SAUSAGE VOLUME OF THE RANDOM STRING AND SURVIVAL IN A MEDIUM OF POISSON TRAPS

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ABSTRACT. We provide asymptotic bounds on the survival probability of a moving polymer in an environment of Poisson traps. Our model for the polymer is the vector-valued solution of a stochastic heat equation driven by additive spacetime white noise; solutions take values in $\mathbb{R}^d, d \geq 1$. We give upper and lower bounds for the survival probability in the cases of hard and soft obstacles. Our bounds decay exponentially with rate proportional to $T^{d/(d+2)}$, the same exponent that occurs in the case of Brownian motion. The exponents also depend on the length J of the polymer, but here our upper and lower bounds involve different powers of J.

Secondly, our main theorems imply upper and lower bounds for the growth of the Wiener sausage around our string. The Wiener sausage is the union of balls of a given radius centered at points of our random string, with time less than or equal to a given value.

1. Introduction

The model of particles performing random diffusive motion in a region containing randomly located traps is known as the trapping problem (see [6] for review). Particle motion is typically Brownian motion in $\mathbb{R}^d$ or a random walk in $\mathbb{Z}^d$. The traps are placed in a Poissonian manner and the particle gets annihilated on encountering a trap. The main question of interest in such models is the "Survival Probability" of the particle. We refer the reader to [11] and references there in for a review of the problem of Brownian motion among Poissonian obstacles, to [8] and references there in for a review of the problem of a random walk in a random potential and to [2] for a review of Random walks among mobile and immobile traps.

There is an extensive literature about such trapping problems, see the references in the preceding paragraph. These results often depend on refined estimates for the

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eigenvalues of the Laplacian or potential theory. On the other hand, the process we consider takes values in function space, and we found it impossible to analyze the situation using existing techniques. Indeed, carrying over finite-dimensional potential theoretic arguments to the infinite dimensional case is often difficult or impossible. We will consider a Gaussian process, but we did not find any Gaussian tools which were relevant to trapping problems.

In this article we will study the annealed survival probability of a random string in a Poissonian trap environment. Let $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P}_0)$ be a filtered probability space on which $\dot{\mathbf{W}} = \dot{\mathbf{W}}(t,x)$ is a d-dimensional random vector whose components are i.i.d. two-parameter white noises adapted to $\mathcal{F}_t$. We consider a *random string* $\mathbf{u}(t,x) \in \mathbb{R}^d$, which is the solution to the following stochastic heat equation (SHE)

$$\begin{aligned} \partial_t \mathbf{u}(t,x) &= \frac{1}{2}\partial_x^2 \mathbf{u}(t,x) + \dot{\mathbf{W}}(t,x) \\ \mathbf{u}(0,x) &= \mathbf{u}_0(x) \end{aligned} \tag{1.1}$$

on the circle $x \in [0,J]$, having endpoints identified, and $t \in [0,T]$. The initial profile $\mathbf{u}_0$ is assumed to be continuous. Note that we will use boldface letters to denote vector-valued quantities.

We will be interested in the evolution of the random string in a field of obstacles centered at points coming from an independent Poisson point process. More precisely, let $(\Omega_1, \mathcal{G}, \mathbb{P}_1)$ be a second probability space on which is defined a Poisson point process $\boldsymbol{\eta}$ with intensity ν given by

$$\boldsymbol{\eta}(\omega_1) = \sum_{i\geq 1} \delta_{\boldsymbol{\xi}_i(\omega_1)}, \quad \omega_1 \in \Omega_1,$$

with points $\{\boldsymbol{\xi}_i(\omega_1)\}_{i\geq 1} \subset \mathbb{R}^d$.

The obstacles will be formed via a potential $\mathrm{V} : \mathbb{R}^d \times \Omega_1 \to [0,\infty]$

$$\mathrm{V}(\mathbf{z}, \boldsymbol{\eta}) = \sum_{i\geq 1} \mathrm{H}(\mathbf{z} - \boldsymbol{\xi}_{\boldsymbol{i}}),$$

where $\mathrm{H} : \mathbb{R}^d \to [0,\infty]$ is a non-negative, measurable function whose support of H is contained in the *closed* ball $B(\mathbf{0}, a)$ of radius $0 < a \leq 1$ centered at $\mathbf{0}$.

We will work in the product space $(\Omega \times \Omega_1, \mathcal{F} \times \mathcal{G}, \mathbb{P}_0 \times \mathbb{P}_1)$ along with the filtration $(\mathcal{F}_t \times \mathcal{G})_{t\geq 0}$. We will write $\mathbb{E}$ for the expectation with respect to $\mathbb{P} := \mathbb{P}_0 \times \mathbb{P}_1$, and $\mathbb{E}_i$ for the expectation with respect to $\mathbb{P}_i$ for $i = 0, 1$. Our main quantity of interest is the quenched and the annealed survival probabilities given by

$$S_{T,\eta}(\omega_1) = \mathbb{E}_0\left[\exp\left(-\int_0^T\int_0^J \mathrm{V}\Big(\mathbf{u}(s,x),\boldsymbol{\eta}(\omega_1)\Big)dxds\right)\right],\ \text{and}$$

$$S_T = \mathbb{E}\left[\exp\left(-\int_0^T\int_0^J \mathrm{V}\left(\mathbf{u}(s,x),\boldsymbol{\eta}\right)dxds\right)\right]$$

respectively. Sometimes we will write $S_T^{\mathrm{H},J,\nu}$ and $S_{T,\boldsymbol{\eta}}^{\mathrm{H},J,\nu}$ to emphasize the dependence on H, J,ν.

1.1. **Main Result.** Our first result on *hard* obstacles, i.e. the string is killed immediately on contact and the only way it can survive is to avoid them.

Theorem 1.1 (Hard obstacles)**.** *Consider the solution to* (1.1) *with* $d\geq 2$ *and* $J\geq 1$, *and let* ν *and* a *be as above. Then the following hold in the case* $H(\cdot)\equiv\infty\cdot\mathbf{1}_{B(\mathbf{0},a)}$

(1) *(Lower bound) There exist positive constants*
  (a) C_0,C_1,C_2 *independent of* T,J *such that for* $T\geq C_0J^{2+\frac{d}{2}}$

$$S_T^{\mathrm{H},J,\nu}\geq C_1\exp\left(-C_2\left(\frac{T}{J}\right)^{\frac{d}{d+2}}\right). \tag{1.2}$$

  (b) C_3,C_4,C_5 *independent of* T,J *such that for* $T\leq C_3J^{2+\frac{d}{2}}$

$$S_T^{\mathrm{H},J,\nu}\geq C_4\exp\left(-C_5\left(TJ\right)^{\frac{d}{d+6}}\right). \tag{1.3}$$

(2) *(Upper bound) There exist positive constants* C_6,C_7 *independent of* T,J *such that for all* $T>0,J\geq 1$

$$S_T^{\mathrm{H},J,\nu}\leq C_6\exp\left(-\frac{C_7}{1+|\log J|}\left(\frac{T}{J^2}\right)^{\frac{d}{d+2}}\right). \tag{1.4}$$

In the case of hard obstacles we immediately see that the survival of the string is only possible if the string avoids the obstacles. Thus the"sausage of radius a around string up to time T" should be devoid of any Poisson points. Indeed it is easy to check using standard properties of the Poisson random variable that

$$S_T^{\mathrm{H},J,\nu}=\mathbb{E}\exp\left(-\nu\left|\mathcal{S}_T^J(a)\right|\right), \tag{1.5}$$

where

$$\mathcal{S}_T^J(a)=\bigcup_{\substack{0\leq s\leq T,\\ 0\leq y\leq J}}\left\{\mathbf{u}(s,y)+B(\mathbf{0},a)\right\}, \tag{1.6}$$

is the sausage of radius a around $\mathbf{u}$. Thus Theorem 1.1 also provides bounds on the exponential moments of the volume of the sausage of radius a around the string up to time T.

We next turn our attention to the case of *soft* obstacles, i.e. H does not take the value ∞. We make the following specific assumptions on H.

Assumption 1.1. *There is a* $\mathcal{C} > 0$ *such that* $H(\mathbf{x}) \geq \mathcal{C} \cdot \mathbf{1}_{B(\mathbf{0},\frac{a}{2})}(\mathbf{x})$.

We note that under this assumption there is a positive probability of survival even if the string interacts with the obstacle environment. We are now ready to state our result in this setting.

Theorem 1.2 (Soft obstacles)**.** *Consider the solution to* (1.1) *with* $d \geq 2$ *and* $J \geq 1$, *let* $\nu > 0$ *be as above and let* H *be a soft obstacle satisfying Assumption 1.1. Then*

(1) *(Lower bound) There exist positive constants*

(a) C_0, C_1, C_2 *independent of* T, J *such that for* $T \geq C_0 J^{2+\frac{d}{2}}$

$$S_T^{\mathrm{H},J,\nu} \geq C_1 \exp\left(-C_2 \left(\frac{T}{J}\right)^{\frac{d}{d+2}}\right). \tag{1.7}$$

(b) C_3, C_4, C_5 *independent of* T, J *such that for* $T \leq C_3 J^{2+\frac{d}{2}}$

$$S_T^{\mathrm{H},J,\nu} \geq C_4 \exp\left(-C_5 \left(TJ\right)^{\frac{d}{d+6}}\right). \tag{1.8}$$

(2) *(Upper bound) Fix* $\beta > 0$. *There exist positive constants* C_6, C_7 *independent of* T, J *such that for all* $T > 0, J \geq 1$

$$S_T^{\mathrm{H},J,\nu} \leq C_6 \exp\left(-\frac{C_7}{J^{3+\beta}(1+|\log J|)} \left(\frac{T}{J^2}\right)^{\frac{d}{d+2}}\right). \tag{1.9}$$

We conclude this sub-section with a few remarks.

Remark 1.1. (i) *If we set* $J = 1$ *in (1.2) and (1.4) or in (1.7) and (1.9) then for large enough* $T > 0$, *in both the hard and soft obstacle cases we have*

$$C_1 \exp\left(-C_2 T^{\frac{d}{d+2}}\right) \leq S_T^{\mathrm{H},1,\nu} \leq C_3 \exp\left(-C_4 T^{\frac{d}{d+2}}\right),$$

for some constants $C_1, C_2, C_3, C_4 > 0$. *Thus the bounds in our results (Theorems 1.1 and 1.2) are optimal in the case when* $J = 1$ *for large enough* $T > 0$.

(ii) *Due to* (1.5)*, Theorem* 1.1 *immediately gives us bounds on* $\mathbb{E}\exp\left(-\nu\left|\mathcal{S}_T^J(a)\right|\right)$*. Further, as we observe above at* $J=1$*, from Theorem* 1.1 *that we have for sufficiently large* $T>0$

$$C_1\exp\left(-C_2T^{\frac{d}{d+2}}\right)\le\mathbb{E}\exp\left(-\nu\left|\mathcal{S}_T^1(a)\right|\right)\le C_3\exp\left(-C_4T^{\frac{d}{d+2}}\right),$$

for some constants $C_1,C_2,C_3,C_4>0$*. The exponent of* T *matches that of the asymptotics of the volume of the Brownian Sausage. Since in our case* $\mathbf{u}$ *is the solution of a stochastic PDE, this seems to be a new result of independent interest.*

(iii) *The constants mentioned in Theorem* 1.1 *and Theorem* 1.2 *are all independent of* T,J *but do depend on* $\nu,a,$ *and* $\mathcal{C}$*. The bound* $0<a\le 1$ *is used for technical convenience.*

1.2. **Overview of Proof.** We will say that $\mathbf{u}$ is a solution to (1.1) if it satisfies,

$$\mathbf{u}(t,x)=\int_0^J G^{(J)}(t,x-y)\mathbf{u}_0(y)dy+\int_{[0,t]\times[0,J]}G^{(J)}(t-s,x-y)\mathbf{W}(dsdy), \tag{1.10}$$

where $[0,J]$ is the circle with endpoints identified and $G^{(J)}:\mathbb{R}_+\times[0,J]\to\mathbb{R}$ is the fundamental solution of the heat equation

$$\begin{aligned}\partial_t G^{(J)}(t,x)&=\frac{1}{2}\,\partial_x^2 G^{(J)}(t,x),\\ G^{(J)}(0,x)&=\delta(x).\end{aligned}$$

Furthermore, the final integral in (1.10) can be regarded as either a Wiener integral or a white noise integral in the sense of Walsh [13].

