ESHT Lecture 1 ① 10/15/18

A spectrum (as originally defined) is a sequence $\{X_n\}$ of pointed spaces with maps

$$\Sigma X_n \xrightarrow{\varepsilon_n^X} X_{n+1} \dashv X_n \xrightarrow{\eta_n^X} \Omega X_{n+1}$$

Possible generalization

1) Replace $\mathcal{T}$ by model category $\mathcal{M}$
2) Replace (Σ, Ω) by a Quillen endopair, ~~e.g. $K\wedge$ and~~ e.g. $K\wedge -$ and $\mathcal{M}(K, -)$)

Reformulation in terms of enriched category theory.

Let $\mathcal{J}^{\mathbb{N}}$ be the $\mathcal{T}$-enriched (pointed topological) category with objects $n \in \mathbb{N}$ and morphism objects

$$\mathcal{J}(m,n) = \begin{cases} S^{n-m} & \text{for } n \geq m \\ * & \text{for } n < m \end{cases}$$

and composition morphisms

$$i_{m,n,p} : \mathcal{J}(n,p) \wedge \mathcal{J}(m,n) \longrightarrow \mathcal{J}(m,p)$$

$$S^{p-n} \wedge S^{n-m} \xrightarrow{\cong} S^{p-m}$$

The usual homeo

Then a spectrum is a $\mathcal{T}$-enriched functor $\mathcal{J}^N \xrightarrow{X} \mathcal{T}$

~~and~~ $n \longmapsto X_n$

Functoriality requires the structure maps Σ^X_n

SUBTLE POINT: $\mathcal{J}^N$ is monoidal (under +) but NOT ~~symmetric~~ monoidal

Proof: Consider the Yoneda functor

$$\mathcal{L}^0 : \mathcal{J}^N \longrightarrow \mathcal{T}$$

$$n \longmapsto \mathcal{J}^N(0, n) = S^n$$

It is strictly monoidal

If $\mathcal{J}^N$ were symmetric, we would have

$$\begin{tikzcd}
S^m \wedge S^n \arrow[d, "\mathcal{L}^0(\gamma_{m,n})"'] \arrow[dr] & \\
S^n \wedge S^m \arrow[r] & S^{m+n}
\end{tikzcd}$$

but ~~the only~~ since $\mathcal{J}^N(m+n, m+n) = S^0$,

the only vertical maps in the image of

This matters because of the following

<u>Day Convolution Thm 1970</u>

Let $(\mathcal{V}, \wedge, S)$ be a closed SMC and $(\mathcal{J}, +, 0)$ a small SMC enriched over $\mathcal{V}$. Then the category of functors $[\mathcal{J}, \mathcal{V}]$ is a closed SMC. Given two such functors X and Y, we have

$$\begin{tikzcd} \mathcal{J} \times \mathcal{J} \arrow[r, "X \times Y"] \arrow[dr, "+"'] & \mathcal{V} \times \mathcal{V} \arrow[r, "\wedge"] & \mathcal{V} \\ & \mathcal{J} \arrow[ur, dashed, "X \wedge Y"'] & \end{tikzcd}$$

where $X \wedge Y$ is the left Kan extension.

This does NOT apply to $[\mathcal{J}^{N}, \mathcal{T}] = \mathcal{S}p$ because $\mathcal{J}^{N}$ is not symmetric. Hence $\mathcal{S}p$ does not have a good smash.

(4)

We can fix this by modifying $\mathcal{J}$.

Def a) $\mathcal{J}^{\Sigma}$ is the $\mathcal{T}$ cat with ~~of~~ objects in $\mathbb{N}$ as before with

$$\mathcal{J}^{\Sigma}(m,n) = \Sigma_{n+} \underset{\Sigma_{n-m}}{\wedge} S^{n-m}$$

e.g. $\mathcal{J}^{\Sigma}(n,n) = \Sigma_{n+}$ instead of S^0.

