The Chromatic Conjectures
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Chromatic Homotopy Theory: Journey to the Frontier
University of Colorado
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Background
Periodic families
The chromatic resolution

Bousfield localization

Bousfield equivalence

The chromatic tower
Harmonic and dissonant spectra
Chromatic convergence
The chromatic resolution and the chromatic tower

Some conjectures
The nilpotence and periodicity theorems
The telescope conjecture

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Annals of Mathematics, 106 (1977), 469–516

Periodic phenomena in the Adams-Novikov spectral sequence

By Haynes R. Miller, Douglas C. Ravenel, and W. Stephen Wilson
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Periodic families

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- It was known that for each $t > 0$, the composite
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\[ \Sigma^{2p^n - 2} V(n - 1) \xrightarrow{} V(n - 1) \xrightarrow{} V(n) \text{ for } 1 \leq n \leq 3. \]

That was in 1973. To this day nobody has constructed \( V(4) \).

In each case there is a lower bound on the prime \( p \).

In 2010, Lee Nave showed that \( V((p+1)/2) \) does not exist.
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What would happen if we replace $I_n = (p, \ldots, v_{n-1})$ by a smaller invariant regular ideal with $n$ generators,
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Are there any periodic maps that are not detected by $BP$-theory?

What would happen if we replace $I_n = (p, \ldots, v_{n-1})$ by a smaller invariant regular ideal with $n$ generators, and look for a self map inducing multiplication by some power of $v_n$?
The chromatic resolution

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The $E_2$-term of the Adams-Novikov spectral sequence converging to the $p$-local stable homotopy groups of spheres is

$$E_2^{s,t} = \text{Ext}_{BP_* (BP)}^{s,t}(BP_*, BP_*),$$
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$$E^{s,t}_2 = \text{Ext}^{s,t}_{BP_*(BP)}(BP_*, BP_*),$$

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$$0 \rightarrow BP_* \rightarrow M^0 \rightarrow M^1 \rightarrow M^2 \rightarrow M^3 \rightarrow \cdots,$$

the chromatic resolution.
The chromatic resolution (continued)

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This leads to a trigraded chromatic spectral sequence converging to the bigraded Adams-Novikov \( E_2 \)-term, with

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We used the term CHROMATIC because each column (value of \( n \)) displays periodic families of elements with varying frequencies,
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\[
\begin{array}{cccccc}
0 & \to & N^0 & \xrightarrow{p^{-1}} & M^0 & \to & N^1 & \to & 0 \\
& & \| & & \| & & \|
\end{array}
\]

\[
\begin{array}{cccccc}
BP_* & \to & BP_* \otimes \mathbb{Q} & \to & BP_*/(p^\infty)
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\end{array}
\]

- For \( n > 0 \), \( M^n \) is obtained from \( N^n \) by inverting \( v_n \).
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0 & \to & N^0 & \overset{p^{-1}}{\to} & M^0 & \overset{\text{BP}_*}{\to} & \to N^1 & \to 0 \\
& & \| & & \| & & \| & \\
& & BP_* & & BP_* \otimes \mathbb{Q} & & BP_*/(p^\infty) & \\
\end{array}
\]

- For \( n > 0 \), \( M^n \) is obtained from \( N^n \) by inverting \( v_n \). There is a short exact sequence

\[
\begin{array}{cccccc}
0 & \to & N^n & \overset{v_n^{-1}}{\to} & M^n & \overset{\text{BP}_*/(p^\infty, \ldots v_{n-1})}{\to} & \to N^{n+1} & \to 0 \\
& & \| & & \| & & \| & \\
& & \text{BP}_*/(p^\infty, \ldots v_{n-1}) & & \text{BP}_*/(p^\infty, \ldots v_n) & & \text{BP}_*/(p^\infty, \ldots v_{n-1}) & \\
\end{array}
\]
The chromatic resolution (continued)

\[
\begin{array}{cccccc}
0 & \rightarrow & N^n & \xrightarrow{v_n^{-1}} & M^n & \rightarrow & N^{n+1} & \rightarrow & 0 \\
& & \parallel & & \parallel & & \parallel & & \\
& & BP_*/(p^\infty, \ldots, v_{n-1}^\infty) & & BP_*/(p^\infty, \ldots, v_n^\infty) & & \vdots \\
& & v_n^{-1}BP_*/(p^\infty, \ldots, v_{n-1}^\infty) & & \\
\end{array}
\]
The chromatic resolution (continued)

\[ \begin{array}{ccccccccc}
0 & \rightarrow & N^n & \overset{v_n^{-1}}{\rightarrow} & M^n & \rightarrow & N^{n+1} & \rightarrow & 0 \\
\| & & \| & & \| & & \| & & \\
BP_* / (p^\infty, \ldots v_{n-1}^\infty) & & BP_* / (p^\infty, \ldots v_n^\infty) & & v_n^{-1} BP_* / (p^\infty, \ldots v_{n-1}^\infty) & & \\
\end{array} \]

