A spectrum (as originally defined)

is a sequence $\ldots X^n \rightarrow X^{n+1} \rightarrow \ldots$ with maps

Possible generalization:

1) Replace $\ldots X^n \rightarrow X^{n+1} \rightarrow \ldots$ by model category $\mathcal{M}$

2) Replace $(\Sigma, \Omega)$ by a Quillen endofunctor, e.g. $\mathcal{K}$ and $\mathcal{K}^*$ and $\mathcal{M}$ and $\mathcal{M}^*$.

Reformulation in terms of enriched category theory.

Let $\mathcal{F}$ be the $\mathcal{K}$-enriched (pointed topological) category, with objects $n \in \mathbb{N}$ and morphism objects

$$J(m, n) = \begin{cases} \subseteq & n = m \\ \varnothing & n < m \\ * & n > m \end{cases}$$

$\otimes$ and composition morphisms

$$j_{m,n,\phi} : J(n, p) \otimes J(m, n) \rightarrow J(m, p)$$

The usual axioms.
Then a spectrum $X$ is $T$-enriched function $n \mapsto X_n$

Functoriality requires the structural maps $\varepsilon^X_n$

**SUBTLE POINT**: $\mathbb{J}^n$ is monoidal (under $+$) but **NOT** symmetric monoidal.

Proof: Consider the Yoneda function $\mathbf{f} : \mathbb{J}^n \to \mathbb{Y}$

\[
\mathbb{J}^n \ni i \mapsto \mathbb{J}^n(0,n) = S^n
\]

It is strictly monoidal.

If $\mathbb{J}^n$ were symmetric, we would have

\[
\mathbb{J}^n(2,m) \cong \mathbb{J}^n(m,2)
\]

but the only possible $\mathbb{J}^n(m+n, m+n) = S^0$ is the only vertical map in the image of
This matters because of the following.

**Day Convolution Theorem 1970**

Let \((\mathcal{C}, \otimes, I)\) be a closed \(\mathcal{C}\)

and \((\mathcal{D}, +, 0)\) a small SMC

enriched over \(\mathcal{V}\). Then the

category of functors \([\mathcal{D}, \mathcal{V}]\)

is a closed SMC. Given \(F\) in \(\mathcal{V}\)

two such functors \(X\) and \(Y\), we

have

\[
\begin{array}{ccc}
F \times F & \xrightarrow{x \times y} & V \\
\downarrow \quad & \quad & \downarrow \\
V & \rightarrow & V \\
\end{array}
\]

where \(x \times y\) is the left Kan

extension.

This does **not** apply to \([\mathcal{D}, \mathcal{V}] = \mathcal{D}\)
because \(\mathcal{D}\) is not symmetric.

Hence \(\mathcal{D}\) does not have a good smash.
we can fix this by modifying

\[ f(m, n) = \sum_{n=m}^{n-m} \]

\[ e.g. f(n, n) = \sum_{n+} \text{ instead of } \sum_{n-} \]

Then \([f^2, 05] = x0^2\) is the

category of symmetric spectra

studied by HSS

b) \(j^0\) is similar with

\[ j^0(m, n) = O(n) \times x \sum_{n-m} \]

\[ e.g. j^0(n, n) = O(n) + \]

Then \([j^0, 05] = x0^0\), the

category of orthogonal spectra

of Mandell-May

Describe \(j(m, n)\) as Thom space
Using equivariant
Replace $T$ by $T_e$, or $T_0$
Replace $J = J^0$ by $J_e$. Its
objects are reps $V$ of $G$
Its morphism objects are
$J_e(V, W) = J^0((V, V), (W, W))$
with $G$-action.
The Stiefel manifold $O(N, W)$
of orthogonal embedding has
a $G$-action given by

This induces an action on the Thom
space $J_e(V, W)$

Prop. The composition morphism
$J_0, V, W : J_e(V, W) \times J_e(U, V) \to J_e(U, W)$
Model Structure

Def Let M be a model category and J a small cat. M^J is the category of functors J \rightarrow M, i.e., of J-shaped diagrams in M.

Proof M^J has a model structure in which a map X \rightarrow Y is a weak homotopy if X \rightarrow Y is one for each j \in J. Cofibrations in M^J are defined in terms of left lifting.

If f : X \rightarrow Y is a cofibration in M^J, each g_j in one in M, but this condition is NOT sufficient.

Def The Homotopy This is the PROJECTIVE model structure on M^J.
i) Project product axiom

If \( g \in G \) is a walkable when grounded
and it is trivial if either \( g \) or \( g' \) is

ii) \([\text{We do not assume } \delta \text{ cofibrant}]\)

let \( g : QS \rightarrow S \) be walkable
replacement. Let \( \delta \) strongly

product (on either side) with cofibrant \( X \) is a weak

Examples: \( M = S^0 \) and \( S^0 \) with

Such a Quillen ring is enriched
over itself, so we can speak
of maps of monoidal objects
Suppose $J$ is enriched over a Dunken ring $M$. Denote the functor category by $[J, M]$.

Then for $j \in J$ we have the Yoneda functor $J^*$ defined by $J^*(k) = J(j, k)$, an object in $M$.

Then suppose $M$ is cofibrantly generated with $\mathcal{C}$ in $\mathcal{J}$ and $\mathcal{J}$.

Then $[\mathcal{J}, M]$ is cofibrantly generated with generating sets $U$ and $V$.

Define tensors and colimits.