String cobordism at the prime 3

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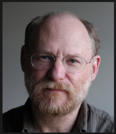

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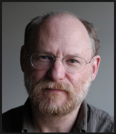

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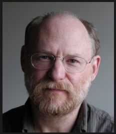

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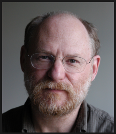

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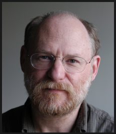

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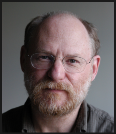

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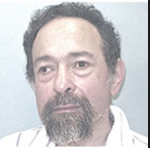

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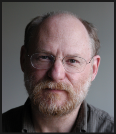

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THE 7-CONNECTED COBORDISM RING AT $p = 3$

MARK A. HOVEY AND DOUGLAS C. RAVENEL

ABSTRACT. In this paper, we study the cobordism spectrum $MO\langle 8\rangle$ at the prime 3. This spectrum is important because it is conjectured to play the role for elliptic cohomology that Spin cobordism plays for real K-theory. We show that the torsion is all killed by 3, and that the Adams-Novikov spectral sequence collapses after only 2 differentials. Many of our methods apply more generally.

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There is a corresponding splitting of the spectrum $MSO_{(2)}$ into a wedge of integer and mod 2 Eilenberg-Mac Lane spectra. The Adams spectral sequence for *MSO* collapses from E_2.

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It is easy to work out the Adams spectral sequence for the hypothetical spectrum X with

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where $v_n \in \pi_{2(2^n-1)}$ (in Adams filtration 1) is related to the generator of π_*BP of the same name. Recall that π_*bo has torsion in dimensions congruent to 1 and 2 modulo 8.

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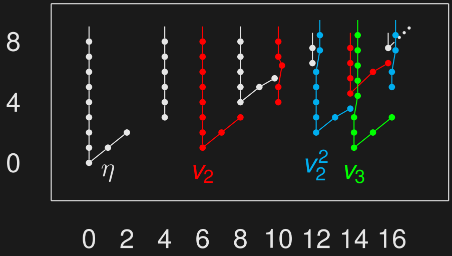

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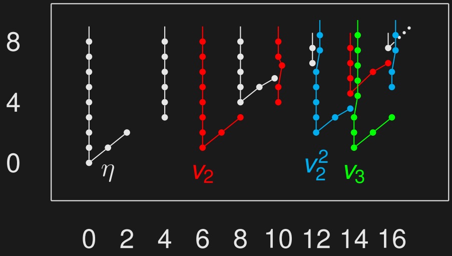

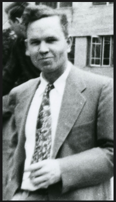

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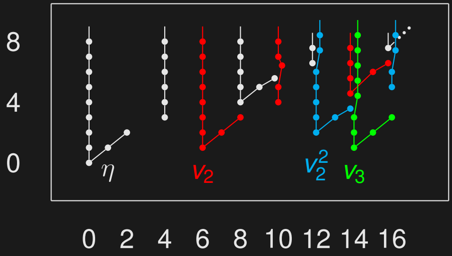

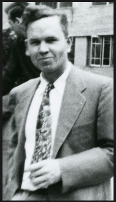

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We have seen that H_*MSU has an A_*-comodule summand isomorphic to

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The space $BO\langle 8\rangle_{(3)}$ is a Wilson space, meaning that is has both torsion free homology and torsion free homotopy. Such spaces are classified by Steve Wilson in a 1973 paper. Their homology groups are described in the 1977 "Hopf ring" paper of Wilson and the third author.

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- In particular, for such k, $BP\langle n\rangle_k$ is a factor of BP_k and of $BP\langle n'\rangle_k$ for each $n' > n$.

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- These two products make the graded space $E_\bullet$ into a graded ring object in the category of coalgebras, a Hopf ring. The star and circle products are related by the Hopf ring distributive law, in which they correspond respectively to addition and multiplication.

