2,y)\big)\varphi_1(y)dydr$$

$$+ \int_{-\infty}^{t_0} \int_0^1 \big(G_{t-r}(x_1,y) - G_{t-r}(x_2,y)\big)\hat{W}(dydr)$$

$$+ a \int_{-\infty}^{t_0} \int_0^1 \big(G_{t-r}(x_1,y) - G_{t-r}(x_2,y)\big)\varphi_1(y)dydr$$

$$= \int_{-\infty}^t \int_0^1 \big(G_{t-r}(x_1,y) - G_{t-r}(x_2,y)\big)\hat{W}(dydr)$$

$$+ a \int_{-\infty}^t \int_0^1 \big(G_{t-r}(x_1,y) - G_{t-r}(x_2,y)\big)\varphi_1(y)dydr$$

$$=: \mathcal{J}_1(t,x_1,x_1) + a\mathcal{J}_2(t,x_1,x_1).$$

Note that $\mathcal{J}_1(t,\cdot,\cdot)$ is a centered Gaussian random field with the following covariance functional

$$C_t(x_1,x_2) := E\left[\int_{-\infty}^t \int_0^1 (G_{t-r}(x_1,y) - G_{t-r}(x_2,y))^2 dydr\right].$$

By a change of variable $s = t - r$ we get

(6.2)
$$C_t(x_1,x_2) = E\left[\int_0^\infty \int_0^1 (G_s(x_1,y) - G_s(x_2,y))^2 dyds\right]$$
$$= C_0(x_1,x_2).$$

So we have

$$\mathcal{J}_1(t,\cdot,\cdot) \stackrel{\mathcal{D}}{=} \mathcal{J}_1(0,\cdot,\cdot).$$

We also get by the change of variable $s = t - r$ that

$$\mathcal{J}_2(t,x_1,x_2) = a \int_{-\infty}^t \int_0^1 \big[G_{t-r}(x_1,y) - G_{t-r}(x_2,y)\big]\varphi_1(y)dydr$$

$$= a \int_0^\infty \int_0^1 \big[G_s(x_1,y) - G_s(x_2,y)\big]\varphi_1(y)dydr$$

$$= \mathcal{J}_2(0,x_1,x_2),$$

and Lemma 2.6 follows. $\qquad\square$

7. Proofs of Lemmas 2.7–2.9

Proof of Lemma 2.7. The proof of the upper bound is standard (see e.g. Proposition 3.7 in Chapter III.4 of [19]).

Next we prove the lower bound for

$$E\left[(u(t,x_1) - u(t,x_2))^2\right] = \int_0^\infty \int_0^1 [G_t(x_1,y) - G_t(x_2,y)]^2 \, dy dt.$$

From (2.4), (2.5) and (2.15) we have

(7.1) $$G_t(x,y) = 1 + 2\sum_{n=1}^\infty e^{-\pi^2 n^2 t} \cos(\pi nx) \cos(\pi ny).$$

By repeating similar steps as in (6.1) and using the orthogonality of $(\varphi_n)_{n \geq 1}$ we get

(7.2)
$$\begin{aligned}
\mathcal{J} &:= \int_0^\infty \int_0^1 [G_t(x_1,y) - G_t(x_2,y)]^2 \, dy dt \\
&= 2\sum_{n=1}^\infty \int_0^\infty e^{-2\pi^2 n^2 t} [\cos(\pi nx_1) - \cos(\pi nx_2)]^2 \, dt \\
&= \frac{1}{\pi^2} \sum_{n=1}^\infty \frac{1}{n^2} [\cos(\pi nx_1) - \cos(\pi nx_2)]^2.
\end{aligned}$$

Let $x = (x_1 + x_2)/2$ and $h = x_2 - x_1$. It suffices to prove our estimate for $h < \delta_0$ where $\delta_0 > 0$ is some small number to be chosen later. By symmetry we may assume $x \in [0, 1/2]$, otherwise we reflect the configuration about $1/2$. Also note that $0 \leq x_1 = x - h/2$ so $x \geq h/2$. Let $\delta_1 > 0$ be a small number to be chosen later, and let $M = [2h^{-1}(1 - \delta_1)]$ where $[\cdot]$ denotes the greatest integer function.

