

SMALL BALL PROBABILITIES AND A SUPPORT THEOREM FOR THE STOCHASTIC HEAT EQUATION

SIVA ATHREYA, MATHEW JOSEPH, AND CARL MUELLER

ABSTRACT. We consider the following stochastic partial differential equation on $t \geq 0, x \in [0, J], J \geq 1$ where we consider $[0, J]$ to be the circle with end points identified:

$$\partial_t \mathbf{u}(t, x) = \frac{1}{2} \partial_x^2 \mathbf{u}(t, x) + \mathbf{g}(t, x, \mathbf{u}) + \sigma(t, x, \mathbf{u}) \dot{\mathbf{W}}(t, x),$$

and $\dot{\mathbf{W}}(t, x)$ is 2-parameter d -dimensional vector valued white noise and σ is function from $\mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}^d$ to space of symmetric $d \times d$ matrices which is Lipschitz in \mathbf{u} . We assume that σ is uniformly elliptic and that \mathbf{g} is uniformly bounded. Assuming that $\mathbf{u}(0, x) \equiv \mathbf{0}$, we prove small-ball probabilities for the solution u . We also prove a support theorem for solutions, when $\mathbf{u}(0, x)$ is not necessarily zero.

1. INTRODUCTION

In this article we study small-ball probabilities and support theorems for solutions to the stochastic heat equation given by

$$(1.1) \quad \partial_t \mathbf{u}(t, x) = \frac{1}{2} \partial_x^2 \mathbf{u}(t, x) + \mathbf{g}(t, x, \mathbf{u}) + \sigma(t, x, \mathbf{u}) \dot{\mathbf{W}}(t, x).$$

where, $t \in \mathbb{R}_+, x \in \mathbb{R}$, $\dot{\mathbf{W}}(t, x) = (\dot{\mathbf{W}}_1(t, x), \dots, \dot{\mathbf{W}}_d(t, x))$ is d -dimensional space-time white noise, with $d \geq 1$ and σ is a function from $\mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}^d$ to space of symmetric $d \times d$ matrices. Assuming that σ is Lipschitz in \mathbf{u} and uniformly elliptic, \mathbf{g} is uniformly bounded, and $\mathbf{u}(0, x) \equiv \mathbf{0}$, our main result Theorem 1.1 provides upper and lower bounds for the small ball probabilities of the solution to (1.1). This result gives bounds on the probability that the profile $\mathbf{u}(t, \cdot)$ stays close to the zero profile up to time T , see Theorem 1.1 for the precise statement. As a consequence of the above result we are able to prove a *support theorem*, which provides similar bounds

2010 *Mathematics Subject Classification.* Primary, 60H15; Secondary, 60G17, 60G60.

Key words and phrases. heat equation, white noise, stochastic partial differential equations, small ball, support.

on the probability that the profile $\mathbf{u}(t, \cdot)$ stays close to a twice differentiable function up to time T , see Theorem 1.2 for the precise statement.

Small ball problems have been well studied and have a long history in probability theory, see [LS01] for a survey. More precisely, for a stochastic process X_t starting at 0, we are interested in the probability that X_t stays near its starting point for a long time, that is,

$$P\left(\sup_{0 \leq t \leq T} |X_t| < \varepsilon\right)$$

where $\varepsilon > 0$ is small. When X_t is Brownian motion, small ball estimates follow from the reflection principle or from the study of eigenvalues, among other techniques. Donsker and Varadhan [DV75] obtained small ball estimates for a wide class of Markov processes as a result of their theory of large deviations of local time. For other processes, the complexity of the small ball estimates are well-known. Moreover, in general small ball probabilities are usually harder to estimate than the the large deviation probability that X_t achieves unusually large values, that is

$$P\left(\sup_{0 \leq t \leq T} |X_t| > \lambda\right)$$

for large values of λ .

For some class of Gaussian processes small ball probabilities can be determined. Often these results are given in terms of metric entropy estimates which are hard to explicitly compute [KL93]. One exceptional case is the Brownian sheet, for which fairly sharp small ball estimates are explicitly known, see Bass [Bas88] and Talagrand [Tal94]. In [DKM⁺09], Xiao provides small ball estimates for Gaussian processes that satisfy a certain condition which is related to the Gaussian concept of local nondeterminism. In [LS01], an overview of known results on Gaussian processes and references on other processes such as fractional Brownian motion are given.

There has not been much exploration of small ball probabilities in the context of stochastic PDEs. Lototsky [Lot17] has studied small ball problems for a linear SPDE with additive white noise, where the solution is a Gaussian process. Martin [Mar04] followed the approach of Talagrand [Tal94] to study the following stochastic wave equation.

$$(1.2) \quad \partial_t^2 u = \partial_x^2 u + f(u) + g(t, x)\dot{W}(t, x)$$

where $\dot{W}(t, x)$ is two-parameter white noise and $g(t, x)$ is a deterministic function. Without $f(u)$, the solution u would be a Gaussian process of the type studied by Bass [Bas88] and Talagrand [Tal94]. Although (1.1) and (1.2) have similar multiplicative

noise terms, in our case the noise coefficient $\sigma(t, x, \mathbf{u})$ depends on the solution, and this dependence takes us away from Gaussian processes setting.

Small ball probabilities have many applications, for e.g. they play key role in studying the small scale behavior of Gaussian processes, such as the Hausdorff dimension of the range (see [Bas88], [Tal95]). Another key application is the support theorem. For e.g., in the case of Brownian motion an application of Girsanov Theorem yields the classical support theorem (See [Bas95, Proposition 6.5 and Theorem 6.6 on pages 59-60]). Once we have obtained small probability estimates for solution (1.1) then we use the Girsanov Theorem for SPDE to obtain a support theorem for solution.

We are now ready to state our main results.

1.1. Main Results. For any $\mathbf{u} \in \mathbb{R}^d$ we shall denote $|\mathbf{u}|$ to be the standard Euclidean norm on \mathbb{R}^d and $\langle \mathbf{u}, \mathbf{v} \rangle$ denote the inner product between $\mathbf{u}, \mathbf{v} \in \mathbb{R}^d$. $\mathcal{M}_d(\mathbb{R})$ will denote the space of symmetric $d \times d$ matrices over real numbers. Let $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ be a filtered Probability space on which $\dot{\mathbf{W}} = \dot{\mathbf{W}}(t, x)$ is a d -dimensional random vector whose components are i.i.d. two-parameter white noises adapted to \mathcal{F}_t .

We consider vector-valued solutions $\mathbf{u}(t, x) \in \mathbb{R}^d$, to the following stochastic heat equation (SHE)

$$(1.3) \quad \begin{aligned} \partial_t \mathbf{u}(t, x) &= \frac{1}{2} \partial_x^2 \mathbf{u}(t, x) + \mathbf{g}(t, x, \mathbf{u}(t, x)) + \sigma(t, x, \mathbf{u}(t, x)) \dot{\mathbf{W}}(t, x). \\ \mathbf{u}(0, x) &= \mathbf{u}_0(x) \equiv \mathbf{0}, \end{aligned}$$

on the circle with $x \in [0, J]$ and endpoints identified, and for the function $\sigma : \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}^d \rightarrow \mathcal{M}_d(\mathbb{R})$. We assume that $\mathbf{g} : \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ is uniformly bounded, $\sigma(t, x, \mathbf{u})$ is Lipschitz continuous in the third variable, that is there is a constant $\mathcal{D} > 0$ such that for all $t \geq 0$, $x \in [0, J]$, $\mathbf{u}, \mathbf{v} \in \mathbb{R}^d$,

$$(1.4) \quad |\sigma(t, x, \mathbf{u}) - \sigma(t, x, \mathbf{v})| \leq \mathcal{D} |\mathbf{u} - \mathbf{v}|.$$

We will further assume that the functions σ is uniformly elliptic, that is there are constants $\mathcal{C}_1, \mathcal{C}_2, > 0$ such that for all $t \geq 0$, $x \in [0, J]$, $\mathbf{u} \in \mathbb{R}^d$, $\mathbf{y} \in \mathbb{R}^d$ with $|\mathbf{y}| = 1$,

$$(1.5) \quad \mathcal{C}_1 \leq \langle \mathbf{y}, \sigma(t, x, \mathbf{u}) \mathbf{y} \rangle \leq \mathcal{C}_2.$$

The above implies that the matrix valued function σ is positive definite everywhere (in particular invertible) and that all the eigenvalues of σ are uniformly bounded above and away from 0.

As is usual in stochastic differential equations, (1.3) is not well-posed as written, because the solution \mathbf{u} is not differentiable and $\dot{\mathbf{W}}$ only exists as a generalized function. We take (1.3) to be shorthand for the mild form:

$$(1.6) \quad \begin{aligned} \mathbf{u}(t, x) &= \int_0^J G(t, x - y) \mathbf{u}_0(y) + \int_0^t \int_0^J G(t - s, x - y) \mathbf{g}(s, y, \mathbf{u}(s, y)) dy ds \\ &+ \int_0^t \int_0^J G(t - s, x - y) \sigma(s, y, \mathbf{u}(s, y)) \mathbf{W}(dy ds) \end{aligned}$$

where $G : \mathbb{R}_+ \times [0, J] \rightarrow \mathbb{R}$ is the fundamental solution of the heat equation

$$\begin{aligned} \partial_t G(t, x) &= \frac{1}{2} \partial_x^2 G(t, x) \\ G(0, x) &= \delta(x). \end{aligned}$$

where $[0, J]$ is the circle with endpoints identified. Furthermore, the final integral in (1.6) is a white noise integral in the sense of Walsh [Wal86]. We give more information about the heat kernel in Section 3.1. Given an initial profile \mathbf{u}_0 that is continuous then it is well known that there exists a unique strong solution to (1.3) (see for example section 6, page 23 of [DKM⁺09]; the proof there can be easily modified to cover (1.3)). We are now ready to state the main results of this paper.

Theorem 1.1. *Consider the solution to (1.3) and let the assumptions (1.4) and (1.5) hold. Then*

- (a) *There is a $\mathcal{D}_0(J, \mathcal{C}_1, \mathcal{C}_2) > 0$ and positive constants $\mathbf{C}_0, \mathbf{C}_1, \mathbf{C}_2, \mathbf{C}_3$ depending only on $d, \mathcal{C}_1, \mathcal{C}_2$ and ε_0 additionally depending on $\sup_{t, x, \mathbf{u}} |\mathbf{g}(t, x, \mathbf{u})|$ such that for any $\mathcal{D} < \mathcal{D}_0$ and all $0 < \varepsilon < \varepsilon_0, T > 0$ we have*

$$(1.7) \quad \mathbf{C}_0 \exp\left(-\mathbf{C}_1 \frac{TJ}{\varepsilon^6}\right) \leq P\left(\sup_{\substack{0 \leq t \leq T \\ x \in [0, J]}} |\mathbf{u}(t, x)| \leq \varepsilon\right) \leq \mathbf{C}_2 \exp\left(-\mathbf{C}_3 \frac{TJ}{\varepsilon^6}\right).$$

- (b) *For any \mathcal{D} and $0 < \delta < 1$, there exist positive constants $\mathbf{C}_0, \mathbf{C}_1, \mathbf{C}_2, \mathbf{C}_3$ depending only on $d, \mathcal{C}_1, \mathcal{C}_2$ and ε_0 additionally depending on $J, \mathcal{D}, \delta, \sup_{t, x, \mathbf{u}} |\mathbf{g}(t, x, \mathbf{u})|$ such that for all $0 < \varepsilon < \varepsilon_0, T > 0$ we have*

$$(1.8) \quad \mathbf{C}_0 \exp\left(-\mathbf{C}_1 \frac{TJ^{1+(\delta/2)}}{\varepsilon^{6+\delta}}\right) \leq P\left(\sup_{\substack{0 \leq t \leq T \\ x \in [0, J]}} |\mathbf{u}(t, x)| \leq \varepsilon\right) \leq \mathbf{C}_2 \exp\left(-\mathbf{C}_3 \frac{TJ}{(1 + J\mathcal{D}^2)\varepsilon^6}\right).$$

As stated earlier in the introduction, in [DKM⁺09], page 168, Theorem 5.1, Xiao proves a result similar to Theorem 1.1 in the Gaussian case, including a term ε^{-6} in the exponent. His argument builds on techniques from Gaussian processes, in particular Robeva and Pitt [RP04]. Xiao's condition C3', which is related to the Gaussian concept of local nondeterminism, is not always easy to verify. His result does not seem to carry over to (1.1) even in the case where the coefficients are functions of t, x but not of \mathbf{u} , and then the solution is a Gaussian process. In contrast, we make key use of the Markov property of (1.1), which allows us to extend our results to the non-Gaussian case in which the equation has coefficients that depend on the solution.

Before stating our next result we define the class of predictable functions.