The first reduction in the proof is to reduce to the case $J=1$. We will do this by deriving a scaling relation for $S_T^{\mathrm{H},J,\nu}$. Consider

$$\mathbf{v}(t,x):=J^{-\frac{1}{2}}\mathbf{u}(J^2t,Jx),$$

defined for $x\in[0,1]$ with endpoints identified, and $t\in[0,TJ^{-2}]$. The initial profile is $\mathbf{v}(0,x)=\mathbf{v}_0(x)=J^{-1/2}\mathbf{u}_0(Jx)$. It was proved in Lemma 2.2 of [3] that $\mathbf{v}$ satisfies

$$\begin{aligned}\partial_t\mathbf{v}&=\frac{1}{2}\partial_x^2\mathbf{v}+\dot{\widetilde{\mathbf{W}}},\quad t\in[0,TJ^{-2}]\\ \mathbf{v}(0,x)&=\mathbf{v}_0(x),\quad x\in[0,1]\end{aligned}$$

for some other white noise $\dot{\widetilde{\mathbf{W}}}$. Now it is easily checked

$$\begin{aligned} S_T^{\mathrm{H},J,\nu} &= \mathbb{E}\left[\exp\left(-\int_0^T\int_0^J \sum_{i\geq 1} \mathrm{H}\left(\mathbf{u}(s,x)-\boldsymbol{\xi}_i\right) dxds\right)\right] \\ &= \mathbb{E}\left[\exp\left(-\int_0^{\frac{T}{J^2}}\int_0^1 \sum_{i\geq 1} J^3\mathrm{H}\left(J^{\frac12}\left(\mathbf{v}(\tilde{s},\tilde{x})-\frac{\boldsymbol{\xi}_i}{J^{\frac12}}\right)\right) d\tilde{x}d\tilde{s}\right)\right] \end{aligned}$$

Define

$$\widetilde{\mathrm{H}}(\cdot) := J^3\mathrm{H}(J^{\frac12}\cdot), \qquad \tilde{\boldsymbol{\xi}}_i := \frac{\boldsymbol{\xi}_i}{J^{\frac12}}, \qquad \text{and} \qquad \widetilde{\nu} := \nu J^{\frac{d}{2}}. \tag{1.11}$$

It is easily seen that $\widetilde{\mathrm{H}}$ is supported in the ball $B(\mathbf{0}, aJ^{-\frac12})$. The points $\tilde{\boldsymbol{\xi}}$ form a Poisson point process of intensity $\widetilde{\nu}$. Setting $\widetilde{T} = TJ^{-2}$ we obtain

$$S_T^{\mathrm{H},J,\nu} = S_{\widetilde{T}}^{\widetilde{\mathrm{H}},1,\widetilde{\nu}}. \tag{1.12}$$

Remark 1.2 **(Important).** *For the rest of article will focus on $J = 1$, so (1.10) becomes*

$$\mathbf{u}(t,x) = \int_0^1 G(t,x-y)\mathbf{u}_0(y)\,dy + \mathbf{N}(t,x),$$

where

$$\mathbf{N}(t,x) = \int_{[0,t]\times[0,1]} G(t-s,x-y)\mathbf{W}(dsdy)$$

is the noise term. For simplicity of notation, we have also removed the superscript in $G^{(1)}$. We will work with $S_T^{\mathrm{H},1,\nu}$ and finally use the scaling relation (1.12) to obtain the bounds for $S_T^{\mathrm{H},J,\nu}$

The key strategy for proving the lower bound for survival probability in Theorem 1.1 Theorem 1.2 is to obtain an optimal configuration for the traps ξ so that the string does not get killed. This configuration has an area free of traps in a ball of radius α_T around the origin and the string under this potential is made to stay inside this ball till time T. The probability of obtaining such a configuration is of the order $\exp(-C_1(\alpha_T + a)^d)$. In the regime 1(b) of Theorems 1.1 and 1.2 we can use the small ball probability estimates of Theorem 1.1 in [3] to obtain a lower bound of $\exp(-C_2\frac{TJ}{\alpha_T^6})$ on $\mathbb{P}_0\left(\sup_{t\leq T,\, x\in[0,1]} |\mathbf{u}(t,x)| \leq \alpha_T\right)$. Optimizing α_T in the product we obtain the lower bound in the regime 1(b) of Theorems 1.1 and 1.2. In the regime 1(a) of the theorems, Theorem 1.1 in [3] is not applicable, and therefore we decompose the string into two components, namely *center of mass* and *radius* of $\mathbf{u}$ respectively. More precisely let

$$
\begin{aligned}
\mathbf{X}_t &= \int_0^1 \mathbf{u}(t,x)\,dx, \text{ (Center of Mass)} \\
\mathbf{R}_t &= \sup_{x\in[0,1]} |\mathbf{u}(t,x) - \mathbf{X}_t|\,, \text{ (Radius)}.
\end{aligned}
\tag{1.13}
$$

We show that $\mathbf{X}_t$ and $\mathbf{R}_t$ are independent. Then separately we consider the events $|\mathbf{X}_t| \le \frac{\alpha_T}{2},\ t \le T$ and $\mathbf{R}_t \le \frac{\alpha_T}{2},\ t \le T$, and show that the probability of their intersection is bounded below by $\exp(-C_2(\frac{T}{\alpha_T^2}))$. Optimizing over α_T and the scaling relations discussed above yield 1(a) in Theorems 1.1 and 1.2. We present the details in Section 2.

Unlike the lower bound, the proof of upper bound differs from the classical setting of random walks or that of Brownian motion. The proof techniques in those models depend on potential theory and eigenvalues of the Laplacian, and both of these are much harder to study for infinite dimensional processes such as the random string. We are thus forced to go back to first principles, which perhaps explains the fact that our upper and lower bounds do not completely match. It is also important to note that while the upper and lower bounds match for the case $J = 1$, the scaling relations in (1.11) imply that they don't carry over to the general case via space-time scaling.

Following Remark 1.2 we first obtain an upper bound for $S_T^{\mathrm{H},1,\nu}$. Recall

$$
S_T^{\mathrm{H},1,\nu} = \mathbb{E}\exp\left(-\nu\left|\mathcal{S}_T^1(a)\right|\right), \tag{1.14}
$$

where $\mathcal{S}_T^1(a)$ is the sausage of radius a around $\mathbf{u}$, that is

$$
\mathcal{S}_T^1(a) = \bigcup_{\substack{0\le s\le T,\\ 0\le y\le 1}} \left\{\mathbf{u}(s,y) + B(\mathbf{0},a)\right\}.
$$

We will explain the strategy for the proof in the case of hard obstacles, the argument for the soft obstacles not being very different. Due to (1.14), an upper bound on the partition function $S_T^{\mathrm{H},1,\nu}$ essentially boils down to obtaining a lower bound on the volume of the sausage $\mathcal{S}_T^1(a)$ around $\mathbf{u}$. For this, we consider the sausage around $\mathbf{u}(t) = \mathbf{u}(t,\cdot)$, that is

$$
\mathcal{S}^1(a;t) = \bigcup_{0\le y\le 1} \left\{\mathbf{u}(t,y) + B(\mathbf{0},a)\right\}, \tag{1.15}
$$

so that

$$
\mathcal{S}_T^1(a) = \bigcup_{0\le t\le T} \mathcal{S}^1(a;t).
$$

We will identify times at which $\mathcal{S}^1(a;t)$ do not intersect, so the sum of the volumes of these fixed time sausages will provide the desired lower bound.

We will consider a set of stopping times τ_i (see (3.3)) such that the center of mass at these time points, $\mathbf{X}_{\tau_i}$, are separated by at least 4Λ from each other (where Λ is suitably chosen, see Lemma 3.10). Using known results on the volume of the Wiener sausage, we show that the number of τ_i before time T should be at least $C_1 T^{\frac{d}{d+2}}$ with probability $1-\exp(-C_2 T^{\frac{d}{d+2}})$ for some constants C_1, C_2 (see Lemma 3.2).

Now, let

$$\mathbf{N}(s,t;x) := \int_{[s,t]\times[0,1]} G(t-r, x-y)\mathbf{W}(drdy),$$

which represents the noise term from time s to t. Then $\mathbf{N}(s,t)$ will represent the function from $x \in [0,1]$ to $\mathbb{R}^d$. For $s<t$, we use the Markov property for $\mathbf{u}$ to write

$$\mathbf{u}(t) = G_{t-s} * \mathbf{u}(s) + \mathbf{N}(s,t).$$

If $s \ll t$ then the first term is almost a constant because of the smoothening effect of the Laplacian. The volume of the sausage around $G_{t-s} * \mathbf{u}(s)$ will then be approximately a^d. We show in Lemma 3.11, using the independence of $\mathbf{X}_t$ and $\mathbf{R}_t$, that with probability $\geq \frac{1}{2}$ the range (see (3.8) for precise definition) of $\mathbf{N}(s,t)$ is at most Λ and the volume of the sausage of radius a around $\mathbf{N}(s,t)$ is of order at least $a^{d-2+\epsilon}$.

Using this we shall construct a subset $\{T_i\}$ of $\{\tau_i\}$, such that:

(a) the $\{T_i\}$ are at least distance L apart (see (3.9));
(b) the range of each $\mathbf{N}(T_{i-1}, T_i])$ is less than equal to Λ ;
(c) the volume of the sausage of radius a around each $\mathbf{N}(T_{i-1}, T_i)$ is of order at least $a^{d-2+\epsilon}$; and
(d) the number of $\{T_i\}$ before time T is at least $C_3 T^{\frac{d}{d+2}}$ with probability $1 - \exp(-C_4 T^{\frac{d}{d+2}})$ for some constants C_3, C_4 (see Lemma 3.2).

These conditions with appropriate choice of Λ will imply that the sausages around $\{\mathbf{u}(T_i)\}$ will be disjoint and have volume $a^{d-2+\epsilon}$ each. Therefore formally speaking

(see proof for precise details)

$$
\begin{aligned}
S_T^{\mathrm{H},1,\nu} &\le \mathbb{P}_0\left(\#T_i \le C_3\frac{T^{\frac{d}{d+2}}}{L}\right) + \mathbb{E}_0\left[\exp\left(-\nu\left|\bigcup_{j=0}^{C_3\frac{T^{\frac{d}{d+2}}}{L}} \mathcal{S}\left(\frac{a}{2};\mathbf{N}(T_{j-1},T_j)\right)\right|\right)\right] \\
&\le \exp\left(-C_4\frac{T^{\frac{d}{d+2}}}{L}\right) + \exp\left(-C_5 a^{d-2+\gamma}\frac{T^{\frac{d}{d+2}}}{L}\right) \\
&\le \exp\left(-C_4\frac{T^{\frac{d}{d+2}}}{L}\right) + \exp\left(-C_5 a^{d-2+\gamma}\frac{T^{\frac{d}{d+2}}}{L}\right).
\end{aligned}
$$

Then using the scaling mentioned above we obtain

$$
S_T^{\mathrm{H},J,\nu} \le \exp\left(-\frac{C_6(T/J^2)^{\frac{d}{d+2}}}{E+3|\log(a/J^{\frac{1}{2}})|}\right) + \exp\left(-\frac{C_7(T/J^2)^{\frac{d}{d+2}}J^{1-\frac{\gamma}{2}}}{E+3|\log(a/J^{\frac{1}{2}})|}\right),
$$

for some $\gamma > 0$. We note that after scaling the first term above will be the dominating term and will provide the upper bound in Theorem 1.1 part 2. We derive the best possible estimates for volume of the sausage as this might allow future improvements in our bounds on the first term.

We conclude this section with some open questions. The upper and lower bounds do not match in Theorem 1.1 and Theorem 1.2 in J. This is a gap that seems hard to fix given the paucity of techniques and tools available in the infinite dimensional setting. The quenched survival probability is also of keen interest. Here the geometry of the string and its topology will come into play. We do not consider this here. We also do not explore large deviations for the volume of the sausage as there is no obvious ergodicity to establish a limiting value of a Lyapunov exponent.

Convention: We will use C to denote constants whose value might change from line to line. Sometimes we will indicate dependence of constants on parameters by putting the parameters in parentheses, for example $C(d), C(\nu, d)$ etc. The notation $C_1, C_2, \cdots$ will be used to denote constants whose value remain fixed throughout a lemma, proposition, theorem etc. Such constants might be used later in which case it will be clear from the context.

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2. Proof of the lower bound in Theorems 1.1 and 1.2

As indicated above we will use the same strategy for the lower bound for the survival probability for both the *hard* and *soft* obstacle case. We begin with the proof of 1(b) first.

Proof of lower bounds 1(b) in Theorem 1.1 and Theorem 1.2. Following Remark 1.2 we find the lower bound for $S_T^{\mathrm{H},1,\nu}$. As indicated earlier for the string to survive in a *hard* obstacle environment, it must avoid the obstacles. We will use the same strategy of survival for the *soft* obstacle as well. Let

$$\mathcal{O} = \bigcup_{i\geq 1} B(\boldsymbol{\xi}_i, a)$$

be the obstacle set. For $T > 0$, let $\alpha \equiv \alpha_T > 0$ be a parameter which will be chosen to devise the optimal strategy. Due to the support of H being in a ball of radius a one obtains

$$\begin{aligned} S_T^{\mathrm{H},1,\nu} &\geq \mathbb{P}(\mathcal{B}_T \cap \mathcal{C}_T) \\ &= \mathbb{P}_0(\mathcal{B}_T)\mathbb{P}_1(\mathcal{C}_T), \end{aligned} \tag{2.1}$$

where

$$\begin{aligned} \mathcal{B}_T &= \left\{ \sup_{\substack{s\in[0,T] \\ x\in[0,1]}} |\mathbf{u}(s,x)| \leq \alpha \right\}, \\ \mathcal{C}_T &= \Big\{ \text{there are no } \boldsymbol{\xi}_i \text{ in the ball } B\left(\mathbf{0}, \alpha + a\right) \Big\}. \end{aligned}$$

It is important to observe here that the above argument does not depend on whether the obstacles are hard or soft. Clearly

$$\mathbb{P}_1(\mathcal{C}_T) = \exp\left(-\nu c_d (\alpha + a)^d\right) \tag{2.2}$$

for some dimension dependent constant c_d.

Using Theorem 1.1 in [3] we have that there exist $C_0 > 0, C_1 > 0$ and $\epsilon_0 > 0$ such that if $\alpha_T < \epsilon_0\sqrt{J}$ then

$$P\left(\sup_{\substack{0\le s\le T\\ y\in[0,J]}} |\mathbf{u}(s,y)| \le \varepsilon\right) \ge C_0 \exp\left(-C_1\frac{TJ}{\varepsilon^6}\right)$$

Therefore using the above we have

$$\begin{aligned} S_T^{\mathrm{H},J,\nu} &\ge C_0 \exp\left(-\nu c_d(\alpha+a)^d\right)\exp\left(-C_1\frac{TJ}{\alpha^6}\right) \\ &\ge C_0 \exp\left(-\nu c_d 2^d a^d\right)\exp\left(-\nu c_d 2^d \alpha^d - C_1\frac{TJ}{\alpha^6}\right) \end{aligned}$$

A simple calculus computation shows that the maximum of the exponent in the attained at $\alpha = (C_2TJ)^{\frac{1}{d+6}}$ for some $C_2(d,\nu) > 0$ so that

$$S_T^{\mathrm{H},J,\nu} \ge C_0 \exp\left(-\nu c_d 2^d a^d\right)\exp\left(-(C_2TJ)^{\frac{d}{d+6}}\right). \tag{2.3}$$

The choice of α is valid provided

$$(C_2TJ)^{\frac{1}{d+6}} < \epsilon_0\sqrt{J} \iff T < \frac{\epsilon_0^{d+6}}{C_2}J^{\frac{d}{2}+2}$$

Thus there is a constant $C_3(d,\nu)$ independent of J, T, such that if $T \le C_3J^{2+\frac{d}{2}}$ then there are $C_4(d,\nu), C_5(d,\nu) > 0$ such that

$$S_T^{\mathrm{H},J,\nu} \ge C_4 \exp\left(-C_5(TJ)^{\frac{d}{d+6}}\right)$$

□

We need a couple of technical results before we begin the proof of 1(a). The following lemma is crucial.

