Model category structures for equivariant spectra

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\mathcal{I} \times \mathcal{I} & \xrightarrow{X \times Y} \mathcal{V} \times \mathcal{V} \xrightarrow{\otimes} \mathcal{V} \\
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\downarrow & & \downarrow \oplus & & \downarrow \\
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The functor \(X \otimes Y\) is the left Kan extension of the composite \(\otimes (X \times Y)\) along \(\oplus\).
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The functor \(X \otimes Y\) is the left Kan extension of the composite \(\otimes(X \times Y)\) along \(\oplus\). It exists because \(\mathcal{I} \times \mathcal{I}\) is small and \(\mathcal{V}\) is cocomplete.
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For a finite group $G$, let $T^G$ be the category of pointed $G$-spaces and equivariant maps. In the Bredon model structure a map $f : X \to Y$ is a fibration or a weak equivalence if the map $f^H : X^H \to Y^H$ of fixed point sets is one for each subgroup $H$. 
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We use the term equifibrant to describe this happy state of affairs. We need an equifibrant model structure on the category of $G$-spectra.
Three ways to construct new model categories from old ones

Given a model category $M$ and a small category $J$, we define the projective model structure on the functor category $M^J$ as follows. A map (aka natural transformation) $f: X \to Y$ between functors is a weak equivalence or a fibration if $f_j: X_j \to Y_j$ is one for each object $j$ in $J$. Cofibrations are defined in terms of left lifting properties.
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Dan Kan
1928-2013
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Suppose we have a diagram of small categories enriched over \( \mathcal{T}^G \) (to be named later),

\[
\begin{array}{cccc}
\mathcal{J}^+ & \xleftarrow{k_+} & \tilde{\mathcal{J}}^+ \\
\mathcal{J}^G & \xleftarrow{k} & \tilde{\mathcal{J}}^G \\
\mathcal{J}^G & \xleftarrow{i} & \tilde{\mathcal{J}}^G \\
\end{array}
\]

Then we get a diagram of enriched functor categories

\[
\begin{array}{cccc}
\left[\mathcal{J}^G, \mathcal{T}^G\right] & \xleftarrow{i!} & \left[\tilde{\mathcal{J}}^G, \mathcal{T}^G\right] \\
\left[\mathcal{J}^G, \mathcal{T}^G\right] & \xleftarrow{k^\ast} & \left[\tilde{\mathcal{J}}^G, \mathcal{T}^G\right] \\
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\]

where \( k^\ast \) and \( k^+ \) are induced by precomposition, and \( i! \) and \( \tilde{i}! \) are induced by left Kan extension.

The category \( \mathcal{J}^G \) is chosen so that the functor category \( \left[\mathcal{J}^G, \mathcal{T}^G\right] \) is that of orthogonal \( G \)-spectra and equivariant maps.
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Suppose we have a diagram of small categories enriched over $T^G$ (to be named later),

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\begin{array}{ccc}
J^*_G & \xrightarrow{k} & \tilde{J}^*_G \\
\downarrow i & & \downarrow \tilde{i} \\
J_G & \xrightarrow{k} & \tilde{J}_G
\end{array}
$$

Then we get a diagram of enriched functor categories

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\left[ J^*_G, T^G \right] & \xrightarrow{i^*} & \left[ \tilde{J}^*_G, T^G \right] \\
\downarrow k^* & & \downarrow \tilde{k}^* \\
\left[ J_G, T^G \right] & \xrightarrow{k} & \left[ \tilde{J}_G, T^G \right]
\end{array}
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where $k^*$ and $\tilde{k}^*$ are induced by precomposition, and $i^*$ and $\tilde{i}^*$ are induced by left Kan extension.