More history: Wilson spaces and Hopf rings (continued)

For $x \in \pi_m E$, we get an element

$$[x] \in H_0 E_{-m},$$

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$$H_{2k}\mathbf{C}P^\infty \ni \beta_k \longmapsto b_k \in H_{2k}E_2.$$

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where β_k is the usual generator of $H_{2k}\mathbf{C}P^\infty$. b_k is known to be decomposable under the star product when k is not a power of p.

More history: Wilson spaces and Hopf rings (continued)

We are interested in elements of the form

$$[v^I]b^J = [v_1^{i_1} \dots v_n^{i_n}]b_1^{j_0} b_p^{j_1} \dots \in H_{2m} BP\langle n\rangle_{2k}$$

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It is known that $H_* BP\langle n\rangle_{2k}$ for $k \leq e_n$ is generated by such elements as a ring under the star product,

More history: Wilson spaces and Hopf rings (continued)

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It is known that $H_*BP\langle n\rangle_{2k}$ for $k \leq e_n$ is generated by such elements as a ring under the star product, subject to the Hopf ring relation, which is related to the formal group law. For example, it implies that for each $t \geq 0$,

$$[v_1]b_{p^t}^p = -b_{p^t}^{*p} \in H_{2p^{t+1}}BP\langle n\rangle_2.$$

More history: Wilson spaces and Hopf rings (continued)

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More history: Wilson spaces and Hopf rings (continued)

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$$\begin{aligned}
y_8 &= b_1^4 \quad \text{with } y_8^3 = 0 \\
x_{12} &= b_1^3 b_3 & x_{16} &= b_1^2 b_3^2 \\
y_{20} &= b_1 b_3^3 \quad \text{with } y_{20}^3 = 0 \\
x_{24} &= b_1^3 b_9 & y_{24} &= b_3^4 - b_1^3 b_9 \quad \text{with } y_{24}^3 = 0 \\
x_{28} &= b_1^2 b_3 b_9 & x_{32} &= b_1 b_3^2 b_9 \\
&\vdots \\
x_{52} &= [v_1] b_1^2 b_3^2 b_9^2, \quad \text{the first appearance of } [v_1]
\end{aligned}$$

$H_*BO\langle 8\rangle$ and $H_*MO\langle 8\rangle$ (continued)

We find that

$$H_*BO\langle 8\rangle \cong P(x_{4m} : m \geq 3, 2m \neq 1 + 3^n) \otimes \Gamma(y_{2(1+3^n)} : n \geq 0),$$

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where $\Gamma(y)$ denotes the divided power algebra on y, which is dual to the polynomial algebra on the dual of y. For example,

$$\Gamma(y_8) \cong P(y_8, y_{24}, y_{72}, \dots)/(y_{8\cdot 3^i}^3),$$

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It is not hard to work out the right action of the mod 3 Steenrod algebra $\mathcal{A}$ on $H_*BO\langle 8\rangle$,

$H_*BO\langle 8\rangle$ and $H_*MO\langle 8\rangle$ (continued)

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Two change of rings isomorphisms

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$$\mathrm{Ext}_{\mathcal{E}_*}(\mathbf{Z}/3, \mathbf{Z}/3) \cong P(a_0, a_1, \dots) =: V.$$

Two change of rings isomorphisms

We want to study the 3-primary Adams spectral sequence for $MO\langle 8\rangle$. Recall that

$$\mathcal{A}_* \cong E(\tau_0, \tau_1, \ldots) \otimes P(\zeta_1, \zeta_2, \ldots),$$

with $|\tau_n| = 2\cdot 3^n - 1$ and $|\zeta_n| = 2\cdot 3^n - 2$. The dual of the subalgebra $\mathcal{P} \subseteq \mathcal{A}$ generated by the Steenrod reduced power operations is

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Here a_n corresponds to $v_n \in \pi_* BP$, where $v_0 = 3$. It has Adams filtration 1 and topological dimension $2(3^n - 1)$.