Lemma 2.1. *With* $\mathbf{X}_t$ *and* $\mathbf{R}_t$ *as in* (1.13)*, we have*

(a) $\mathbf{X}_t$ *is a standard Brownian motion starting at* $\int_0^1 \mathbf{u}_0(x)dx$.
(b) $\mathbf{X}_t$ *and* $\mathbf{R}_t$ *are independent.*

Proof. It is easily checked that

$$\mathbf{X}_t = \int_0^1 \mathbf{u}_0(x)dx + \int_0^t \mathbf{W}(dyds)$$

is a standard Brownian motion, and

$$\mathbf{u}(t,x) - \mathbf{X}_t = \int_0^1 [G(t, x-y) - 1]\, \mathbf{u}_0(y)dy + \int_{[0,t]\times[0,1]} [G(t-s, x-y) - 1]\, \mathbf{W}(dsdy),$$

where $G = G^{(1)}$ is the heat kernel on the unit circle. Both these processes are Gaussian. The components of $\mathbf{X}_t$ and $\mathbf{u}(t,x) - \mathbf{X}_t$ are uncorrelated since

$$\int_0^t \int_0^1 [G(t-s, x-y) - 1]\ dyds = 0.$$

The second part of the lemma immediately follows. □

We will also need

Proposition 2.1. *Assume* $\sup_x |\mathbf{u}_0(x)| \le \frac{\alpha}{2}$. *Then there are constants* $0 < C_0 < 1$ *and* $K_0 > 0$ *such that for all* $\alpha \ge K_0$

$$\mathbb{P}_0\left(\sup_{\substack{s\le\alpha^2\\ x\in[0,1]}} |\mathbf{u}(s,x)| \le \alpha,\ \sup_{x\in[0,1]} |\mathbf{u}(\alpha,x)| \le \frac{\alpha}{2} \right) \ge C_0. \tag{2.4}$$

Proof. Let us consider first the case $\mathbf{u}_0 \equiv \mathbf{0}$. From the previous lemma

$$\mathbb{P}_0\left(\sup_{\substack{s\le\alpha^2\\ x\in[0,1]}} |\mathbf{u}(s,x)| \le \frac{\alpha}{2} \right) \ge \mathbb{P}_0\left(\sup_{s\le\alpha^2} |\mathbf{R}_s| \le \frac{\alpha}{4} \right) \mathbb{P}_0\left(\sup_{s\le\alpha^2} |\mathbf{X}_s| \le \frac{\alpha}{4} \right).$$

Since the last term is a positive constant independent of T, it is enough to show that there is a $K_0 > 0$ such that

$$\sup_{\alpha\ge K_0} \mathbb{P}_0\left(\sup_{s\le\alpha^2} |\mathbf{R}_s| > \frac{\alpha}{4} \right) < 1.$$

Now

$$\mathbb{P}_0\left(\sup_{s\le\alpha^2} |\mathbf{R}_s| > \frac{\alpha}{4} \right) = \mathbb{P}_0\left(\sup_{\substack{s\le\alpha^2\\ x\in[0,1]}} |\mathbf{u}(s,x) - \mathbf{X}_s| > \frac{\alpha}{4} \right). \tag{2.5}$$

By splitting the time interval into subintervals of length 1 we have the bound

$$\begin{aligned}
&\mathbb{P}_0\left(\sup_{\substack{s\le\alpha^2\\ x\in[0,1]}} |\mathbf{u}(s,x)-\mathbf{X}_s| > \frac{\alpha}{4}\right)\\
&\le \sum_{k=0}^{[\alpha^2]+1} \mathbb{P}_0\left(\sup_{\substack{s\in[k,k+1]\\ x\in[0,1]}} |\mathbf{u}(s,x)-\mathbf{X}_s| > \frac{\alpha}{4}\right)\\
&\le \sum_{k=0}^{[\alpha^2]+1} \mathbb{P}_0\left(|\mathbf{u}(k,0)-\mathbf{X}_k| > \frac{\alpha}{8}\right)\\
&\qquad + \sum_{k=0}^{[\alpha^2]+1} \mathbb{P}_0\left(\sup_{\substack{s\in[i,i+1]\\ x\in[0,1]}} |\,[\mathbf{u}(k,0)-\mathbf{X}_k]-[\mathbf{u}(s,x)-\mathbf{X}_s]\,| > \frac{\alpha}{8}\right)
\end{aligned} \tag{2.6}$$

Using the standard Fourier decomposition of $G(t,x)$ (see Section 3 of [3]) one obtains that each coordinate of $\mathbf{u}(i,0)-\mathbf{X}_i$ has variance

$$\begin{aligned}
\int_0^k\int_0^1 [G(s,y)-1]^2\, dyds &= \int_0^k\int_0^1 G^2(s,y)\,dyds - k\\
&= \int_0^k \sum_{l\ge1} \exp\left(-(2\pi l)^2 s\right)\, ds\\
&\le C.
\end{aligned}$$

See Lemma 3.1 of [3] for details. Therefore for large α

$$\sum_{k=0}^{[\alpha^2]+1} \mathbb{P}_0\left(|\mathbf{u}(k,0)-\mathbf{X}_k| > \frac{\alpha}{8}\right) \le \exp\left(-C\alpha^2\right).$$

Now we turn to the last term in (2.6). Consider the process

$$\mathbf{M}(s,x) = [\mathbf{u}(k,0)-\mathbf{X}_k] - [\mathbf{u}(s,x)-\mathbf{X}_s], \quad s\in[k,k+1],\ x\in[0,1].$$

Note that $\mathbf{M}(k,0)=0$. A quick calculation gives

$$\mathbf{M}(s,x)-\mathbf{M}(s,\tilde{x}) = \int_{[0,s]\times[0,1]} [G(s-r,\tilde{x}-y)-G(s-r,x-y)]\,\mathbf{W}(drdy)$$

whose components have variance less than $C|x-y|$ (see Lemma 3.1 of [3] for details). Similarly for $k \le s < \tilde{s} \le k+1$ we obtain

$$\begin{aligned}\mathbf{M}(s,x) - \mathbf{M}(\tilde{s},x) = \int_{[0,s]\times[0,1]} \left[G(\tilde{s}-r,x-y) - G(s-r,x-y)\right]\mathbf{W}(drdy) \\ + \int_{[s,\tilde{s}]\times[0,1]} G(\tilde{s}-r,x-y)\mathbf{W}(drdy) + \left[\mathbf{X}_s - \mathbf{X}_{\tilde{s}}\right].\end{aligned}$$

Following Lemma 3.1 of [3] we obtain that the components have variance less than $C\sqrt{\tilde{s}-s}$. Note that $\tilde{s} \le s+1$ so that the variance of the components of $\mathbf{X}_{\tilde{s}} - \mathbf{X}_s$ are also bounded by $C\sqrt{\tilde{s}-s}$.

Therefore the conditions of Lemma 3.4 of [3] are satisfied, and we obtain for large α

$$\sum_{k=0}^{[\alpha^2]+1} \mathbb{P}_0\left(\sup_{\substack{s\in[k,k+1]\\ x\in[0,1]}} \left|[\mathbf{u}(k,0) - \mathbf{X}_k] - [\mathbf{u}(s,x) - \mathbf{X}_s]\right| > \frac{\alpha}{8}\right) \le \exp\left(-C\alpha^2\right).$$

Returning to (2.5) we obtain

$$\mathbb{P}_0\left(\sup_{s\le\alpha^2} |\mathbf{R}_s| > \frac{\alpha}{2}\right) \le \exp\left(-C\alpha^2\right) < 1,$$

uniformly in $\alpha \ge K_0$ for some $K_0 > 0$. This completes the proof of the proposition in the case that $\mathbf{u}_0 \equiv \mathbf{0}$.

In the general case $\sup_x |\mathbf{u}_0(x)| \le \frac{\alpha}{2}$, we apply a Girsanov change of measure argument. Consider the measure $\mathbb{Q}_0$ given by

$$\frac{d\mathbb{Q}_0}{d\mathbb{P}_0} = \exp\left(-\int_{[0,\alpha^2]\times[0,1]} \frac{(G_s * \mathbf{u}_0)(y)\cdot\mathbf{W}(dsdy)}{\alpha^2} - \frac{1}{2}\int_0^{\alpha^2}\int_0^1 \frac{|G_s*\mathbf{u}_0)(y)|^2}{\alpha^4}\right),$$

where $G_s * \mathbf{u}_0$ is the convolution of $G(s,\cdot)$ with $\mathbf{u}_0$. Under the measure $\mathbb{Q}_0$,

$$\widetilde{\mathbf{W}}(dsdy) = \mathbf{W}(dsdy) + \frac{(G_s*\mathbf{u}_0)(y)}{\alpha^2}dsdy$$

is a white noise (see [1]). Moreover for $0 \le t \le \alpha^2$

$$\mathbf{u}(t,x) = \left(1 - \frac{t}{\alpha^2}\right)(G_t * \mathbf{u}_0)(x) + \int_{[0,t]\times[0,1]} G(t-s,x-y)\widetilde{\mathbf{W}}(dsdy), \tag{2.7}$$

and the first term is 0 at time $t = \alpha^2$. The case of $\mathbf{u}_0 \equiv \mathbf{0}$ shows

$$\mathbb{Q}_0\left(\sup_{\substack{t\le\alpha^2\\ x\in[0,1]}}\left|\int_{[0,t]\times[0,1]} G(t-s,x-y)\widetilde{\mathbf{W}}(dsdy)\right|\le\frac{\alpha}{2}\right)\ge\tilde{C}_0,$$

for some $\tilde{C}_0 > 0$. An application of the Cauchy-Schwarz inequality gives

$$\begin{aligned}
&\mathbb{P}_0\left(\sup_{\substack{t\le\alpha^2\\ x\in[0,1]}}\left|\int_{[0,t]\times[0,1]} G(t-s,x-y)\widetilde{\mathbf{W}}(dsdy)\right|\le\frac{\alpha}{2}\right)\\
&\ge\mathbb{Q}_0\left(\sup_{\substack{t\le\alpha^2\\ x\in[0,1]}}\left|\int_{[0,t]\times[0,1]} G(t-s,x-y)\widetilde{\mathbf{W}}(dsdy)\right|\le\frac{\alpha}{2}\right)^{1/2}\cdot\mathbb{E}_0\left[\left(\frac{d\mathbb{Q}_0}{d\mathbb{P}_0}\right)^2\right]^{-1/2}\\
&\ge\tilde{C}_0^{1/2}\exp\left(-\frac{1}{4}\int_0^{\alpha^2}\int_0^1\frac{\alpha^2}{4\alpha^4}dyds\right)\\
&=\tilde{C}_0^{1/2}\exp\left(-\frac{1}{16}\right).
\end{aligned}$$

The first term in (2.7) is at most $\frac{\alpha}{2}$, and so the above lower bound is also a lower bound for the probability in (2.4). This completes the proof of the proposition. □

Proof of lower bound in Theorems 1.1 and 1.2 1(b): Following Remark 1.2 we find the lower bound for $S_T^{\mathrm{H},1,\nu}$. As indicated earlier for the string to survive in a *hard* obstacle environment, it must avoid the obstacles. We will use the same strategy of survival for the *soft* obstacle as well. Let

$$\mathcal{O}=\bigcup_{i\ge1}B(\boldsymbol{\xi}_i,a)$$

be the obstacle set. For $T > 0$, let $\alpha \equiv \alpha_T > 0$ be a parameter which will be chosen to devise the optimal strategy. Due to the support of H being in a ball of radius a one obtains

$$\begin{aligned}
S_T^{\mathrm{H},1,\nu}&\ge\mathbb{P}(\mathcal{B}_T\cap\mathcal{C}_T)\\
&=\mathbb{P}_0(\mathcal{B}_T)\mathbb{P}_1(\mathcal{C}_T),
\end{aligned}\tag{2.8}$$

where

$$\mathcal{B}_T = \left\{ \sup_{\substack{s\in[0,T] \\ x\in[0,1]}} |\mathbf{u}(s,x)| \le \alpha \right\},$$

$$\mathcal{C}_T = \Big\{ \text{there are no } \boldsymbol{\xi}_i \text{ in the ball } B\left(\mathbf{0}, \alpha + a\right) \Big\}.$$

It is important to observe here that the above argument does not depend on whether the obstacles are hard or soft. Clearly

$$\mathbb{P}_1(\mathcal{C}_T) = \exp\left(-\nu c_d(\alpha+a)^d\right) \tag{2.9}$$

for some dimension dependent constant c_d.

We will next estimate $\mathbb{P}_0(\mathcal{B}_T)$ by using Proposition 2.1. Indeed, an application of the Markov property and (2.4) yields

$$\mathbb{P}_0(\mathcal{B}_T) \ge \mathbb{P}_0\left(\sup_{\substack{s\le\alpha^2 \\ x\in[0,1]}} |\mathbf{u}(s,x)| \le \alpha,\ \sup_{x\in[0,1]} |\mathbf{u}(\alpha,x)| \le \frac{\alpha}{2} \right)^{\frac{T}{\alpha^2}} \ge \exp\left(\frac{T}{\alpha^2}\log C_0\right), \tag{2.10}$$

for $\alpha^2 \ll T$.