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Suppose we have a diagram of small categories enriched over $\mathcal{T}^G$ (to be named later),

$$
\begin{array}{ccc}
\mathcal{I}_G & \xrightarrow{k} & \mathcal{I}_G \\
\downarrow{i} & & \downarrow{\tilde{i}} \\
\mathcal{I} & \xrightarrow{k} & \mathcal{I}
\end{array}
$$

Then we get a diagram of enriched functor categories

$$
\begin{array}{ccc}
[\mathcal{I}_G, \mathcal{T}^G] & \xleftarrow{k^*} & [\mathcal{I}_G, \mathcal{T}^G] \\
\downarrow{i} & & \downarrow{\tilde{i}} \\
Sp^G & \longleftarrow{k^*} & [\mathcal{I}_G, \mathcal{T}^G]
\end{array}
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\downarrow{i} & & \downarrow{i} \\
\mathcal{J}_{\mathcal{G}} & \xrightarrow{k} & \mathcal{J}_{\mathcal{G}}
\end{array}
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\[
\begin{array}{ccc}
[\mathcal{J}_G^+, \mathcal{T}^G] & \xleftarrow{k^*} & [\mathcal{J}_G^+, \mathcal{T}^G] \\
\downarrow{i^*_i} & & \downarrow{i^*_i} \\
Sp^G & \xrightarrow{k^*} & [\mathcal{J}_G^+, \mathcal{T}^G]
\end{array}
\]

where $k^*$ and $k^*_+$ are induced by precomposition,
The main construction

Suppose we have a diagram of small categories enriched over $\mathcal{T}^G$ (to be named later),

$$
\begin{array}{ccc}
\mathcal{J}_{+}^{G} & \xrightarrow{k_{+}} & \mathcal{J}_{+}^{G} \\
\mathcal{J}_{G} & \xrightarrow{k} & \mathcal{J}_{G} \\
i & \downarrow & \downarrow \\
\mathcal{J}_{G}^+ & \xrightarrow{i} & \mathcal{J}_{G}
\end{array}
$$

Then we get a diagram of enriched functor categories

$$
\begin{array}{ccc}
[\mathcal{J}_{+}^{G}, \mathcal{T}^G] & \xleftarrow{k_{+}^*} & [\mathcal{J}_{+}^{G}, \mathcal{T}^G] \\
i_{!} & \downarrow & \downarrow \tilde{i}_{!} \\
Sp^G & \xrightarrow{k^*} & [\mathcal{J}_{G}, \mathcal{T}^G] & \xleftarrow{k_{!}^*} & [\mathcal{J}_{G}, \mathcal{T}^G]
\end{array}
$$

where $k^*$ and $k_{+}^*$ are induced by precomposition, and $i_{!}$ and $\tilde{i}_{!}$ are induced by left Kan extension.
The main construction

Suppose we have a diagram of small categories enriched over $\mathcal{T}^G$ (to be named later),

\[
\begin{array}{ccc}
\mathcal{J}^+ & \xrightarrow{k} & \mathcal{J}^+ \\
\downarrow i & & \downarrow \sim i \\
\mathcal{J}^G & \xrightarrow{k} & \mathcal{J}^G \\
\end{array}
\]

Then we get a diagram of enriched functor categories

\[
\begin{array}{ccc}
[\mathcal{J}^+, \mathcal{T}^G] & \xleftarrow{k^*} & [\mathcal{J}^+, \mathcal{T}^G] \\
\downarrow i_i & & \downarrow \sim i_i \\
Sp^G & \xleftarrow{k^*} & [\mathcal{J}^G, \mathcal{T}^G] \\
\end{array}
\]

where $k^*$ and $k^+_+$ are induced by precomposition, and $i_i$ and $\sim i_i$ are induced by left Kan extension. The category $\mathcal{J}^G$ is chosen so that the functor category $[\mathcal{J}^G, \mathcal{T}^G]$ is that of orthogonal $G$-spectra and equivariant maps.
The main construction (continued)

Now we proceed as follows.
The main construction (continued)

\[ [\mathcal{I}_G^+, \mathcal{T}_G] \leftarrow k^* \rightarrow [\widetilde{\mathcal{I}}_G^+, \mathcal{T}_G^+] \]

\[ \mathcal{I}_G \xrightarrow{k} \widetilde{\mathcal{I}}_G \xleftarrow{i} \]

\[ Sp^G = [\mathcal{I}_G, \mathcal{T}_G] \leftarrow k^* \rightarrow [\widetilde{\mathcal{I}}_G, \mathcal{T}_G^+] \]

Now we proceed as follows.

