ECHT Minicourse

What is the telescope conjecture?
Lecture 1
An algebraic prelude to chromatic homotopy theory

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December 5, 2023

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The best reference for this material is the 1974 book *Characteristic classes* by Milnor and Stasheff, [MS74].

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$$H_k BU(n) \cong H_{k+2n} MU(n).$$

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Since $E(\gamma_{n+1}^{\mathbb{C}})$ restricts under the map $BU(n) \to BU(n+1)$ to the bundle $\epsilon_1^{\mathbb{C}} \oplus \gamma_n^{\mathbb{C}}$, where $\epsilon_1^{\mathbb{C}}$ is the trivial line bundle,

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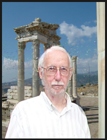
-   Localizing at a prime p gives a splitting

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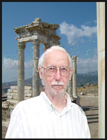
- 

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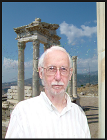
- 

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BP is the Brown-Peterson spectrum, first constructed in 1966.

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$$MU_* \otimes \mathbb{Q} \cong \mathbb{Q}[[\mathbb{C}P^1], [\mathbb{C}P^2], \ldots].$$

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$$MU^*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty) \cong MU^*[\![x \otimes 1, 1 \otimes x]\!].$$

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This power series is a linchpin of the theory. It is easily seen to have the following three properties:

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- Euler's elliptic integral addition formula:

$$F(x, y) = \frac{x\sqrt{1 - y^4} + y\sqrt{1 - x^4}}{1 + x^2y^2} \in \mathbb{Z}[1/2][\![x, y]\!].$$

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Lazard's classification in characteristic p

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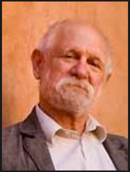

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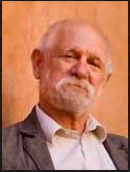

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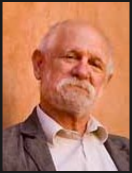

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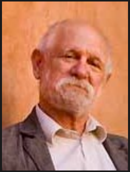

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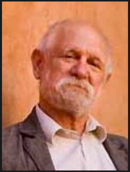

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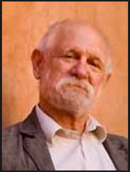

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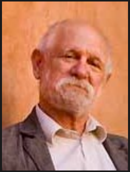

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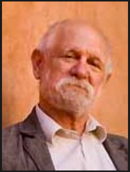

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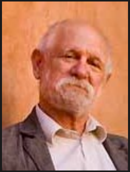

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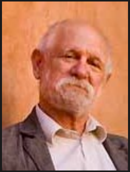

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- Consider the ideal $I_h = (v_0, v_1, \ldots, v_{h-1}) \subset L$, where $v_0 = p$. Then the ascending chain of ideals

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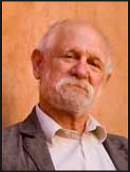

Lazard proved that two formal group laws over $\overline{\mathbb{F}}_p$ are isomorphic if and only if they have the same height. Jack Morava realized this has profound implications for homotopy theory. We will say much more about this later.

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Billboard by Yuri Sulyma

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Rather than getting into the nuts and bolts of these objects, which are discussed thoroughly in the green book, we will present a conceptual picture of them.

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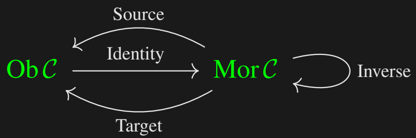

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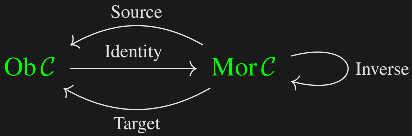

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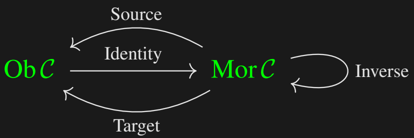

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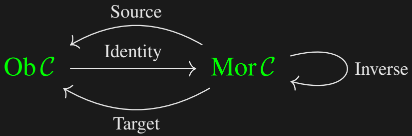

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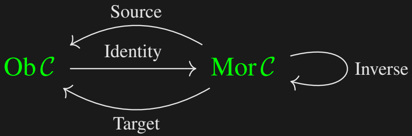

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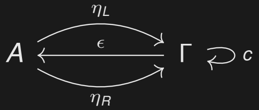

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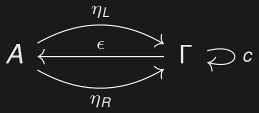

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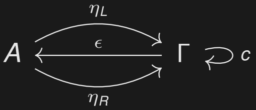

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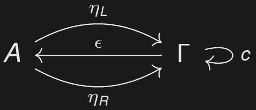

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THANK YOU!

References

[MS74] John W. Milnor and James D. Stasheff.
Characteristic classes.
Princeton University Press, Princeton, N. J., 1974.
Annals of Mathematics Studies, No. 76.

[Rav86] Douglas C. Ravenel.
Complex cobordism and stable homotopy groups of spheres, volume 121 of *Pure and Applied Mathematics*.
Academic Press Inc., Orlando, FL, 1986.
Second edition available online at author's home page.