Using (2.9) and (2.10) in (2.8) we have for $\alpha \ge K_0$

$$\begin{aligned} S_T^{\mathrm{H},1,\nu} &\ge \exp\left(-\nu c_d(\alpha+a)^d\right)\exp\left(\frac{T}{\alpha^2}\log C_0\right) \\ &\ge \exp\left(-\nu c_d 2^d a^d\right)\exp\left(-\nu c_d 2^d\alpha^d + \frac{T}{\alpha^2}\log C_0\right). \end{aligned}$$

For general $J \ge 1$ we use (1.12), as well as the fact that $\widetilde{a} = aJ^{-\frac{1}{2}}$, $\widetilde{\nu} = \nu J^{\frac{d}{2}}$ to obtain

$$S_T^{\mathrm{H},J,\nu} \ge \exp\left(-\nu c_d 2^d a^d\right)\exp\left(-\nu J^{\frac{d}{2}} c_d 2^d \alpha^d + \frac{T}{J^2\alpha^2}\log C_0\right).$$

A simple calculus computation shows that the maximum of the exponent in the second term is attained at $\alpha = C(d,\nu)\left(\frac{T}{J^{2+\frac{d}{2}}}\right)^{\frac{1}{d+2}}$ so that

$$S_T^{\mathrm{H},J,\nu} \ge \exp\left(-\nu c_d 2^d a^d\right)\exp\left(-C_1(d,\nu)\left(\frac{T}{J}\right)^{\frac{d}{d+2}}\right), \tag{2.11}$$

for a constant $C_1(d,\nu)$ independent of J, T, as long as $\alpha \geq K_0$ or equivalently $T \geq C_2(d,\nu)J^{2+\frac{d}{2}}$. □

3. Preliminaries for the upper bounds in Theorems 1.1 and 1.2

In this section we prove several preliminary lemmas required for the proof. We begin with Section 3.1, where we define the stopping times $\{\tau_i\}$ precisely and show that there are order of $T^{\frac{d}{d+2}}$ such times in $[0,T]$ with very high probability. In Section 3.2, we define the crucial stopping times $\{T_i\}$ at which we will consider the volume of the sausage around $\mathbf{u}(T_i,\cdot)$. We will choose the $\{T_i\}$ from the $\{\tau_i\}$ so that

(1) $\mathbf{u}(T_i,\cdot)$ has a larger volume than the sausage of radius $a/2$ around $\mathbf{N}\left(T_{i-1},T_i\right)$,
(2) the volume of the sausage around $\mathbf{N}(T_{i-1},T_i)$ of radius $\frac{a}{2}$ is at least $\mathcal{C}_\gamma a^{d-2+\gamma}$, and
(3) the range of $\mathbf{N}(T_{i-1},T_i)$ has volume less than or equal to Λ.

Then in Section 3.4 we show is that there are sufficiently many times $\{T_i\}$, and they are far enough apart to ensure that the sausages at these times do not intersect and the gaps between these times have finite mean. Finally we conclude with Section 3.5 where we prove some estimates needed for the soft obstacle case.

3.1. **Using Estimates of the Wiener Sausage.** For $\Lambda > 1$ (to be chosen specifically in Lemma 3.10), it is useful to consider the sausage of radius 4Λ around the center of mass $\mathbf{X}_T$:

$$\mathcal{X}_T(4\Lambda) := \bigcup_{0\leq t\leq T} \left\{\mathbf{X}_s + B(\mathbf{0},4\Lambda)\right\}. \tag{3.1}$$

This is the well studied Wiener sausage.

Lemma 3.1 ([12], [4], [7], [10])**.** *Let $d \geq 2$ and $\Lambda > 1$. There exists $C(d,\Lambda) > 0$ such that for $T > 0$ we have*

$$\mathbb{P}_0\left(|\mathcal{X}_T(4\Lambda)| \leq T^{\frac{d}{d+2}}\right) \leq \exp\left(-C(d,\Lambda)T^{\frac{d}{d+2}}\right). \tag{3.2}$$

Let $\tau_0 = 0$ and consider consecutive stopping times τ_i defined as

$$\tau_{i+1} = \inf\left\{t > \tau_i : \operatorname{dist}\left(\mathbf{X}_t, \bigcup_{k=0}^{i} \mathbf{X}_{\tau_k}\right) \geq 4\Lambda\right\}. \tag{3.3}$$

Note that $\mathbf{X}_{\tau_i}$ is on the boundary of the region $\bigcup_{k=0}^{i-1} B(\mathbf{X}_{\tau_k}, 4\Lambda)$. Let

$$\#(T) := |\{i \geq 1 : \tau_i \leq T\}| \tag{3.4}$$

be the number of τ_i's with $i \geq 1$ before time T.

Lemma 3.2. *Let $d \geq 2$ and $\Lambda > 1$. There exists $C_d > 0$ such that for $T > 0$ we have*

$$\mathbb{P}_0\left(\#(T) \leq \frac{C_d T^{\frac{d}{d+2}}}{\Lambda^d}\right) \leq \exp\left(-C(d,\Lambda)T^{\frac{d}{d+2}}\right), \tag{3.5}$$

where $C(d,\Lambda)$ is the same as in Lemma 3.1.

Proof. We first claim that

$$\bigcup_{i=0}^{\#(T)+1} B\left(\mathbf{X}_{\tau_i}, 8\Lambda\right) \supset \mathcal{X}_T(4\Lambda). \tag{3.6}$$

Clearly we need to just consider the behavior of the sausage for the time points strictly between τ_i and τ_{i+1} for any fixed i. By the definition of τ_{i+1} the path $\mathbf{X}_t$, $\tau_i \leq t \leq \tau_{i+1}$ is inside $\bigcup_{k=0}^{i} B(\mathbf{X}_{\tau_k}, 4\Lambda)$. Therefore any such $\mathbf{X}_t$ is within 4Λ distance of some $\mathbf{X}_{\tau_k}, k \leq i$. It then follows that $B(\mathbf{X}_t, 4\Lambda) \subset B(\mathbf{X}_{\tau_k}, 8\Lambda)$. The claim (3.6) then follows immediately.

Using (3.6) and the formula for the volume of a d sphere of radius 8Λ

$$(\#(T)+2)\frac{\pi^{\frac{d}{2}}(8\Lambda)^d}{\Gamma(\frac{d}{2}+1)} \geq |\mathcal{X}_T(4\Lambda)|. \tag{3.7}$$

Therefore

$$\begin{aligned}
\mathbb{P}_0\left(\#(T) \leq \frac{C_d T^{\frac{d}{d+2}}}{\Lambda^d}\right) &\leq \mathbb{P}_0\left((\#(T)+2)\frac{\pi^{\frac{d}{2}}(8\Lambda)^d}{\Gamma(\frac{d}{2}+1)} \leq \frac{C_d 16^d \pi^{\frac{d}{2}} T^{\frac{d}{d+2}}}{\Gamma(\frac{d}{2}+1)}\right) \\
&\leq \mathbb{P}_0\left(|\mathcal{X}_T(4\Lambda)| \leq \frac{C_d 16^d \pi^{\frac{d}{2}} T^{\frac{d}{d+2}}}{\Gamma(\frac{d}{2}+1)}\right) \\
&\leq \exp\left(-C(d,\Lambda)T^{\frac{d}{d+2}}\right),
\end{aligned}$$

if we choose

$$C_d^{-1} = \frac{16^d \pi^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2}+1\right)}.$$

This completes the proof of the lemma. ∎

3.2. **Stopping times for string.** For a $\mathbb{R}^d$ valued function $\mathbf{f}$ defined on $[0,1]$, we denote the *range* of $\mathbf{f}$ by

$$\mathcal{R}(\mathbf{f}) := \sup_{x,y\in[0,1]} |\mathbf{f}(x) - \mathbf{f}(y)| \tag{3.8}$$

We will need the following lemma

Lemma 3.3. *Let* $\mathbf{f} : [0,1] \to \mathbb{R}^d$. *We have for* $t \geq 1$

$$\mathcal{R}(G_t * \mathbf{f}) \leq 4de^{-4\pi^2 t} \left\| \mathbf{f} - \int_0^1 \mathbf{f}(x)dx \right\|_2 \leq 4de^{-4\pi^2 t}\mathcal{R}(\mathbf{f})$$

Proof. We expand each component $\mathbf{f}$ in the Fourier basis

$$f_j(x) = \sum_k a_k^{(j)} e^{i2\pi kx}.$$

Then

$$\sup_{x,y} \Big| (G_t * f_j)(x) - (G_t * f_j)(y) \Big| = \sup_{x,y} \left| \sum_{k\neq 0} e^{-4\pi^2 k^2 t} a_k^{(j)} \left[e^{i2\pi kx} - e^{i2\pi ky} \right] \right|$$

$$\leq 4e^{-4\pi^2 t} \left(\sum_{k\neq 0} \left[a_k^{(j)} \right]^2 \right)^{1/2}$$

Now observe that $\int_0^1 f_j(x)dx$ is the zeroth Fourier coefficient of f_j so that

$$\left(\sum_{k\neq 0} \left[a_k^{(j)} \right]^2 \right)^{1/2} = \left\| f_j - \int_0^1 f_j(x)dx \right\|_2 \leq \left\| \mathbf{f} - \int_0^1 \mathbf{f}(x)dx \right\|_2.$$

Clearly $\mathcal{R}(\mathbf{f})$ is an upper bound for the right hand side. Finally note that

$$\mathcal{R}(G_t * \mathbf{f}) \leq \sum_{i=1}^{d} \mathcal{R}(G_t * f_j)$$

which gives the factor d in the bound. □

For the rest of this article, we let

$$\delta = \frac{a}{100}, \qquad L = E + 3|\log a|, \tag{3.9}$$

where E is a large enough constant.

Remark 3.1. *The constant E chosen above is independent of any of the parameters in the model, e.g. T, J, a, ν. It is chosen so that L satisfies the following:*

- $4de^{-4\pi^2 L} \le \delta$ *(see Lemma 3.3).*
- $L \ge \mathrm{D} + 2|\log a|$, *where* D *is the constant appearing in Lemma 3.9.*
- $e^{4\pi^2 L} \ge \frac{8C_0 d^{\frac{3}{2}}}{\delta}$, *where* C_0 *is the constant appearing in* (3.21).
- L *is large enough so that the last inequality in* (3.24) *holds.*

Recall the stopping times τ_i defined in (3.3). Let $T_0 = 0$ and define the sequence of stopping times

$$\begin{aligned}
&S_1 = \{t \ge T_0 + L :\ \mathcal{R}\left(G_{t-T_0} * \mathbf{u}_0\right) \le \delta\} \\
&T_1 = \min\{\tau_j :\ \tau_j \ge S_1\}, \\
&S_2 = \inf\Big\{t \ge T_1 + L :\ \mathcal{R}\Big(G_{t-T_1} * \mathbf{N}(T_0, T_1)\Big) \le \delta\Big\}, \\
&T_2 = \min\{\tau_j :\ \tau_j \ge S_2\}, \\
&S_3 = \inf\Big\{t \ge T_2 + L :\ \mathcal{R}\Big(G_{t-T_2} * \mathbf{N}(T_1, T_2)\Big) \le \delta\Big\}, \\
&T_3 = \min\{\tau_j :\ \tau_j \ge S_3\}, \\
&\ \vdots \qquad \vdots \qquad \vdots
\end{aligned} \tag{3.10}$$

The reason for introducing the stopping times T_i will be clear below. Inductively

$$\begin{aligned}
\mathbf{u}(T_i) &= G_{T_i - T_{i-1}} * \mathbf{u}\big(T_{i-1}\big) + \mathbf{N}(T_{i-1}, T_i) \\
&= G_{T_i - T_{i-1}} * \big[G_{T_{i-1} - T_{i-2}} * \mathbf{u}\big(T_{i-2}\big) + \mathbf{N}(T_{i-2}, T_{i-1})\big] + \mathbf{N}(T_{i-1}, T_i) \\
&= G_{T_i - T_{i-2}} * \mathbf{u}\big(T_{i-2}\big) + G_{T_i - T_{i-1}} * \mathbf{N}(T_{i-2}, T_{i-1}) + \mathbf{N}(T_{i-1}, T_i) \\
&\ \vdots \qquad \vdots \qquad \vdots \\
&= G_{T_i - T_0} * \mathbf{u}(T_0) + G_{T_i - T_1} * \mathbf{N}(T_0, T_1) + G_{T_i - T_2} * \mathbf{N}(T_1, T_2) + \cdots + \mathbf{N}(T_{i-1}, T_i)
\end{aligned}$$

Definition 3.1. *For a $\mathbb{R}^d$ valued function $\mathbf{f}$ on $[0,1]$ we denote*

$$\mathcal{S}(a; \mathbf{f}) := \bigcup_{0 \le y \le 1} \{\mathbf{f}(y) + B(\mathbf{0}, a)\}$$

to be the sausage of radius a around $\mathbf{f}$.

The following lemma is crucial for the upper bound.

Lemma 3.4. *We have*

$$\Big|\mathcal{R}\big(\mathbf{u}(T_i)\big) - \mathcal{R}\big(\mathbf{N}(T_{i-1}, T_i)\big)\Big| \le 2\delta. \tag{3.11}$$

and

$$\left|\mathcal{S}^1\,(a;T_i)\right| \geq \left|\mathcal{S}\Big(a/2;\;\mathbf{N}\,(T_{i-1},T_i)\,\Big)\right|. \tag{3.12}$$

Proof. Denote by

$\mathcal{G} := G_{T_i-T_0}*\mathbf{u}(T_0)+G_{T_i-T_1}*\mathbf{N}(T_0,T_1)+G_{T_i-T_2}*\mathbf{N}(T_1,T_2)+\cdots+G_{T_i-T_{i-1}}*\mathbf{N}(T_{i-2},T_{i-1}).$

It is easily checked that when $\mathbf{f}=\mathbf{g}+\mathbf{h}$ then $|\mathcal{R}(\mathbf{f})-\mathcal{R}(\mathbf{g})|\leq\mathcal{R}(\mathbf{h})$. Therefore, with our choice of δ and L, and by using Lemma 3.3, we obtain

$$\left|\mathcal{R}\big(\mathbf{u}(T_i)\big)-\mathcal{R}\big(\mathbf{N}(T_{i-1},T_i)\big)\right| \leq \mathcal{R}(\mathcal{G}) \leq \sum_{i\geq 1}\delta^i \leq 2\delta.$$

To prove (3.12) , note that

$$\mathbf{u}(T_i,x) = [\mathcal{G}(0)+\mathbf{N}\,(T_{i-1},T_i;x)] + [\mathcal{G}(x)-\mathcal{G}(0)],$$

and so the ball of radius $a/2$ around $[\mathcal{G}(0)+\mathbf{N}\,(T_{i-1},T_i;x)]$ is contained in the ball of radius a around $\mathbf{u}(T_i,x)$ (note $\mathcal{R}(\mathcal{G})\leq a/2$ by our choice of δ). Consequently the sausage of radius $a/2$ around $\mathbf{N}\,(T_{i-1},T_i)$ has a smaller volume than the sausage of radius a around $\mathbf{u}(T_i)$. □