(i) Start with the projective model structure on \([\widetilde{\mathcal{I}}_G^+, \mathcal{T}_G^+].\)
The main construction (continued)

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(i) Start with the projective model structure on $[\tilde{J}_G^+, T^G]$. It is equifibrant, while the projective model structure on $[J_G, T^G]$ is not.
The main construction (continued)

Now we proceed as follows.

(i) Start with the projective model structure on $[\hat{\mathcal{I}}_G^+, \mathcal{T}^G]$. It is equifibrant, while the projective model structure on $[\mathcal{I}_G, \mathcal{T}^G]$ is not.

(ii) The composite functor $i_! k^*_+ = k^* \tilde{i}_!$ is a left adjoint,
The main construction (continued)

\[
\begin{array}{cccccc}
\mathcal{I}_G^+, \mathcal{T}^G & \xleftarrow{k^*} & \tilde{\mathcal{I}}_G^+, \tilde{\mathcal{T}}^G \\
\downarrow i & & \downarrow \tilde{i} \\
\mathcal{I}_G & \xrightarrow{k} & \tilde{\mathcal{I}}_G \\
\downarrow & & \downarrow \\
\mathcal{J}_G & \xrightarrow{k} & \tilde{\mathcal{J}}_G \\
\end{array}
\]

\[Sp^G = [\mathcal{I}_G, \mathcal{T}^G] \xleftarrow{k^*} [\tilde{\mathcal{I}}_G, \tilde{\mathcal{T}}^G]\]

Now we proceed as follows.

(i) Start with the projective model structure on \([\tilde{\mathcal{I}}_G^+, \tilde{\mathcal{T}}^G]\). It is equifibrant, while the projective model structure on \([\mathcal{I}_G, \mathcal{T}^G]\) is not.

(ii) The composite functor \(i_! k^* = k^* \tilde{i}_!\) is a left adjoint, so we can use the Kan transfer theorem to get a model structure on \(Sp^G\).
The main construction (continued)

\[
\begin{array}{ccc}
\mathcal{I}_G^+, \mathcal{T}^G & \xleftarrow{k^*} & \mathcal{I}_G^+, \mathcal{T}^G \\
\mathcal{I}_G & \xrightarrow{k} & \mathcal{I}_G \\
\mathcal{J}_G & \xrightarrow{i} & \tilde{\mathcal{I}}_G \\
Sp^G = [\mathcal{I}_G, \mathcal{T}^G] & \xleftarrow{k^*} & [\tilde{\mathcal{I}}_G, \mathcal{T}^G] \\
\end{array}
\]

Now we proceed as follows.

(i) Start with the projective model structure on $[\tilde{\mathcal{I}}_G^+, \mathcal{T}^G]$. It is equifibrant, while the projective model structure on $[\mathcal{I}_G, \mathcal{T}^G]$ is not.

(ii) The composite functor $i_!k^* = k^*\tilde{i}_!$ is a left adjoint, so we can use the Kan transfer theorem to get a model structure on $Sp^G$. This transferred model structure is also equifibrant.
The main construction (continued)

Now we proceed as follows.

(i) Start with the projective model structure on $[\tilde{\mathcal{I}}_G^+, \mathcal{T}^G]$. It is equifibrant, while the projective model structure on $[\mathcal{I}_G, \mathcal{T}^G]$ is not.

