ECHT Minicourse

What is the telescope conjecture?

Lecture 4

v_h-periodic families and telescopes

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December 14, 2023

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We have left out a motivating development in the stable homotopy groups of spheres: the discovery in the early 70s of periodic families known as Greek letter elements. We will describe these now.

Greek letter elements

Recall the *h*th Greek letter sequence,

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These α_ts comprise a v_1-periodic family.

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To repeat, the α sequence,

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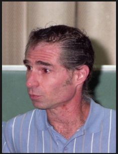

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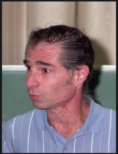

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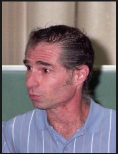

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The following was conjectured in [Rav84] and proved by Ethan Devinatz, Mike Hopkins and Jeff Smith in [DHS88].

Class Invariance Theorem

The Bousfield equivalence class of a p-local finite spectrum is determined by its type.



In particular any p-local finite spectrum X with nontrivial rational homology is Bousfield equivalent to $\mathbb{S}_{(p)}$.

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Jeremy, Tomer, myself, Ishan and Robert at Oxford University, June 9, 2023.
Photo by Matteo Barucco.

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THANK YOU!

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