Definition 1.1. *Let S be the set consisting of functions $f : \mathbb{R} \times [0, J] \times \Omega \rightarrow \mathbb{R}^d$ of the form*

$$f(x, t, \omega) = X(\omega) \cdot 1_A(x) \cdot 1_{(a,b]}(t),$$

with $0 < a < b < \infty$, $A \subset \mathbb{R}$, X an \mathcal{F}_a measurable random variable, and consider the predictable sigma-algebra \mathcal{P} generated by all functions in S . A function $\mathbf{h}(t, x, \omega) : \mathbb{R}_+ \times \mathbb{R} \times \Omega \rightarrow \mathbb{R}^d$ is said to be predictable if it is measurable with respect to \mathcal{P} . We will say a predictable function $\mathbf{h} \in \mathcal{PC}_b^2$, if with probability one \mathbf{h} , $\partial_t \mathbf{h}$, and $\partial_x^2 \mathbf{h}$ are uniformly bounded by a deterministic constant \mathbf{H} .

Theorem 1.2. *Consider the solution to (1.3) and let the assumptions (1.4) and (1.5) hold. Let $\mathbf{u}_0, \mathbf{h} \in \mathcal{PC}_b^2$ and assume*

$$\sup_{x \in [0, J]} |\mathbf{u}_0(x) - \mathbf{h}(0, x)| < \varepsilon/2$$

almost surely. Then

- (a) *There is a $\mathcal{D}_0(J, \mathcal{C}_1, \mathcal{C}_2) > 0$ and positive constants $\mathbf{C}_0, \mathbf{C}_1, \mathbf{C}_2, \mathbf{C}_3$ depending only on $d, \mathcal{C}_1, \mathcal{C}_2$ and ε_0 additionally depending on $\mathbf{H}, \sup_{t,x,\mathbf{u}} |\mathbf{g}(t, x, \mathbf{u})|$ such that for any $\mathcal{D} < \mathcal{D}_0$ and all $0 < \varepsilon < \varepsilon_0$, $T > 0$ we have*
- (1.9)

$$\mathbf{C}_0 \exp\left(-\mathbf{C}_1 \frac{TJ}{\varepsilon^6}\right) \leq P\left(\sup_{\substack{0 \leq t \leq T \\ x \in [0, J]}} |\mathbf{u}(t, x) - \mathbf{h}(t, x)| \leq \varepsilon\right) \leq \mathbf{C}_2 \exp\left(-\mathbf{C}_3 \frac{TJ}{\varepsilon^6}\right).$$

- (b) *For any \mathcal{D} and $0 < \delta < 1$, there exist positive constants $\mathbf{C}_0, \mathbf{C}_1, \mathbf{C}_2, \mathbf{C}_3$ depending only on $d, \mathcal{C}_1, \mathcal{C}_2$ and ε_0 additionally depending on $J, \mathcal{D}, \delta, \mathbf{H}, \sup_{t,x,\mathbf{u}}$*

$\mathbf{g}(t, x, \mathbf{u})$ | such that for all $0 < \varepsilon < \varepsilon_0$, $T > 0$ we have

$$(1.10) \quad \mathbf{C}_0 \exp\left(-\frac{\mathbf{C}_1 T J^{1+(\delta/2)}}{\varepsilon^{6+\delta}}\right) \leq P\left(\sup_{\substack{0 \leq t \leq T \\ x \in [0, J]}} |\mathbf{u}(t, x) - \mathbf{h}(t, x)| \leq \varepsilon\right) \\ \leq \mathbf{C}_2 \exp\left(-\frac{\mathbf{C}_3 T J}{(1 + J\mathcal{D}^2)\varepsilon^6}\right).$$

Our result is similar to the support theorem for Brownian motion given in [Bas95, Proposition 6.5 and Theorem 6.6 on pages 59-60]. Theorem 1.2 says for any *nice* function \mathbf{h} there is a positive probability that the solution \mathbf{u} gets *close* (within ε) to \mathbf{h} provided that the initial profile \mathbf{u}_0 is close (within $\varepsilon/2$) to $\mathbf{h}(0, \cdot)$. Support theorems, of a slightly different flavour, for (1.3) have been studied in the literature. In [BMSS95], a support theorem is proven for (1.6) when $d = 1$ and $J = 1$. Suppose $S(h)$ denotes the solution (1.6) when the white noise \dot{W} is replaced by $\dot{h} \in \mathcal{H}$, where $\mathcal{H} = \{h : [0, T] \times [0, 1] \rightarrow \mathbb{R} : h \text{ is absolutely continuous and } \dot{h} \in L^2([0, T] \times [0, 1])\}$.

When $u(0, x)$ is Hölder α with $\alpha < \frac{1}{2}$, g has bounded derivatives up to order 3, and σ is a Lipschitz function they show that the support of $\mathbb{P} \circ u^{-1}$ is the closure in the Hölder topology of the set $\{S(h) : h \in \mathcal{H}\}$.

We will now make a couple of remarks. These could be of independent interest.

Remark 1.1. (a) For Theorem 1.1, our assumptions on σ , \mathbf{g} need only hold until \mathbf{u} exits from the ε -ball and respectively until \mathbf{u} exits from the ε -ball around \mathbf{h} for Theorem 1.2.

(b) It will be clear from our proofs of part (a) in Theorem 1.1 and Theorem 1.2 that in fact $\mathcal{D}_0(J, \mathcal{C}_1, \mathcal{C}_2) = D_0 J^{-\frac{1}{2}}$ where D_0 depends on $\mathcal{C}_1, \mathcal{C}_2$ only. For part (b) of Theorem 1.1 and Theorem 1.2 we can choose $\varepsilon_0 = e_0 \cdot (J\mathcal{D}^2)^{-\frac{2}{5}}$ where e_0 depends only on $J, \mathcal{C}_1, \mathcal{C}_2, \sup_{t, x, \mathbf{u}} |\mathbf{g}(t, x, \mathbf{u})|$, and \mathbf{H} .

(c) Theorem 1.1 shows that under certain conditions. $\varepsilon^6 \log P\left(\sup_{x \in [0, J], t \leq T} |\mathbf{u}(t, x)| \leq \varepsilon\right)$ is bounded away from 0 and ∞ as $\varepsilon \rightarrow 0$. An important question is whether the limit exists (the limit is then called the small ball constant). This is in general a very difficult question (see the discussion in in Section 6 of [LS01]).

As noted earlier in the introduction, sharp small ball estimates for Gaussian processes are obtained using their special structures. Gaussian processes have many detailed properties and these do not hold for the stochastic heat equation (1.3). Therefore the

proofs of our above results will not follow by translating techniques used in proving small-ball probabilities in the literature.

However, by *freezing* the coefficient $\sigma(t, x, \mathbf{u})$ we may approximate u by a Gaussian random field, at least in a small time region. One of the key arguments of the paper is in showing that the error in the approximation can be well controlled if the time interval where the coefficient is frozen is suitably chosen.

The stochastic heat equation also has a Markov property with respect to the time parameter t , and this property plays an essential role in the proof of Theorem 1.1. Thanks to this property, we are able to reduce our analysis to the behaviour of the solution in small time intervals of order ε^4 . Roughly speaking, we show that the probability that the solution remains within ε in this small time increment is like $\exp(-C\varepsilon^{-2})$; this is the content of Proposition 2.1. Since there are $O(\varepsilon^{-4})$ such time intervals, we get our result.

The rest of the paper is organized as follows. In Section 2 we state the key Proposition 2.1 required to prove our main results, and reduce our problem to the case $\mathbf{g} \equiv \mathbf{0}$ and $J = 1$ using a couple of lemmas. We also explain how Theorem 1.2 follows from Theorem 1.1. Then in Section 3 we provide some heat kernel estimates which yield critical tail bounds on the noise term in Lemma 3.4. We conclude the paper with Section 4 where we prove Proposition 2.1 and then Theorem 1.1.

Convention on constants: Throughout the paper C denotes a positive constant whose value may change from line to line. All other constants will be denoted by C_1, C_2, \dots . These are positive with their precise values being not important. The dependence of constants on parameters when relevant will be denoted by special symbols or by mentioning the parameters in brackets, for e.g. $\mathcal{C}, C_1(\sigma, J)$.

2. SOME REDUCTIONS AND THE KEY PROPOSITION

We first explain how Theorem 1.2 follows immediately from Theorem 1.1. We next discuss how the analysis of Theorem 1.1 can be reduced to the case $\mathbf{g} \equiv \mathbf{0}$ and $J = 1$. Finally we state the key Proposition 2.1 which is the main ingredient in the proof of Theorem 1.1 and whose proof will occupy the majority of the paper.

2.1. Proof of Theorem 1.2. Before we proceed to the proof of Theorem 1.1, let us discuss how Theorem 1.2 follows from Theorem 1.1. Let

$$H = \frac{1}{2} \partial_x^2 - \partial_t$$

be the heat operator on $(t, x) \in \mathbb{R}_+ \times [0, J]$ where as usual, $[0, J]$ is the circle with endpoints identified. Let $\mathbf{h}_0(x) = \mathbf{h}(0, x)$ and let

$$\begin{aligned} \mathbf{w}(t, x) &= \mathbf{u}(t, x) - \mathbf{u}_0(x) - \mathbf{h}(t, x) + \mathbf{h}_0(x) \\ \mathbf{g}_1(t, x, \mathbf{w}) &= -\mathbf{g}(t, x, \mathbf{u}) - H\mathbf{u}_0(x) - H\mathbf{h}(t, x) + H\mathbf{h}_0(x) \\ \sigma_1(t, x, \mathbf{w}) &= \sigma(t, x, \mathbf{u}). \end{aligned}$$

We see that σ_1 is Lipschitz in \mathbf{w} with the same Lipschitz constant as σ . Furthermore, by the assumptions of Theorem 1.2, \mathbf{g}_1 is uniformly bounded by a deterministic constant, almost surely.

Then \mathbf{w} satisfies

$$\begin{aligned} \partial_t \mathbf{w}(t, x) &= \frac{1}{2} \partial_x^2 \mathbf{w}(t, x) - \mathbf{g}_1(t, x, \mathbf{w}) + \sigma_1(t, x, \mathbf{w}) \dot{\mathbf{W}}(t, x), \\ \mathbf{w}(0, x) &= \mathbf{0}. \end{aligned}$$

Now $\sup_x |\mathbf{u}_0(x) - \mathbf{h}_0(x)| \leq \varepsilon/2$, and so $\sup_x |\mathbf{w}(t, x)| \leq \varepsilon/2$ implies $\sup_x |\mathbf{u}(t, x) - \mathbf{h}(t, x)| \leq \varepsilon$.

Thus Theorem 1.2 follows from applying Theorem 1.1 to \mathbf{w} . □

The rest of the paper will be focused on the proof of Theorem 1.1.

2.2. Reduction to the case $\mathbf{g} \equiv \mathbf{0}$. We show that it is enough to prove Theorem 1.1 when $\mathbf{g} \equiv \mathbf{0}$. We will need the following Girsanov lemma and moment estimate on the Radon-Nikodym derivative.

Lemma 2.1. *Let $\mathbf{f} : [0, \infty) \times \mathbb{R}^d \times \Omega \rightarrow \mathbb{R}^d$ be a predictable function which is uniformly bounded by $M > 0$. Let P_t be P restricted to \mathcal{F}_t and consider the measure Q_t given by*

$$(2.1) \quad \frac{dQ_t}{dP_t} = \exp \left(Z_t^{(1)} - \frac{1}{2} Z_t^{(2)} \right),$$

where

$$\begin{aligned} Z_t^{(1)} &:= \int_0^t \int_0^J \mathbf{f}(s, x) \cdot \mathbf{W}(dx ds), \\ Z_t^{(2)} &:= \int_0^t \int_0^J |\mathbf{f}(s, x)|^2 dx ds, \end{aligned}$$

and $\mathbf{f} \cdot \mathbf{W}$ is the dot product of \mathbf{f} and \mathbf{W} in \mathbb{R}^d . Then

(a) Under the measure Q_t ,

$$\widetilde{\mathbf{W}}(s, x) := \dot{\mathbf{W}}(s, x) - \mathbf{f}(s, x), \quad x \in [0, J], \quad s \in [0, t]$$

is a d dimensional vector of i.i.d. two-parameter white noise.

(b) Furthermore

$$(2.2) \quad 1 \leq E \left[\left(\frac{dQ_t}{dP_t} \right)^2 \right] \leq \exp(M^2 t J).$$

Proof. (a) A stronger version of the statement can be found in [All98] but this is enough for our purposes. While [All98] deals only with $d = 1$, the extension below to higher dimensions is immediate.