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(iii) Expand the transferred class of weak equivalences on $Sp^G$ to that of stable equivalences.
The main construction (continued)

\[
\begin{align*}
\mathcal{J}_G^+, \mathcal{T}^G & \leftarrow k_+^* & \mathcal{J}_G^+, \mathcal{T}^G \\
\downarrow i \downarrow \downarrow i & \mathcal{J}_G^+, \mathcal{T}^G & \downarrow \downarrow \downarrow i \\
\mathcal{J}_G, \mathcal{T}^G & \leftarrow k^* & \mathcal{J}_G, \mathcal{T}^G \\
\end{align*}
\]

\[Sp^G = [\mathcal{J}_G, \mathcal{T}^G] \leftarrow k^* \mathcal{J}_G, \mathcal{T}^G \]

Now we proceed as follows.

(i) Start with the projective model structure on \(\mathcal{J}_G^+, \mathcal{T}^G\). It is equifibrant, while the projective model structure on \([\mathcal{J}_G, \mathcal{T}^G]\) is not.

(ii) The composite functor \(i_! k_+^* = k^* \tilde{i}_!\) is a left adjoint, so we can use the Kan transfer theorem to get a model structure on \(Sp^G\). This transferred model structure is also equifibrant.

(iii) Expand the transferred class of weak equivalences on \(Sp^G\) to that of stable equivalences and apply Bousfield localization.
Defining the four small categories

\[ \mathcal{I}_G \] is the Mandell-May category.

Mike Mandell

Peter May
Defining the four small categories

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Let \( O(V, W) \) be the (possibly empty) Stiefel manifold of isometric embeddings (which need not be equivariant) of \( V \) into \( W \).
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Let \( O(V, W) \) be the (possibly empty) Stiefel manifold of isometric embeddings (which need not be equivariant) of \( V \) into \( W \). For each such embedding \( f : V \leftrightarrow W \) one has the orthogonal compliment \( V^\perp \) of \( f(V) \) in \( W \),
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Let \( O(V, W) \) be the (possibly empty) Stiefel manifold of isometric embeddings (which need not be equivariant) of \( V \) into \( W \). For each such embedding \( f : V \leftrightarrow W \) one has the orthogonal compliment \( V^\perp \) of \( f(V) \) in \( W \), which is the fiber of our vector bundle over \( O(V, W) \).
Defining the four small categories (continued)

The morphism space $\mathcal{J}_G(V, W)$ is the Thom space of a certain vector bundle over the embedding space $O(V, W)$. 
Defining the four small categories (continued)

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The positive Mandell-May category $\mathcal{I}_G^+$ is the full subcategory of representations $V$ for which the invariant subspace $V^G$ is nontrivial.
Defining the four small categories (continued)

~\text{J}_G\text{is the equifibrant Mandell-May category. Its objects are finite dimensional orthogonal representations of finite }G\text{-sets. For a }G\text{-set }T\text{ there is a category }B_TG\text{ whose objects are the elements of }T\text{, and for each }\(t, \gamma\) \in T \times G\text{ there is a morphism that sends }t\text{ to }\gamma t\text{. This category is a split groupoid. A representation }V\text{ of }T\text{ is a functor from }B_TG\text{ to the category of finite dimensional real orthogonal vector spaces. If }T = G/H\text{, such a functor is equivalent to an orthogonal representation of }H\text{. In general for each orbit of }T\text{ we get a representation of its isotropy group.}
Defining the four small categories (continued)

\( \tilde{I}_G \) is the equifibrant Mandell-May category.
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Defining the four small categories (continued)

$\widetilde{\mathcal{J}}_G$ is the equifibrant Mandell-May category. Its objects are finite dimensional orthogonal representations of finite $G$-sets. For a $G$-set $T$ there is a category $\mathcal{B}_T G$ whose objects are the elements of $T$, and for each $(t, \gamma) \in T \times G$ there is a morphism that sends $t$ to $\gamma t$. 
Defining the four small categories (continued)

