University of Glasgow

Rankin Lecture

What is the telescope conjecture?

A walking tour of modern homotopy theory

Doug Ravenel
University of Rochester

24 May, 2022

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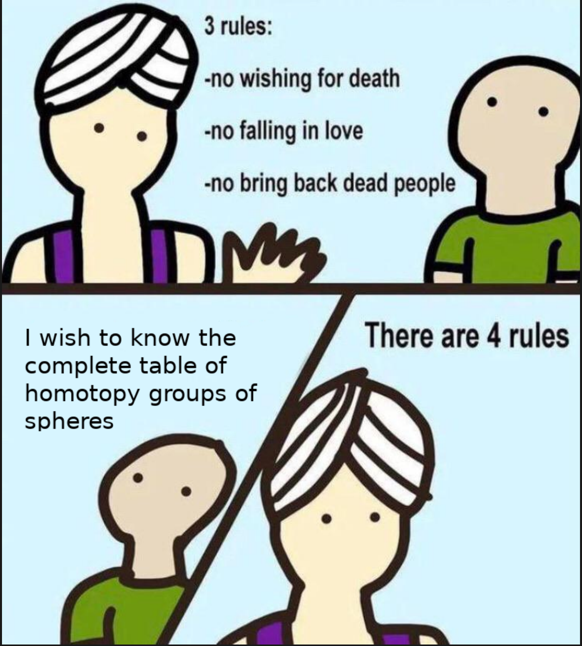

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Image courtesy of Wikipedia

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Image courtesy of Dror Bar-Natan

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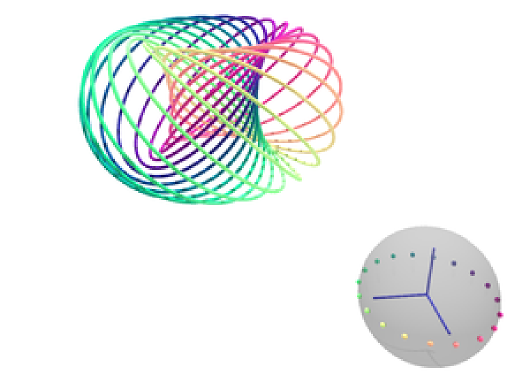
https://nilesjohnson.net/hopf.html



Stable homotopy groups and number theory

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$\pi_k S$	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/24$	0	0	$\mathbb{Z}/2$	$\mathbb{Z}/240$

k	8	9	10	11	12	13	14	15
$\pi_k S$	$(\mathbb{Z}/2)^2$	$(\mathbb{Z}/2)^3$	$Z/6$	$\mathbb{Z}/2 \oplus \mathbb{Z}/504$	0	$\mathbb{Z}/3$	$(\mathbb{Z}/2)^2$	$\mathbb{Z}/2 \oplus \mathbb{Z}/480$

Note the large cyclic subgroups in dimensions congruent to 3 mod 4. They are known to exist in all such dimensions. Let a_m denote the order of this summand in dimension $4m - 1$.

Doug Ravenel

Stable homotopy groups and number theory

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m	1	2	3	4	5	6	7	8
a_m	24	240	504	480	264	65,520	24	16,320

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Does anybody recognize these numbers?

Stable homotopy and number theory (continued)

Again, a_m is the order of a known cyclic summand of the group $\pi_{4m-1}S$.

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$$\zeta(-11) = -\frac{B_{12}}{12} = \frac{691}{32{,}760} = \frac{2 \cdot 691}{a_6}.$$

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Why are these numbers in the homotopy groups of spheres?

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THE STATE OF THE ART

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THE STATE OF THE ART

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THE STATE OF THE ART

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FOR THIS WE NEED A NEW ALGEBRAIC APPARATUS.

Formal group laws

The algebra that follows is related to the topology that preceded it, as will be explained later.

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Algebraic motivation: In a 1-dimensional analytic lie group, the multiplication could be described by a function of two variables with similar properties.

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Formal group laws (continued)

Definition

A *formal group law over a ring F* is a power series $F(x,y) \in R[\![x,y]\!]$ with $F(x,0) = F(0,x) = x$, $F(y,x) = F(x,y)$, and $F(F(x,y),z) = F(x,F(y,z))$.

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- EULER'S ELLIPTIC INTEGRAL ADDITION FORMULA,

$$F(x,y) = \frac{x\sqrt{1-y^4} + y\sqrt{1-x^4}}{1+x^2y^2} \in \mathbb{Z}[1/2][[x,y]].$$

The Lazard ring L

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A formal group law over a ring R is a power series

$$F(x,y) = \sum_{i,j} a_{i,j} x^i y^j \in R[[x,y]]$$

(so $a_{i,j} \in r$)

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Formal group laws were studied by Michel Lazard in 1955. He considered the ring $L = \mathbb{Z}[a_{i,j}]/(\sim)$, in which the relations are those implied by the three defining properties of the power series $F(x,y)$. This makes F equivalent to a ring homomorphism $\theta : L \to R$.