Remark 3.2. *Equations* (3.11) *and* (3.12) *show that the range of* $\mathbf{u}(T_i)$ *is close to that of* $\mathbf{N}(T_{i-1},T_i)$*, and a lower bound on the volume of the sausage around* $\mathbf{u}(T_i)$ *is given by the volume of a smaller sausage around* $\mathbf{N}(T_{i-1},T_i)$ *. As we will see later* $\mathbf{N}(T_{i-1},T_i)$ *form a weakly dependent sequence. In particular, using a weak form of law of large numbers, we will see that there is a subset of* $O(T^{\frac{d}{d+2}})$ *many* T_i*'s where the volume of the sausage around* $\mathbf{N}(T_{i-1},T_i)$ *is large and where the range of* $\mathbf{N}(T_{i-1},T_i)$ *is small. Because of* (3.11) *and* (3.12) *the same holds for* $\mathbf{u}(T_i)$ *on this subset. The small range guarantees that the sausages at these* T_i*'s are disjoint, and thus a lower bound for* $\mathcal{S}^1_T(a)$ *is obtained by adding the volumes of the sausages at the* T_i*'s on this subset.*

3.3. Volume of the sausage around $\mathbf{N}(T_{i-1},T_i)$ and Range of $\mathbf{N}(T_{i-1},T_i)$.

Our first objective will be to give a lower bound on the probability that the volume of the sausage of radius $a/2$ around $\mathbf{N}(0,t)$ is at least $Ca^{d-2+\epsilon}$ (see Lemma 3.9). As indicated in Remark 3.2, we will show later that there are sufficiently many i such that the sausages around $\mathbf{N}(T_{i-1},T_i)$ have at least this volume. Let

$$\mathbf{N}(t;x,y) = \mathbf{N}(t,x)-\mathbf{N}(t,y), \tag{3.13}$$

and write

$$\mathbf{N}(t;x,y) = \mathbf{N}^{(1)}(t;x,y)-\mathbf{N}^{(2)}(t;x,y),$$

where

$$\mathbf{N}^{(1)}(t;x,y) = \int_{(-\infty,t]\times[0,1]} \left[G(t-s,x,z) - G(t-s,y,z)\right] \mathbf{W}(dsdz), \text{ and}$$
$$\mathbf{N}^{(2)}(t,x,y) = \int_{(-\infty,0]\times[0,1]} \left[G(t-s,x,z) - G(t-s,y,z)\right] \mathbf{W}(dsdz)$$

It is easy to see that $\mathbf{N}^{(1)}(t;\cdot,\cdot)$ is a stationary process in t. The reason for considering the differences $\mathbf{N}(t,x) - \mathbf{N}(t,y)$ instead of $\mathbf{N}(t,x)$ is that the term

$$\int_{(-\infty,t]\times[0,1]} G(t-s,x,z)\mathbf{W}(dzds)$$

would not be convergent. Note also that the volume of a sausage around $\mathbf{N}(0,t)$ is the same as the volume of the sausage around $\mathbf{N}(t;\cdot,0)$. The lemma below shows that $\mathbf{N}^{(2)}$ becomes smaller with increasing t, and hence the main contribution of $\mathbf{N}(t,x,y)$ comes from $\mathbf{N}^{(1)}(t;x,y)$ for large t. The process $\mathbf{N}^{(1)}(t;\cdot,0)$ behaves locally like Brownian motion (see Lemma 3.6, and so we can use some techniques (see for example Lemma 3.7) which work for Brownian motion, to obtain a lower bound on the sausage around $\mathbf{N}^{(1)}(t;\cdot,0)$.

Lemma 3.5. *There exists a constant $C_1 > 0$ such that for any $\lambda > 0$*

$$\mathbb{P}_0\left(\sup_{x,y\in[0,1]} \left|\mathbf{N}^{(2)}(t;x,y)\right| > \lambda\right) \le 2\exp\left(-\frac{e^t\lambda^2}{C_1}\right). \tag{3.14}$$

Proof. We first observe

$$\begin{aligned}
\mathbb{E}_0\left[\left|\mathbf{N}^{(2)}(t;x,y)\right|^2\right] &= \int_{-\infty}^0 \int_0^1 \left[G(t-s,x,z) - G(t-s,y,z)\right]^2 dzds \\
&= \int_{-\infty}^0 ds \sum_{k\ge1} e^{-k^2(t-s)} \left|1 - e^{i\cdot 2\pi k(x-y)}\right|^2 \\
&\le \sum_{k\ge1} \frac{e^{-k^2t}}{k^2} \left|1 \wedge (k|x-y|)\right|^2 \\
&\le Ce^{-t}|x-y|.
\end{aligned}$$

An application of the Burkholder-Davis-Gundy inequality then gives us

$$\mathbb{E}_0\left[\left|\mathbf{N}^{(2)}(t;x,y)\right|^{2k}\right] \le C^k(2\sqrt{2})^{2k}k^ke^{-kt}|x-y|^k$$

for all positive integers $k \geq 1$. From the argument leading up to inequality (6.9) in [5] one then obtains

$$\mathbb{E}_0\left[\sup_{x,y\in[0,1]}\left|\mathbf{N}^{(2)}(t;x,y)\right|^{2k}\right] \leq C^k e^{-kt}k^k, \quad k \geq 2.$$

Therefore there exists a constant $C_1 > 0$ such that

$$\mathbb{E}_0\left[\sup_{x,y\in[0,1]}\exp\left(\frac{\exp(t)\cdot|\mathbf{N}^{(2)}(t;x,y)|^2}{C_1}\right)\right] \leq 2.$$

The inequality (3.14) follows immediately from this. □

The following lemma indicates that the process $\mathbf{N}^{(1)}(t;\cdot,0)$ behaves locally like Brownian motion. It will be used in Lemma 3.7 below.

Lemma 3.6. *There are constants $C_1, C_2 > 0$ such that for all $x, y \in [0,1]$ and all $t \geq 1$ one has*

$$C_1 d(x,y) \leq \int_0^t \int_0^1 [G(s,x,z) - G(s,y,z)]^2\, dzds \leq C_2 d(x,y),$$

where $d(x,y)$ is the distance between x and y on the torus $\mathbb{T} = [0,1)$.

Proof. Using the Fourier decomposition of $G(s,x)$ we obtain

$$\begin{aligned}
&\int_0^t ds \int_0^1 dz\, [G(s,x,z) - G(s,y,z)]^2 \\
&= C\int_0^t ds \sum_{k\geq 1}\exp\left(-(2\pi k)^2 s\right)\left|1-\exp\left[i(2\pi k)d(x,y)\right]\right|^2 \qquad (3.15)\\
&= C\sum_{k\geq 1}\frac{[1-\exp(-(2\pi k)^2 t)]}{k^2}\left|1-\exp\left[i(2\pi k)d(x,y)\right]\right|^2.
\end{aligned}$$

Now use $|1-e^{iz}| \leq 1 \wedge |z|$ and $|1-e^{-4\pi^2k^2t}| \leq 1$ to obtain that the above is less than

$$C\sum_{k\geq 1}\frac{1}{k^2}\left|1 \wedge [kd(x,y)]\right| \leq Cd(x,y).$$

The final inequality is obtained by splitting the sum according to whether $k \leq d(x,y)^{-1}$ or $k > d(x,y)^{-1}$.

Next we turn to the lower bound. For this we only consider the sum in (3.15) for $1 \le k \le \frac{1}{2d(x,y)}$. Now $|1-e^{iz}| \ge C|z|$ for all $z \in [0,\pi]$ and some $C > 0$. Therefore since $t \ge 1$

$$\int_0^t ds \int_0^1 dz \, [G(s,x,z) - G(s,y,z)]^2 \ge C \sum_{k=1}^{[2d(x,y)]^{-1}} \frac{k^2 d(x,y)^2}{k^2} \ge C d(x,y).$$

This completes the proof of the lemma. □

Before proceeding we recall

Definition 3.2. *The lower Minkowski dimension of a set A is*

$$\underline{\dim}_M(A) := \liminf_{\epsilon \to 0} \frac{\log N_\epsilon(A)}{\log(\epsilon^{-1})},$$

where $N_\epsilon(A)$ is the minimum number of balls of radius ϵ needed to cover A.

We will need

Lemma 3.7. *For $t \ge 1$*

$$\underline{\dim}_M \left[\text{Range}\big(\mathbf{N}^{(1)}(t;\cdot,0)\big)\right] \ge 2 \quad a.s.$$

Proof. We first recall that for any set A we have

$$\underline{\dim}_M(A) \ge \dim_H(A),$$

where $\dim_H$ is the Hausdorff dimension (see page 115 of [9]). We use the *energy method* (Theorem 4.27 in [9]) to get a lower bound on the Hausdorff dimension of the range. Let μ_t be the occupation measure of $\mathbf{N}^{(1)}(t;\cdot,0)$:

$$\int_{\mathbb{R}^d} f(\mathbf{x}) d\mu_t(\mathbf{x}) = \int_0^1 f\left(\mathbf{N}^{(1)}(t;x,0)\right) dx.$$

We just need to show that for any $0 < \alpha < 2$

$$\mathbb{E}_0 \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{d\mu_t(\mathbf{x}) d\mu_t(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^\alpha} = \mathbb{E}_0 \int_0^1 \int_0^1 \frac{dx dy}{|\mathbf{N}^{(1)}(t;x,0) - \mathbf{N}^{(1)}(t;y,0)|^\alpha} < \infty.$$

Now $\mathbf{N}^{(1)}(t,x,0) - \mathbf{N}^{(1)}(t,y,0)$ is a Gaussian random variable with mean 0 and variance

$$\int_0^\infty \int_0^1 [G(s,x,z) - G(s,y,z)]^2 \, dz ds.$$

From Lemma 3.6 this expression is bounded above and below by a constant multiple of $d(x,y)$. Therefore it follows that

$$\mathbb{E}_0 \int_0^1 \int_0^1 \frac{dxdy}{|\mathbf{N}^{(1)}(t;x,0) - \mathbf{N}^{(1)}(t;x,0)|^\alpha} \leq C \int_0^1 \int_0^1 \frac{dxdy}{d(x,y)^{\alpha/2}} < \infty,$$

as required. Consequently

$$\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{d\mu_t(\mathbf{x}) d\mu_t(\mathbf{y})}{|\mathbf{x} - \mathbf{y}|^\alpha} < \infty \quad \text{a.s.}$$

It thus follows from Theorem 4.27 in [9] that $\underline{\dim}_M \left[\text{Range}\big(\mathbf{N}^{(1)}(t;\cdot,0)\big)\right] \geq 2$ a.s. as required. □

The lower bound in the lower Minkowski dimension gives a lower bound on the number of balls of radius a which intersect the range of $\mathbf{N}^{(1)}(t;\cdot,0)$, and consequently a lower bound on the volume of the sausage of radius a around $\mathbf{N}^{(1)}(t;\cdot,0)$.

Lemma 3.8. *Fix an arbitrary $0 < \gamma < 1$. There exists a $\tilde{\mathcal{C}}_\gamma > 0$ such that for any $t \in \mathbb{R}$ and any $0 < a \leq 1$*

$$\mathbb{P}_0 \left(|\mathbb{S}\left(a;\ \mathbf{N}^{(1)}(t;\cdot,0)\right)| \geq \tilde{\mathcal{C}}_\gamma a^{d-2+\gamma} \right) \geq \frac{4}{5}.$$

Proof. First note that the process $\mathbf{N}^{(1)}(t;\cdot,0)$ is stationary in t. Partition $\mathbb{R}^d$ into cubes of side length a. Let $\tilde{N}_a(\mathbf{N}^{(1)},t)$ be the number of cubes through which $\mathbf{N}^{(1)}(t;\cdot,0)$ passes. By Lemma 3.7 we obtain almost surely

$$\tilde{N}_a(\mathbf{N}^{(1)},t) \geq \left(\frac{1}{a}\right)^{2-\gamma} \quad \text{for all } a \text{ small enough.}$$

Therefore there exists a positive random variable $A(\omega)$ which is finite almost surely such that

$$\tilde{N}_a(\mathbf{N}^{(1)},t) \geq A(\omega) \left(\frac{1}{a}\right)^{2-\gamma} \quad \text{for all } 0 < a \leq 1.$$

We now choose the largest possible subcollection of these $\tilde{N}_a(\mathbf{N}^{(1)},t)$ cubes such that no two cubes are adjacent (that is, share a common edge). Let $N_a^*(\mathbf{N}^{(1)},t)$ be the number of cubes in this subcollection. Clearly there exists a constant C_d such that almost surely

$$N_a^*(\mathbf{N}^{(1)},t) \geq A(\omega) C_d \left(\frac{1}{a}\right)^{2-\gamma} \quad \text{for all } 0 < a \leq 1.$$

From each of these $N_a^*(\mathbf{N}^{(1)},t)$ cubes choose any point in the range of $\mathbf{N}^{(1)}(t,\cdot,0)$. The union of the balls of radius a around these points is contained in the sausage of radius a around $\mathbf{N}^{(1)}(t,\cdot,0)$, so that

$$|\mathcal{S}\left(a;\ \mathbf{N}^{(1)}(t,\cdot,0)\right)| \geq A(\omega)\tilde{C}_d a^{d-2+\gamma} \qquad \text{for all } 0<a\leq 1,$$

for some other constant $\tilde{C}_d>0$. The constant $\mathcal{C}_\gamma>0$ is chosen so that

$$\mathbb{P}_0\left(A(\omega)\tilde{C}_d \geq \tilde{\mathcal{C}}_\gamma\right) \geq \frac{4}{5}.$$

This completes the proof of the lemma. □

We now use Lemma 3.5 on the smallness of $\mathbf{N}^{(2)}(t;\cdot,0)$ to control the volume of the sausage around $\mathbf{N}(0,t)$.

Lemma 3.9. *Fix an arbitrary* $0<\gamma<1$. *There are constants* $\mathcal{C}_\gamma>0$ *and* $\mathrm{D}>0$ *such that for any* $0<a\leq 1$ *and* $t\geq \mathrm{D}+2|\log a|$

$$\mathbb{P}_0\left(|\mathcal{S}\left(\frac{a}{2};\ \mathbf{N}(0,t)\right)| \geq \mathcal{C}_\gamma a^{d-2+\gamma}\right) \geq \frac{3}{4}.$$

Proof. We shall use (3.14).

$$\mathbb{P}_0\left(\sup_{x,y\in[0,1]}\left|\mathbf{N}^{(2)}(t;x,y)\right| > \frac{a}{4}\right) \leq 2\exp\left(-\frac{e^t a^2}{16C_1}\right).$$

If we choose D large enough the right hand side above is at most $\frac{1}{20}$. Therefore,

$$\mathbb{P}_0\left(\mathcal{R}\left(\mathbf{N}^{(2)}(t;\cdot,0)\right) \leq \frac{a}{4}\right) \geq \frac{19}{20}.$$

As a consequence of this we obtain by a similar argument as in Lemma 3.4 that

$$|\mathcal{S}\left(\frac{a}{2};\ \mathbf{N}(0,t)\right)| \geq |\mathcal{S}\left(\frac{a}{4};\ \mathbf{N}^{(1)}(t;\cdot,0)\right)|.$$

We now use Lemma 3.8 to complete the proof. □

Our second objective in this subsection is (following Remark 3.2) analyze the probability that the range of $\mathbf{N}(0,t)$ is small (see Lemma 3.10).

