Homotopy 2023
In Celebration of Paul Goerss

Northwestern University

Hiking in the Alps: C_p-fixed points of Lubin-Tate spectra

Doug Ravenel
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Fortunately Mark Behrens took some careful notes for us.

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For any closed subgroup $H \subseteq \mathbb{G}_n$, one also has a homotopy fixed point spectrum E_n^{hH} under $S^0_{K(n)}$. $\mathbb{G}_n$ is known to have a subgroup of order p when $p-1$ divides n. Our goal is to study $E_{(p-1)f}^{hC_p}$ for positive integers f.

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- The symbol ^ at the end denotes completion with respect to the maximal ideal $I_n = (p, u_1, \dots u_{n-1})$.

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In short, we start with a graded polynomial local ring, invert each of its specified generators, and then complete at its graded maximal ideal. We will come back to this later.

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The extended Morava stabilizer group $\mathbb{G}_n$ is related to the automorphism group $\mathbb{S}_n$ of the Honda height n formal group law F_n over $\mathbb{F}_{p^n}$.

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$$\mathrm{Gal}(\mathbb{F}_{p^n}, \mathbb{F}_p) \cong \mathrm{Gal}(W, \mathbb{Z}_p) \cong C_n.$$

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The main tool for computing the homotopy groups of the homotopy fixed point spectrum of E^{hG} for a group G acting on a spectrum E is the homotopy fixed point spectral sequence

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leading to the expected

$$H^i(C_p; \mathbb{Z}_p) = \begin{cases} \mathbb{Z}_p & \text{for } i = 0 \\ \mathbb{Z}/p & \text{for } i > 0 \text{ even} \\ 0 & \text{otherwise.} \end{cases}$$

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We will now describe $\pi_* E_n$ for $n = (p-1)f$ as a module over the group ring WC_p, where $W = W(\mathbb{F}_{p^n})$.

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For the case $H = C_p$, R_n is isomorphic as a WC_p-algebra to

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and that $\overline{\rho} \otimes \overline{\rho} \equiv \mathbb{Z}$.

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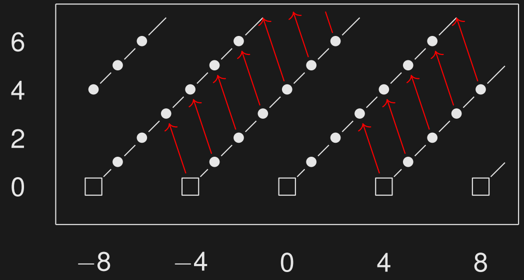
It follows that the E_2-term of the homotopy fixed point spectral sequence is

$$E_2^{s,t} = H^s(C_2; \pi_t E_2) = \begin{cases} \mathbb{Z}_2 & \text{for } s = 0 \text{ and } t \text{ divisible by } 4 \\ 0 & \text{for } s = 0 \text{ and } t \equiv 2 \bmod 4 \\ \mathbb{Z}/2 & \text{for } s > 0,\ t \text{ even,} \\ & \qquad \text{and } s \equiv t \bmod 2 \\ 0 & \text{otherwise.} \end{cases}$$

The homotopy fixed point spectral sequence for $\pi_* KO$



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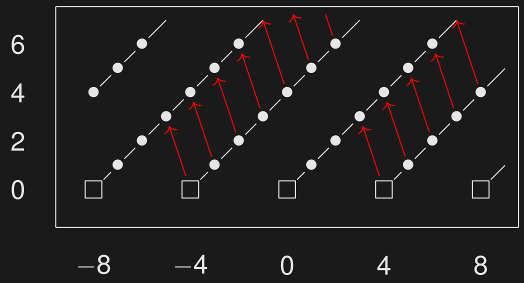

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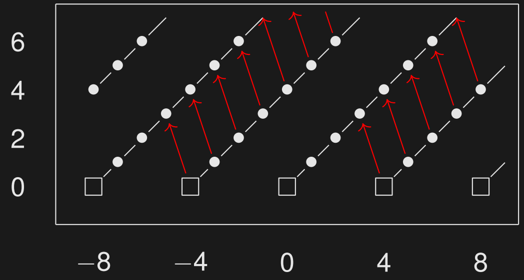

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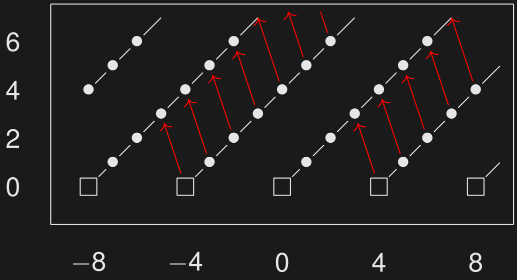

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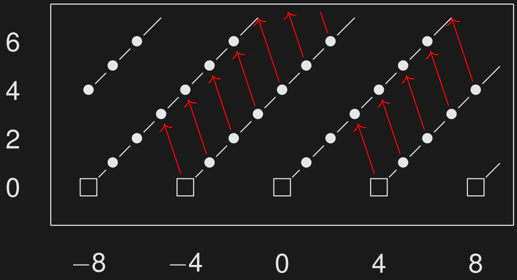

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The indicated d_3s can be established by equivariant methods, or by the requirement that the spectral sequence must converge to the known value of $\pi_* KO$.