Two change of rings isomorphisms (continued)

There is a Cartan-Eilenberg spectral sequence converging to our Adams E_2-page with

$$E_1^{*,*,*} \cong \mathrm{Ext}_{\mathcal{P}_*}\left(\mathbf{Z}/3, \mathrm{Ext}_{\mathcal{E}_*}\left(\mathbf{Z}/3, H_*MO\langle 8\rangle\right)\right) \cong \mathrm{Ext}_{\mathcal{P}_*}\left(\mathbf{Z}/3, H_*MO\langle 8\rangle \otimes V\right). \tag{1}$$

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The coaction of $\mathcal{E}_*$ on $H_* MO\langle 8\rangle$ is trivial since the latter is concentrated in even dimensions.

Two change of rings isomorphisms (continued)

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The coaction of $\mathcal{E}_*$ on $H_*MO\langle 8\rangle$ is trivial since the latter is concentrated in even dimensions. This leads to the second isomorphism of (1).

Two change of rings isomorphisms (continued)

Let

$$J = (x_{12}^3, x_{16}^3, x_{52}, x_{160}, \dots) \subseteq H_* MO\langle 8\rangle,$$

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the change of rings ideal. One can show that

$$\operatorname{Ext}_{\mathcal{P}_*}(\mathbf{Z}/3, H_* MO\langle 8\rangle) \cong \operatorname{Ext}_{\mathcal{P}(1)_*}(\mathbf{Z}/3, H_* MO\langle 8\rangle/J),$$

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36 48 52 160

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is dual to the subalgebra $\mathcal{P}(1) \subseteq \mathcal{P}$ generated by the Steenrod operations P^1 and P^3. This is a major simplification.

Two change of rings isomorphisms (continued)

Recall

$$\mathrm{Ext}_{\mathcal{P}_*}(\mathbf{Z}/3, H_*MO\langle 8\rangle) \cong \mathrm{Ext}_{\mathcal{P}(1)_*}(\mathbf{Z}/3, L),$$

where $L = H_*MO\langle 8\rangle/J$ and $\mathcal{P}(1)_* = P(\zeta_1, \zeta_2)/(\zeta_1^9, \zeta_2^3)$.

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$$\mathrm{Ext}_{\mathcal{P}(1)_*}(\mathbf{Z}/3, L) \cong \mathrm{Ext}_{\mathcal{P}(1)'_*}(\mathbf{Z}/3, L'),$$

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where $\mathcal{P}(1)' = \mathcal{P}(1)/T$ is commutative with dual

$$\mathcal{P}(1)'_* = P(\zeta_1)/(\zeta_1^9),$$

and $L' \subseteq L$ is the subring on which Q acts trivially.

The Adams spectral sequence for $MO\langle 8\rangle$

Similarly in the Adams spectral sequence for $MO\langle 8\rangle$,

$$\begin{aligned} E_2 &= \mathrm{Ext}_{\mathcal{P}_*}(\mathbf{Z}/3, H_* MO\langle 8\rangle \otimes V) \\ &\cong \mathrm{Ext}_{\mathcal{P}(1)_*}(\mathbf{Z}/3, L \otimes V) \\ &\cong \mathrm{Ext}_{\mathcal{P}(1)'_*}(\mathbf{Z}/3, (L \otimes V)') \end{aligned}$$

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where $\mathcal{P}(1)'_* = P(\zeta_1)/\zeta_1^9$ and

$$(L \otimes V)' := \ker Q \subseteq L \otimes V.$$

The Adams spectral sequence for $MO\langle 8\rangle$ (continued)

Here is the first $P(1)'$-summand of L'.