\( \widetilde{J}_G \) is the equifibrant Mandell-May category. Its objects are finite dimensional orthogonal representations of finite \( G \)-sets. For a \( G \)-set \( T \) there is a category \( BTG \) whose objects are the elements of \( T \), and for each \((t, \gamma) \in T \times G\) there is a morphism that sends \( t \) to \( \gamma t \). This category is a split groupoid.
Defining the four small categories (continued)

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If \(T = G/H\), such a functor is equivalent to an orthogonal representation of \(H\).
Defining the four small categories (continued)

\( \overline{\mathcal{J}}_G \) is the equifibrant Mandell-May category. Its objects are finite dimensional orthogonal representations of finite \( G \)-sets. For a \( G \)-set \( T \) there is a category \( B_T G \) whose objects are the elements of \( T \), and for each \((t, \gamma) \in T \times G\) there is a morphism that sends \( t \) to \( \gamma t \). This category is a split groupoid.

A representation \( V \) of \( T \) is a functor from \( B_T G \) to the category of finite dimensional real orthogonal vector spaces.

If \( T = G/H \), such a functor is equivalent to an orthogonal representation of \( H \). In general for each orbit of \( T \) we get a representation of its isotropy group.
Defining the four small categories (continued)

Recall that Mandell-May morphism objects involved orthogonal embeddings \( V \hookrightarrow W \).
Defining the four small categories (continued)

Recall that Mandell-May morphism objects involved orthogonal embeddings $V \leftrightarrow W$. An orthogonal embedding $f : (S, V) \rightarrow (T, W)$ consists of the following data.

• For each $t \in T$ an element $f(t) \in S$ such that $\dim V f(t) \leq \dim W t$.

• For each $t \in T$ an orthogonal embedding $f_t : V f(t) \rightarrow W t$.

We call the map $f : T \rightarrow S$ a choice. It need not be equivariant. We say the embedding $f$ is chosen by $f$.

For a given $(S, V)$ and $(T, W)$, there may be no choices. Such orthogonal embeddings can be composed in an obvious way. We denote the space of all such embeddings chosen by $f$ by $O((S, V), (T, W)) f$. It is a product of ordinary Stiefel manifolds.
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We denote the space of all such embeddings chosen by $\bar{f}$ by
Defining the four small categories (continued)

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- For each $t \in T$ an element $\tilde{f}(t) \in S$ such that $\dim V_{\tilde{f}(t)} \leq \dim W_t$.
- For each $t \in T$ an orthogonal embedding $f_t : V_{\tilde{f}(t)} \hookrightarrow W_t$.

We call the map $\tilde{f} : T \to S$ a choice. It need not be equivariant. We say the embedding $f$ is chosen by $\tilde{f}$. For a given $(S, V)$ and $(T, W)$, there may be no choices.

Such orthogonal embeddings can be composed in an obvious way.

We denote the space of all such embeddings chosen by $\tilde{f}$ by

$$O((S, V), (T, W))_{\tilde{f}}.$$
Defining the four small categories (continued)

Recall that Mandell-May morphism objects involved orthogonal embeddings \( V \hookrightarrow W \). An orthogonal embedding \( f : (S, V) \rightarrow (T, W) \) consists of the following data.

- For each \( t \in T \) an element \( \bar{f}(t) \in S \) such that \( \dim V_{\bar{f}(t)} \leq \dim W_t \).
- For each \( t \in T \) an orthogonal embedding \( f_t : V_{\bar{f}(t)} \hookrightarrow W_t \).

We call the map \( \bar{f} : T \rightarrow S \) a choice. It need not be equivariant. We say the embedding \( f \) is chosen by \( \bar{f} \). For a given \((S, V)\) and \((T, W)\), there may be no choices.

Such orthogonal embeddings can be composed in an obvious way.