TMF at $p = 3$

Here is the homotopy fixed point spectral sequence for $E_2^{hC_3}$

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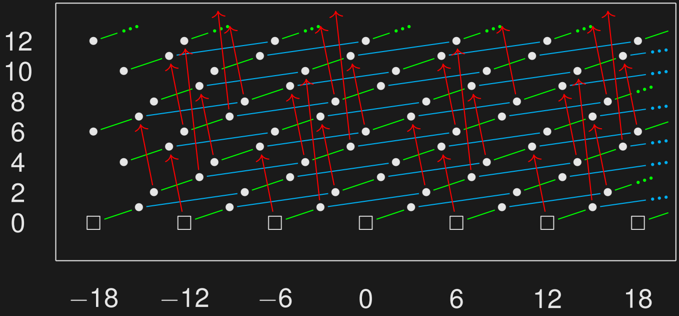

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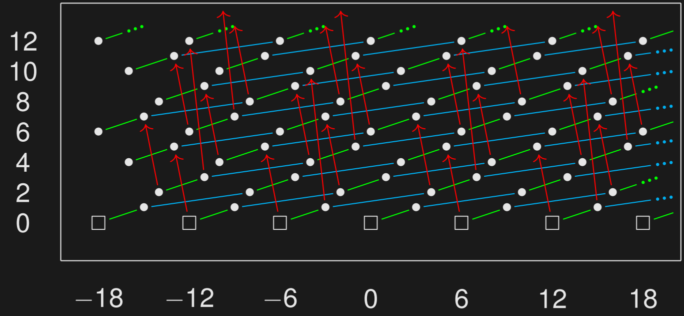

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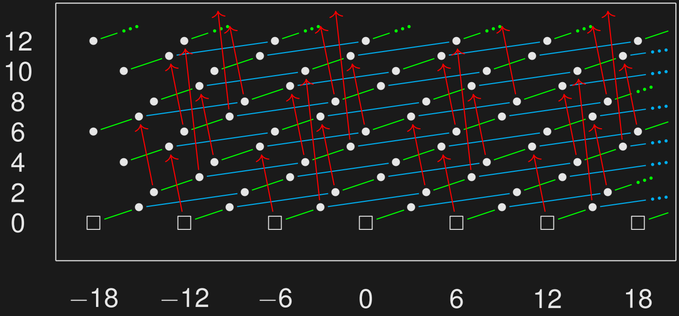

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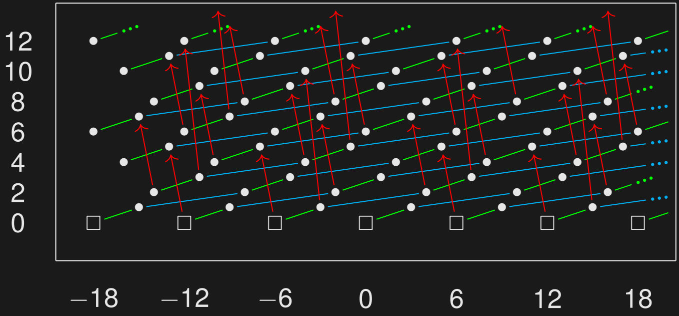

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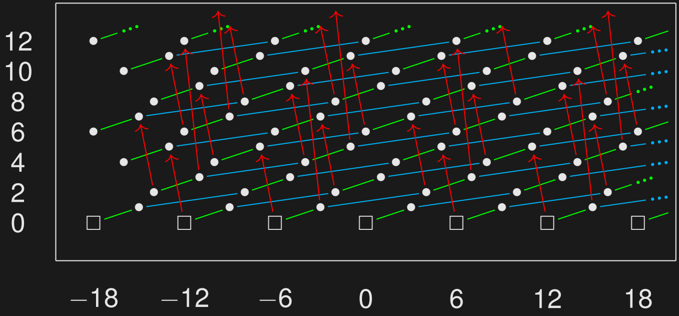