$$
\begin{array}{ccccccc}
0 & & 12 & & 24 & & \\
1 & \xleftarrow[P^3]{-1} & x_{12} & \xleftarrow[P^3]{} & x_{12}^2 + \overline{y}_{24} & & \\
 & & \Big\downarrow{\scriptstyle P^1} & & \Big\downarrow{\scriptstyle P^1} & & \\
 & & y_8 & \xleftarrow[P^3]{} & \overline{y}_{20} - y_8 x_{12} & \xleftarrow[P^3]{-1} & x_{12}\overline{y}_{20} + y_8(x_{12}^2 - \overline{y}_{24}), \\
 & & 8 & & 20 & & 32
\end{array}
$$

The Adams spectral sequence for $MO\langle 8\rangle$ (continued)

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$$
\begin{array}{ccccccc}
{\color{red}0} & & {\color{red}12} & & {\color{red}24} & & \\
1 & \xleftarrow[P^3]{-1} & x_{12} & \xleftarrow[P^3]{} & x_{12}^2 + \overline{y}_{24} & & \\
 & & \big\downarrow P^1 & & \big\downarrow P^1 & & \\
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 & & {\color{red}8} & & {\color{red}20} & & {\color{red}32}
\end{array}
$$

where $\overline{y}_{20} = y_{20} + y_8 x_{12}$, and $\overline{y}_{24} = y_{24} - y_8 x_{16}$.

The Adams spectral sequence for $MO\langle 8\rangle$ (continued)

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$$
\begin{array}{ccccc}
0 & & 12 & & 24 \\
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 & & 8 & & 20 & & 32
\end{array}
$$

where $\overline{y}_{20} = y_{20} + y_8 x_{12}$, and $\overline{y}_{24} = y_{24} - y_8 x_{16}$. Here is the next one, which is free.

$$
\begin{array}{ccccc}
24 & & 36 & & 48 \\
\overline{y}_{24} & \xleftarrow{-1} & x_{12}\overline{y}_{24} + y_8^2\overline{y}_{20} & \longleftarrow & x_{12}^2\overline{y}_{24} \\
\downarrow & & \downarrow & & \downarrow \\
\overline{y}_{20} & \xleftarrow{-1} & x_{12}\overline{y}_{20} + y_8\overline{y}_{24} & \longleftarrow & x_{12}^2\overline{y}_{20} - y_8 x_{12}\overline{y}_{24} \\
\downarrow & & \downarrow & & \downarrow \\
y_8^2 & \xleftarrow{-1} & -y_8\overline{y}_{20} + y_8^2 x_{12} & \longleftarrow & y_8 x_{12}\overline{y}_{20} + y_8^2(x_{12}^2 - \overline{y}_{24}) \\
16 & & 28 & & 40
\end{array}
$$

Introduction

MSU at $p = 2$

Wilson spaces and Hopf rings

$H_* BO\langle 8\rangle$ and $H_* MO\langle 8\rangle$

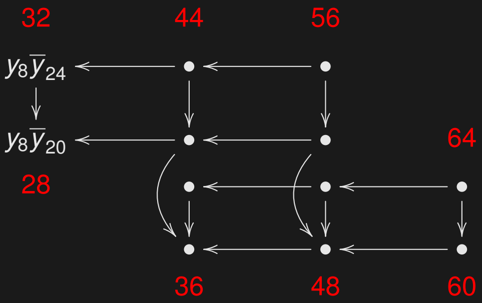
Two change of rings isomorphisms

The Adams spectral sequence for $MO\langle 8\rangle$

Carl McTague
Vitaly Lorman
Doug Ravenel

The Adams spectral sequence for $MO\langle 8\rangle$ (continued)

Here is a third one.