(b) Since dQ_t/dP_t is a Radon-Nikodym derivative,

$$1 = E \left[\frac{dQ_t}{dP_t} \right] = E \left[\exp \left(Z_t^{(1)} - \frac{1}{2} Z_t^{(2)} \right) \right]$$

and replacing \mathbf{f} by $2\mathbf{f}$ in the definitions of $Z_t^{(1)}, Z_t^{(2)}$, we get

$$(2.3) \quad 1 = E \left[\exp \left(2Z_t^{(1)} - 2Z_t^{(2)} \right) \right].$$

Next, note that $0 \leq Z_t^{(2)} \leq M^2 t J$ and therefore

$$(2.4) \quad 1 \leq \exp \left(Z_t^{(2)} \right) \leq \exp \left(M^2 t J \right).$$

Combining (2.3) and (2.4), we get

$$\begin{aligned} E \left[\left(\frac{dQ_t}{dP_t} \right)^2 \right] &= E \left[\exp \left(2Z_t^{(1)} - Z_t^{(2)} \right) \right] \\ &= E \left[\exp \left(2Z_t^{(1)} - 2Z_t^{(2)} \right) \cdot \exp \left(Z_t^{(2)} \right) \right], \end{aligned}$$

and we obtain (2.2) from (2.3) and (2.4). \square

Using the above lemma we now explain how it is enough to prove Theorem 1.1 when $\mathbf{g} \equiv \mathbf{0}$. Consider the event

$$A = \left\{ \sup_{s \leq T, y \in [0, J]} |\mathbf{u}(s, y)| \leq \varepsilon \right\}.$$

Consider (1.3) with $\mathbf{g} \equiv \mathbf{0}$, and write

$$\begin{aligned} \partial_t \mathbf{u}(t, x) &= \frac{1}{2} \partial_x^2 \mathbf{u}(t, x) + \sigma(t, x, \mathbf{u}(t, x)) \dot{\mathbf{W}}(t, x) \\ &= \frac{1}{2} \partial_x^2 \mathbf{u}(t, x) + \mathbf{g}(t, x, \mathbf{u}(t, x)) \\ &\quad + \sigma(t, x, \mathbf{u}(t, x)) \left(\dot{\mathbf{W}}(t, x) - \sigma^{-1}(t, x, \mathbf{u}(t, x)) \mathbf{g}(t, x) \right). \end{aligned}$$

Let

$$\mathbf{f}(t, x) = \sigma^{-1}(t, x, \mathbf{u}(t, x)) \mathbf{g}(t, x),$$

and note that \mathbf{f} is uniformly bounded by some $M > 0$ by the assumptions on \mathbf{g} and (1.5). Define Q_T as in (2.1). From Lemma 2.1, we know that $\widetilde{\mathbf{W}}(s, x) := \dot{\mathbf{W}}(s, x) - \mathbf{f}(s, x)$, $x \in [0, J]$, $s \in [0, T]$ is a white noise under Q_T . Therefore the distribution of \mathbf{u} under Q_T corresponds to the case when \mathbf{g} is present in (1.3). Now

$$\begin{aligned} (2.5) \quad Q_T(A) &= E \left[\mathbf{1}_A \frac{dQ_T}{dP_T} \right] \\ &\leq \sqrt{P(A)} \cdot \sqrt{E \left[\left(\frac{dQ_T}{dP_T} \right)^2 \right]} \\ &\leq \sqrt{P(A)} \cdot \exp \left(\frac{M^2 T J}{2} \right). \end{aligned}$$

This explains how we get a similar upper bound for nonzero \mathbf{g} as when $\mathbf{g} \equiv \mathbf{0}$ with different constants. For the lower bound consider instead (1.3) with nonzero \mathbf{g} . We can write

$$\partial_t \mathbf{u}(t, x) = \frac{1}{2} \partial_x^2 \mathbf{u}(t, x) + \sigma(t, x, \mathbf{u}(t, x)) \left(\dot{\mathbf{W}}(t, x) + \sigma^{-1}(t, x, \mathbf{u}(t, x)) \mathbf{g}(t, x) \right).$$

Now let $\mathbf{f}(t, x) = -\sigma^{-1}(t, x, \mathbf{u}(t, x)) \mathbf{g}(t, x)$ and define Q_T as before. Note that in this case $\widetilde{\mathbf{W}} = \dot{\mathbf{W}} + \mathbf{f}$ is a white noise under Q_T , and so \mathbf{u} has the distribution of (1.3) with $\mathbf{g} \equiv \mathbf{0}$. Follow the same argument as in (2.5) to obtain a similar lower bound for nonzero \mathbf{g} as when $\mathbf{g} \equiv \mathbf{0}$.

2.3. Reduction to the case $J = 1$. Due to the previous subsection we can now assume $\mathbf{g} \equiv \mathbf{0}$. Let $\mathbf{u}(t, x)$ be the solution to (1.3) with $\mathbf{g} \equiv \mathbf{0}$. We now reduce to the case $J = 1$. Consider the function

$$(2.6) \quad \mathbf{v}(t, z) = J^{-1/2} \mathbf{u}(J^2 t, Jz), \quad t \geq 0, z \in [0, 1],$$

with initial profile $\mathbf{v}_0(z) = J^{-1/2} \mathbf{u}_0(Jz)$, $z \in [0, 1]$.

Let us denote the heat kernel by $G^{(J)}(t, x)$ to emphasize the dependence on J . The scaling relation below is immediate.

$$(2.7) \quad G^{(1)}(J^{-2}t, J^{-1}x) = J \cdot G^{(J)}(t, x), \quad x \in [0, J], t \geq 0.$$

The following distributional identity for white noise is well known.

$$(2.8) \quad \mathbf{W}^{J^2, J}(dy ds) := J^{-3/2} \mathbf{W}(Jdy J^2 ds) \stackrel{\mathcal{D}}{=} \mathbf{W}(dy ds),$$

where $\stackrel{\mathcal{D}}{=}$ denotes equality in distribution.

Lemma 2.2. *The random field $\mathbf{v}(t, x) : t \geq 0, x \in [0, 1]$ is the mild solution to the stochastic heat equation on $[0, 1]$ with white noise $\dot{\mathbf{W}}^{J^2, J}$:*

$$(2.9) \quad \begin{aligned} \partial_t \mathbf{v}(t, x) &= \frac{1}{2} \partial_x^2 \mathbf{v}(t, x) + \sigma^{(J)}(t, x, \mathbf{v}(t, x)) \cdot \dot{\mathbf{W}}^{J^2, J}(t, x) \\ \mathbf{v}(0, x) &= \mathbf{v}_0(x), \end{aligned}$$

where

$$\sigma^{(J)}(t, x, \mathbf{u}) := \sigma(J^2 t, Jx, J^{1/2} \mathbf{u}).$$

Proof. From (1.6) one obtains

$$\begin{aligned} \mathbf{v}(t, z) &= J^{-1/2} \int_0^J G^{(J)}(J^2 t, Jz - y) \mathbf{u}_0(y) dy \\ &\quad + J^{-1/2} \int_0^{J^2 t} \int_0^J G^{(J)}(J^2 t - s, Jz - y) \sigma(s, y, \mathbf{u}(s, y)) \mathbf{W}(dy ds) \\ &= J^{-3/2} \int_0^J G^{(1)}(t, z - J^{-1}y) \mathbf{u}_0(y) dy \\ &\quad + J^{-3/2} \int_0^{J^2 t} \int_0^J G^{(1)}(t - J^{-2}s, z - J^{-1}y) \sigma(s, y, \mathbf{u}(s, y)) \mathbf{W}(dy ds) \\ &= \int_0^1 G^{(1)}(t, z - w) \mathbf{v}_0(w) dw \\ &\quad + \int_0^t \int_0^1 G^{(1)}(t - r, z - w) \sigma(J^2 r, Jw, J^{1/2} \mathbf{v}(r, w)) \mathbf{W}^{J^2, J}(dw dr), \end{aligned}$$

where we have used (2.7) for the second equality, and (2.8) for the last equality. \square

The reduction to $J = 1$ then follows easily using Lemma 2.2. Indeed, write $s = J^2t$, $y = Jx$ and note that

$$\begin{aligned} P \left(\sup_{\substack{0 \leq s \leq T \\ y \in [0, J]}} |\mathbf{u}(s, y)| \leq \varepsilon \right) &= P \left(\sup_{\substack{0 \leq J^2 t \leq T \\ Jx \in [0, J]}} J^{-1/2} |\mathbf{u}(J^2 t, Jx)| \leq \varepsilon J^{-1/2} \right) \\ &= P \left(\sup_{\substack{0 \leq t \leq T J^{-2} \\ x \in [0, 1]}} |\mathbf{v}(t, x)| \leq \varepsilon J^{-1/2} \right). \end{aligned}$$

Assuming we have Theorem 1.1 for $J = 1$ we will obtain the result for general J from the above.

Remark 2.1 (Important). Note that the function $\sigma^{(J)}$ satisfies (1.5) with the same $\mathcal{C}_1, \mathcal{C}_2$. However the Lipschitz constant for $\sigma^{(J)}$ is $J^{1/2}\mathcal{D}$, and not \mathcal{D} . This is the reason for the somewhat strange expressions for the upper bounds in (1.8) and (1.10).

Remark 2.2. Thanks to the above reductions, we will assume for the rest of the article that $J = 1$ and $\mathbf{g} \equiv \mathbf{0}$.

2.4. Key proposition. We divide the time interval $[0, T]$ into increments of length $c_0\varepsilon^4$ where $c_0 = c_0(\mathcal{C}_1, \mathcal{C}_2)$ is chosen so that

$$(2.10) \quad 0 < c_0 < \max \left\{ \left(\frac{\mathbf{K}_2}{36 \log(\mathbf{K}_1) \mathcal{C}_2^2} \right)^2, 1 \right\}.$$

Above \mathbf{K}_1 and \mathbf{K}_2 are universal constants specified in the statement of Lemma 3.4. Consider time points

$$t_n = nc_0\varepsilon^4, \quad n \geq 0,$$

and let $I_n := [t_n, t_{n+1}]$ be the n th time increment. Let

$$n_1 := \min\{n \geq 1 : t_n > T\}$$

be the smallest n for which the time interval I_n is completely outside $[0, T]$.

We shall similarly consider a discrete set of spatial points separated by $c_1\varepsilon^2$, where $c_1^2 = \theta c_0$ with $\theta = \theta(\mathcal{C}_1, \mathcal{C}_2) > 0$ is chosen so that

$$(2.11) \quad \theta \geq \max \left\{ 2, 4 \log \left(\frac{1}{2c_0} \right) \right\} \quad \text{and} \quad \frac{C_{10}}{C_8} \sum_{k \geq 1} \exp \left(-\frac{\theta k^2}{8} \right) < \frac{1}{6}.$$

The constants C_8 and C_{10} are specified in the statement of Lemma 4.1 and depend on \mathcal{C}_1 and \mathcal{C}_2 only. Consider discrete spatial points

$$x_n = nc_1\varepsilon^2, \quad n \geq 0,$$

and let $J_n := [x_n, x_{n+1}]$ be the n th space increment. Let

$$n_2 := \min\{n \geq 1 : x_n > 1\}$$

be the smallest n for which the space interval J_n is completely outside $[0, 1]$. Note that

$$(2.12) \quad n_2 \leq (c_1\varepsilon^2)^{-1} + 1.$$

We will first define a sequence of sets which we can use to prove the lower bounds in (1.7) and (1.8). Let $A_{-1} = \Omega$ and for $n \geq 0$ define events

$$(2.13) \quad A_n = \left\{ |\mathbf{u}(t_{n+1}, x)| \leq \frac{\varepsilon}{3} \quad \forall x, \text{ and } |\mathbf{u}(t, x)| \leq \varepsilon \quad \forall t \in I_n, x \in [0, 1] \right\}.$$

Thus A_n is the event that $\mathbf{u}(t, \cdot)$ is everywhere of modulus at most ε in the time interval I_n , and is everywhere of modulus at most $\varepsilon/3$ at the terminal time t_{n+1} . We will next define a sequence of sets which we use to prove the upper bounds in (1.7) and (1.8). Denote by

$$p_{ij} = (t_i, x_j),$$

the left hand corner of the box $I_i \times J_j$. Let $F_{-1} = \Omega$ and for $n \geq 0$, define

$$F_n = \{|\mathbf{u}(p_{nj})| \leq \varepsilon \text{ for all } j \leq n_2 - 2\}.$$

By the above constructions of A_n and F_n , the proposition below along with the *Markov property* will be the key step in the proof Theorem 1.1.