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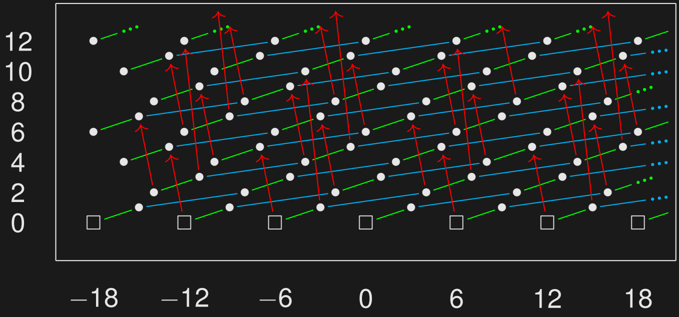

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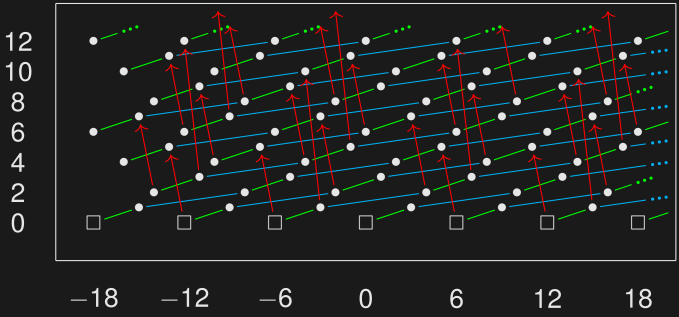

This pattern of differentials is 18-periodic.

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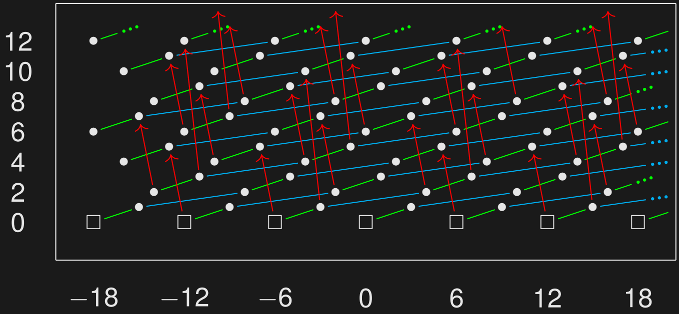

This pattern of differentials is 18-periodic. A comparable homotopy fixed point spectral sequence for *TMF* is 72-periodic.

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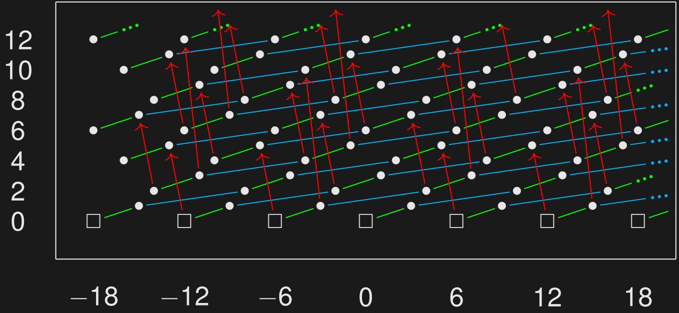

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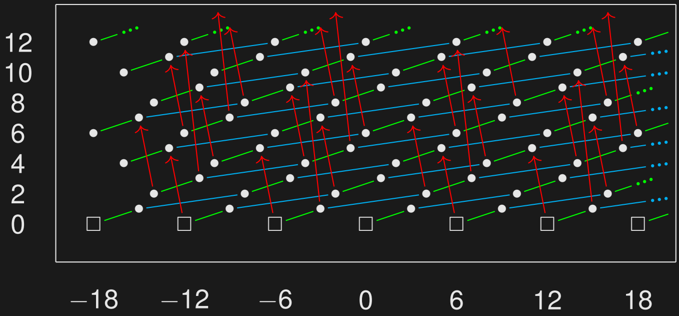

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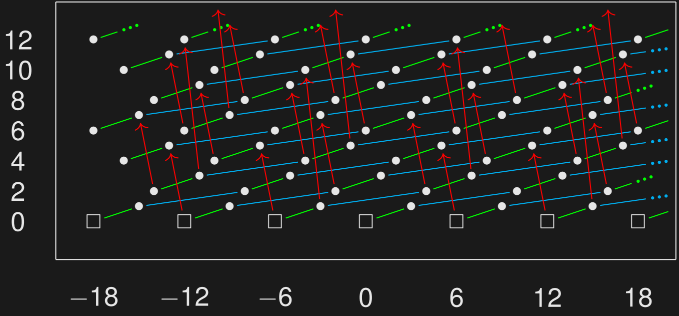

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Hiking in the Alps: C_p-fixed points of Lubin-Tate spectra