Lemma 3.10. *There exists* $\Lambda>1$ *such that for all* $t\geq L$

$$\mathbb{P}_0\left(\mathcal{R}(\mathbf{N}(0,t)) \leq \Lambda\right) \geq \frac{3}{4}. \tag{3.16}$$

Proof. Since $\mathbf{N}^{(1)}(t;\cdot,0)$ is stationary in t, we can choose $\Lambda > 1$ such that

$$\mathbb{P}_0\left(\mathcal{R}\left(\mathbf{N}^{(1)}(t;\cdot,0)\right) \le \frac{\Lambda}{2}\right) \ge \frac{4}{5}.$$

In the proof of Lemma 3.9 we have seen that with probability at least $\frac{19}{20}$ one has

$$\mathcal{R}\left(\mathbf{N}^{(2)}(t;\cdot,0)\right) \le \frac{a}{4}.$$

Therefore with probability at least $\frac{3}{4}$ we have

$$\mathcal{R}\Big(\mathbf{N}(0,t)\Big) \le \mathcal{R}\Big(\mathbf{N}^{(1)}(t;\cdot,0)\Big) + \mathcal{R}\Big(\mathbf{N}^{(2)}(t;\cdot,0)\Big) \le \Lambda.$$

This completes the proof. □

Remark 3.3. *We fix and use a Λ as in Lemma 3.10 for the rest of the article. In particular the τ_i's defined in (3.3) are defined in terms of this particular choice of Λ.*

Our final objective is to show that there are sufficiently many T_i's such that $\mathcal{R}\left(\mathbf{N}(T_{i-1},T_i)\right) \le \Lambda$ and $\mathcal{S}\Big(\frac{a}{2};\mathbf{N}\left(T_{i-1},T_i\right)\Big) \ge \mathcal{C}_\gamma a^{d-2+\gamma}$ (See Lemma 3.11). Before we proceed, we will need

Definition 3.3. *Let $\tilde{\mathcal{F}}_t$ be the filtration generated by white noise*

$$\tilde{\mathcal{F}}_t = \sigma\left\{\mathbf{W}(A\times[r,s]);\ A\subset[0,1],\, 0\le r,s\le t\right\}.$$

Let $\mathcal{G}_i$ denote the σ-algebra generated by the white noise up to time T_i:

$$\mathcal{G}_i = \Big\{\mathrm{A}\in\mathcal{F} : \mathrm{A}\cap\{T_i\le t\}\in\tilde{\mathcal{F}}_t\Big\}$$

Let

$$\mathcal{H}_i = \mathcal{G}_i \vee \sigma\left\{\mathbf{X}_t;\ t\ge 0\right\}$$

the σ-algebra generated by the white noise up to time T_i and the center of mass process.

Lemma 3.11. *Let Λ be as in Lemma 3.10. We have*

$$\mathbb{P}_0\left(\mathcal{R}\left(\mathbf{N}(T_{i-1},T_i)\right)\le\Lambda,\ \left|\mathcal{S}\Big(\frac{a}{2};\,\mathbf{N}(T_{i-1},T_i)\Big)\right| \ge \mathcal{C}_\gamma a^{d-2+\gamma}\,\middle|\,\mathcal{H}_{i-1}\right) \ge \frac{1}{2}. \tag{3.17}$$

Proof. It is enough to show

$$\mathbb{P}_0\left(\mathcal{R}\left(\mathbf{N}\left(T_{i-1},T_i\right)\right)\le\Lambda\,\middle|\,\mathcal{H}_{i-1}\right) \ge \frac{3}{4}, \tag{3.18}$$

$$\mathbb{P}_0\left(\left|\mathcal{S}\Big(\frac{a}{2};\,\mathbf{N}\left(T_{i-1},T_i\right)\Big)\right| \ge \mathcal{C}_\gamma a^{d-2+\gamma}\,\middle|\,\mathcal{H}_{i-1}\right) \ge \frac{3}{4}. \tag{3.19}$$

We have from Lemma 3.10

$$\mathbb{P}_0\Big(\mathcal{R}\left(\mathbf{N}\left(0,t\right)\right) \leq \Lambda\Big) \geq \frac{3}{4},$$

uniformly in $t \geq L$. Then observe

$$\begin{aligned}
&\mathbb{P}_0\Big(\mathcal{R}\left(\mathbf{N}\left(T_{i-1},T_i\right)\right) \leq \Lambda \,\Big|\, \mathcal{H}_{i-1}\Big) \\
&= \int_0^\infty \int_{s+L}^\infty \mathbb{P}_0\left(T_{i-1} \in ds,\ T_i \in dt,\ \mathcal{R}\Big(\mathbf{N}(s,t)\Big) \leq \Lambda \,\Big|\, \mathcal{H}_{i-1}\right) \\
&= \int_0^\infty \int_{s+L}^\infty \mathbf{1}\left\{T_{i-1} \in ds,\ T_i \in dt\right\} \cdot \mathbb{P}_0\left(\mathcal{R}\Big(\mathbf{N}(s,t)\Big) \leq \Lambda\right).
\end{aligned}$$

The second equality follows from an argument similar to Lemma 2.1, In fact the event $\{T_{i-1} \in ds,\ T_i \in dt\}$ is measurable with respect to the sigma field $\mathcal{H}_{i-1}$, while $\mathcal{R}(\mathbf{N}(s,t))$ depends on

$$\sigma\left(\int_s^{\tilde{t}} \int_0^1 \left[G_{t-r}(x,z) - G_{t-r}(0,z)\right] \mathbf{W}(dzdr),\ x \in [0,1],\ \tilde{t} \geq s\right),$$

which is independent of $\mathcal{H}_{i-1}$. From this we obtain (3.18). Similarly, to show (3.19) we use Lemma 3.9, and integrate over the realizations of T_{i-1} and T_i. This completes the proof of the lemma. □

Consequently, due to (3.11) and (3.12) and using Lemma 3.11, there are sufficiently many T_i such that $\mathcal{R}\left(\mathbf{u}(T_i)\right) \leq \Lambda + 2\delta$ and $|\mathbb{S}^1\left(a;T_i\right)| \geq \mathcal{C}_\gamma a^{d-2+\gamma}$.

3.4. **Sufficiently many T_i far apart.**

Lemma 3.12. *There exists $C_2 > 0$ such that for all $t \geq L$*

$$\mathbb{P}_0\left[S_{i+1} - T_i > t \,\Big|\mathcal{H}_{i-1}\right] \leq \frac{C_2 e^{-8\pi^2 t}}{\delta^2}.$$

In particular for any $\eta < 8\pi^2$ there exists $C_3(\eta) > 0$ such that

$$\mathbb{E}_0\left[\exp\left(\eta(S_{i+1} - T_i)\right)\ \Big|\mathcal{H}_{i-1}\right] \leq e^{\eta L} + \frac{C_3(\eta)}{\delta^2} \tag{3.20}$$

Proof. Recall from (3.10)

$$S_{i+1} = \inf\left\{t \geq T_i + L : \mathcal{R}\Big(G_{t-T_i} * \mathbf{N}(T_{i-1},T_i)\Big) \leq \delta\right\}.$$

The event $\{S_{i+1} - T_i > t\}$ implies that $\mathcal{R}\Big(G_t * \mathbf{N}(T_{i-1}, T_i)\Big) > \delta$, and in light of Lemma 3.3, it further implies

$$\left\|\mathbf{N}(T_{i-1}, T_i) - \int_0^1 \mathbf{N}(T_{i-1}, T_i;\, x)dx\right\|_2 > \frac{\delta}{4d} e^{4\pi^2 t}.$$

Using the subscript j to denote the components of $\mathbf{N}(T_{i-1}, T_i)$ it follows that there exists a $1 \le j \le d$ such that

$$\left\|\mathbf{N}_j(T_{i-1}, T_i) - \int_0^1 \mathbf{N}_j(T_{i-1}, T_i;\, x)dx\right\|_2 > \frac{\delta}{4d^{\frac{3}{2}}} e^{4\pi^2 t}.$$

We can formally write each component $\mathbf{W}_j$ of the white noise as $\mathbf{W}_j(dydr) = \sum_{k\in\mathbb{Z}} e^{\imath(2\pi k y)} dB_k(r)dy$, where the B_k's are independent standard complex Brownian motions (that is $B_k = \frac{R_k}{\sqrt{2}} + \imath\frac{C_k}{\sqrt{2}}$ where R_k, C_k are standard real Brownian motions) with $\bar{B_k} = B_{-k}$. This can be seen by integrating both sides with test functions and computing the second moments. Since

$$G_t(x, y) = \sum_{l\in\mathbb{Z}} e^{-2\pi^2 l^2 t} e^{\imath 2\pi l(x-y)},$$

the kth Fourier coefficient of $\mathbf{N}_j(s, \tilde{s}),\ k \neq 0$ is

$$a_k = \int_s^{\tilde{s}} e^{-2\pi^2 k^2(\tilde{s}-r)} dB_k(r)$$

Furthermore we have $\bar{a_k} = a_{-k}$ and a_k is independent of $a_{\tilde{k}}$ if $\tilde{k} \neq k, -k$. Now

$$\mathrm{E}\left[\sum_{k\neq 0} |a_k|^2\right] = \sum_{k\neq 0} \frac{1}{2\pi^2 k^2}\left(1 - e^{-2\pi^2 k^2(\tilde{s}-s)}\right),$$

and so there exist positive constants C_0, C_1 such that

$$\mathrm{E}\left[\left(\sum_{k\neq 0} |a_k|^2\right)^{\frac{1}{2}}\right] \le C_0 \quad \text{and} \quad \mathrm{Var}\left[\left(\sum_{k\neq 0} |a_k|^2\right)^{\frac{1}{2}}\right] \le C_1$$

uniformly in s and $\tilde{s}$. Therefore

$$\begin{aligned}
&\mathbb{P}_0\left[S_{i+1} - T_i > t \,\middle|\, \mathcal{H}_{i-1}\right] \\
&\le d \cdot \mathbb{P}_0\left[\left\|\mathbf{N}_j(T_{i-1}, T_i) - \int_0^1 \mathbf{N}_j(T_{i-1}, T_i; x)dx\right\|_2 > \frac{\delta}{4d^{\frac{3}{2}}} e^{4\pi^2 t} \,\middle|\, \mathcal{H}_{i-1}\right] \qquad (3.21) \\
&\le \frac{d \cdot C_1}{\left(\frac{\delta}{4d^{\frac{3}{2}}} e^{4\pi^2 t} - C_0\right)^2}.
\end{aligned}$$

The second part of the lemma follows from

$$\mathbb{E}_0\left[\exp\left(\eta(S_{i+1}-T_i)\right)\,\middle|\,\mathcal{H}_{i-1}\right] \le e^{\eta L} - \int_L^\infty e^{\eta t}\mathbb{P}_0\left[S_{i+1}-T_i > t\,\middle|\,\mathcal{H}_{i-1}\right]\,dt,$$

and the above tail bound. □

We conclude this section regarding the spacings of S_i and T_i. This will be crucially used on a specific subset of T_i's to show the upper bound.

Lemma 3.13. *There is a constant $\widetilde{C} > 0$ such that for any $A_4 > 0$ and L as in (3.9) we have*

$$\mathbb{P}_0\left(\sum_{i=1}^{A_4\frac{T^{\frac{d}{d+2}}}{L}} (S_i - T_{i-1}) > \widetilde{C}A_4 T^{\frac{d}{d+2}}\right) \le \exp\left(-\frac{\widetilde{C}A_4}{2}T^{\frac{d}{d+2}}\right).$$

Proof. With the choice of $\eta = 1$ in (3.20) we obtain

$$\begin{aligned}\mathbb{P}_0\left(\sum_{i=1}^{A_4\frac{T^{\frac{d}{d+2}}}{L}} (S_i - T_{i-1}) > \widetilde{C}A_4 T^{\frac{d}{d+2}}\right) &\le \mathbb{E}_0\exp\left(\sum_{i=1}^{A_4\frac{T^{\frac{d}{d+2}}}{L}} (S_i - T_{i-1}) - \widetilde{C}A_4 T^{\frac{d}{d+2}}\right)\\ &\le \left(e^L + \frac{C_3}{\delta^2}\right)^{A_4\frac{T^{\frac{d}{d+2}}}{L}} \exp\left(-\widetilde{C}A_4 T^{\frac{d}{d+2}}\right).\end{aligned}$$

The lemma follows by a large choice of the constant $\widetilde{C}$ above. □

3.5. **Estimates for Soft obstacles.** We will need a few lemmas which lead up Proposition 3.2. This is a key proposition that will be used in the proof of the upper bound in Theorem 1.2.

Proposition 3.1. *There is a $C_3 > 0$ such that for all $s_0 \le 1$*

$$\mathbb{P}_0\left(\sup_{s\le s_0}\sup_{x\in[0,1]} |\mathbf{N}(0,t+s;x) - \mathbf{N}(0,t;x)| > \lambda\right) \le \exp\left(-\frac{C_3^2\lambda^2}{\sqrt{s_0}}\right)$$

uniformly in t.

The proof of the above proposition follows from a sequence of lemmas. Define for $s,\, t \geq 0$ and $x, y \in [0,1]$

$$\mathbf{Z}(t,s;x,y) = \Big\{ \mathbf{N}(0,t+s;x) - \mathbf{N}(0,t;x) \Big\} - \Big\{ \mathbf{N}(0,t+s;y) - \mathbf{N}(0,t;y) \Big\}$$

Lemma 3.14. *There is a constant $C_1 > 0$ such that*

$$\mathbb{P}_0\left(|\mathbf{Z}(t,s;x,y)| > \lambda \right) \leq \exp\left(-\frac{C_1^2 \lambda^2}{\sqrt{s} \wedge |x-y|} \right)$$

uniformly in t.