The Adams spectral sequence for $MO\langle 8\rangle$ (continued)

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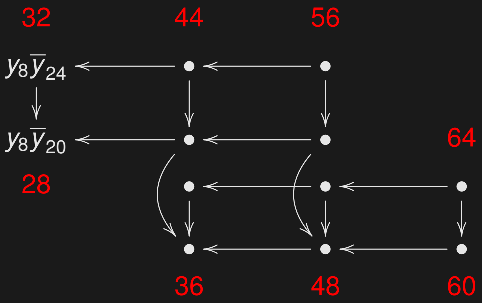

This one is isomorphic to the first one tensored with a rank 2 module in the first column.

The Adams spectral sequence for $MO\langle 8\rangle$ (continued)

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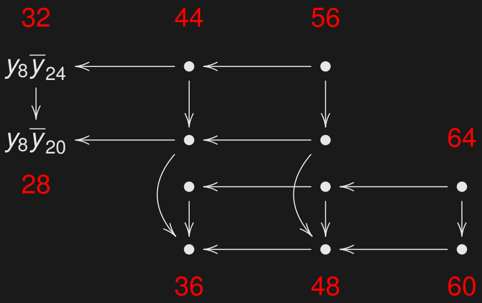

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In each case the Ext group is easy to compute.

The Adams spectral sequence for $MO\langle 8\rangle$ (continued)

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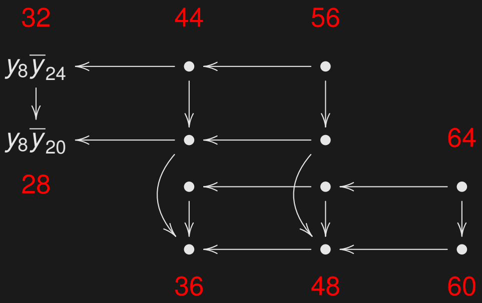

This one is isomorphic to the first one tensored with a rank 2 module in the first column.

In each case the Ext group is easy to compute. It turns out that both L' and $(L \otimes V)'$ decompose as a direct sum of $\mathcal{P}(1)'$-modules of these three types.

The Adams spectral sequence for $MO\langle 8\rangle$ (continued)

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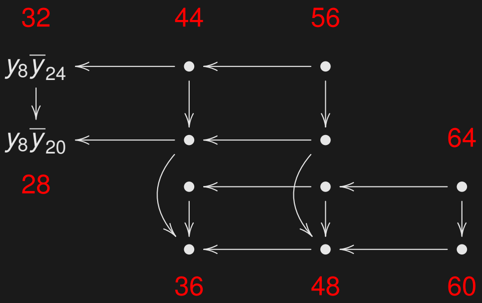
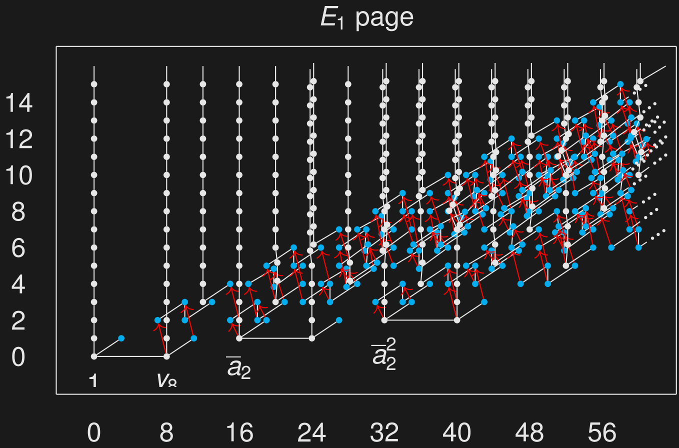


This one is isomorphic to the first one tensored with a rank 2 module in the first column.