Recall the following trigonometric identities.

$$\cos(a) - \cos(b) = -2\sin\left(\frac{a-b}{2}\right)\sin\left(\frac{a+b}{2}\right)$$

$$\sum_{n=1}^M \sin^2(nx) = \frac{1}{4}\left[2M + 1 - \frac{\sin((2M+1)x)}{\sin(x)}\right]$$

Continuing to estimate the above integral $\mathcal{J}$, we get (since $h < 1/4$) and using the above trigonometric identities,

$$\mathcal{J} = \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \sin^2\left(n\pi 2^{-1}h\right) \sin^2\left(n\pi x\right)$$

$$\geq \frac{4}{\pi^2} \sum_{n=1}^{M} \frac{1}{n^2} \sin^2\left(n\pi 2^{-1}h\right) \sin^2\left(n\pi x\right)$$

$$\geq Ch^2 \sum_{n=1}^{M} \sin^2\left(n\pi x\right)$$

$$= Ch^2 \left[2M + 1 - \frac{\sin((2M+1)\pi x)}{\sin(\pi x)}\right]$$

For the second to the third line, we require

$$(7.3) \qquad \pi(1 - \delta_1) - 1 \leq M\pi 2^{-1}h \leq \pi(1 - \delta_1)$$

which holds by the definition of M. This implies that for $1 \leq n \leq M$ we have $\sin(n\pi 2^{-1}h) \geq cnh$ for some $c > 0$ not depending on n.

Now we wish to show that the term in square brackets above is of order M. We need to show that for some small number $\delta_2 > 0$ to be chosen later,

$$(7.4) \qquad \frac{\sin((2M+1)\pi x)}{\sin(\pi x)} \leq 2M(1 - \delta_2).$$

First, $\sin((2M+1)\pi x) \leq 1$. We also want a lower bound on $\sin(\pi x)$. Recall that $h/2 \leq x \leq 1/2$ and $h < \delta_0$. Since $\sin(x)$ is increasing on $x \in [h/2, 1/2]$, if $\delta_3 > 0$ then we can choose δ_0 small enough so that

$$\sin(\pi x) \geq \sin(\pi 2^{-1}h) \geq \pi 2^{-1}h(1 - \delta_3).$$

Thus if we assume equality holds in (7.3) then we have

$$\frac{\sin((2M+1)\pi x)}{\sin(\pi x)} \leq \frac{1}{\pi 2^{-1}h(1 - \delta_3)}$$

$$= M \cdot \frac{1}{(M\pi 2^{-1}h)(1 - \delta_3)}$$

$$\leq M \cdot \frac{1}{[\pi(1 - \delta_1) - 1](1 - \delta_3)},$$

which verifies (7.4), provided $\delta_1, \delta_2, \delta_3$ are small enough.

The end result is that for δ_2 small enough, there exists $\widetilde{C} > 0$ such that,

$$\mathcal{J} \geq Ch^2 M \delta_2 \geq 2\widetilde{C}h\delta_2 = 2\widetilde{C}|x_1 - x_2|\delta_2, \quad \text{for all } |x_1 - x_2| \leq \delta_0,$$

where we have used (7.3) in the second inequality. □

Proof of Lemma 2.8. Let $0 \leq x_1 \leq x_2 \leq 1$. Then we have

$$\mathcal{D}(x_1, x_2) := (F)(x_1) - (F)(x_2)$$

$$= \frac{\sqrt{2}}{\pi^2} \big(\cos(\pi x_1) - \cos(\pi x_2) \big)$$

$$= \frac{2\sqrt{2}}{\pi^2} \sin\left(\frac{\pi}{2}(x_1 + x_2) \right) \sin\left(\frac{\pi}{2}(x_2 - x_1) \right).$$