The chromatic resolution

\[ 0 \rightarrow BP_* \rightarrow M^0 \rightarrow M^1 \rightarrow M^2 \rightarrow M^3 \rightarrow \cdots \]
The chromatic resolution (continued)

\[
\begin{array}{cccccc}
0 & \rightarrow & N^n & \xrightarrow{\nu_{n-1}} & M^n & \rightarrow & N^{n+1} & \rightarrow & 0 \\
\| & & BP_*/(p^\infty, \ldots, \nu_{n-1}^\infty) & \| & BP_*/(p^\infty, \ldots, \nu_n^\infty) & & \| \\
& & \nu_{n-1} BP_*/(p^\infty, \ldots, \nu_{n-1}^\infty) & & & & \\
\end{array}
\]

The chromatic resolution

\[
0 \rightarrow BP_* \rightarrow M^0 \rightarrow M^1 \rightarrow M^2 \rightarrow M^3 \rightarrow \cdots
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is obtained by splicing together these short exact sequence for all \( n \geq 0 \).
The chromatic resolution (continued)

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\begin{array}{cccccc}
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& & BP_* / (p^\infty, \ldots v_{n-1}^\infty) & \overset{v_n^{-1}}{\rightarrow} & BP_* / (p^\infty, \ldots v_n^\infty) & & & & \\
& & v_n^{-1} BP_* / (p^\infty, \ldots v_{n-1}^\infty) & & & & & & \\
\end{array}
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This construction is purely algebraic.
The chromatic resolution (continued)

\[ \begin{array}{cccccc}
0 & \rightarrow & N^n & \rightarrow & M^n & \rightarrow & N^{n+1} & \rightarrow & 0 \\
\| & & \| & & \| & & \|
\end{array} \]

\[ \begin{array}{c}
BP_*/(p^\infty, \ldots v_{n-1}^\infty) \\
\nu_n^{-1}BP_*/(p^\infty, \ldots v_n^\infty)
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This construction is purely algebraic. It takes place in the category of \( BP_*(BP) \)-comodules.

IS THERE A SIMILAR CONSTRUCTION IN THE STABLE HOMOTOPY CATEGORY?
The chromatic resolution (continued)

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\[ 0 \rightarrow \text{BP}_* \rightarrow M^0 \rightarrow M^1 \rightarrow M^2 \rightarrow M^3 \rightarrow \cdots \]

IS THERE A SIMILAR CONSTRUCTION, AND THE BEAUTIFUL ALGEBRA THAT GOES ALONG WITH IT, IN THE STABLE HOMOTOPY CATEGORY?
The chromatic resolution (continued)

\[ 0 \rightarrow N^n \xrightarrow{v_n^{-1}} M^n \rightarrow N^{n+1} \rightarrow 0 \]

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**IS THERE A SIMILAR CONSTRUCTION, AND THE BEAUTIFUL ALGEBRA THAT GOES ALONG WITH IT, IN THE STABLE HOMOTOPY CATEGORY?**

**OR IS IT JUST AN ARTIFACT OF COMPLEX COBORDISM THEORY?**
The chromatic resolution (continued)

\[ 0 \to N^n \xrightarrow{v^{-1}_n} M^n \to N^{n+1} \to 0 \]

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This question occupied me for several years.
The chromatic resolution (continued)

LOCALIZATION WITH RESPECT TO CERTAIN PERIODIC HOMOLOGY THEORIES

By Douglas C. Ravenel*

This paper represents an attempt, only partially successful, to get at what appear to be some deep and hitherto unexamined properties of the stable homotopy category. This work was motivated by the discovery of the pervasive manifestation of various types of periodicity in the $E_2$-term of the Adams-Novikov spectral sequence converging to the stable homotopy groups of spheres. In section 3 of [34] and section 8 of [41], we introduced the chromatic spectral sequence, which converges to the above $E_2$-term. Unlike most spectral sequences, its input is in some sense more interesting than its output, as the former displays many appealing patterns which are somewhat hidden in the latter (see section 8 of [41] for a more detailed discussion). It is not so much a computational aid (although it has been used [34] for computing the Novikov 2-line) as a conceptual tool for understanding certain qualitative aspects of the Novikov $E_2$-term.