We denote the space of all such embeddings chosen by \( \bar{f} \) by

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It is a product of ordinary Stiefel manifolds.
Defining the four small categories (continued)

Given an orthogonal embedding

\[(S, V) \xrightarrow{f} (T, W),\]
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the orthogonal complement $f^\perp$ of $f$ is the direct sum of the orthogonal complements of $f_t(V_{\tilde{f}(t)})$ in $W_t$. Using these direct sums as fibers, we get a vector bundle over the space $O((S, V), (T, W))_{\tilde{f}}$. 
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the orthogonal complement \(f^\perp\) of \(f\) is the direct sum of the orthogonal complements of \(f_t(V_{\bar{f}(t)})\) in \(W_t\). Using these direct sums as fibers, we get a vector bundle over the space \(O((S, V), (T, W))_{\bar{f}}\) of embeddings chosen by \(\bar{f}\). We denote its Thom space by

\[\widetilde{J}_G((S, V), (T, W))_{\bar{f}}.\]
Defining the four small categories (continued)

Given an orthogonal embedding

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It is a smash product of ordinary Mandell-May morphism spaces.
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It is a smash product of ordinary Mandell-May morphism spaces.

The morphism object in $\tilde{\mathcal{I}}_G$ is

$$\tilde{\mathcal{I}}_G((S, V), (T, W)) := \bigvee_{\bar{f}: T \rightarrow S} \tilde{\mathcal{I}}_G((S, V), (T, W))_{\bar{f}},$$
Defining the four small categories (continued)

Given an orthogonal embedding

$$(S, V) \xrightarrow{f} (T, W),$$

the orthogonal complement $f^\perp$ of $f$ is the direct sum of the orthogonal complements of $f_t(V_{\bar{f}(t)})$ in $W_t$. Using these direct sums as fibers, we get a vector bundle over the space $O((S, V), (T, W))_{\bar{f}}$ of embeddings chosen by $\bar{f}$. We denote its Thom space by

$$\tilde{\mathcal{I}}_G((S, V), (T, W))_{\bar{f}}.$$

It is a smash product of ordinary Mandell-May morphism spaces.

The morphism object in $\tilde{\mathcal{I}}_G$ is

$$\tilde{\mathcal{I}}_G((S, V), (T, W)) := \bigvee_{\bar{f}: T \rightarrow S} \tilde{\mathcal{I}}_G((S, V), (T, W))_{\bar{f}},$$

the one point union over all possible choices $\bar{f}$. 
Defining the four small categories (continued)

The morphism object in $\widetilde{\mathcal{I}}_G$ is

$$\widetilde{\mathcal{I}}_G((S, V), (T, W)) := \bigvee_{\tilde{f}: T \to S} \widetilde{\mathcal{I}}_G((S, V), (T, W))_{\tilde{f}},$$

the one point union over all possible choices.
The morphism object in $\widetilde{J}_G$ is

$$\widetilde{J}_G((S, V), (T, W)) := \bigvee_{\tilde{f}: T \to S} \widetilde{J}_G((S, V), (T, W))_{\tilde{f}},$$

the one point union over all possible choices.

This category is symmetric monoidal under Cartesian product,
Defining the four small categories (continued)

The morphism object in $\widetilde{\mathcal{I}}_G$ is

$$
\widetilde{\mathcal{I}}_G((S, V), (T, W)) := \bigvee_{\tilde{f}: T \to S} \widetilde{\mathcal{I}}_G((S, V), (T, W))_{\tilde{f}},
$$

the one point union over all possible choices.

This category is symmetric monoidal under Cartesian product, so the functor category $[\widetilde{\mathcal{I}}_G, \mathcal{T}^G]$ is closed symmetric monoidal by the Day Convolution Theorem.
Defining the four small categories (continued)

The morphism object in \( \tilde{\mathcal{I}}_G \) is

\[
\tilde{\mathcal{I}}_G((S, V), (T, W)) := \bigvee_{\tilde{f}: T \to S} \tilde{\mathcal{I}}_G((S, V), (T, W))_{\tilde{f}}
\]

the one point union over all possible choices.