We will use the following lower bounds on the sine function:

$$\sin(\pi x) \geq \frac{x}{2}, \quad \text{for all } 0 \leq x \leq \frac{1}{2},$$

$$\sin(\pi x) \geq 1 - x, \quad \text{for all } \frac{1}{2} \leq x \leq 1.$$

We get that

$$\sin\left(\frac{\pi}{2}(x_2 - x_1) \right) \geq \frac{x_2 - x_1}{4}, \quad \text{for all } 0 \leq x_1 \leq x_2 \leq 1.$$

For $0 \leq x_1 \leq x_2 \leq 1$ we also have

$$\sin\left(\frac{\pi}{2}(x_2 + x_1) \right) \geq \begin{cases} \frac{x_2 + x_1}{4}, & \text{for } x_1 + x_2 \leq 1, \\ 1 - \frac{x_2 + x_1}{2}, & \text{for } 1 < x_1 + x_2 \leq 2. \end{cases}$$

We get that for all $0 \leq x_1 \leq x_2 \leq 1$

$$\mathcal{D}(x_1, x_2) \geq \begin{cases} \frac{\sqrt{2}}{8\pi^2}(x_2 - x_1)(x_2 + x_1), & \text{for } x_1 + x_2 \leq 1, \\ \frac{\sqrt{2}}{2\pi^2}(x_2 - x_1)\left(1 - \frac{x_2 + x_1}{2}\right), & \text{for } 1 < x_1 + x_2 \leq 2. \end{cases}$$

□

Proof of Lemma 2.9. First we bound $\mathcal{I}_2(a)$. Recall that

$$\mathcal{I}_2(a) = \int_0^1 \int_{x_1}^1 \mathbf{1}_{\{x_1 + x_2 > 1\}}(x_2 - x_1)^{-1/2}$$

$$\times \exp\left(-C_4 a^2 \left(1 - \frac{x_2 + x_1}{2}\right)^2 (x_2 - x_1) \right) dx_1 dx_2.$$

Using the transformation

$$x_2' - x_1' = x_2 - x_1,$$

$$x_2' + x_1' = 1 - \frac{x_2 + x_1}{2},$$

we get that

$$\mathcal{I}_2(a)$$
$$\leq \frac{1}{2} \int_{-1}^{1} \int_{x_1'}^{1} (x_2' - x_1')^{-1/2} \exp\left(-C_4 a^2 (x_1' + x_2')^2 (x_2' - x_1')\right) dx_1' dx_2',$$

where the inequality is due to an enlargement of the integration region.

Letting $x_1' = x$ and $x_2' - x_1' = h$ we get that

$$(7.5) \qquad \mathcal{I}_2(a) \leq \frac{1}{2} \int_{-1}^{1} \int_{0}^{1} h^{-1/2} \exp\left(-C_4 a^2 (2x + h)^2 h\right) dh dx.$$

We distinct between the following cases.

Case I: $0 \leq a \leq 1$.
Let $y = 2a^{2/3}x$, $g = a^{2/3}h$. Then we get from (7.5) that

$$\mathcal{I}_2(a) \leq \frac{1}{4a} \int_{-a^{2/3}}^{a^{2/3}} \int_{0}^{2a^{2/3}} g^{-1/2} \exp\left(-\frac{C_4}{2}(y + g)^2 g\right) dg dy$$
$$\leq \frac{1}{4a} \int_{-a^{2/3}}^{a^{2/3}} \int_{0}^{2a^{2/3}} g^{-1/2} dg dy$$
$$\leq C\frac{1}{a},$$

where C does not depend on a. We therefore derived the bound (i) for $\tilde{\mathcal{I}}_2(a)$.