Since the Novikov $E_2$-term is a reasonably good approximation to sta-
Bousfield localization

\[ 0 \to N^n \xrightarrow{v_n^{-1}} M^n \to N^{n+1} \to 0 \]

\[ 0 \to BP_* \to M^0 \to M^1 \to M^2 \to M^3 \to \cdots \]
Bousfield localization

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\[ 0 \rightarrow BP_* \rightarrow M^0 \rightarrow M^1 \rightarrow M^2 \rightarrow M^3 \rightarrow \cdots \]

It would be nice if each short exact sequence above were the \( BP_* \) homology of a cofiber sequence of spectra.
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\[
BP_* M_n \cong M^n \quad \text{and} \quad BP_* N_n \cong N^n.
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This was easy enough for $n = 0$. 
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\[
\begin{align*}
0 & \to N^n \xrightarrow{\nu_n^{-1}} M^n \to N^{n+1} \to 0 \\
0 & \to BP_* \to M^0 \to M^1 \to M^2 \to M^3 \to \cdots
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This was easy enough for \( n = 0 \). We knew then how to invert a prime \( p \) homotopically. The resulting \( N^1 \) is the Moore spectrum for the group \( \mathbb{Q}/\mathbb{Z}(p) \). But how would we invert \( v_1 \) to do the next step?
Bousfield localization (continued)

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Pete Bousfield
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Suppose we have a generalized homology theory represented by a spectrum $E$. We say a spectrum $Z$ is $E$-local if, whenever $f : A \to B$ is an $E_*$-equivalence, that is a map inducing an isomorphism $E_*A \to E_*B$, then the induced map

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is also an isomorphism.
Theorem (Bousfield localization of spectra 1979)

For a given $E$ there is a coaugmented functor $L_E$ such that for any spectrum $X$, $L_E X$ is $E$-local and the map $X \rightarrow L_E X$ is an $E_\ast$-equivalence.
It turns out that when $E$ and $X$ are both connective, then $L_E X$ can be described in arithmetic terms.
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Things can be much more interesting when either $E$ or $X$ (or both) fail to be connective.
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It is closely related to the fancier Morava spectrum $E_n$, but the latter had not been invented yet. It turns out that both lead to the same localization functor.
Bousfield equivalence

Recall that a spectrum $Z$ is $E$-local if, whenever $f : A \to B$ is an $E_\ast$-equivalence, that is a map inducing an isomorphism $E_\ast A \to E_\ast B$, then the induced map

$$f^* : [B, Z] \to [A, Z]$$

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The Chromatic Conjectures

- The chromatic tower
- Harmonic and dissonant spectra
- The nilpotence and periodicity theorems
- The telescope conjecture

Background

- Periodic families
- The chromatic resolution

Bousfield localization

- The chromatic resolution and the chromatic tower

Some conjectures

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It follows that the maximal Bousfield class is that of the sphere spectrum $S$, and the minimal one is that of a point $\ast$.

It is easy to check that wedges and smash products of Bousfield classes are well defined, that is we can define

$$\langle E \rangle \vee \langle F \rangle := \langle E \vee F \rangle \quad \text{and} \quad \langle E \rangle \wedge \langle F \rangle := \langle E \wedge F \rangle.$$
Bousfield equivalence (continued)

\[ \langle E \rangle \vee \langle F \rangle := \langle E \vee F \rangle \quad \text{and} \quad \langle E \rangle \wedge \langle F \rangle := \langle E \wedge F \rangle. \]
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\[ \langle E \rangle \lor \langle F \rangle := \langle E \lor F \rangle \quad \text{and} \quad \langle E \rangle \land \langle F \rangle := \langle E \land F \rangle. \]

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The collection of classes with complements forms a Boolean algebra \( \mathbb{BA} \).
Bousfield equivalence (continued)

Theorem (Formal properties of Bousfield classes)
**Bousfield equivalence (continued)**

<table>
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---

**Background**
- Periodic families
- The chromatic resolution

**Bousfield localization**

**Bousfield equivalence**
- The chromatic tower
- Harmonic and dissonant spectra
- Chromatic convergence
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### Theorem (Some Bousfield equivalence classes)

1. $\langle S \rangle = \langle S_Q \rangle \vee \bigvee_{p \text{ prime}} \langle S/p \rangle$, where $S_Q$ is the rational Moore spectrum and $S/p$ is the mod $p$ Moore spectrum.