Proof. We first give an upper bound on $\mathbb{E}_0\left[\mathbf{Z}_i^2(t,s;x,y) \right]$, for any fixed coordinate $\mathbf{Z}_i$ of $\mathbf{Z}$. This is easily seen to be equal to

(3.22)
$$\int_0^t \int_0^1 \left[G(t+s-r,x,z) - G(t+s-r,y,z) - G(t-r,x,z) + G(t-r,y,z) \right]^2 dz dr$$

(3.23)
$$+ \int_t^{t+s} \int_0^1 \left[G(t+s-r,x,z) - G(t+s-r,y,z) \right]^2 dz dr$$

Let us first look at (3.23). This is bounded by

$$\begin{aligned}
\int_0^s dr \sum_{k\geq 1} e^{-(2\pi k)^2 r} \Big| 1 - \exp\left(i(2\pi k)(x-y) \right) \Big|^2 \\
&\leq C \sum_{k \geq 1} \frac{1 - e^{-(2\pi k)^2 s}}{k^2} \Big[1 \wedge |2\pi k(x-y)| \Big]^2 \\
&\leq C \sum_{k \geq 1} \frac{1 \wedge (2\pi k)^2 s}{k^2} \Big[1 \wedge |2\pi k(x-y)| \Big]^2
\end{aligned}$$

In the case that $\sqrt{s} \leq |x-y|$, the above is bounded by

$$C \sum_{k=1}^{\frac{1}{2\pi|x-y|}} \frac{k^2 s}{k^2} k^2 |x-y|^2 + C \sum_{k=\frac{1}{2\pi|x-y|}+1}^{\frac{1}{2\pi\sqrt{s}}} s + C \sum_{k=\frac{1}{2\pi\sqrt{s}}+1}^{\infty} \frac{1}{k^2} \leq C\sqrt{s}.$$

In the case that $|x-y| \leq \sqrt{s}$ we obtain a bound

$$C \sum_{k=1}^{\frac{1}{2\pi\sqrt{s}}} \frac{k^2 s}{k^2} k^2 |x-y|^2 + C \sum_{k=\frac{1}{2\pi\sqrt{s}}+1}^{\frac{1}{2\pi|x-y|}} \frac{k^2|x-y|^2}{k^2} + C \sum_{k=\frac{1}{2\pi|x-y|}+1}^{\infty} \frac{1}{k^2} \leq C|x-y|.$$

Let us next consider the term (3.22). This is bounded by

$$\int_0^t dr \sum_{k\geq 1} \left[\exp\left(-\frac{(2\pi k)^2(t+s-r)}{2}\right) - \exp\left(-\frac{(2\pi k)^2(t-r)}{2}\right)\right]^2 [1\wedge |2\pi k(x-y)|]^2$$
$$\leq \int_0^t dr \sum_{k\geq 1} \exp\left(-\pi^2 k^2 r\right) \left|1-\exp\left(-2\pi^2 k^2 s\right)\right|^2 [1\wedge |2\pi k(x-y)|]^2$$
$$\leq C\int_0^t dr \sum_{k\geq 1} \frac{\left[1\wedge 2\pi^2 k^2 s\right]^2}{k^2} [1\wedge |2\pi k(x-y)|]^2$$

Therefore a similar bound as that for (3.23) holds for (3.22). The conclusion of our arguments is that

$$\mathbb{E}_0\left[\mathbf{Z}_i^2(t,s;x,y)\right] \leq C\left[\sqrt{s}\wedge |x-y|\right]$$

Since $\mathbf{Z}(t,s;x,y)$ is Gaussian we obtain the lemma by standard arguments. □

By similar arguments (see Lemma 3.3 in [3]) one has

Lemma 3.15. *There is a constant $\tilde{C}_1 > 0$ such that for all $s \leq 1$*

$$\mathbb{P}_0\left(|\mathbf{N}(0,t+s;0) - \mathbf{N}(0,t;0)| > \lambda\right) \leq \exp\left(-\frac{\tilde{C}_1^2\lambda^2}{\sqrt{s}}\right)$$

uniformly in t.

Let $\mathbb{D}_n$ denote the collection of dyadic points of the form $\frac{k}{2^n}$ in $[0,1]$. For any dyadic point $x\in\mathbb{D}_n$, we can find a sequence $0=p_0,p_1,\cdots,p_m=x$ of points such that p_i,p_{i+1} are nearest neighbors in some $\mathbb{D}_k$, $k\leq n$, and there are at most 2 points in any $\mathbb{D}_k$. Now

$$\mathbf{N}(0,t+s;x) - \mathbf{N}(0,t;x) = \Big[\mathbf{N}(0,t+s;0) - \mathbf{N}(0,t;0)\Big] + \sum_{i=1}^m \mathbf{Z}(t,s;p_{i+1},p_i)$$

From this and a chaining argument, similar to that of Lemma 3.4 in [3] we obtain

Lemma 3.16. *There is a $C_2 > 0$ such that for all $s\leq 1$*

$$\mathbb{P}_0\left(\sup_{x\in[0,1]} |\mathbf{N}(0,t+s;x) - \mathbf{N}(0,t;x)| > \lambda\right) \leq \exp\left(-\frac{C_2^2\lambda^2}{\sqrt{s}}\right).$$

uniformly in t.

We clearly have for $s,\ \tilde{s} \le 1$

$$\left| \sup_{x\in[0,1]} |\mathbf{N}(0,t+s;x) - \mathbf{N}(0,t;x)| - \sup_{x\in[0,1]} |\mathbf{N}(0,t+\tilde{s};x) - \mathbf{N}(0,t;x)| \right| \le \sup_{x\in[0,1]} |\mathbf{N}(0,t+s;x) - \mathbf{N}(0,t+\tilde{s};x)|,$$

and just as in the above lemma we have for $\tilde{s} \le s \le 1$

$$\mathbb{P}_0 \left(\sup_{x\in[0,1]} |\mathbf{N}(0,t+s,x) - \mathbf{N}(0,t+\tilde{s},x)| > \lambda \right) \le \exp\left(-\frac{C_2^2 \lambda^2}{\sqrt{s-\tilde{s}}} \right).$$

Therefore a chaining argument gives us Proposition 3.1. □

Now we return to the case of soft obstacles. We will need the following

Proposition 3.2. *Fix any $\eta > 0$. There are constant $0 < C_6(\eta) < 1$ and $C_7(\eta) > 0$ such that for $t \ge L$ we have*

$$\mathbb{P}_0 \Bigg(\mathcal{R}\left(\mathbf{N}(0,t)\right) \le \frac{a}{8}, \sup_{s\le C_6 a^{4+\eta}} \sup_{x\in[0,1]} |\mathbf{N}(0,t+s,x) - \mathbf{N}(0,t,x)| \le \frac{a}{16}, \quad \textit{and} \sup_{s\le C_6 a^{4+\eta}} |\mathbf{X}_{t+s} - \mathbf{X}_t| \le \frac{a}{16} \Bigg) \ge \frac{a^2}{C_7}.$$

Proof. We first give a lower bound on $\mathbb{P}_0\left(\mathcal{R}(\mathbf{N}(0,t)) \le \frac{a}{2}\right)$. Clearly $\mathcal{R}(\mathbf{N}(0,t)) = \sup_{x,y} |\mathbf{N}(t;x,y)|$, where $\mathbf{N}(t;x,y)$ is defined in (3.13). We have

$$\mathbf{N}(t;x,y) = \mathbf{N}^{(1)}(t;x,y) - \mathbf{N}^{(2)}(t;x,y).$$

Lemma 3.5 gives

$$\mathbb{P}_0 \left(\sup_{x,y\in[0,1]} |\mathbf{N}^{(2)}(t;x,y)| \ge \frac{a}{16} \right) \le 2\exp\left(-\frac{e^t a^2}{256 C_1} \right).$$

We next obtain an upper bound on the tail probabilities of $\sup_{x,y\in[0,1]} |\mathbf{N}^{(1)}(t;x,y)|$. Using ideas analogous to Proposition 3.1, but now we use Lemma 3.6 instead of the bounds on (3.22) and (3.23), we obtain

$$\mathbb{P}_0 \left(\sup_{x,y\in[0,1]} |\mathbf{N}^{(1)}(t;x,y)| \ge \frac{a}{16} \right) \le \exp\left(-\frac{a^2}{256 C_4} \right)$$

for some $C_4 > 0$. Therefore

$$\begin{aligned} \mathbb{P}_0\left(\mathcal{R}\left(\mathbf{N}(0,t)\right) \le \frac{a}{2}\right) &= 1 - \mathbb{P}_0\left(\mathcal{R}\left(\mathbf{N}(0,t)\right) > \frac{a}{4}\right) \\ &\ge 1 - \mathbb{P}_0\left(\sup_{x,y\in[0,1]} |\mathbf{N}^{(1)}(t;x,y)| \ge \frac{a}{16}\right) \\ &\quad - \mathbb{P}_0\left(\sup_{x,y\in[0,1]} |\mathbf{N}^{(2)}(t;x,y)| \ge \frac{a}{16}\right) \\ &\ge \frac{a^2}{C_5}. \end{aligned} \tag{3.24}$$

for some $C_5 > 0$. By Proposition 3.1 and standard results on Brownian motion the quantities

$$\mathbb{P}_0\left(\sup_{s\le C_6 a^{4+\eta}} \sup_{x\in[0,1]} |\mathbf{N}(0,t+s;x) - \mathbf{N}(0,t;x)| \le \frac{a}{16}\right)$$

and

$$\mathbb{P}_0\left(\sup_{s\le C_6 a^{4+\eta}} |\mathbf{X}_{t+s} - \mathbf{X}_t| \le \frac{a}{16}\right)$$

are both at least $1 - \exp\left(-\frac{C}{\sqrt{C_6} a^{\frac{\eta}{2}}}\right)$, *uniformly* in t. The proposition is proved by combining the above with (3.24). □

4. The proof of upper bound in Theorems 1.1 and 1.2

As explained earlier (in Remark 1.2) due to the scaling relations, we will first obtain an upper bound for $S_T^{H,1,\nu}$. We will consider the hard obstacle case first and then modify its proof suitably to handle the soft obstacle case.

Proof of Upper bound in Theorem 1.1. Recall from (1.5) that

$$S_T^{\mathrm{H},1,\nu} = \mathbb{E}_0 \exp\left(-\nu \left|\mathcal{S}_T^1(a)\right|\right)$$

where $\mathcal{S}_T^1(a)$ is the sausage of radius a around $\mathbf{u}$, that is

$$\mathcal{S}_T^1(a) = \bigcup_{\substack{0\le s\le T,\\ 0\le y\le 1}} \left\{\mathbf{u}(s,y) + B(\mathbf{0},a)\right\}.$$

The upper bound on $S_T^{H,1,\nu}$ essentially involves finding a lower bound on the volume of the sausage.

Recall from (3.4) that $\#(T) = |\{i \geq 1 : \tau_i \leq T\}|$, counts the number of τ_i's before time T, and for $\Lambda > 1$ from Lemma 3.2 that there are positive constants $A_1(d,\Lambda)$, $B_1(d,\Lambda)$ such for all $T > 0$

$$\mathbb{P}_0\left(\#(T) \leq A_1 T^{\frac{d}{d+2}}\right) \leq \exp\left(-B_1 T^{\frac{d}{d+2}}\right). \tag{4.1}$$

Therefore

$$\begin{aligned}&\mathbb{E}_0 \exp\left(-\nu \left|\mathcal{S}_T^1(a)\right|\right)\\ &\leq \exp\left(-B_1 T^{\frac{d}{d+2}}\right) + \mathbb{E}_0\left[\exp\left(-\nu\left|\mathcal{S}_T^1(a)\right|\right)\cdot \mathbf{1}\left\{\#(T) > A_1 T^{\frac{d}{d+2}}\right\}\right].\end{aligned}$$

Now let

$$\#_1(T) := \left|\left\{i \leq A_1 T^{\frac{d}{d+2}} : \tau_{i+1} - \tau_i \geq \Lambda^2\right\}\right|.$$

Clearly $\tau_{i+1} - \tau_i$ is more than the time it takes for the Brownian motion $\mathbf{X}_t$ to leave a ball of radius 4Λ centered at $\mathbf{X}_{\tau_i}$. Therefore the sequence $\tau_{i+1} - \tau_i$ stochastically dominates an i.i.d. sequence $\mathcal{T}_i$, where $\mathcal{T}_i$ is distributed as the time it takes for a Brownian motion starting at $\mathbf{0}$ to leave a ball of radius 4Λ. Moreover

$$\mathbb{P}\left(\mathcal{T}_i \geq \Lambda^2\right) = p > 0,$$

where p is independent of any of the parameters. Therefore by standard large deviation theory, there are positive constants $A_2(p,d,\Lambda), B_2(p,d,\Lambda)$ such that

$$\mathbb{P}_0\left(\#_1(T) < A_2 T^{\frac{d}{d+2}}\right) \leq \exp\left(-B_2 T^{\frac{d}{d+2}}\right). \tag{4.2}$$

Consider the event

$$\mathcal{A}_1 := \left\{\#(T) > A_1 T^{\frac{d}{d+2}}, \quad \#_1(T) > A_2 T^{\frac{d}{d+2}}\right\}.$$

Equations (4.1) and (4.2) imply that there is a positive $B_3(d,\Lambda,p)$ such that

$$\mathbb{P}_0\left(\mathcal{A}_1^c\right) \leq \exp\left(-B_3 T^{\frac{d}{d+2}}\right). \tag{4.3}$$

To get a good lower bound on the volume of the sausage, we need a good control on the number of T_i's up to time T. So let

$$\#_2(T) := |\{T_i : T_i \leq T\}|.$$

Our next task is to show that on the event $\mathcal{A}_1$ we must have that $\#_2(T)$ is sufficiently large.