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Formal group laws were studied by Michel Lazard in 1955. He considered the ring $L=\mathbb{Z}[a_{i,j}]/(\sim)$, in which the relations are those implied by the three defining properties of the power series $F(x,y)$. This makes F equivalent to a ring homomorphism $\theta : L\to R$. Hence L is the ground ring for the universal formal group law.

the Lazard ring I(continued)

Lazard's ring L is $\mathbb{Z}[a_{i,j}]/(\sim)$, where the relations are those implied by the three defining properties of a formal group law. It is the ground ring for the universal formal group law G.

the Lazard ring l(continued)

Lazard's ring L is $\mathbb{Z}[a_{i,j}]/(\sim)$, where the relations are those implied by the three defining properties of a formal group law. It is the ground ring for the universal formal group law G. This means that any formal group law F over any ring R is induced from G via a ring homomorphism $\theta : L \to R$.

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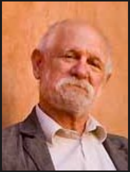

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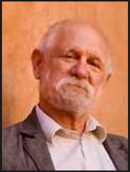

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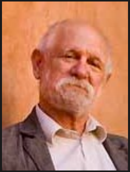

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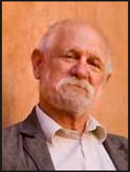

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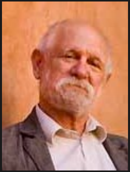

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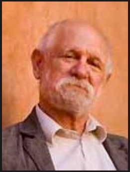

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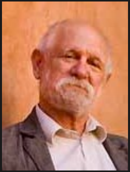

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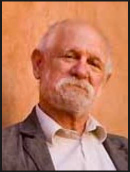

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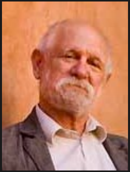

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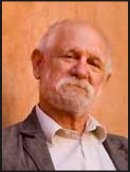

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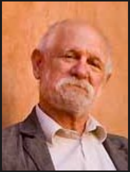

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Let V denote the "vector space" of ring homomorphisms $\theta : L \to \overline{\mathbb{F}}_p$.

- Each point in V corresponds to a formal group law over $\overline{\mathbb{F}}_p$.
- V has an action of $\Gamma = \Gamma_{\overline{\mathbb{F}}_p}$ for which each orbit is an isomorphism class of formal group laws over $\overline{\mathbb{F}}_p$. Hence there is one orbit for each height.

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$$V = V_1 \supset V_2 \supset V_3 \supset \cdots$$

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Four Fields medalists

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Bringing the algebra to the topology (continued)

COMPLEX BORDISM

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Bringing the algebra to the topology (continued)

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Doug Ravenel

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The "height 1 orbit" of $\pi_* S$ consists of this family and similar ones linked by $v_1^{p^j}$ for various j.

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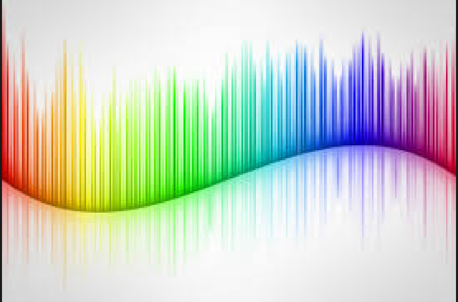
It is like a spectrum in the sense of astronomy.



Telescopes and periodic families

WHAT IS A TELESCOPE?

Doug Ravenel

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$$X_{i-1} \times I \ni (x_{i-1}, 1) \rightsquigarrow (f_i(x_{i-1}), 0) \in X_i \times I$$

the right end of one cylinder — the left end of the next one

Telescopes and periodic families (continued)

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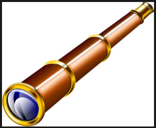

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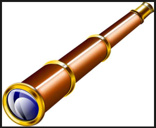

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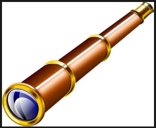

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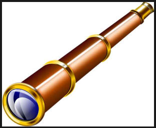

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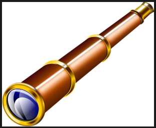

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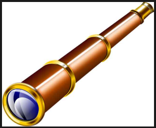

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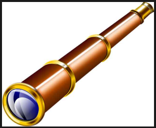

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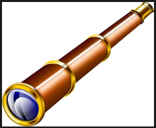

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The height one case was proved around 1980 by Mark Mahowald for $p = 2$ and Haynes Miller for odd primes.

The telescope conjecture (continued)

FULL DISCLOSURE

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LOCALIZATION WITH RESPECT TO CERTAIN PERIODIC HOMOLOGY THEORIES

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In the fall of 1989 there was a homotopy theory program at MSRI in Berkeley.

MSRI
Mathematical Sciences Research Institute

The telescope conjecture (continued)

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Doug Ravenel

The telescope conjecture (continued)

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San Francisco earthquake of October 17, 1989

Doug Ravenel

The telescope conjecture (continued)

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DISCLAIMER: Having bet on both sides of this question, my credibility now stands at ZERO.

THANK YOU!

Doug Ravenel