Special Algebraic Topology Seminar

What is the telescope conjecture?

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1 Early 70s

1.1 Morava K-theory

Two developments in the early 70s

1. Morava K-theory

In the early 70's Jack Morava discovered the eponynumous spectra $K(n)$. I was lucky enough to spend a lot of time listening to him explain their inner workings.

$K(0)$ is rational cohomology. For each $n > 0$ and each prime p, there is a nonconnective complex oriented p-local spectrum $K(n)$ with

$$\pi_* K(n) = \mathbb{Z}/p[v_n^{\pm 1}] \qquad \text{where } |v_n| = 2(p^n - 1).$$

It is related to height n formal group laws, and $K(n)_*(K(n))$ is related to the Morava stabilizer group $\mathbb{G}_n$. It is a p-adic Lie group and the automorphism group of a height n formal group law over a suitable field of characteristic p.

1.2 Smith-Toda complexes

Two developments in the early 70s (continued)

2. Smith-Toda complexes

In 1973 Toda constructed the p-local finite spectrum $V(n)$, a CW-complex having 2^{n+1} cells with

$$BP_* V(n) = BP_*/(v_0 = p, v_1, \ldots v_n),$$

and a cofiber sequence

$$\Sigma^{|v_n|} V(n-1) \xrightarrow{w_n} V(n-1) \longrightarrow V(n)$$

for $1 \leq n \leq 3$ and $p \geq 2n+1$. We know that $K(n)_* V(n-1) \neq 0$ and w_n is a $K(n)$-equivalence. These lead to the construction of the v_n-periodic families aka Greek letter elements

$$\alpha_t \in \pi_{t|v_1|-1}\mathbb{S} \qquad \text{for } p \geq 3$$
$$\beta_t \in \pi_{t|v_2|-2p}\mathbb{S} \qquad \text{for } p \geq 5$$
$$\gamma_t \in \pi_{t|v_3|-2p^2-2p+1}\mathbb{S} \qquad \text{for } p \geq 7$$

2 Is there more?

2.1 Algebraic answer

Is there more?

The Adams-Novikov spectral sequence

$$\alpha_t \in \pi_{t|v_1|-1}\mathbb{S} \qquad \text{for } p \geq 3$$
$$\beta_t \in \pi_{t|v_2|-2p}\mathbb{S} \qquad \text{for } p \geq 5$$
$$\gamma_t \in \pi_{t|v_3|-2p^2-2p+1}\mathbb{S} \qquad \text{for } p \geq 7$$

These are nicely displayed in the E_2-term the Adams-Novikov spectral sequence. In it there are similar families for all n.

In 1977 Haynes Miller, Steve Wilson and I constructed the chromatic spectral sequence converging to the above E_2-term. It organizes things into layers so that in the nth layer everything is v_n-periodic. The structure of this nth layer is controlled by the cohomology of the nth Morava stabilizer group $\mathbb{G}_n$.

The chromatic filtration

Later we learned that the stable homotopy category itself is similarly organized. The key tool here is Bousfield localization, which conveniently appeared in 1978.

Let Sp denote the category of spectra. Given a homology theory E_*, Bousfield constructed an endofunctor $L_E : \mathrm{Sp} \to \mathrm{Sp}$ whose image category $L_E\mathrm{Sp}$ is stable homotopy as seen through the eyes of E-theory.

We are interested in the case $E = K(n)$. $L_{K(n)}\mathrm{Sp}$ is much easier to deal with than Sp itself. For example, we can compute $\pi_* L_{K(2)} V(1)$, but have no hope of computing $\pi_* V(1)$.

2.2 The Hopkins-Smith periodicity theorem

The Hopkins-Smith periodicity theorem

Recall the cofiber sequence

$$\Sigma^{|v_n|} V(n-1) \xrightarrow{w_n} V(n-1) \longrightarrow V(n)$$

for $1 \leq n \leq 3$ and $p \geq 2n+1$. Since $K(n)_* w_n$ is an isomorphism, all iterates of w_n are essential. This means that the homotopy colimit of the following is noncontractible.

$$V(n-1) \xrightarrow{w_n} \Sigma^{-|v_n|} V(n-1) \xrightarrow{w_n} \Sigma^{-2|v_n|} V(n-1) \xrightarrow{w_n} \dots$$

We call this the v_n-periodic telescope $w_n^{-1} V(n-1)$, often denoted by $T(n)$. The telescope conjecture says it is $L_{K(n)} V(n-1)$. The former is more closely related to the homotopy groups of spheres, while the latter is more computationally acccessible.

The Hopkins-Smith periodicity theorem (continued)

$$\Sigma^{|v_n|}V(n-1) \xrightarrow{w_n} V(n-1) \longrightarrow V(n)$$

Can we generalize this to $n > 3$? Not exactly.

However, in 1998 Mike Hopkins and Jeff Smith published the following.

Periodicity Theorem . *Let X be a p-local type n, finite spectrum, meaning that $K(n)_*X \neq 0$ and $K(m)_*X = 0$ for $m < n$. Then for some $d > 0$ (and divisible by $|v_n|$) there is a map*

$$w : \Sigma^d X \to X \qquad \text{where } K(n)_*w \text{ is an isomorphism.}$$

The Hopkins-Smith periodicity theorem (continued)

Periodicity Theorem . *Let X be a p-local type n finite spectrum, meaning that $K(n)_*X \neq 0$ and $K(m)_*X = 0$ for $m < n$. Then for some $d > 0$ (and divisible by $|v_n|$) there is a self-map*

$$w : \Sigma^d X \to X \qquad \text{where } K(n)_*w \text{ is an isomorphism.}$$

$V(n-1)$ is an early example of a finite spectrum of type n.

The theorem implies that the cofiber of w has type $n+1$. As before we can form a v_n-periodic telescope $w^{-1}X$. It is independent of the choice of w.

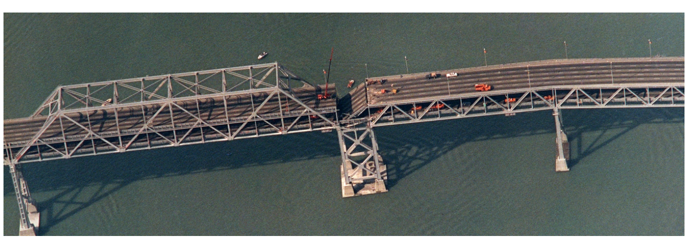
Again the telescope conjecture equates the geometrically appealing telescope $w^{-1}X$ with the computationally accessible Bousfield localization $L_{K(n)}X$.

3 The telescope conjecture

Historical note

When I stated the telescope conjecture in 1984, it was known to be true for $n = 0$ and $n = 1$. The latter is due to Mahowald for $p = 2$ and Miller for $p > 2$. Thus the statement for $n > 1$ seemed to be favored by Occam's Razor.

However, while I was visiting MSRI in 1989, something happened that led me to believe it is false for $n \geq 2$.



San Francisco earthquake of October 17, 1989

THANK YOU!