On the event

$$\mathcal{A}_2 = \left\{\sum_{i=1}^{A_4 \frac{T^{\frac{d}{d+2}}}{L}} (S_i - T_{i-1}) \leq \widetilde{C} A_4 T^{\frac{d}{d+2}}\right\},$$

the number of τ_i with $\tau_{i+1} - \tau_i \geq \Lambda^2$ in the union of all the intervals $[T_{i-1}, S_i]$, $i = 1, 2 \cdots, A_4 \frac{T^{\frac{d}{d+2}}}{L}$ is less than $\frac{\widetilde{C} A_4 T^{\frac{d}{d+2}}}{\Lambda^2}$. On the event $\mathcal{A}_1$ we have $\#_1(T) > A_2 T^{\frac{d}{d+2}}$. Therefore with the choice of A_4 such that

$$\frac{\widetilde{C} A_4}{\Lambda^2} = \frac{A_2}{4}, \tag{4.4}$$

the intersection of $\mathcal{A}_1$ and $\mathcal{A}_2$ contains at least $\frac{A_2}{2} T^{\frac{d}{d+2}}$ many τ_i's before time T outside of the union of intervals $[T_{i-1}, S_i]$, $i = 1, 2 \cdots, A_4 \frac{T^{\frac{d}{d+2}}}{L}$. Note carefully that any T_j is the smallest τ_i immediately following S_j, and therefore this guarantees that there are at least $A_4 \frac{T^{\frac{d}{d+2}}}{L}$ many T_j's up to time T. Then using Lemma 3.13 and (4.3) we have that

$$\mathbb{P}_0\left(\#_2(T) < A_4 \frac{T^{\frac{d}{d+2}}}{L}\right) \leq \mathbb{P}_0(\mathcal{A}_1^c) + \mathbb{P}_0(\mathcal{A}_2^c) \leq \exp\left(-B_4 T^{\frac{d}{d+2}}\right), \tag{4.5}$$

for some $B_4(d, \Lambda, p) > 0$.

Therefore on the event $\mathcal{A}_3 := \mathcal{A}_1 \cap \mathcal{A}_2$ we have that $\#_2(T) \geq A_4 \frac{T^{\frac{d}{d+2}}}{L}$. We now count the T_i's before time T with large sausage volumes at $T_i's$, as in Lemma 3.11. Namely,

$$\#_3(T) := \left|\left\{ i \leq A_4 \frac{T^{\frac{d}{d+2}}}{L} : \mathcal{R}\left(\mathbf{N}(T_{i-1}, T_i)\right) \leq \Lambda, \ \left|\mathcal{S}\left(\frac{a}{2}; \mathbf{N}(T_{i-1}, T_i)\right)\right| \geq \mathcal{C}_\gamma a^{d-2+\gamma} \right\}\right|.$$

It now follows from (3.17), for $\Lambda > 1$ as in Lemma 3.10, that for any $A_5 > 0$

$$\begin{aligned}
\mathbb{P}_0\left(\#_3(T) \leq A_5 \frac{T^{\frac{d}{d+2}}}{L}\right) &= \mathbb{P}_0\left(\exp\left(-\#_3(T)\right) \geq \exp\left(-A_5 \frac{T^{\frac{d}{d+2}}}{L}\right)\right) \\
&\leq \left(\frac{1}{2} + e^{-1}\right)^{A_4 \frac{T^{\frac{d}{d+2}}}{L}} \exp\left(A_5 \frac{T^{\frac{d}{d+2}}}{L}\right) \\
&\leq \exp\left(A_4 \frac{T^{\frac{d}{d+2}}}{L} \log\left(\frac{1}{2} + e^{-1}\right) + A_5 \frac{T^{\frac{d}{d+2}}}{L}\right).
\end{aligned}$$

We now choose

$$A_5 = -\frac{A_4}{2} \log\left(\frac{1}{2} + e^{-1}\right),$$

to obtain

$$\mathbb{P}_0\left(\#_3(T) \le A_5 \frac{T^{\frac{d}{d+2}}}{L}\right) \le \exp\left(-B_5 \frac{T^{\frac{d}{d+2}}}{L}\right), \tag{4.6}$$

for a positive constant $B_5(d, \Lambda, p)$.

Let $n_1 < n_2 < \cdots$ be the indices $j \le A_4 \frac{T^{\frac{d}{d+2}}}{L}$ such that $T_j \le T$ and

$$\mathcal{R}\left(\mathbf{N}(T_{i-1}, T_i)\right) \le \Lambda, \ \left|\mathcal{S}\Big(\frac{a}{2}; \mathbf{N}(T_{i-1}, T_i)\Big)\right| \ge \mathcal{C}_\gamma a^{d-2+\gamma}. \tag{4.7}$$

Thanks to (4.1), (4.2), (4.5) and (4.6), outside of a set of probability $\exp\left(-B_6 \frac{T^{\frac{d}{d+2}}}{L}\right)$, where B_6 is a positive constant depending only on d, Λ, p, there are at least $A_5 \frac{T^{\frac{d}{d+2}}}{L}$ many such n_i's. Further, from (3.11) we have that the *fixed-time* sausages $\mathcal{S}^1(a; T_{n_j})$ are disjoint. Therefore, using (3.12) we have

$$\begin{aligned}
S_T^{\mathrm{H},1,\nu} &= \mathbb{E}_0 \exp\left(-\nu \left|\mathcal{S}_T^1(a)\right|\right) \\
&\le \exp\left(-B_6 \frac{T^{\frac{d}{d+2}}}{L}\right) + \mathbb{E}_0\left[\exp\left(-\nu \left|\bigcup_{j=0}^{A_5 \frac{T^{\frac{d}{d+2}}}{L}} \mathcal{S}^1(a; T_{n_j})\right|\right)\right] \\
&\le \exp\left(-B_6 \frac{T^{\frac{d}{d+2}}}{L}\right) + \mathbb{E}_0\left[\exp\left(-\nu \left|\bigcup_{j=0}^{A_5 \frac{T^{\frac{d}{d+2}}}{L}} \mathcal{S}(\frac{a}{2}; \mathbf{N}(T_{n_{j-1}}, T_{n_j})\right|\right)\right]
\end{aligned}$$

Now applying (4.7),

$$S_T^{\mathrm{H},1,\nu} \le \exp\left(-B_6 \frac{T^{\frac{d}{d+2}}}{L}\right) + \exp\left(-\nu A_5 C_\gamma a^{d-2+\gamma} \frac{T^{\frac{d}{d+2}}}{L}\right).$$

Note that the exponents in both the terms on the right hand side match in the case $J = 1$.

Finally we apply the scaling (1.12) to get an upper bound for $S_T^{H,J,\nu}$. We thus obtain

$$S_T^{\mathrm{H},J,\nu} \le \exp\left(-\frac{B_6 (T/J^2)^{\frac{d}{d+2}}}{E + 3|\log(a/J^{\frac{1}{2}})|}\right) + \exp\left(-\nu A_5 C_\gamma a^{d-2+\gamma} \frac{(T/J^2)^{\frac{d}{d+2}} J^{1-\frac{\gamma}{2}}}{E + 3|\log(a/J^{\frac{1}{2}})|}\right).$$

It is clear that the first term is the leading term. □

We now turn to the case of soft obstacles. As before we first find an upper bound for $S_T^{H,1,\nu}$. We will explain how the argument differs from the case of hard obstacles.

The proof of upper bound in Theorem 1.2 . We will now look for a subsequence T_{n_j} of the T_i's such that the entire string stays in a small ball of radius $\frac{3a}{8}$ during the time interval $[T_{n_j}, T_{n_j} + a^{4+\epsilon}]$, and there exists a Poisson point within distance $\frac{a}{8}$ of the center of mass (See Definition 4.8). Assumption 1.1 then guarantees there is a contribution of at least $\mathcal{C}a^{4+\epsilon}$ to the integral in S_T during this time period.

We then follow the argument of Proof of upper bound in Theorem 1.1. The only difference now is in the definition of $\#_3(T)$, which in the present case becomes

$$\begin{aligned} \#_3(T) = \Bigg| \Bigg\{ i \le A_4 \frac{T^{\frac{d}{d+2}}}{L} : \mathcal{R}(\mathbf{N}(T_{j-1}, T_j)) \le \frac{a}{8}, \\ \sup_{s \le C_6 a^{4+\eta}} \sup_{x \in [0,1]} |\mathbf{N}(T_{j-1}, T_j + s, x) - \mathbf{N}(T_{j-1}, T_j, x)| \le \frac{a}{16}, \\ \sup_{s \le C_6 a^{4+\eta}} |\mathbf{X}_{T_j + s} - \mathbf{X}_{T_j}| \le \frac{a}{16}, \\ \text{and there is a Poisson point within distance } \frac{a}{8} \text{ of } \mathbf{X}_{T_j} \Bigg\} \Bigg| \end{aligned} \tag{4.8}$$

Therefore $\#_3(T)$ is a sum of Bernoulli random variables with probability of success $\tilde{p}$ satisfying

$$\tilde{p} \ge \frac{a^2}{C_7} \left(1 - \exp\left(-\nu \frac{a^d}{8^d} \right) \right) \ge C_8 \nu a^{d+2}$$

for some constant $0 < C_8 < 1$.

Let $Z_1, Z_2, \cdots$ be i.i.d. Bernoulli random variables with success probability $p_* = C_8 \nu a^{d+2}$. By standard large deviation theory

$$\begin{aligned} \mathbb{P}_0 \left(\#_3(T) \le A_4 \frac{T^{\frac{d}{d+2}}}{L} \cdot \frac{p_*}{2} \right) &\le \mathbb{P}_0 \left(\sum_{i=1}^{\frac{A_4 T^{\frac{d}{d+2}}}{L}} Z_i \le A_4 \frac{T^{\frac{d}{d+2}}}{L} \cdot \frac{p_*}{2} \right) \\ &\le \exp\left(-B_7 \frac{T^{\frac{d}{d+2}}}{L} \cdot \nu a^{d+2} \right), \end{aligned} \tag{4.9}$$

for some constant $B_7(d, \Lambda, p) > 0$.

Let $n_1 < n_2 < \cdots$ be the indices j such that the event in the right hand side of (4.8) occurs. Note that at these times T_{n_k} it follows from the proof of (3.11)

$$\begin{aligned}\sup_{s\le C_6 a^{4+\eta}} \mathcal{R}(\mathbf{u}(T_{n_k}+s)) &\le \sup_{s\le C_6 a^{4+\eta}} \mathcal{R}(\mathbf{N}(T_{n_k-1}, T_{n_k}+s)) + 2\delta \\ &\le \mathcal{R}(\mathbf{N}(T_{n_k-1}, T_{n_k})) + \frac{a}{8} + 2\delta \\ &\le \frac{a}{4} + 2\delta\end{aligned}$$

Moreover we have that the center of mass satisfies $\sup_{s\le C_6 a^{4+\eta}} \left|\mathbf{X}_{T_{n_k}+s} - \mathbf{X}_{T_{n_k}}\right| \le \frac{a}{16}$. Therefore

$$\begin{aligned}&\sup_{s\le C_6 a^{4+\eta}} \left|\mathbf{u}(T_{n_k}+s,x) - \mathbf{X}_{T_{n_k}}\right| \\ &\quad\le \sup_{s\le C_6 a^{4+\eta}} \left|\mathbf{u}(T_{n_k}+s,x) - \mathbf{X}_{T_{n_k}+s}\right| + \sup_{s\le C_6 a^{4+\eta}} \left|\mathbf{X}_{T_{n_k}+s} - \mathbf{X}_{T_{n_k}}\right| \\ &\quad\le \sup_{s\le C_6 a^{4+\eta}} \mathcal{R}(\mathbf{u}(T_{n_k}+s)) + \sup_{s\le C_6 a^{4+\eta}} \left|\mathbf{X}_{T_{n_k}+s} - \mathbf{X}_{T_{n_k}}\right| \\ &\quad\le \frac{5a}{16} + 2\delta \\ &\quad\le \frac{3a}{8}.\end{aligned}$$

Thus the entire string lies within a ball of radius $\frac{3a}{8}$ centered at $\mathbf{X}_{T_{n_k}}$ for the duration $[T_{n_k}, T_{n_k} + C_6 a^{4+\eta}]$. Since there is a Poisson point within distance $\frac{a}{8}$ of $\mathbf{X}_{T_{n_k}}$ the string will be entirely contained within distance $\frac{a}{2}$ of the Poisson point during the time interval $[T_{n_k}, T_{n_k} + C_6 a^{4+\eta}]$. Therefore using the bounds in the proof of the upper bound in Theorem 1.1 along with (4.9) we obtain for $T > 0$

$$\begin{aligned}S_T^{\mathrm{H},1,\nu} &= \mathbb{E}\left[\exp\left(-\int_0^T \int_0^1 \mathrm{V}(\mathbf{u}(s,x),\eta)\,ds\,dx\right)\right] \\ &\le \exp\left(-B_7 \frac{T^{\frac{d}{d+2}}}{L} \cdot \nu a^{d+2}\right) + \exp\left(-B_8 \frac{T^{\frac{d}{d+2}}}{L}\right) \\ &\quad + \mathbb{E}\left[\exp\left(-\sum_{j=1}^{\frac{A_5}{L} T^{\frac{d}{d+2}} \nu a^{d+2}} \int_{T_{n_j}}^{T_{n_j}+C_6 a^{4+\eta}} \int_0^1 \mathrm{V}(\mathbf{u}(s,x),\eta)\,ds\,dx\right) \cdot\right],\end{aligned}$$

where $A_5 = \frac{A_4 C_8}{2}$. Now we use Assumption 1.1 to obtain an upper bound

$$S_T^{\mathrm{H},1,\nu} \leq \exp\left(-B_7 \frac{T^{\frac{d}{d+2}}}{L} \cdot \nu a^{d+2}\right) + \exp\left(-B_8 \frac{T^{\frac{d}{d+2}}}{L}\right) + \exp\left(-\mathfrak{C} A_5 C_6 \nu a^{d+6+\eta} \frac{T^{\frac{d}{d+2}}}{L}\right)$$

Finally we use (1.12) to get an upper bound for $S_T^{\mathrm{H},J,\nu}$. We obtain for $T > 0$

$$S_T^{\mathrm{H},J,\nu} \leq \exp\left(-\frac{B_7 \nu a^{d+2} (T/J^2)^{\frac{d}{d+2}}}{J\left(E + 3|\log(a/J^{\frac{1}{2}})|\right)}\right) + \exp\left(-\frac{\mathfrak{C} A_5 C_6 \nu a^{d+6+\eta} (T/J^2)^{\frac{d}{d+2}}}{J^{3+\frac{\eta}{2}}\left(E + 3|\log(a/J^{\frac{1}{2}})|\right)}\right).$$

It is clear that the second term dominates the first term for large J.

□

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