This category is symmetric monoidal under Cartesian product, so the functor category \([\tilde{\mathcal{I}}_G, \mathcal{T}^G]\) is closed symmetric monoidal by the Day Convolution Theorem.

The ordinary Mandell-May category \( \mathcal{I}_G \) is the full subcategory of \( \tilde{\mathcal{I}}_G \) with objects of the form \((G/G, V)\).
Defining the four small categories (continued)

The morphism object in $\tilde{\mathcal{I}}_G$ is

$$\tilde{\mathcal{I}}_G((S, V), (T, W)) := \bigvee_{\tilde{f}: T \to S} \tilde{\mathcal{I}}_G((S, V), (T, W))_{\tilde{f}},$$

the one point union over all possible choices.

This category is symmetric monoidal under Cartesian product, so the functor category $[\tilde{\mathcal{I}}_G, T^G]$ is closed symmetric monoidal by the Day Convolution Theorem.

The ordinary Mandell-May category $\mathcal{I}_G$ is the full subcategory of $\tilde{\mathcal{I}}_G$ with objects of the form $(G/G, V)$.

The positive equifibrant Mandell-May category $\tilde{\mathcal{I}}_G^+$ is the full subcategory with objects $(T, V)$ in which the representation for each orbit of $T$ has a nontrivial invariant vector.
The main construction again

\[ \mathcal{J}_G^+; \mathcal{T}^G \leftarrow k^* \rightarrow \widetilde{\mathcal{J}}_G^+; \mathcal{T}^G \]

\[ \mathcal{J}_G \leftarrow i_! \rightarrow \widetilde{\mathcal{J}}_G \]

\[ \mathcal{J}_G \leftarrow k \rightarrow \widetilde{\mathcal{J}}_G \]

\[ S\mathcal{P}^G = [\mathcal{J}_G; \mathcal{T}^G] \leftarrow k^* \rightarrow [\widetilde{\mathcal{J}}_G; \mathcal{T}^G] \]
The main construction again

\[
\begin{array}{c}
\left[ \mathcal{I}_G^+, \mathcal{T}^G \right] & \xleftarrow{k^*} & \left[ \tilde{\mathcal{I}}_G^+, \tilde{T}^G \right] \\
\downarrow i & & \downarrow \tilde{i} \\
\mathcal{I}_G & \xrightarrow{k_+} & \tilde{\mathcal{I}}_G \\
\downarrow i & & \downarrow \tilde{i} \\
\mathcal{I}_G^+ & \xrightarrow{k_+} & \tilde{\mathcal{I}}_G^+ \\
\downarrow i & & \downarrow \tilde{i} \\
\mathcal{I}_G & \xrightarrow{k} & \tilde{\mathcal{I}}_G \\
\downarrow i & & \downarrow \tilde{i} \\
\mathcal{I}_G & \xrightarrow{k^*} & \tilde{\mathcal{I}}_G \\
\downarrow i & & \downarrow \tilde{i} \\
\mathcal{I}_G & \xrightarrow{k} & \tilde{\mathcal{I}}_G \\
\downarrow i & & \downarrow \tilde{i} \\
\mathcal{I}_G & \xrightarrow{k^*} & \tilde{\mathcal{I}}_G \\
\downarrow i & & \downarrow \tilde{i} \\
\mathcal{I}_G & \xrightarrow{k} & \tilde{\mathcal{I}}_G \\
\downarrow i & & \downarrow \tilde{i} \\
\mathcal{I}_G & \xrightarrow{k^*} & \tilde{\mathcal{I}}_G \\
\downarrow i & & \downarrow \tilde{i} \\
\mathcal{I}_G & \xrightarrow{k} & \tilde{\mathcal{I}}_G \\
\downarrow i & & \downarrow \tilde{i} \\
\mathcal{I}_G & \xrightarrow{k^*} & \tilde{\mathcal{I}}_G \\
\downarrow i & & \downarrow \tilde{i} \\
\mathcal{I}_G & \xrightarrow{k} & \tilde{\mathcal{I}}_G \\
\end{array}
\]