Proposition 2.1. *Fix $\mathbf{g} \equiv \mathbf{0}$, $J = 1$. Consider the solution to (1.3) with $\mathbf{u}_0(x) \equiv 0$ and let the assumptions (1.4) and (1.5) hold. Then*

- (a) *For all $\mathcal{D} > 0$, there exist constants $\varepsilon_0(\mathcal{C}_1, \mathcal{C}_2, \mathcal{D}) > 0$ and $C_4, C_5 > 0$ depending only on $\mathcal{C}_1, \mathcal{C}_2$ such that for $0 < \varepsilon < \varepsilon_0$ and $n \geq 1$*

$$(2.14) \quad P\left(F_n \mid \bigcap_{k=0}^{n-1} F_k\right) \leq C_4 \exp\left(-\frac{C_5}{(1 + \mathcal{D}^2)\varepsilon^2}\right).$$

- (b) *There is a $\mathcal{D}_0(\mathcal{C}_1, \mathcal{C}_2) > 0$ and constants ε_0, C_6, C_7 depending only on $\mathcal{C}_1, \mathcal{C}_2$ such that for any $\mathcal{D} < \mathcal{D}_0$, $0 < \varepsilon < \varepsilon_0$ and $n \geq 0$*

$$(2.15) \quad P\left(A_n \mid \bigcap_{k=-1}^{n-1} A_k\right) \geq C_6 \exp(-C_7\varepsilon^{-2}).$$

The majority of the work in the paper will be to prove the above Proposition. In the last section we will explain how Theorem 1.1 follows from this.

3. PRELIMINARIES

In this section we state and prove some preliminaries with regard to (1.3) that we will need for the proof of Proposition 2.1. Recall that we are restricting ourselves to $J = 1$ and $\mathbf{g} \equiv \mathbf{0}$.

3.1. Heat Kernel Estimates. In this subsection we prove a few preliminary results about the heat kernel $G(t, x)$ which was mentioned in the introduction. G is given by

$$G(t, x) = \sum_{n \in \mathbb{Z}} (2\pi t)^{-1/2} \exp\left(-\frac{(x+n)^2}{2t}\right).$$

The following lemma is well-known for $x \in \mathbb{R}$, see for example [DKM⁺09], Lemma 4.3, page 126. We give a brief proof for the case $x \in [0, 1]$ (the circle).

Lemma 3.1. *There exists a constant $C_0 > 0$ such that for all $0 < s < t \leq 1$ with $|t - s| \leq 1$ and $x, y \in [0, 1]$, we have*

$$(3.1) \quad \int_0^t \int_0^1 [G(s, x - z) - G(s, y - z)]^2 dz ds \leq C_0 |x - y|,$$

$$(3.2) \quad \int_s^t \int_0^1 G^2(t - r, z) dz dr \leq C_0 |t - s|^{1/2},$$

$$(3.3) \quad \int_0^s \int_0^1 [G(t - r, z) - G(s - r, z)]^2 dz dr \leq C_0 |t - s|^{1/2}.$$

Proof. From the standard Fourier decomposition we have

$$G(t, x) = \sum_{k \in \mathbb{Z}} \exp\left(-\frac{(2\pi k)^2 t}{2}\right) \cdot \exp(i(2\pi k)x).$$

Using the orthogonality of $\{\exp(i(2\pi k)z) : k \in \mathbb{Z}\}$ in $L^2([0, 1])$, it is immediate that there is a $c_1 > 0$ such that

$$\begin{aligned} & \int_0^t \int_0^1 [G(s, x - z) - G(s, y - z)]^2 dz ds \\ &= C \int_0^t ds \sum_{k \geq 1} \exp(- (2\pi k)^2 s) \cdot |1 - \exp(i(2\pi k)(x - y))|^2 \\ &\leq C \sum_{k \geq 1} \frac{1}{k^2} \cdot |1 \wedge (2\pi k)|x - y||^2 \\ &\leq C|x - y|. \end{aligned}$$

The second inequality above is obtained by using Fubini's theorem, integrating over s and using $|1 - e^{ix}| \leq 2 \wedge |x|$. The last inequality above is obtained by splitting the sum according to whether k is less than or greater than $(2\pi|x - y|)^{-1}$. Thus we have obtained (3.1).

As for (3.2) we integrate over z first and using orthogonality again we obtain

$$\begin{aligned} & \int_s^t \int_0^1 G^2(t - r, z) dz dr \\ &\leq C \int_0^{t-s} dr \sum_{k \geq 0} \exp(- (2\pi k)^2 r) \\ &\leq C(t - s) + C \sum_{k \geq 1} \frac{1}{(2\pi k)^2} \left[1 - \exp(- (2\pi k)^2 (t - s)) \right] \\ &\leq C(t - s) + C \sum_{k \geq 1} \frac{1}{(2\pi k)^2} \left[1 \wedge (2\pi k)^2 (t - s) \right] \\ &\leq C(t - s)^{1/2}. \end{aligned}$$

We obtain the last inequality above by splitting the earlier sum according to whether $(2\pi k)^2(t - s)$ is less than or greater than 1. Thus we have obtained (3.2).

For (3.3), using orthogonal basis as above we have that

$$\begin{aligned} & \int_0^s \int_0^1 [G(t - r, z) - G(s - r, z)]^2 dz dr \\ &= \int_0^s dr \sum_{k \geq 1} \exp(- (2\pi k)^2 r) \cdot \left[1 - \exp(- (2\pi k)^2 (t - s)) \right]. \end{aligned}$$

Using Fubini and integrating each of the terms over r , we obtain (3.3). \square

For $x \in [0, 1]$, the circle with end points identified, define

$$(3.4) \quad x_* = \begin{cases} x, & 0 \leq x \leq \frac{1}{2}, \\ x - 1, & \frac{1}{2} < x \leq 1. \end{cases}$$

We will need the following comparison between the heat kernel on the circle $[0, 1]$ with the heat kernel on \mathbb{R} .

Lemma 3.2. *There is a $C_G > 0$ such that for all $t \leq 1$*

$$G(t, x) \leq \frac{C_G}{\sqrt{2\pi t}} \exp\left(-\frac{x_*^2}{2t}\right), \quad x \in [0, 1].$$

Proof. As we are working on the circle $[0, 1]$ we have $G(t, x) = G(t, x_*)$. It is immediate to observe that $|x_*| \leq \min(|x_*|, |x_* - 1|, |x_* + 1|)$, and

$$-(x_* + k)^2 \leq -x_*^2 - \frac{k^2}{2} \quad \text{if } |k| \geq 2.$$

Therefore,

$$\begin{aligned} G(t, x_*) &\leq \frac{3}{\sqrt{2\pi t}} \exp\left(-\frac{x_*^2}{2t}\right) + \sum_{|k| \geq 2} \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{x_*^2}{2t} - \frac{k^2}{4t}\right) \\ &\leq \frac{3}{\sqrt{2\pi t}} \exp\left(-\frac{x_*^2}{2t}\right) + \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{x_*^2}{2t}\right) \sum_{|k| \geq 2} e^{-k^2/4}, \end{aligned}$$

uniformly for $t \leq 1$. This completes the proof. \square

3.2. Noise Term Estimates. Recall that any solution \mathbf{u} to (1.3) with $J = 1, \mathbf{g} \equiv \mathbf{0}$ satisfies

(3.5)

$$\mathbf{u}(t, x) = \int_0^1 G(t, x - y) \mathbf{u}_0(y) dy + \int_0^t \int_0^1 G(t - s, x - y) \sigma(s, y, \mathbf{u}(s, y)) \mathbf{W}(dy ds),$$

where $x - y$ denotes subtraction modulo 1. The above is known as the mild form of the solution. We refer the reader to [Wal86] or [DKM⁺09] for a discussion of the stochastic integral with respect to white noise, and a treatment of the mild form. The Lipschitz assumption on σ guarantees that a unique strong solution exists [DKM⁺09, Theorem 6.4, page 26].

We shall denote the second term of (3.5), i.e. *noise term*, by

$$(3.6) \quad \mathbf{N}(t, x) := \int_0^t \int_0^1 G(t - s, x - y) \sigma(s, y, \mathbf{u}(s, y)) \mathbf{W}(dy ds).$$

Lemma 3.3. *There exist constants C_1, C_2 depending on the dimension d such that for all $0 < s < t < 1$, $x, y \in [0, 1]$ and $\lambda > 0$,*

$$P(|\mathbf{N}(t, x) - \mathbf{N}(t, y)| > \lambda) \leq C_1 \exp\left(-\frac{C_2 \lambda^2}{\mathcal{C}_2^2 |x - y|}\right)$$

$$P(|\mathbf{N}(t, x) - \mathbf{N}(s, x)| > \lambda) \leq C_1 \exp\left(-\frac{C_2 \lambda^2}{\mathcal{C}_2^2 |t - s|^{1/2}}\right)$$

Proof. One can use Lemma 3.1 and follow the argument in Corollary 4.5 on page 127 of [DKM⁺09] to obtain the result. We sketch the details. First note that it is enough to prove the above inequalities for each of the components of \mathbf{N} . Let us focus on the first coordinate N_1 . For the first inequality one observes that for $s \in [0, t]$

$$\int_0^s \int_0^1 [G(t-r, x-z) - G(t-r, y-z)] \sigma(r, z, \mathbf{u}(r, z)) \mathbf{W}(dzdr)$$

is an \mathcal{F}_t -martingale whose value at time t is $\mathbf{N}(t, x) - \mathbf{N}(t, y)$. Thus N_1 is also a martingale. Any one dimensional martingale is a time-changed Brownian motion and (3.1) gives a bound of $C\mathcal{C}_2^2|x - y|$ on the time change. One then uses the reflection principle to get the bound on the probability. Next consider the martingale

$$\mathbf{M}_q = \int_0^q \int_0^1 G(t-r, x-z) \sigma(r, z, \mathbf{u}(r, z)) \mathbf{W}(dzdr) - \mathbf{N}(s, x), \quad s \leq q \leq t.$$

The second bound can be proved similarly using (3.2) and (3.3) to get a bound of $C\mathcal{C}_2^2|t - s|^{1/2}$ on the time change. \square

Lemma 3.4. *There exist constants $\mathbf{K}_1, \mathbf{K}_2 > 0$ depending on the dimension d such that for all $\alpha, \lambda, \varepsilon > 0$ we have*

$$(3.7) \quad P\left(\sup_{\substack{0 \leq t \leq \alpha \varepsilon^4 \\ x \in [0, \varepsilon^2]}} |\mathbf{N}(t, x)| > \lambda \varepsilon\right) \leq \frac{\mathbf{K}_1}{1 \wedge \sqrt{\alpha}} \exp\left(-\mathbf{K}_2 \frac{\lambda^2}{\mathcal{C}_2^2 \sqrt{\alpha}}\right).$$

Proof. Let us first consider the case $\alpha \geq 1$. For $n \geq 0$, define the grid

$$\mathbb{G}_n = \left\{ \left(\frac{j}{2^{2n}}, \frac{k}{2^n} \right) : 0 \leq j \leq \alpha \varepsilon^4 2^{2n}, 0 \leq k \leq \varepsilon^2 2^n \right\}.$$

Let

$$n_0 = \lceil \log_2(\alpha^{-1/2} \varepsilon^{-2}) \rceil.$$

For $n < n_0$, the grid \mathbb{G}_n consists simply of the point $(0, 0)$. For $n \geq n_0$, the grid \mathbb{G}_n has at most $4\alpha\varepsilon^4 2^{2n} \cdot \varepsilon^2 2^n \leq 4 \cdot 2^{3(n-n_0)}$ many points. Fix

$$K = \left(4 \sum_{n \geq 0} 2^{-n/4} \right)^{-1}.$$

Consider the events

$$A(n, \lambda) = \{ |\mathbf{N}(p) - \mathbf{N}(q)| \leq \lambda K \varepsilon 2^{-(n-n_0)/4} \text{ for all } p, q \in \mathbb{G}_n, \text{ nearest neighbors} \}.$$

Thanks to Lemma 3.3, a union bound gives for $n \geq n_0$

$$\begin{aligned} P(A^c(n, \lambda)) &\leq C_1 2^{3(n-n_0)} \exp\left(-C_2 \frac{\lambda^2 K^2}{\mathfrak{C}_2^2} \varepsilon^2 2^{n_0} 2^{(n-n_0)/2}\right) \\ &\leq C_1 2^{3(n-n_0)} \exp\left(-C_2 \frac{\lambda^2 K^2}{\mathfrak{C}_2^2 \sqrt{\alpha}} 2^{(n-n_0)/2}\right). \end{aligned}$$

Let $A(\lambda) = \cap_{n \geq n_0} A(n, \lambda)$. Therefore

$$P(A(\lambda)^c) \leq \sum_{n \geq n_0} P(A(n, \lambda)^c) \leq \exp\left(-C_3 \frac{\lambda^2 K^2}{\mathfrak{C}_2^2 \sqrt{\alpha}}\right)$$

for some constant $C_3 > 0$, as long as $\lambda^2/\sqrt{\alpha} \geq \tilde{C}_3 \mathfrak{C}_2^2$ for some \tilde{C}_3 large enough. Since the left hand side is a probability it is at most 1. Therefore we can conclude that there exist constants $C_4, C_5 > 0$ such that for all $\alpha \geq 1$ and $\lambda > 0$ we have

$$P(A(\lambda)^c) \leq C_4 \exp\left(-C_5 \frac{\lambda^2 K^2}{\mathfrak{C}_2^2 \sqrt{\alpha}}\right).$$

Now consider a point $(t, x) \in [0, \alpha\varepsilon^4] \times [0, \varepsilon^2]$ which is in a grid \mathbb{G}_n for some $n \geq n_0$. From arguments similar to page 128 of [DKM⁺09] we can find a sequence of points $(0, 0) = p_0, p_1, \dots, p_m = (t, x)$ of points in \mathbb{G}_n such that each pair p_{i-1}, p_i are nearest neighbors in some grid \mathbb{G}_k , $n_0 \leq k \leq n$, and at most 4 such pairs are nearest neighbors in any given grid \mathbb{G}_k . Therefore on the event $A(\lambda)$ we have

$$\begin{aligned} |\mathbf{N}(t, x)| &\leq \sum_{k=1}^m |\mathbf{N}(p_{j-1}) - \mathbf{N}(p_j)| \\ &\leq 4 \sum_{n \geq n_0} \lambda K \varepsilon 2^{-(n-n_0)/4} \leq \lambda \varepsilon. \end{aligned}$$

This points $(t, x) \in \mathbb{G}_n$ are dense in $[0, \alpha\varepsilon^4] \times [0, \varepsilon^2]$, and therefore we have (3.7) in the case $\alpha \geq 1$.