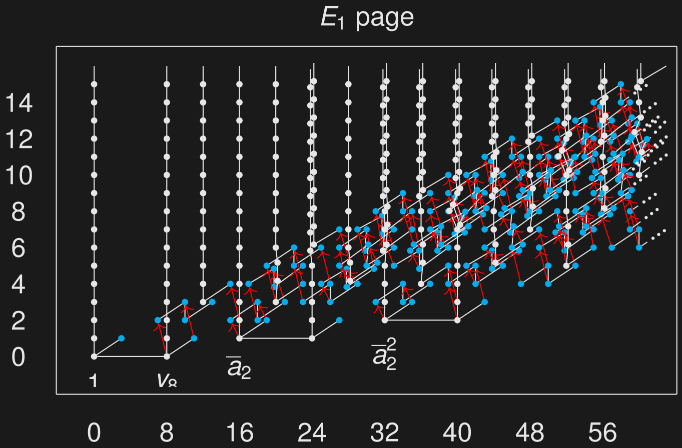
In each case the Ext group is easy to compute. It turns out that both L' and $(L \otimes V)'$ decompose as a direct sum of $\mathcal{P}(1)'$-modules of these three types. Each free summand of L' corresponds to summand of the spectrum $MO\langle 8\rangle$ equivalent to a suspension of BP.

Carl McTague
Vitaly Lorman
Doug Ravenel

The Adams spectral sequence for $MO\langle 8\rangle$ (continued)



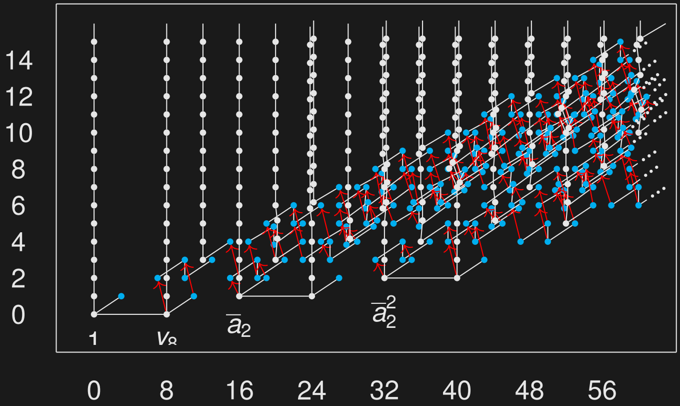
The Adams spectral sequence for $MO\langle 8\rangle$ (continued)



This chart shows Adams d_1s and d_2s in for the subalgebra of L' generated by y_8, x_{12}, $\overline{y}_{20}$ and $\overline{y}_{24}$.

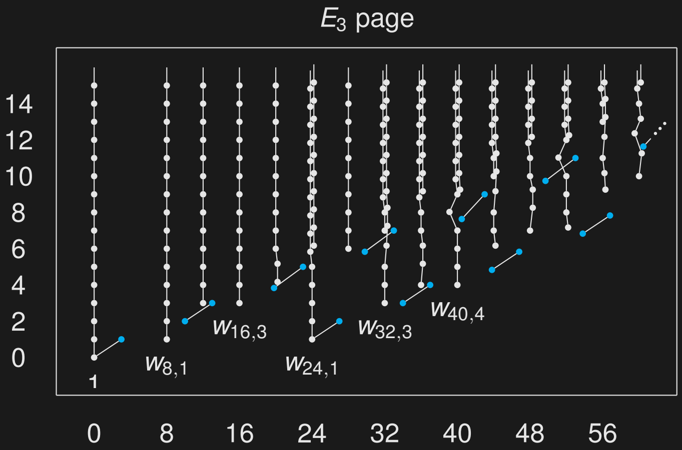
The Adams spectral sequence for $MO\langle 8\rangle$ (continued)