Case II: $a > 1$.
We split the right-hand side of (7.5) into two regions:

$$\frac{1}{2} \int_{-1}^{1} \int_{0}^{1} h^{-1/2} \exp\left(-C_4 a^2 (2x + h)^2 h\right) dh dx$$

$$(7.6) \qquad \begin{aligned} &= \frac{1}{2} \int_{-1}^{1} \int_{0}^{\frac{1}{a^2}} h^{-1/2} \exp\left(-C_4 a^2 (2x + h)^2 h\right) dh dx \\ &\quad + \frac{1}{2} \int_{-1}^{1} \int_{\frac{1}{a^2}}^{1} h^{-1/2} \exp\left(-C_4 a^2 (2x + h)^2 h\right) dh dx \\ &=: \frac{1}{2}\left(\mathcal{I}_{2,1}(a) + \mathcal{I}_{2,2}(a)\right). \end{aligned}$$

We first deal with $\mathcal{I}_{2,1}(a)$. By making the same change of variable as in case I, that is $y = 2a^{2/3}x$ and $g = a^{2/3}h$, we get that

$$\mathcal{I}_{2,1}(a) = \frac{1}{2a} \int_{-2a^{2/3}}^{2a^{2/3}} \int_0^{a^{-4/3}} g^{-1/2} \exp\left(-C_4(y+g)^2 g\right) dg dy$$

(7.7)
$$\leq \frac{1}{2a} \int_{-2a^{2/3}}^{2a^{2/3}} \int_0^{a^{-4/3}} g^{-1/2} dg dy$$

$$= \frac{1}{2a} \int_{-2a^{2/3}}^{2a^{2/3}} 2a^{-2/3} dy$$

$$= \frac{4}{a}.$$

Now we deal with $\mathcal{I}_{2,2}(a)$. Define

$$\sigma^2 = \frac{1}{8C_4 h a^2},$$

and using the Gaussian density we get

$$\int_{-\infty}^{\infty} \exp\left(-C_4 a^2 \left(2x+h\right)^2 h\right) dx = \int_{-\infty}^{\infty} \exp\left(-\frac{(x+h/2)^2}{2\sigma^2}\right) dx$$

$$= \sqrt{2\pi}\sigma.$$

It follows that

$$\mathcal{I}_{2,2}(a) = \int_{\frac{1}{a^2}}^1 h^{-1/2} \int_{-\infty}^{\infty} \exp\left(-C_4 a^2 \left(2x+h\right)^2 h\right) dx dh$$

$$= \int_{\frac{1}{a^2}}^1 h^{-1/2} \sqrt{2\pi}\sigma dh$$

(7.8)
$$= \sqrt{\frac{2\pi}{8C_4}} \frac{1}{a} \int_{\frac{1}{a^2}}^1 h^{-1} dh$$

$$= C \frac{\log a}{a},$$

where C is independent of a. From (7.5)–(7.8) we get (ii) for $\mathcal{I}_2(a)$.

Next, we bound $\mathcal{I}_1(a)$. By setting $h = x_2 - x_1$ and $x = x_1$ we get

$$\mathcal{I}_1(a) \leq \int_0^1 \int_0^1 \mathbf{1}_{\{2x+h\leq 1\}} h^{-1/2} \exp\left(-C_3 a^2 (x_1+x_2)^2 h\right) dh dx$$

$$\leq \int_0^1 \int_0^1 h^{-1/2} \exp\left(-C_3 a^2 (x_1+x_2)^2 h\right) dh dx.$$

We can therefore bound $\mathcal{I}_1(a)$ similarly as we bounded the right hand side of (7.5), so we are done. $\qquad\square$

Appendix A. Some heuristic ideas

For intuitive purposes, we define R so that the processes in this section are restricted to a ball of radius R.

A.1. **Flory's argument for self-avoiding walk in two dimensions.** Since this is an imprecise argument, we will freely assume that the walk $(S_n)_{n \in \mathbf{N}_0}$ is a Brownian motion $B(t)$. Large deviation theory suggests that probabilities such as $Q_T(A)$ are dominated by a "most likely path" $(X(t))_{t \in [0,T]}$, with approximate probability

$$(\text{A.1}) \qquad \exp\left(-\beta \int_{\mathbf{R}^2} \ell_T^X(y)^2 dy\right) \exp\left(-\frac{1}{2} \int_0^T |X'(t)|^2 dt\right)$$

where the second exponential is, roughly speaking, the probability of a Brownian motion path whose increments are independent normal variables. Here we take ℓ_t^X to be the usual local time for X.