2. $\langle BP \rangle \geq \langle H/p \rangle \vee \bigvee_{n \geq 0} \langle K(n) \rangle$, where $H/p$ is the mod $p$ Eilenberg-Mac Lane spectrum and $K(n)$ is the $n$th Morava K-theory.

3. $\langle E(n) \rangle = \langle E_n \rangle = \bigvee_{0 \leq i \leq n} \langle K(i) \rangle$. In each case, the smash product of any two of the wedge summands on the right is contractible.
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The Chromatic Conjectures

Doug Ravenel

The chromatic tower

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- Periodic families
- The chromatic resolution

Bousfield localization

Bousfield equivalence

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- Harmonic and dissonant spectra
- Chromatic convergence
- The chromatic resolution and the chromatic tower

Some conjectures
- The nilpotence and periodicity theorems
- The telescope conjecture

The localization functor $L_E$ is determined by the Bousfield class $\langle E \rangle$. When $\langle E \rangle \geq \langle F \rangle$, there is a natural transformation $L_E \Rightarrow L_F$. For a fixed prime $p$, let $L_n = L_E(n)$. Then for any spectrum $X$ we get a diagram

$$X \to L_\infty X \to \cdots \to L_n X \to L_{n-1} X \to \cdots \to L_1 X \to L_0 X.$$ 

This is the chromatic tower of $X$. Here $L_\infty$ denotes localization with respect to the Bousfield class $\bigcup_{n \geq 0} \langle K(n) \rangle$. 

1.23
The chromatic tower

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The Chromatic Conjectures

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The chromatic tower (continued)

The chromatic tower of a $p$-local spectrum $X$ is the diagram

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This raises some questions:

- When is the map $X \rightarrow L_{\infty}X$ an equivalence? When it is, we say $X$ is harmonic. We call $L_{\infty}X$ the harmonic localization of $X$.
- When is the map $X \rightarrow \text{holim} L_nX$ an equivalence? This is the chromatic convergence question.
- Can we describe $BP_\ast L_nX$ in terms of $BP_\ast X$? This is the localization question.
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- When is the map $X \to \text{holim} L_n X$ an equivalence? This is the chromatic convergence question.
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- Can we describe $BP_* L_n X$ in terms of $BP_* X$? This is the localization question.
Harmonic and dissonant spectra

Recall that $L_\infty$ denotes localization with respect to the Bousfield class $\bigvee_{n \geq 0} \langle K(n) \rangle$. A $p$-local spectrum is harmonic if $X \simeq L_\infty X$. It is dissonant if $L_\infty X \simeq \ast$, meaning that $K(n) \ast X = 0$ for all $n$. It follows from the definitions that there are no essential maps from a dissonant spectrum to a harmonic one.

In the 1984 paper I showed that

- Every $p$-local finite spectrum is harmonic.
- A $p$-local connective spectrum $X$ is harmonic when $BP^\ast X$ has finite projective dimension as a $BP^\ast$-module.
- The mod $p$ Eilenberg-Mac Lane spectrum $H/p$ is dissonant. The same is true for any spectrum whose homotopy groups are all torsion and bounded above.
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It turns out that $L_n BP$ is easy to analyze, and this makes it easy to understand the spectrum $X \wedge L_n BP$.

**Theorem (The localization conjecture)**

For any spectrum $X$, $BP \wedge L_n X \simeq X \wedge L_n BP$.

In particular, when $E(n-1)^\ast X = 0$, $BP^\ast L_n X = v_n BP^\ast X$.

It follows that the chromatic resolution can be realized as desired.

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I ended the 1984 paper with a list of conjectures, all but one of which (the telescope conjecture) were proved within 15 years, most by Hopkins and various collaborators. I have already mentioned some of them. I will state some more of them here as theorems.