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Some group theory

In terms of the algebra $\mathrm{End}(F_2)$ at $p = 3$, let $\omega \in W$ be a primitive 8th root of unity, and $i = \omega^2$. Then we have a cube root of unity

$$\zeta = \frac{-1 - \omega F}{2} \quad \text{with} \quad i\zeta i^{-1} = \zeta^{-1} = \frac{-1 + \omega F}{2}.$$

Let $\phi \in \mathrm{Gal}(\mathbb{F}_9 : \mathbb{F}_3)$ be the Frobenius element. Then $\omega\phi$ commutes with ζ and has order 4. The group $\langle i, \omega\phi \rangle$ is isomorphic to Q_8, and the group $C_3 \rtimes Q_8$ is the group G_{24} of Goerss-Henn-Mahowald-Rezk.

TMF at $p = 3$ *(continued)*

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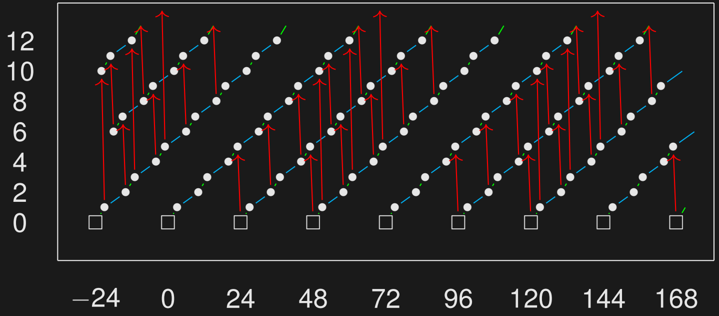

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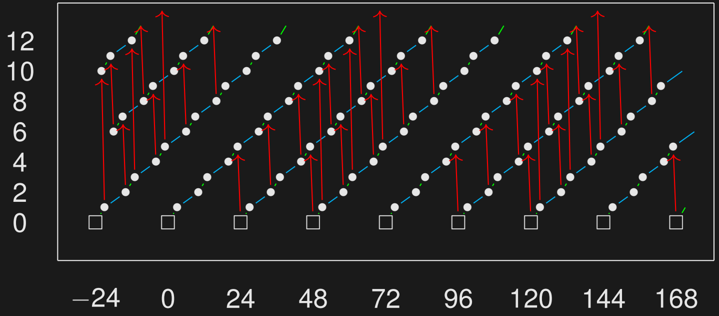

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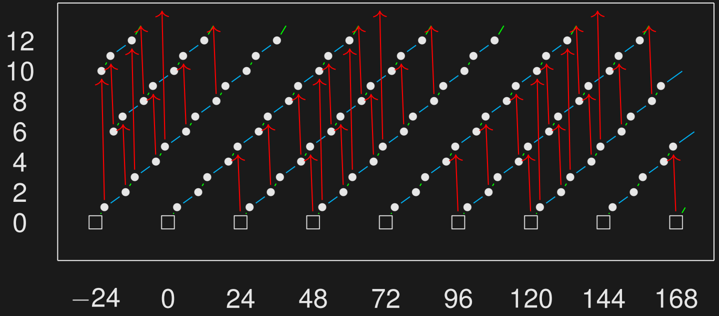

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x	β_1	$\beta_{3/3}$	β_4	$\beta_{6/3}$	$\beta_{9,9},\ \beta_7$	$\beta_{9/3,2}$	β_{10}
$\lvert x\rvert$	10	34	58	82	106	130	154

Doug Ravenel

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$$\alpha_1 \in E_2^{1,2p-2}, \quad \beta_1 \in E_2^{2,2p^2-2p}, \quad \text{and} \quad \Delta \in E_2^{0,2p(p-1)^2},$$

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Here are the dimensions of these elements for small primes.

p	$\lvert\alpha_1\rvert$	$\lvert\beta_1\rvert$	$\lvert\Delta\rvert$
3	3	10	24
5	7	38	160
7	11	82	504

Larger primes (continued)

In the homotopy fixed point spectral sequence for EO_p we have

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Then there are differentials

$$d_{2p-1}\Delta = \alpha_1\beta_1^{p-1} \quad \text{and} \quad d_{2(p-1)^2+1}(\alpha_1\Delta^{p-1}) = \beta_1^{(p-1)^2+1}.$$

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From the Adams-Novikov E_2-term for the sphere spectrum we have

$$\theta_j := \beta_{p^{j-1}/p^{j-1}} \mapsto \beta_1\Delta^{(p^{j-1}-1)/(p-1)} \quad \text{for all } j \geq 1,$$

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and for $p = 5$ only, we have

$$\gamma_3 \mapsto \alpha_1 \beta_1 \Delta^4 \qquad \text{in dimension 685.}$$

THANK YOU

and have a wonderful retirement, Paul!