Let us next consider the case $0 < \alpha < 1$. We divide the interval $[0, \epsilon^2]$ into $\frac{1}{\sqrt{\alpha}}$ intervals each of length $\sqrt{\alpha}\epsilon^2$. A simple union bound and stationarity in x implies that

$$\begin{aligned} P\left(\sup_{\substack{0 \leq t \leq \alpha \epsilon^4 \\ x \in [0, \epsilon^2]}} |\mathbf{N}(t, x)| > \lambda \epsilon\right) &\leq \frac{1}{\sqrt{\alpha}} P\left(\sup_{\substack{0 \leq t \leq (\sqrt{\alpha}\epsilon^2)^2 \\ x \in [0, \sqrt{\alpha}\epsilon^2]}} |\mathbf{N}(t, x)| > \lambda \epsilon\right) \\ &= \frac{1}{\sqrt{\alpha}} P\left(\sup_{\substack{0 \leq t \leq (\sqrt{\alpha}\epsilon^2)^2 \\ x \in [0, \sqrt{\alpha}\epsilon^2]}} |\mathbf{N}(t, x)| > \frac{\lambda}{\alpha^{1/4}} (\alpha^{1/4} \epsilon)\right) \\ &\leq \frac{\mathbf{K}_1}{\sqrt{\alpha}} \exp\left(-\mathbf{K}_2 \frac{\lambda^2}{\mathcal{C}_2^2 \sqrt{\alpha}}\right). \end{aligned}$$

This completes the proof the lemma. \square

Remark 3.1. Suppose the function σ in \mathbf{N} (see equation (3.6)) satisfies

$$|\sigma(s, y, \mathbf{u}(s, y))| \leq C_1 \epsilon$$

then the probability in (3.7) is bounded above by $\frac{\mathbf{K}_1}{1 \wedge \sqrt{\alpha}} \exp\left(-\mathbf{K}_2 \frac{\lambda^2}{C_1^2 \mathcal{C}_2^2 \epsilon^2 \sqrt{\alpha}}\right)$ for the same constants $\mathbf{K}_1, \mathbf{K}_2$ as in (3.7). This can be proved similarly to the above lemma and will be used later.

4. PROOF OF PROPOSITION 2.1

4.1. Proof of Proposition 2.1(a). Let $u_1(t, x)$ be the first coordinate of $\mathbf{u}(t, x)$. Let us define

$$\tilde{F}_n = \{|u_1(p_{nj})| \leq \epsilon \text{ for all } j \leq n_2 - 2\}.$$

Since $F_n \subset \tilde{F}_n$, it is sufficient to prove

$$(4.1) \quad P\left(\tilde{F}_n \mid \bigcap_{k=0}^{n-1} F_k\right) \leq C_4 \exp\left(-\frac{C_5}{(1 + \mathcal{D}^2)\epsilon^2}\right).$$

The following lemma on the random variables $N_1(p_{1k})$ (recall that N_1 is the first coordinate of \mathbf{N}) is crucially used in the proof of the proposition. It shows that the variance of $N_1(p_{1k})$ is of order ϵ^2 , and gives a bound on the decay of correlations between random variables $N_1(p_{1k})$ and $N_1(p_{1k'})$ as $|k - k'|$ increases.

Lemma 4.1. *The random variables $N_1(p_{1k})$ are Gaussian with mean 0. Furthermore there exist constants C_8, C_9, C_{10} depending only on $\mathcal{C}_1, \mathcal{C}_2$ such that*

$$(4.2) \quad C_8 \varepsilon^2 \leq \text{Var}(N_1(p_{1k})) \leq C_9 \varepsilon^2$$

$$(4.3) \quad \text{Cov}(N_1(p_{1k}), N_1(p_{1k'})) \leq C_{10} t_1 \sup_{0 \leq t \leq 2t_1} \frac{1}{\sqrt{t}} \exp\left(-\frac{|x_k - x_{k'}|_*^2}{2t}\right).$$

Furthermore, if $0 < |x_k - x_{k'}| \leq \frac{1}{2}$ and θ is as in (2.11), one obtains

$$(4.4) \quad \text{Cov}(N_1(p_{1k}), N_1(p_{1k'})) \leq C_{10} \varepsilon^2 \exp\left(-\frac{\theta(k - k')^2}{8}\right).$$

Proof. It is immediate that the random variables are mean zero Gaussian. As for the covariance

$$(4.5) \quad \begin{aligned} \text{Cov}(N_1(p_{1k}), N_1(p_{1k'})) &\leq C \mathcal{C}_2^2 \int_0^{t_1} \int_0^1 G(t_1 - s, x_k - y) G(t_1 - s, x_{k'} - y) dy ds \\ &= C \mathcal{C}_2^2 \int_0^{t_1} G(2t_1 - 2s, x_k - x_{k'}) ds \\ &\leq C \mathcal{C}_2^2 t_1 \sup_{0 \leq t \leq 2t_1} G(t, x_k - x_{k'}) \\ &\leq C \mathcal{C}_2^2 t_1 \sup_{0 \leq t \leq 2t_1} \frac{C}{\sqrt{2\pi t}} \exp\left(-\frac{|x_k - x_{k'}|_*^2}{2t}\right). \end{aligned}$$

The last inequality follows by the symmetry of $G(t, x)$ in x and by Lemma 3.2. The bound for the variance obtained above is ∞ which is useless. We instead use the expression after the equality above and Lemma 3.2 to obtain the upper bound in (4.2). The lower bound in (4.2) can be obtained similarly since the components of σ are bounded away from 0 as well.

Let us turn our attention back to (4.5) and consider the situation when $|x_k - x_{k'}| \leq \frac{1}{2}$. In this case we have $|x_k - x_{k'}|_* = |x_k - x_{k'}|$. Thus

$$\begin{aligned} \text{Cov}(N_1(p_{1k}), N_1(p_{1k'})) &\leq C \mathcal{C}_2^2 \varepsilon^2 \sup_{0 \leq t \leq 2c_0} \exp\left(-\frac{c_1^2(k - k')^2}{2t} + \frac{\log(1/t)}{2}\right) \\ &= C \mathcal{C}_2^2 \varepsilon^2 \sup_{s \geq (2c_0)^{-1}} \exp\left(-\frac{1}{2} [c_1^2(k - k')^2 s - \log s]\right). \end{aligned}$$

By our choice of $c_1^2 = \theta c_0$ with θ as in (2.11) we see that the maximum is attained at $(2c_0)^{-1}$ for $k \neq k'$. Indeed the expression inside the exponential is decreasing in the interval $[(2c_0)^{-1}, \infty)$. Moreover the quantity in brackets inside the exponential is

at least $\theta(k - k')^2/4$ as long as $\theta \geq 4 \log[(2c_0)^{-1}]$, which we have assumed in (2.11). This proves (4.4) and completes the proof of the lemma. \square

Proof of Proposition 2.1(a) ($\mathcal{D} \equiv 0$ – The Gaussian case, i.e. deterministic σ). By the Markov property, it is enough to show that

$$P(\tilde{F}_1) \leq C_0 \exp(-C_1 \varepsilon^{-2}),$$

for constants C_0, C_1 depending only on $\mathcal{C}_1, \mathcal{C}_2$, if we start with a deterministic initial profile \mathbf{u}_0 satisfying $|\mathbf{u}_0(x_j)| \leq \varepsilon$ for all $j \leq n_2 - 2$. Let $\tilde{H}_{-1} = \Omega$ and for $j \geq 0$ define the events

$$\tilde{H}_j = \{|u_1(p_{1k})| \leq \varepsilon \text{ for all } k \leq j\}.$$

We will show that

$$(4.6) \quad P(\tilde{H}_j | \tilde{H}_{j-1}) \leq \eta \text{ for all } 0 \leq j \leq n_2 - 2,$$

for some constant $\eta = \eta(\mathcal{C}_1, \mathcal{C}_2) < 1$. Since $n_2 = \lceil c_1^{-1} \varepsilon^{-2} \rceil$, we have that

$$P(\tilde{F}_1) = \prod_{j=0}^{n_2-2} P(\tilde{H}_j | \tilde{H}_{j-1}) \leq \eta^{\lceil c_1^{-1} \varepsilon^{-2} \rceil}$$

as required. Let us therefore turn our attention to proving (4.6). Now

$$u_1(p_{1k}) = [G_{t_1}(\mathbf{u}_0)(x_k)]_1 + N_1(p_{1k}).$$

The term $[G_{t_1}(\mathbf{u}_0)(x_k)]_1$ is the first component of the deterministic term in (3.5), while $N_1(p_{1k})$ are mean zero Gaussian random variables. To obtain (4.6) we will show the existence of some $0 \leq \eta < 1$ such that

$$(4.7) \quad P(|u_1(p_{1j})| \leq \varepsilon | \mathcal{G}_{j-1}) \leq \eta \text{ for all } 0 \leq j \leq n_2 - 2,$$

where \mathcal{G}_j is the σ -algebra generated by the random variables $N_1(p_{1k})$, $0 \leq k \leq j$. The inequality (4.7) gives a uniform bound on the probability of the event $|u_1(p_{1j})| \leq \varepsilon$, given every realization of the random variables $N_1(p_{1k})$, $0 \leq k \leq j - 1$. In particular it gives the same bound on $P(|u_1(p_{1j})| \leq \varepsilon | \tilde{H}_{j-1})$.

Therefore let us turn our attention to proving (4.7).

This will be achieved by producing a uniform (in j) lower bound of order $C\varepsilon^2$ on the conditional variance of $u_1(p_{1j})$ given \mathcal{G}_{j-1} , where C depends only on $\mathcal{C}_1, \mathcal{C}_2$. This will imply that the conditional distribution of $u_1(p_{1j})$ is sufficiently *spread out* and that the event $|u_1(p_{1j})| > \varepsilon$ has nonvanishing probability.

By general properties of Gaussian random vectors we can decompose

$$(4.8) \quad u_1(p_{1j}) = [G_{t_1}(\mathbf{u}_0)(x_k)]_1 + X + Y,$$

where X is the conditional expectation of $N_1(p_{1j})$ given \mathcal{G}_{j-1} . Furthermore

$$(4.9) \quad X = \sum_{k=0}^{j-1} \beta_k^{(j)} N_1(p_{1k})$$

for some coefficients $\beta_l^{(j)}$. The variance of the random variable $Y = N_1(p_{1j}) - X$ is precisely the conditional variance of $N_1(p_{1j})$ given \mathcal{G}_{j-1} , and is *also* the conditional variance of $u_1(p_{1j})$ given \mathcal{G}_{j-1} .