E_1 page

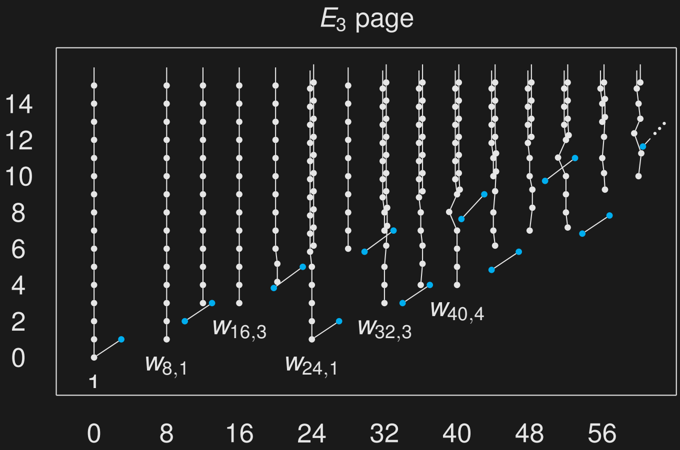


This chart shows Adams d_1s and d_2s in for the subalgebra of L' generated by y_8, x_{12}, $\overline{y}_{20}$ and $\overline{y}_{24}$. The 48-dimensional class $\overline{a}_2^3$ is exluded to avoid clutter.

The Adams spectral sequence for $MO\langle 8\rangle$ (continued)

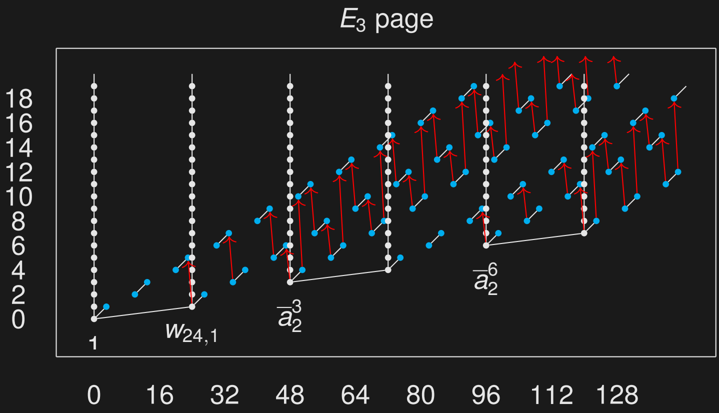


The Adams spectral sequence for $MO\langle 8\rangle$ (continued)



This chart shows the resulting E_3 page with torsion elements shown in blue.

The Adams spectral sequence for $MO\langle 8\rangle$ (continued)



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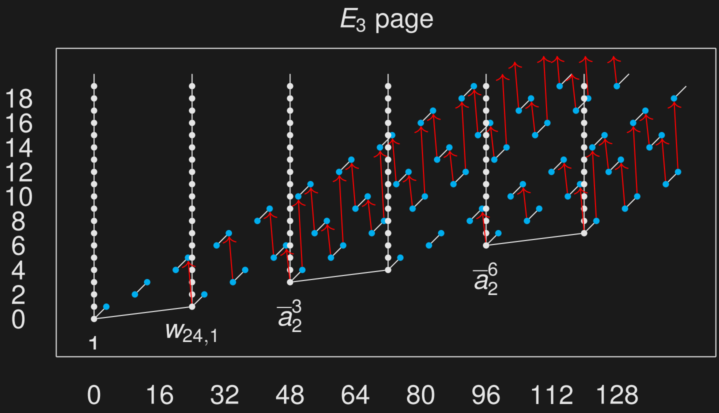
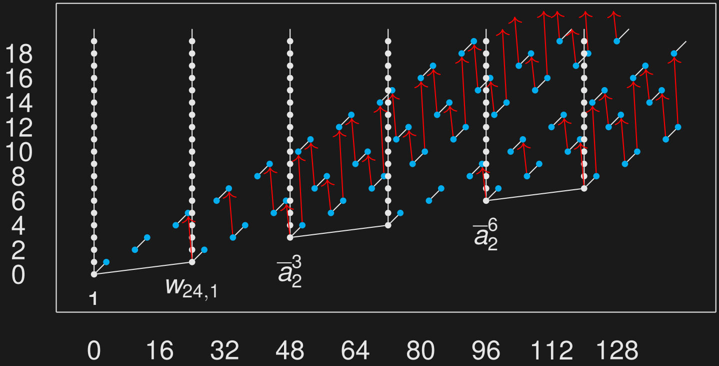


This is the previous chart with $\bar{a}_2^3$ tensored in.