Nilpotence Theorem (Devinatz-Hopkins-Smith 1988)

1. For a finite spectrum $X$, a map $\Sigma^d X \to X$ is nilpotent iff $MU^*(\Sigma^d X)$ is nilpotent.
2. For a finite spectrum $X$, a map $g: X \to Y$ is smash nilpotent if the map $MU \wedge g$ is null homotopic.
3. Let $R$ be a connective ring spectrum of finite type, and let $h: \pi_\ast R \to MU^* R$ be the Hurewicz map. Then $\alpha \in \pi_\ast R$ is nilpotent when $h(\alpha) = 0$.
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Some conjectures (continued)

If it were the case that $\langle MU \rangle = \langle S \rangle$, or if $\langle BP \rangle = \langle S(p) \rangle$ for each prime $p$, then the Nilpotence Theorem would follow immediately. However $\langle BP \rangle < \langle S(p) \rangle$, meaning there are $BP\ast$-acyclic $p$-local spectra that are not contractible. In other words $MU$ does NOT "see everything." In fact there are connective $p$-local spectra $T(m)$ for $m \geq 0$ with $BP\ast T(m) \sim = BP\ast [t_1, t_2, \ldots, t_m]$ (so $T(0) = S(p)$) and $\langle T(0) \rangle > \langle T(1) \rangle > \langle T(2) \rangle \cdots > \langle BP \rangle$. 

The Chromatic Conjectures

Doug Ravenel

Background
Periodic families
The chromatic resolution

Bousfield localization

Bousfield equivalence

The chromatic tower
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The chromatic resolution and the chromatic tower

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$$\langle T(0) \rangle > \langle T(1) \rangle > \langle T(2) \rangle \cdots > \langle BP \rangle.$$
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Nilpotence Theorem (Devinatz-Hopkins-Smith 1988)

For a finite spectrum $X$, a map $f : \Sigma^d X \to X$ is nilpotent iff $MU_*(f)$ is nilpotent.
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This means that such a map can be periodic (the opposite of being nilpotent) only if it detected as such by $MU$-homology. In the $p$-local case, the internal properties of $MU$-theory imply that $f$ must induce a nontrivial isomorphism in some Morava $K$-theory $K(n)_\ast$. 
Some conjectures (continued)

**Periodicity Theorem (Hopkins-Smith 1998)**

Let $X$ be a $p$-local finite spectrum of *chromatic type* $n$, meaning

- $K(n-1)^*X = 0$,
- but $K(n)^*X \neq 0$.

Then there is a map $v: \Sigma^d X \to X$ (a $v^n$ self-map) with $K(n)^*(v)$ an isomorphism and $H^*(v;\mathbb{Z}/p) = 0$.

If $n = 0$ then $d = 0$, and when $n > 0$, $d$ is a multiple of $2p^{n-2}$.

Given a second such map $w: \Sigma^e X \to X$, there are positive integers $i$ and $j$ such that $id = je$ and $v^i = w^j$.

In other words, $v$ is asymptotically unique.

It follows that the cofiber of $v$ (or of any of its iterates) is a $p$-local finite spectrum of chromatic type $n+1$.

This means that there are finite complexes of all chromatic types.

Finite complexes of arbitrary chromatic type were first constructed by Steve Mitchell in 1985.

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**Theorem (The class invariance conjecture)**

*The Bousfield class of a $p$-local finite spectrum $X$ is determined by its chromatic type, i.e., the smallest $n$ for which $K(n)^*X \neq 0.*
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**Theorem (The class invariance conjecture)**

*The Bousfield class of a $p$-local finite spectrum $X$ is determined by its chromatic type, i.e., the smallest $n$ for which $K(n)_* X \neq 0$. In particular if $H_* X$ is not all torsion, then $\langle X \rangle = \langle S(p) \rangle$.***
The telescope conjecture

Suppose $X$ is a $p$-local finite spectrum of chromatic type $n$. The Periodicity Theorem says that it has a $v_n$ self-map $v_n: \Sigma^d X \to X$. Let $\hat{X}$ be the associated mapping telescope, meaning the homotopy colimit of $X \xrightarrow{v_n} \Sigma^{-d} X \xrightarrow{v_n} \Sigma^{-2d} X \to \cdots$. Note that it is independent of the choice of $v$. Since $v$ is a $K(n)$-equivalence and therefore an $E(n)$-equivalence, we have maps $X \to \hat{X} \xrightarrow{\lambda} \to L_n X$. 

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The telescope conjecture (continued)

Telescope Conjecture

For any $p$-local spectrum $X$ of chromatic type $n$, the map $\hat{\lambda}: \hat{X} \to L^n X$ is an equivalence.
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For any $p$-local spectrum $X$ of chromatic type $n$, the map $\lambda : \hat{X} \rightarrow L_nX$ is an equivalence.
THANK YOU!