Then $[\mathcal{J}^{\Sigma}, \mathcal{T}] = \mathcal{S}p^{\Sigma}$ is the category of <u>symmetric spectra</u> studied by HSS

Mandell MMSS

b) $\mathcal{J}^{O}$ is similar with

$$\mathcal{J}^{O}(m,n) = O(n)_{+} \underset{O(n-m)}{\wedge} S^{n-m}$$

e.g. $\mathcal{J}^{O}(n,n) = O(n)_{+}$.

Then $[\mathcal{J}^{O}, \mathcal{T}] = \mathcal{S}p^{O}$, the category of orthogonal spectra of Mandell-May

Describe $\mathcal{J}(m,n)$ ~~as~~ as Thom space

Going equivariant

Replace $\mathcal{T}$ by $\mathcal{T}_G$ or $\mathcal{T}^G$

Replace $\mathcal{J} = \mathcal{J}^0$ by $\mathcal{J}_G$. Its objects are reps V of G. Its morphism objects are

$$\mathcal{J}_G(V,W) = \mathcal{J}^0(|V|,|W|)$$

with ~~[illegible]~~ ~~G~~ suitable G-action.

The Stiefel manifold $O(V,W)$ of orthogonal embeddings has a G-action given by

$$\begin{tikzcd} V \arrow[r, "f"] & W \\ V \arrow[u, "\gamma^{-1}"] \arrow[r, "f"'] & W \arrow[u, "\gamma"'] \end{tikzcd} \qquad \gamma(f) = \gamma f \gamma^{-1}$$

This induces an action on the Thom space $\mathcal{J}_G(V,W)$.

Prop The composition morphism

$$\circ_{U,V,W} : \mathcal{J}_G(V,W) \wedge \mathcal{J}_G(U,V) \longrightarrow \mathcal{J}_G(U,W)$$

is equiv.

Model structures

Def Let $\mathcal{M}$ be a model category and J a small cat.

$\mathcal{M}^J$ is the category of functors $J \to \mathcal{M}$, i.e. of J-shaped diagrams in $\mathcal{M}$.

Prop $\mathcal{M}^J$ has a model structure in which a map

$X \xrightarrow{f} Y$ is a weq or fib

if $X_j \xrightarrow{f_j} Y_j$ is one for each $j \in J$. Cofibrations in $\mathcal{M}^J$ are defined in terms of left lifting.

If $f: X \to Y$ is a cofibration in $\mathcal{M}^J$, each f_j is one in $\mathcal{M}$, but this condition is NOT sufficient.

~~Def The Yoneda~~ This is the PROJECTIVE model structure on $\mathcal{M}^J$.

5.3.9 ⑦ 10/15/22

Def. ~~①~~ A Quillen ring $(\mathcal{M}, \wedge, S)$ ~~Suppose the~~ $\mathcal{M}$ ~~is a closed~~ SMC satisfying

i) Pushout product axiom

$f \square g$ is a cofib when f and g are and it is trivial if either f or g is

ii) [We do not assume S cofibrant]

Let $q: QS \to S$ be cofib replacement. Its smash product (on either side) ~~is~~ with cofibrant X is a weq.

Examples: $\mathcal{M} = \mathcal{J}$ and $\mathcal{J}^G$ with Bredon model

~~Such~~ A Quillen ring is enriched over itself, so we can speak of weqs of morphism objects

Suppose $\mathcal{J}$ is enriched over a Quillen ring $\mathcal{M}$. Denote the functor category by $[\mathcal{J}, \mathcal{M}]$. For $j \in \mathcal{J}$ we have the Yoneda functor $\mathcal{L}^j$ defined by $\mathcal{L}^j(k) = \mathcal{J}(j,k)$, an object in $\mathcal{M}$.

<u>Thm</u> Suppose $\mathcal{M}$ is cofibrantly generated with gen sets I and J.

Then $[\mathcal{J}, \mathcal{M}]$ is CG with generating sets

$$\bigcup_{j\in\mathcal{J}} \mathcal{L}^j \square I \quad \text{and} \quad \bigcup_{j\in\mathcal{J}} \mathcal{L}^j \square J$$

Define tensors and cotensors.