Large deviation theory also suggests that the overall most likely path is realized when the two exponential terms are comparable. Now suppose that the walk X is confined to a ball B_R centered at the origin and of radius R. The first exponential in (A.1) will be minimized when $\ell_T^X(y)$ is constant over $y \in B_R$, meaning that $\ell_T^X(y) = T/(\pi R^2)$.

$$(\text{A.2}) \qquad \int_{B_R} \ell_T^X(y)^2 dy = \frac{CT^2}{R^4} \cdot R^2 = \frac{CT^2}{R^2}.$$

As for the second exponential term in (A.1), if X reaches the boundary of B_R (as it must for ℓ_T^X to be constant on B_R), then the most likely path has constant velocity, $X(t) = (R/T)t$. Then

$$(\text{A.3}) \qquad \int_0^T |X'(t)|^2 dt = T \cdot \left(\frac{R}{T}\right)^2 = \frac{R^2}{T}.$$

Equating (A.2) and (A.3), we get

$$R = CT^{3/4}$$

which yields $\nu = 3/4$ as conjectured (see (1.1)).

We have ignored the inconvenient fact that the minimizing paths for the two exponential terms are different.

A.2. **An intuitive justification for Theorem 1.1.** The intuition is quite close to that in Section A.1. Indeed, in the exponential term involving local time we again assume that $\ell_t(y)$ is constant over $y \in [-R, R]$ and so $\ell_t(y) = J/(2R)$. Then,

$$(A.4) \qquad \int_0^T \int_{-R}^R \ell_t^2(y) dy dt = 2TR \left(\frac{J}{2R} \right)^2 = \frac{CTJ^2}{R}.$$

The main difference is that the approximate probability for a white noise $\dot{W}$, intuitively speaking, is $\exp(-\frac{1}{2} \int_0^T \int_0^J (\dot{W}(t, x))^2 dx dt)$, since we think of $(\dot{W}(t, x))_{t \in [0,T], x \in [0,J]}$ as a collection of independent Gaussian variables. But using (1.2) to substitute for $\dot{W}$, we get an approximate probability of

$$\exp \left(-\frac{1}{2} \int_0^T \int_0^J \left[(\partial_t u - \partial_x^2 u)(t, x) \right]^2 dx dt \right).$$

In such problems the minimizer u is often constant in t, giving us

$$\exp \left(-\frac{T}{2} \int_0^J \left[\partial_x^2 u(t, x) \right]^2 dx \right).$$

We might think that the minimizer u has a constant value of $|\partial_x^2 u|$, consistent with the Neumann boundary conditions. Such a function could be

$$u(x) = \begin{cases} ax^2 - aJ^2/4 & \text{if } x \in [0, J/2], \\ a(J - x)^2 - aJ^2/4 & \text{if } x \in [J/2, J]. \end{cases}$$

If we require $u(0) = R$ and $u(J) = -R$ then we get $a = 4R/J^2$ and $|u''(x)| = 2a = 8R/J^2$. Then

$$(A.5) \qquad \frac{T}{2} \int_0^J \left[\partial_x^2 u(t, x) \right]^2 dx = CTJ \left(\frac{R}{J^2} \right)^2 = \frac{CTR^2}{J^3}.$$

Equating (A.4) and (A.5), we get

$$R = CJ^{5/3}$$

which gives the dependence of R on J in Theorem 1.1.

A.3. **Intuition behind the Conjecture.** The intuition in this case is almost the same as in Section A.2. Again, the approximate probability of a path is given as a product of two exponentials as in (A.1). The second exponential is the same as in (A.5), giving an exponent of

$$(A.6) \qquad \qquad \frac{CTR^2}{J^3}$$

For the local time exponential, we are using a two dimensional ball B_R of volume CR^2, so we get an exponent of

$$(A.7) \qquad \int_0^T \int_{-R}^R \ell_t^2(y) dy = CTR^2 \left(\frac{J}{R^2} \right)^2 = \frac{CTJ^2}{R^2}.$$

Equating the terms (A.6) and (A.7), we get

$$R = CJ^{5/4}$$

as in Conjecture 1.1.

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