Let us use the notation SD to denote the standard deviation of a random variable. By Minkowski's inequality

$$\text{SD}(X) \leq \sum_{k=0}^{j-1} |\beta_k^{(j)}| \cdot \text{SD}(N_1(p_{1k})).$$

We will show in the following lemma that $\sum_{k=0}^{j-1} |\beta_k^{(j)}|$ can be made less than 1/2 by our choice of θ . In particular for this choice of θ the standard deviation of X is less than one half the standard deviation of $N_1(p_{1j})$. Therefore, for this choice of θ

$$\text{SD}(N_1(p_{1k})) \leq \text{SD}(X) + \text{SD}(Y) \leq \frac{\text{SD}(N_1(p_{1k}))}{2} + \text{SD}(Y).$$

From (4.2) the variance of $N_1(p_{1j})$ is bounded below by $C_8\varepsilon^2$. We have thus shown that the conditional variance of $N_1(p_{1j})$ given \mathcal{G}_{j-1} , which is also the variance of Y , is uniformly (in j) bounded below by $C_{11}\varepsilon^2$. Recall that the conditional variance of $N_1(p_{1j})$ given \mathcal{G}_{j-1} is also the conditional variance of $u_1(p_{1j})$ given \mathcal{G}_{j-1} . This implies

$$P(|u_1(p_{1j})| \leq \varepsilon | \mathcal{G}_{j-1}) \leq \eta$$

for some $\eta < 1$ uniformly in j . Indeed, for a Gaussian random variable $Z \sim N(\mu, \sigma^2)$ and any $a > 0$ the probability $P(|Z| \leq a)$ is maximized when $\mu = 0$. Therefore

$$\begin{aligned} P(|u_1(p_{1j})| \leq \varepsilon | \mathcal{G}_{j-1}) &\leq P\left(|N(0, 1)| \leq \frac{\varepsilon}{\sqrt{\text{Var}(u_1(p_{1j}) | \mathcal{G}_{j-1})}}\right) \\ &\leq P\left(|N(0, 1)| \leq \frac{1}{\sqrt{C_{11}}}\right). \end{aligned}$$

This completes the proof of the proposition. □

The only ingredient left in the proof of Proposition 2.1 in the Gaussian case is the following lemma.

Lemma 4.2. *Recall the coefficients $\beta_k^{(j)}$ from (4.9). By choosing θ as in (2.11)*

$$\sum_{k=0}^{j-1} |\beta_k^{(j)}| \leq \frac{1}{2} \text{ for all } 0 \leq j \leq n_2 - 2.$$

Proof. The random variable Y , defined in (4.8), has the nice property that it is independent of \mathcal{G}_{j-1} , and therefore

$$\text{Cov}(Y, N_1(p_{1k})) = 0 \text{ for } k = 0, 1, \dots, j-1.$$

It follows that

$$(4.10) \quad \text{Cov}(N_1(p_{1j}), N_1(p_{1l})) = \sum_{k=0}^{j-1} \beta_k^{(j)} \text{Cov}(N_1(p_{1k}), N_1(p_{1l})) \text{ for all } l = 0, 1, \dots, j-1.$$

Consider now the vector $\boldsymbol{\beta} = (\beta_0^{(j)}, \beta_1^{(j)}, \dots, \beta_{j-1}^{(j)})^T$ and

$$\mathbf{y} = \left(\text{Cov}(N_1(p_{1j}), N_1(p_{10})), \dots, \text{Cov}(N_1(p_{1j}), N_1(p_{1,j-1})) \right)^T.$$

Let $\mathbf{S} = ((\text{Cov}(N_1(p_{1k}), N_1(p_{1l})))_{0 \leq k, l \leq j-1})$ be the covariance matrix. The system (4.10) can be succinctly written as $\mathbf{y} = \mathbf{S}\boldsymbol{\beta}$, whence

$$\boldsymbol{\beta} = \mathbf{S}^{-1}\mathbf{y}.$$

Denote by $\|\cdot\|_1$ the ℓ_1 norm on \mathbf{R}^k , and by $\|\cdot\|_{1,1}$ the matrix norm induced on $j \times j$ matrices by the $\|\cdot\|_1$ norm, that is for a matrix \mathbf{A}

$$\|\mathbf{A}\|_{1,1} = \sup_{\mathbf{x} \neq \mathbf{0}} \frac{\|\mathbf{A}\mathbf{x}\|_1}{\|\mathbf{x}\|_1}.$$

It can be shown that $\|\mathbf{A}\|_{1,1} = \max_j \sum_{i=1}^n |a_{ij}|$ (see page 259 of [RB00]). Therefore we have

$$(4.11) \quad \|\boldsymbol{\beta}\|_1 \leq \|\mathbf{S}^{-1}\|_{1,1} \|\mathbf{y}\|_1.$$

Now we write $\mathbf{S} = \mathbf{D}\mathbf{T}\mathbf{D}$ where \mathbf{D} is a diagonal matrix with diagonal entries $\sqrt{\text{Var}(N_1(p_{1k}))}$, and \mathbf{T} is the matrix of correlation coefficients

$$t_{kl} = \frac{\text{Cov}(N_1(p_{1k}), N_1(p_{1l}))}{\sqrt{\text{Var}(N_1(p_{1k}))} \sqrt{\text{Var}(N_1(p_{1l}))}}, \quad 0 \leq k, l \leq j-1.$$

Therefore $\mathbf{S}^{-1} = \mathbf{D}^{-1}\mathbf{T}^{-1}\mathbf{D}^{-1} = \mathbf{D}^{-1}(\mathbf{I} - \mathbf{A})^{-1}\mathbf{D}^{-1}$ for some matrix \mathbf{A} . Thanks to (4.2) and (4.4) we can make $\|\mathbf{A}\|_{1,1} < \frac{1}{3}$ by choosing θ as in the second inequality in (2.11). Therefore

$$\|(\mathbf{I} - \mathbf{A})^{-1}\|_{1,1} \leq \frac{1}{1 - \|\mathbf{A}\|_{1,1}} \leq \frac{3}{2}$$

and so $\|\mathbf{S}^{-1}\|_{1,1} \leq \|\mathbf{D}^{-1}\|_{1,1} \cdot \|\mathbf{T}^{-1}\|_{1,1} \cdot \|\mathbf{D}^{-1}\|_{1,1} \leq 2C_8^{-1}\varepsilon^{-2}$, where C_8 is as in (4.2). Note in obtaining an upper bound on $\|\mathbf{D}^{-1}\|_{1,1}$, we have crucially used the lower bound in (4.2), which in turn is based on the assumption that the components of σ are bounded away from zero. Substituting the above bounds into (4.11) we obtain

$$(4.12) \quad \|\boldsymbol{\beta}\|_1 \leq 2C_8^{-1}\varepsilon^{-2}\|\mathbf{y}\|_1,$$

which can be made less than $1/2$ by choosing θ as in (2.11). \square

Proof of Proposition 2.1(a) (The general case *i.e.* general σ). By the Markov property it is enough to show the bound $P(F_1) \leq C_0 \exp\left(-\frac{C_1}{(1+\mathcal{D}^2)\varepsilon^2}\right)$, for constants C_0, C_1 depending only on $\mathcal{C}_1, \mathcal{C}_2$, starting with a deterministic initial profile \mathbf{u}_0 with $|\mathbf{u}_0(x_j)| \leq \varepsilon$ for all $j \leq n_2 - 2$.

For a point $\mathbf{z} \in \mathbb{R}^d$ define

$$f_\varepsilon(\mathbf{z}) = \begin{cases} \mathbf{z}, & |\mathbf{z}| \leq \varepsilon, \\ \frac{\varepsilon}{|\mathbf{z}|}\mathbf{z}, & |\mathbf{z}| > \varepsilon. \end{cases}$$

In particular $|f_\varepsilon(\mathbf{z})| \leq \varepsilon$. We consider the equation

$$\partial_t \mathbf{v}(t, x) = \frac{1}{2} \partial_x^2 \mathbf{v}(t, x) + \sigma(t, x, f_\varepsilon(\mathbf{v}(t, x))) \cdot \dot{\mathbf{W}}(t, x)$$

with initial profile \mathbf{u}_0 . It is clear that as long as $|\mathbf{u}(t, x)| \leq \varepsilon$ for all x , we have $\mathbf{u}(t, \cdot) = \mathbf{v}(t, \cdot)$. Therefore it enough to prove the proposition for \mathbf{v} .

We compare \mathbf{v} with \mathbf{v}_g where

$$(4.13) \quad \partial_t \mathbf{v}_g = \frac{1}{2} \partial_x^2 \mathbf{v}_g + \sigma(t, x, f_\varepsilon(\mathbf{u}_0(x))) \cdot \dot{\mathbf{W}}(t, x)$$

starting with the same initial profile \mathbf{u}_0 . We decompose $\mathbf{v}(t, x) = \mathbf{v}_g(t, x) + \mathbf{D}(t, x)$, with

$$\mathbf{D}(t, x) = \int_0^t \int_0^1 G(t-s, x-y) [\sigma(s, y, f_\varepsilon(\mathbf{v}(s, y))) - \sigma(s, y, f_\varepsilon(\mathbf{u}_0(y)))] \cdot \mathbf{W}(dy ds).$$

Define

$$H_j = \{|\mathbf{v}(p_{1j})| \leq \varepsilon\},$$

so that $F_1 = \bigcap_{j=0}^{n_2-2} H_j$. Define also the events

$$\begin{aligned} A_{1,j} &= \{|\mathbf{v}_g(p_{1j})| \leq 2\varepsilon\}, \text{ and} \\ A_{2,j} &= \{|\mathbf{D}(p_{1j})| > \varepsilon\}. \end{aligned}$$

If $|\mathbf{D}(p_{1j})| \leq \varepsilon$ and $|\mathbf{v}_g(p_{1j})| > 2\varepsilon$, we must have $|\mathbf{v}(p_{1j})| > \varepsilon$. As a consequence,

$$H_j \subset A_{1,j} \cup A_{2,j}.$$

Therefore

$$P(F_1) \leq P\left(\bigcap_{j=0}^{n_2-2} [A_{1,j} \cup A_{2,j}]\right).$$

The intersection can be expanded as the union of various events. One of the terms in the union is

$$A_{1,0} \cap A_{1,1} \cap A_{1,2} \cap \cdots \cap A_{1,n_2-2}.$$

The remaining events in the union looks like this:

$$A_{1,0} \cap A_{1,1} \cap A_{1,2} \cap \cdots \cap A_{1,k-1} \cap A_{2,k} \cap \cdots .$$

That is, it will involve a run of $\{A_{1,j} : 0 \leq j \leq k-1\}$ and then by $A_{2,k}$ followed by the intersection with other sets. We collect all these events which have the same time k of the first appearance of an $A_{2,j}$. The probability of the union of all these sets is dominated by

$$P(A_{1,0} \cap A_{1,1} \cap A_{1,2} \cdots \cap A_{1,k} \cap A_{2,k}).$$

Thus we obtain the upper bound

$$\begin{aligned} P(F_1) &\leq P\left(\bigcap_{j=0}^{n_2-2} A_{1,j}\right) + \sum_{k=0}^{n_2-2} P(A_{1,0} \cap A_{1,1} \cap A_{1,2} \cdots \cap A_{1,k-1} \cap A_{2,k}) \\ &\leq P\left(\bigcap_{j=0}^{n_2-2} A_{1,j}\right) + \sum_{j=0}^{n_2-2} P(A_{2,j}). \end{aligned}$$

The argument for the Gaussian case shows that

$$(4.14) \quad P\left(\bigcap_{j=0}^{n_2-2} A_{1,j}\right) = P\left(|\mathbf{v}_g(p_{nj})| \leq 2\varepsilon \text{ for all } j \leq n_2-2\right) \leq C_2 \exp(-C_3\varepsilon^{-2}),$$

for constants C_2, C_3 depending only on $\mathcal{C}_1, \mathcal{C}_2$. By the Lipschitz assumption on σ we have

$$|\sigma(t, x, f_\varepsilon(\mathbf{v}(t, x))) - \sigma(t, x, f_\varepsilon(\mathbf{u}_0(x)))| \leq 2\mathcal{D}\varepsilon.$$

An application of (3.7) with \mathbf{N} replaced by \mathbf{D} (see Remark 3.1) gives

$$P(A_{2,j}) \leq \mathbf{K}_1 \exp\left(-\frac{\mathbf{K}_2}{4\mathcal{C}_2^2 \mathcal{D}^2 \varepsilon^2 \sqrt{c_0}}\right).$$

Then by adjusting $\mathbf{K}_1, \mathbf{K}_2$ and using our bound (2.12) on n_2 , we get

$$\begin{aligned} \sum_{j=0}^{n_2-2} P(A_{2,j}) &\leq (n_2 - 2)\mathbf{K}_1 \exp\left(-\frac{\mathbf{K}_2}{4\mathcal{C}_2^2\mathcal{D}^2\varepsilon^2\sqrt{c_0}}\right) \\ &\leq \frac{1}{c_1\varepsilon^2}\mathbf{K}_1 \exp\left(-\frac{\mathbf{K}_2}{4\mathcal{C}_2^2\mathcal{D}^2\varepsilon^2\sqrt{c_0}}\right) \\ &\leq C_4 \exp\left(-\frac{C_5}{8\mathcal{C}_2^2(1+\mathcal{D}^2)\varepsilon^2\sqrt{c_0}}\right) \end{aligned}$$

when ε is small enough, for some constants C_4, C_5 dependent on d, \mathcal{C}_2 . Combining the above bound with (4.14) completes the proof of the proposition for the general case. \square

4.2. Proof of Proposition 2.1(b). We begin by stating the Gaussian correlation inequality which we will need.

Lemma 4.3. *For any convex symmetric sets K, L in \mathbf{R}^d and any centered Gaussian measure μ on \mathbf{R}^d we have*

$$\mu(K \cap L) \geq \mu(K)\mu(L).$$

Proof. cf. [Roy14] and [LaM17]. \square

By the Markov property of the solution $\mathbf{u}(t, \cdot)$ to (1.3) (see page 247 in [DPZ14]), the behavior of $\mathbf{u}(t, \cdot)$ in the interval I_n depends only on the profile $\mathbf{u}(t_n, \cdot)$. Therefore it is enough to prove that there are constants $C_1, C_2 > 0$ such that

$$(4.15) \quad P(A_0) \geq C_1 \exp(-C_2\varepsilon^{-2}),$$

for constants C_1, C_2 depending only $\mathcal{C}_1, \mathcal{C}_2$ when \mathbf{u}_0 satisfies $|\mathbf{u}_0(x)| \leq \varepsilon/3$.