(i) Start with the projective model structure on \( [\mathcal{I}_G^+, \mathcal{T}^G] \).
The main construction again

\[ [\mathcal{I}_G^+, \mathcal{T}_G^+], \quad [\tilde{\mathcal{I}}_G^+, \mathcal{T}_G^+] \]

\[ \begin{array}{ccc}
\mathcal{I}_G & \xrightarrow{k_+} & \tilde{\mathcal{I}}_G \\
i & \downarrow & \downarrow \tilde{i} \\
\mathcal{I}_G & \xrightarrow{k} & \tilde{\mathcal{I}}_G \\
Sp^G = [\mathcal{I}_G, \mathcal{T}_G] & \xleftarrow{k^*} & [\tilde{\mathcal{I}}_G, \mathcal{T}_G] \\
\end{array} \]

(i) Start with the projective model structure on \([\tilde{\mathcal{I}}_G^+, \mathcal{T}_G^+]\).
(ii) Use Kan’s theorem to transfer it to a model structure on \(Sp^G\).
The main construction again

\[
[\mathcal{I}_G^+, \mathcal{T}_G^G] \xleftarrow{k^*} \xrightarrow{k_+} [\tilde{\mathcal{I}}_G^+, \mathcal{T}_G^G]
\]

\[
\begin{array}{ccc}
\mathcal{I}_G & \xrightarrow{k} & \tilde{\mathcal{I}}_G \\
 i & \downarrow & \tilde{i} \\
\mathcal{J}_G & \xrightarrow{k} & \tilde{\mathcal{J}}_G \\
\end{array}
\]

(i) Start with the projective model structure on \([\tilde{\mathcal{I}}_G^+, \mathcal{T}_G^G]\).

(ii) Use Kan’s theorem to transfer it to a model structure on \(Sp^G\). This is the positive equifibrant model structure.
The main construction again

(i) Start with the projective model structure on $[\mathcal{J}_G^+, \mathcal{T}_G]$.  
(ii) Use Kan’s theorem to transfer it to a model structure on $Sp^G$. This is the positive equifibrant model structure. 
(iii) Expand the class of weak equivalences on $Sp^G$ to that of stable equivalences.
The main construction again

(i) Start with the projective model structure on $[\mathcal{J}_G^+, \mathcal{T}^G]$.  
(ii) Use Kan’s theorem to transfer it to a model structure on $Sp^G$. This is the positive equifibrant model structure.  
(iii) Expand the class of weak equivalences on $Sp^G$ to that of stable equivalences and apply Bousfield localization.
The main construction again

(i) Start with the projective model structure on $[\mathcal{I}_G^+, \mathcal{T}^G]$.  
(ii) Use Kan’s theorem to transfer it to a model structure on $Sp^G$. This is the positive equifibrant model structure.  
(iii) Expand the class of weak equivalences on $Sp^G$ to that of stable equivalences and apply Bousfield localization. The result is the positive stable equifibrant model structure.
The main construction again

(i) Start with the projective model structure on $[\mathcal{I}_G^+, \mathcal{T}^G]$. 
(ii) Use Kan’s theorem to transfer it to a model structure on $Sp^G$. This is the positive equifibrant model structure.
(iii) Expand the class of weak equivalences on $Sp^G$ to that of stable equivalences and apply Bousfield localization. The result is the positive stable equifibrant model structure. The positivity condition enables us to define a model structure on the category of equivariant commutative ring spectra.
Model category structures for equivariant spectra

Mike Hill
Mike Hopkins
Doug Ravenel

Enriched category theory
Some equivariant homotopy theory
Three constructions of new model categories
The main construction
Defining the four small categories
Summary

THANK YOU!