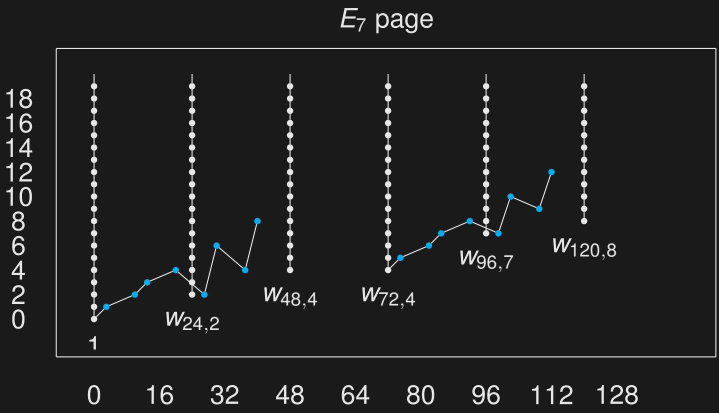
The Adams spectral sequence for $MO\langle 8\rangle$ (continued)

E_3 page



This is the previous chart with $\bar{a}_2^3$ tensored in. It shows a larger range of dimensions with higher Toda type differentials, with more elements removed to avoid clutter.

The Adams spectral sequence for $MO\langle 8\rangle$ (continued)



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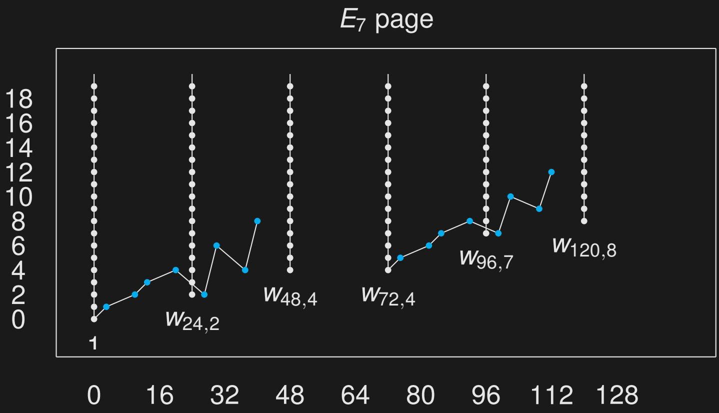
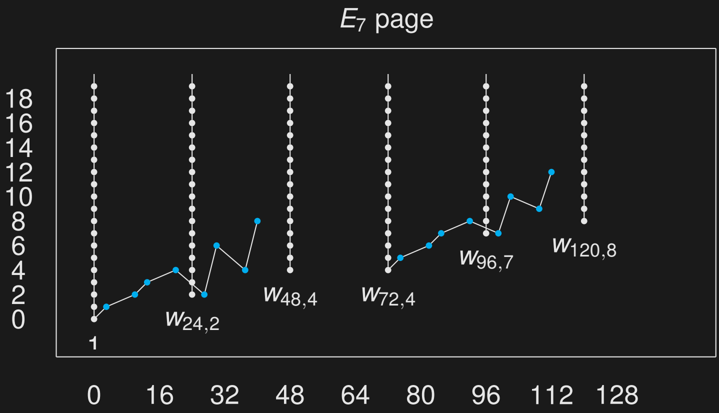


Thus shows the resulting E_∞ page with torsion elements in blue.

The Adams spectral sequence for $MO\langle 8\rangle$ (continued)



Thus shows the resulting E_∞ page with torsion elements in blue. They coincide with Dominic Culver's 2019 description of the 3-primary torsion in $\pi_* tmf$, which is 144-dimensional periodic.

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