Proof of Proposition 2.1(b) ($\mathcal{D} \equiv 0$ – The Gaussian case i.e. deterministic σ). Consider the event

$$(4.16) \quad B_0 = \left\{ |\mathbf{u}(t_1, x)| \leq \frac{\varepsilon}{6} \quad \forall x, \text{ and } |\mathbf{u}(t, x)| \leq \frac{2\varepsilon}{3} \quad \forall t \in I_0, x \in [0, 1] \right\}.$$

We shall prove below a lower bound of the form (4.15) with the event A_0 replaced by B_0 . Since $B_0 \subset A_0$, this implies (4.15).

We employ a standard technique in large deviation theory to obtain lower bounds on probabilities of atypical events. That is, we construct a measure Q , absolutely

continuous with respect to P , under which the event A_0 is likely. Once we have done this, we next control the Radon Nikodym derivative $\frac{dQ}{dP}$. Let us denote

$$G_t(\mathbf{u}_0)(x) = \int_0^1 G(t, x - y) \mathbf{u}_0(y) dy,$$

the space convolution of $G(t, \cdot)$ and \mathbf{u}_0 . Consider the measure Q_t given by

$$\frac{dQ_t}{dP_t} = \exp\left(Z_{t_1}^{(1)} - \frac{1}{2}Z_{t_1}^{(2)}\right)$$

where

$$\begin{aligned} Z_{t_1}^{(1)} &= - \int_0^{t_1} \int_0^1 \sigma^{-1}(s, y) \frac{G_s(\mathbf{u}_0)(y)}{t_1} \mathbf{W}(dyds), \\ Z_{t_1}^{(2)} &= \int_0^{t_1} \int_0^1 \left| \sigma^{-1}(s, y) \frac{G_s(\mathbf{u}_0)(y)}{t_1} \right|^2 dyds. \end{aligned}$$

By Lemma 2.1

$$\dot{\widetilde{\mathbf{W}}}(s, y) := \dot{\mathbf{W}}(s, y) + \sigma^{-1}(s, y) \frac{G_s(\mathbf{u}_0)(y)}{t_1}$$

is a white noise under the measure Q_t . We now write $\mathbf{u}(t, x)$ as

$$\begin{aligned} & \int_0^1 G(t, x - y) \mathbf{u}_0(y) dy \\ (4.17) \quad & + \int_0^t \int_0^1 G(t - s, x - y) \sigma(s, y) \left[\widetilde{\mathbf{W}}(dyds) - \sigma^{-1}(s, y) \frac{G_s(\mathbf{u}_0)(y)}{t_1} dsdy \right] \\ & = \left(1 - \frac{t}{t_1}\right) G_t(\mathbf{u}_0)(x) + \int_0^t \int_0^1 G(t - s, x - y) \sigma(s, y) \widetilde{\mathbf{W}}(dyds) \end{aligned}$$

The first term is equal to 0 at time t_1 and

$$(4.18) \quad \left| \left(1 - \frac{t}{t_1}\right) G_t \mathbf{u}_0(x) \right| \leq \frac{\varepsilon}{3}, \quad x \in [0, 1], \quad t \leq t_1.$$

Denote the second term by

$$\widetilde{\mathbf{N}}(t, x) = \int_0^t \int_0^1 G(t - s, x - y) \sigma(s, y) \widetilde{\mathbf{W}}(dyds).$$

Since $\tilde{\mathbf{W}}$ is a white noise under Q_t we can apply (3.7) to conclude

$$\begin{aligned} Q_t \left(\sup_{\substack{0 \leq t \leq t_1 \\ x \in [0, c_0 \varepsilon^2]}} |\tilde{\mathbf{N}}(t, x)| > \frac{\varepsilon}{6} \right) &= Q_t \left(\sup_{\substack{0 \leq t \leq c_0^{-1}(c_0 \varepsilon^2)^2 \\ x \in [0, c_0 \varepsilon^2]}} |\tilde{\mathbf{N}}(t, x)| > (6\sqrt{c_0})^{-1} \cdot (\sqrt{c_0} \varepsilon) \right) \\ &\leq \mathbf{K}_1 \exp \left(-\frac{\mathbf{K}_2}{36\mathcal{C}_2^2 \sqrt{c_0}} \right), \end{aligned}$$

where we used $\alpha = c_0^{-1}$ and $\lambda = (6\sqrt{c_0})^{-1}$ in (3.7). An application of Lemma 4.3 gives the following.

$$\begin{aligned} Q_t \left(\sup_{\substack{0 \leq t \leq t_1 \\ x \in [0, 1]}} |\tilde{\mathbf{N}}(t, x)| \leq \frac{\varepsilon}{6} \right) &\geq Q_t \left(\sup_{\substack{0 \leq t \leq t_1 \\ x \in [0, c_0 \varepsilon^2]}} |\tilde{\mathbf{N}}(t, x)| \leq \frac{\varepsilon}{6} \right)^{\frac{1}{c_0 \varepsilon^2}} \\ &\geq \left[1 - \mathbf{K}_1 \exp \left(-\frac{\mathbf{K}_2}{36\mathcal{C}_2^2 \sqrt{c_0}} \right) \right]^{\frac{1}{c_0 \varepsilon^2}} \end{aligned}$$

We observed earlier that the first term in (4.17) is bounded by $\varepsilon/3$ and is 0 at time t_1 . Therefore

$$Q_t(B_0) \geq Q_t \left(\sup_{\substack{0 \leq t \leq t_1 \\ x \in [0, 1]}} |\tilde{\mathbf{N}}(t, x)| \leq \frac{\varepsilon}{6} \right).$$

We finally compare $P_t(A_0)$ with $Q_t(A_0)$. To do this we first observe

$$\begin{aligned} E \left(\frac{dQ_t}{dP_t} \right)^2 &= \exp \left(\int_0^{t_1} \int_0^1 \left| \sigma^{-1}(s, y) \frac{G_s(\mathbf{u}_0)(y)}{t_1} \right|^2 dy ds \right) \\ &\leq \exp \left(\frac{1}{c_0 \mathcal{C}_1^2 \varepsilon^2} \right) \end{aligned}$$

The inequality is a consequence of the bound $\max_{|\mathbf{x}|=1} |\sigma^{-1} \mathbf{x}|^2 \leq \lambda_1^{-2} \leq \mathcal{C}^2$, where λ_1 is the smallest eigenvalue of σ . By the Cauchy-Schwarz inequality

$$Q_t(B_0) \leq \sqrt{E \left(\frac{dQ_t}{dP_t} \right)^2} \cdot \sqrt{P_t(B_0)},$$

and from this it follows

$$(4.19) \quad P(B_0) \geq \exp \left(-\frac{1}{c_0 \mathcal{C}_1^2 \varepsilon^2} \right) \exp \left(\frac{2}{c_0 \varepsilon^2} \log \left[1 - \mathbf{K}_1 \exp \left(-\frac{\mathbf{K}_2}{36\mathcal{C}_2^2 \sqrt{c_0}} \right) \right] \right),$$

where $\mathbf{K}_1, \mathbf{K}_2$ are the constants appearing in (3.7). With our choice of c_0 in (2.10), this completes the proof of the proposition in the case of deterministic σ . \square

For the Proof of Proposition 2.1(b) with general σ we compare \mathbf{u} with \mathbf{u}_g , where

$$(4.20) \quad \partial_t \mathbf{u}_g = \frac{1}{2} \partial_x^2 \mathbf{u}_g + \sigma(t, x, \mathbf{u}_0(x)) \dot{\mathbf{W}}(t, x), \quad t \in [0, c_0 \varepsilon^4], \quad x \in [0, 1],$$

with the same initial profile \mathbf{u}_0 . Recall that we are assuming $|\mathbf{u}_0(x)| \leq \varepsilon/3$ for all x . We write

$$(4.21) \quad \mathbf{u}(t, x) = \mathbf{u}_g(t, x) + \mathbf{D}(t, x),$$

where

$$\mathbf{D}(t, x) = \int_0^t \int_0^1 G(t-s, x-y) [\sigma(s, y, \mathbf{u}(s, y)) - \sigma(s, y, \mathbf{u}_0(y))] \mathbf{W}(dy ds).$$

Let us define

$$(4.22) \quad \tilde{B}_0 = \left\{ |\mathbf{u}_g(t_1, x)| \leq \frac{\varepsilon}{6} \quad \forall x, \text{ and } |\mathbf{u}_g(t, x)| \leq \frac{2\varepsilon}{3} \quad \forall t \in I_0, x \in [0, 1] \right\}.$$

Since the solution of (4.20) is Gaussian, we can apply (4.19) applied to \mathbf{u}_g to obtain

$$(4.23) \quad P(\tilde{B}_0) \geq \exp\left(-\frac{1}{c_0 \mathcal{C}_1^2 \varepsilon^2}\right) \exp\left(\frac{2}{c_0 \varepsilon^2} \log \left[1 - \mathbf{K}_1 \exp\left(-\frac{\mathbf{K}_2}{36 \mathcal{C}_2^2 \sqrt{c_0}}\right)\right]\right),$$

where \mathbf{K}_1 and \mathbf{K}_2 are the constants appearing in (3.7).

Proof of Proposition 2.1(b)(general σ). We will again prove (4.15). As discussed in the Gaussian case, we will use the Markov property of the solution $\mathbf{u}(t, \cdot)$ to (1.3) and therefore it is enough to prove that there are constants $C_1, C_2 > 0$ such that

$$(4.24) \quad P(A_0) \geq C_1 \exp(-C_2 \varepsilon^{-2}),$$

for constants C_1, C_2 depending only $\mathcal{C}_1, \mathcal{C}_2$ when \mathbf{u}_0 satisfies $|\mathbf{u}_0(x)| \leq \varepsilon/3$. Define

$$\tau = \inf\{t : |\mathbf{u}(t, x) - \mathbf{u}_0(x)| > 2\varepsilon, \text{ for some } x \in [0, 1]\}.$$

Since $|\mathbf{u}_0(x)| \leq \frac{\varepsilon}{3}$, we must have $\tau > t_1$ on the event A_0 . Moreover on the event $\tau > t_1$ we have $\mathbf{D}(t, x) = \tilde{\mathbf{D}}(t, x)$ for $t \leq t_1$, where

$$\tilde{\mathbf{D}}(t, x) = \int_0^t \int_0^1 G(t-s, x-y) [\sigma(s, y, \mathbf{u}(s \wedge \tau, y)) - \sigma(s, y, \mathbf{u}_0(y))] \mathbf{W}(dy ds).$$

Therefore, thanks to the decomposition (4.21), we have

$$\begin{aligned}
P(A_0) &\geq P\left(\tilde{B}_0 \cap \left\{ \sup_{\substack{0 \leq t \leq t_1 \\ x \in [0,1]}} |\mathbf{D}(t, x)| \leq \frac{\varepsilon}{6} \right\}\right) \\
&= P\left(\tilde{B}_0 \cap \left\{ \sup_{\substack{0 \leq t \leq t_1 \\ x \in [0,1]}} |\tilde{\mathbf{D}}(t, x)| \leq \frac{\varepsilon}{6} \right\}\right) \\
&\geq P(\tilde{B}_0) - P\left(\sup_{\substack{0 \leq t \leq t_1 \\ x \in [0,1]}} |\tilde{\mathbf{D}}(t, x)| > \frac{\varepsilon}{6}\right).
\end{aligned}$$

Let us explain the equality above. On the event $\{\tau > t_1\}$ we have $\sup_{x \leq 1, t \leq t_1} |\mathbf{D}(t, x)| = \sup_{x \leq 1, t \leq t_1} |\tilde{\mathbf{D}}(t, x)|$, whereas on the event $\tilde{B}_0 \cap \{\tau \leq t_1\}$ we have

$$\sup_{t \leq t_1, x \leq 1} |\tilde{\mathbf{D}}(t, x)| \geq \sup_x |\tilde{\mathbf{D}}(\tau, x)| = \sup_x |\mathbf{D}(\tau, x)| \geq \varepsilon,$$

a consequence of $|\mathbf{u}_g(\tau, x)| \leq 2\varepsilon/3$ and $|\mathbf{u}(\tau, x) - \mathbf{u}_0(x)| > 2\varepsilon$ for some x (recall that we are assuming that $|\mathbf{u}_0(x)| \leq \varepsilon/3$ for all x). Now a union bound gives

$$\begin{aligned}
(4.25) \quad P\left(\sup_{\substack{0 \leq t \leq t_1 \\ x \in [0,1]}} |\tilde{\mathbf{D}}(t, x)| > \frac{\varepsilon}{6}\right) &\leq \frac{1}{\sqrt{c_0}\varepsilon^2} P\left(\sup_{\substack{0 \leq t \leq t_1 \\ x \in [0, \sqrt{c_0}\varepsilon^2]}} |\tilde{\mathbf{D}}(t, x)| > \frac{\varepsilon}{6}\right) \\
&= \frac{1}{\sqrt{c_0}\varepsilon^2} P\left(\sup_{\substack{0 \leq t \leq t_1 \\ x \in [0, \sqrt{c_0}\varepsilon^2]}} |\tilde{\mathbf{D}}(t, x)| > \frac{1}{6c_0^{1/4}} \cdot c_0^{1/4} \varepsilon\right) \\
&\leq \frac{\mathbf{K}_1}{\sqrt{c_0}\varepsilon^2} \exp\left(-\frac{\mathbf{K}_2}{144\mathcal{D}^2\mathcal{C}_2^2\varepsilon^2\sqrt{c_0}}\right),
\end{aligned}$$

where we applied Remark 3.1 to $\tilde{\mathbf{D}}$ instead of \mathbf{N} . Thus when ε is small enough we have

$$(4.26) \quad P\left(\sup_{\substack{0 \leq t \leq t_1 \\ x \in [0,1]}} |\tilde{\mathbf{D}}(t, x)| > \frac{\varepsilon}{6}\right) \leq \mathbf{K}_1 \exp\left(-\frac{\mathbf{K}_2}{288\mathcal{D}^2\mathcal{C}_2^2\varepsilon^2\sqrt{c_0}}\right)$$

Using (4.23) and (4.26) we have

$$(4.27) \quad P(A_0) \geq \exp\left(-\frac{1}{c_0 \mathcal{C}_1^2 \varepsilon^2}\right) \exp\left(\frac{2}{c_0 \varepsilon^2} \log \left[1 - \mathbf{K}_1 \exp\left(-\frac{\mathbf{K}_2}{36 \mathcal{C}_2^2 \sqrt{c_0}}\right)\right]\right) \\ - \mathbf{K}_1 \exp\left(-\frac{\mathbf{K}_2}{288 \mathcal{D}^2 \mathcal{C}_2^2 \varepsilon^2 \sqrt{c_0}}\right).$$

Consequently there is a $\mathcal{D}_0(\mathcal{C}_1, \mathcal{C}_2)$ such that if $\mathcal{D} < \mathcal{D}_0$ then there are constants $C_1, C_2 > 0$ depending only on $\mathcal{C}_1, \mathcal{C}_2$ such that

$$(4.28) \quad P(A_0) \geq C_1 \exp\left(-\frac{C_2}{\varepsilon^2}\right).$$

This completes the proof of Proposition 2.1(b). \square

We can now give the proof of Theorem 1.1

4.3. Proof of Theorem 1.1: Recall that we are working with $J = 1$ and $\mathbf{g} \equiv \mathbf{0}$. For the upper bounds in (1.7) and (1.8) we consider the set

$$F = \bigcap_{n=0}^{n_1-2} F_n.$$

Clearly we have

$$\{|\mathbf{u}(t, x)| \leq \varepsilon \text{ for } t \in [0, T], x \in [0, 1]\} \subset F.$$

From Proposition 2.1(a) we obtain

$$P(F) = \prod_{n=1}^{n_1-2} P\left(F_n \mid \bigcap_{k=0}^{n-1} F_k\right) \leq \left[C_4 \exp\left(-\frac{C_5}{(1 + \mathcal{D}^2)\varepsilon^2}\right)\right]^{\frac{T}{c_0 \varepsilon^4}},$$

which proves the upper bounds in Theorem 1.1.

For the lower bound in (1.7) consider the set

$$A := \bigcap_{n=-1}^{n_1-1} A_n.$$

Clearly we have

$$A \subset \{|\mathbf{u}(t, x)| \leq \varepsilon, \text{ for } 0 \leq t \leq T, x \in [0, 1]\}.$$

From Proposition 2.1(b) we obtain for $\mathcal{D} < \mathcal{D}_0$

$$P(A) = \prod_{n=0}^{n_1-1} P\left(A_n \mid \bigcap_{k=-1}^{n-1} A_k\right) \geq \left[C_6 \exp\left(-\frac{C_7}{\varepsilon^2}\right) \right]^{\frac{T}{c_0\varepsilon^4}},$$

which proves the lower bound in (1.7).

To prove the lower bound in (1.8), we now take $t_1 = \varepsilon^{4+2\delta}$ (note that this is smaller than the earlier value of $t_1 = c_0\varepsilon^4$, at least for small ε), and generally $t_n = n\varepsilon^{4+2\delta}$. We consider the events A_n (see (2.13)), B_0 (see (4.16)) and \tilde{B}_0 (see (4.22)), now *with this new value of t_n* . We have as before

$$P(A_0) \geq P(\tilde{B}_0) - P\left(\sup_{\substack{0 \leq t \leq t_1 \\ x \in [0,1]}} |\tilde{\mathbf{D}}(t, x)| > \frac{\varepsilon}{6}\right).$$

Since $t_1 < c_0\varepsilon^{4+\delta}$ for small ε , we have the same lower bound (4.23) for $P(\tilde{B}_0)$:

$$(4.29) \quad P(\tilde{B}_0) \geq \exp\left(-\frac{1}{c_0\mathcal{C}_1^2\varepsilon^2}\right) \exp\left(\frac{2}{c_0\varepsilon^2} \log\left[1 - \mathbf{K}_1 \exp\left(-\frac{\mathbf{K}_2}{36\mathcal{C}_2^2\sqrt{c_0}}\right)\right]\right),$$

because \tilde{B}_0 is a larger event than the *earlier* defined event with $c_0\varepsilon^4$. By a similar argument as (4.25) we obtain

$$P\left(\sup_{\substack{0 \leq t \leq t_1 \\ x \in [0,1]}} |\tilde{\mathbf{D}}(t, x)| > \frac{\varepsilon}{6}\right) \leq \frac{\mathbf{K}_1}{\varepsilon^{2+\frac{\delta}{2}}} \exp\left(-\frac{\mathbf{K}_2}{144\mathcal{D}^2\mathcal{C}_2^2\varepsilon^{2+\frac{\delta}{2}}\sqrt{c_0}}\right).$$

This is much smaller than the lower bound for $P(\tilde{B}_0)$, for small ε . Therefore $P(A_0) \geq C_1 \exp(-C_2\varepsilon^{-2})$ for constants C_1, C_2 depending on $\mathcal{C}_1, \mathcal{C}_2$ only, when ε is small enough. Now with $n_1 = T\varepsilon^{-4-\delta}$ we obtain

$$P(A) = \prod_{n=0}^{n_1-1} P\left(A_n \mid \bigcap_{k=-1}^{n-1} A_k\right) \geq \left[C_6 \exp\left(-\frac{C_7}{\varepsilon^2}\right) \right]^{\frac{T}{\varepsilon^{4+\delta}}},$$

which gives the lower bound in (1.8). □

REFERENCES

- [All98] H. Allouba, *Different types of SPDEs in the eyes of Girsanov's theorem*, Stochastic Anal. Appl. **16** (1998), no. 5, 787–810. MR 1643116

- [BMSS95] Vlad Bally, Annie Millet, and Marta Sanz-Solé, *Approximation and support theorem in Hölder norm for parabolic stochastic partial differential equations*, Ann. Probab. **23** (1995), no. 1, 178–222. MR 1330767
- [Bas88] Richard F. Bass, *Probability estimates for multiparameter Brownian processes*, Ann. Probab. **16** (1988), no. 1, 251–264. MR 920269
- [Bas95] ———, *Probabilistic techniques in analysis*, xii+392. MR 1329542
- [DKM⁺09] Robert Dalang, Davar Khoshnevisan, Carl Mueller, David Nualart, and Yimin Xiao, *A minicourse on stochastic partial differential equations*, Lecture Notes in Mathematics, vol. 1962, Springer-Verlag, Berlin, 2009, Held at the University of Utah, Salt Lake City, UT, May 8–19, 2006, Edited by Khoshnevisan and Firas Rassoul-Agha. MR 1500166 (2009k:60009)
- [DPZ14] Giuseppe Da Prato and Jerzy Zabczyk, *Stochastic equations in infinite dimensions*, second ed., Encyclopedia of Mathematics and its Applications, vol. 152, Cambridge University Press, Cambridge, 2014. MR 3236753
- [DV75] M. D. Donsker and S. R. S. Varadhan, *Asymptotic evaluation of certain Markov process expectations for large time. I. II*, Comm. Pure Appl. Math. **28** (1975), 1–47; *ibid.* **28** (1975), 279–301. MR 386024
- [KL93] James Kuelbs and Wenbo V. Li, *Metric entropy and the small ball problem for Gaussian measures*, J. Funct. Anal. **116** (1993), no. 1, 133–157. MR 1237989
- [LaM17] Rafał Łatała and Dariusz Matlak, *Royen’s proof of the Gaussian correlation inequality*, Geometric aspects of functional analysis, Lecture Notes in Math., vol. 2169, Springer, Cham, 2017, pp. 265–275. MR 3645127
- [Lot17] S. V. Lototsky, *Small ball probabilities for the infinite-dimensional Ornstein-Uhlenbeck process in Sobolev spaces*, Stoch. Partial Differ. Equ. Anal. Comput. **5** (2017), no. 2, 192–219. MR 3640070
- [LS01] W. V. Li and Q.-M. Shao, *Gaussian processes: inequalities, small ball probabilities and applications*, Stochastic processes: theory and methods, Handbook of Statist., vol. 19, North-Holland, Amsterdam, 2001, pp. 533–597. MR 1861734
- [Mar04] A. Martin, *Small ball asymptotics for the stochastic wave equation*, J. Theoret. Probab. **17** (2004), no. 3, 693–703. MR 2091556
- [RB00] A. Ramachandra Rao and P. Bhimasankaram, *Linear algebra*, second ed., Texts and Readings in Mathematics, vol. 19, Hindustan Book Agency, New Delhi, 2000. MR 1781860
- [Roy14] Thomas Royen, *A simple proof of the Gaussian correlation conjecture extended to some multivariate gamma distributions*, Far East J. Theor. Stat. **48** (2014), no. 2, 139–145. MR 3289621
- [RP04] Raina S. Robeva and Loren D. Pitt, *On the equality of sharp and germ σ -fields for Gaussian processes and fields*, Pliska Stud. Math. Bulgar. **16** (2004), 183–205. MR 2070315
- [Tal94] Michel Talagrand, *The small ball problem for the Brownian sheet*, Ann. Probab. **22** (1994), no. 3, 1331–1354. MR 1303647
- [Tal95] ———, *Hausdorff measure of trajectories of multiparameter fractional Brownian motion*, Ann. Probab. **23** (1995), no. 2, 767–775. MR 1334170
- [Wal86] John B. Walsh, *An introduction to stochastic partial differential equations*, École d’été de probabilités de Saint-Flour, XIV—1984, Lecture Notes in Math., vol. 1180, Springer, Berlin, 1986, pp. 265–439. MR 876085 (88a:60114)

SIVA ATHREYA, STATMATH UNIT, INDIAN STATISTICAL INSTITUTE, 8TH MILE MYSORE ROAD,
BANGALORE 560059

Email address: athreya@isibang.ac.in

MATHEW JOSEPH, STATMATH UNIT, INDIAN STATISTICAL INSTITUTE, 8TH MILE MYSORE ROAD,
BANGALORE 560059

Email address: m.joseph@isibang.ac.in

CARL MUELLER, DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ROCHESTER ROCHESTER, NY
14627

Email address: carl.e.mueller@rochester.edu