The HMS Equivariant *sailed proudly out of the harbor, newly fitted with Mackey functor rigging, Mandell-May sails, geometric fixed point guns, a Burnside ring navigational system, homotopy fixed point masts, and free action lifeboats.*

The journal of Captain Greenlees, 1729

The eightfold way: how to build the right model structure on orthogonal G-spectra

Mike Hill UCLA
Mike Hopkins Harvard University
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Equivariant Topology and Derived Algebra
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Norwegian University of Science and Technology
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If J is enriched over $\mathcal{M}$, each morphism object $J(j, k)$ is a more general object in $\mathcal{M}$.

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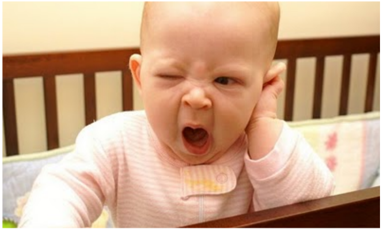

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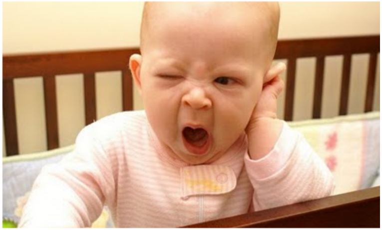

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WHY DO WE CARE ABOUT MODEL STRUCTURES ON FUNCTOR CATEGORIES?

Orthogonal G-spectra as functors

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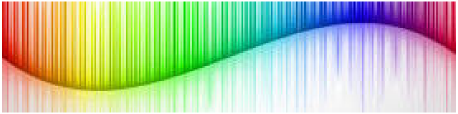

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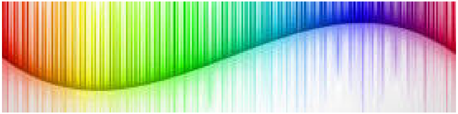

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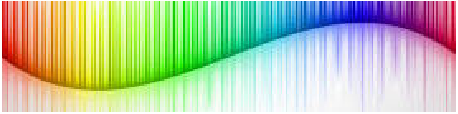

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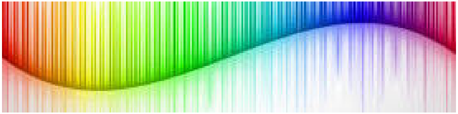

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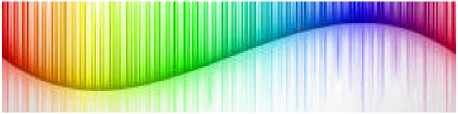

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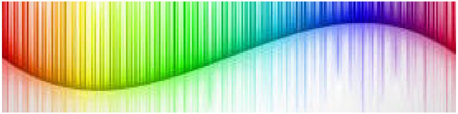

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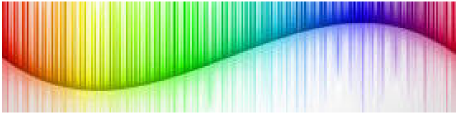

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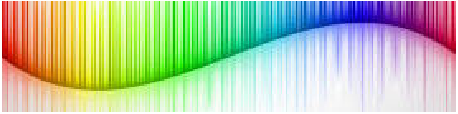

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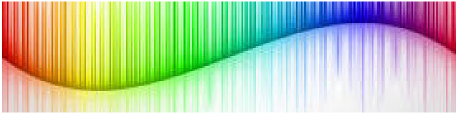

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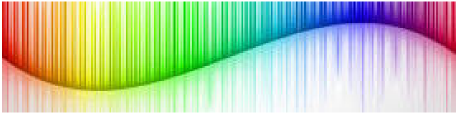

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The morphism object $\mathscr{J}_G(V, W)$ is the Thom space of the orthogonal complement vector bundle over the space $O(V, W)$ of (nonequivariant) orthogonal embeddings of V into W.

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Functor categories
Orthogonal G-spectra as functors
Modifying the model structure
The Crans-Kan transfer theorem
Equifibrant enlargement
Positivization
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where i_H^G is the restriction functor. It needs to be a Quillen adjunction. This means the class of cofibrations in $\mathcal{S}p^G$ needs to be enlarged to include cofibrations induced up from H. When we have this for each H, we say the model structure is equifibrant.

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Mike Hill
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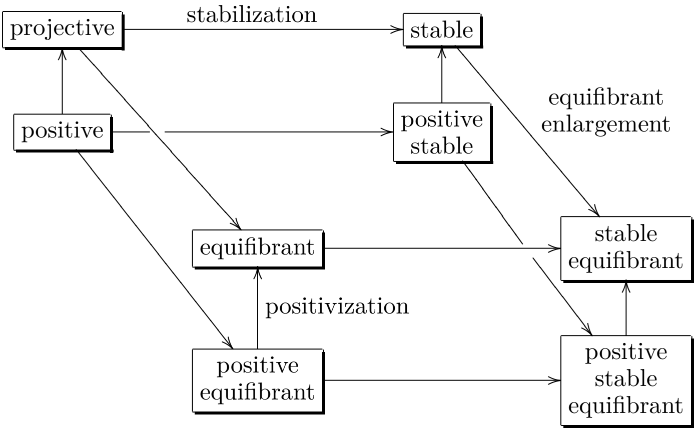
3. It needs to be positivized, a term to be defined later. This is needed to define a model structure on the category of commutative ring spectra. It involves confining the class of cofibrations in a certain way. The sphere spectrum S^{-0} will no longer be cofibrant.

Eight model structures for orthogonal G-spectra

Eight model structures for orthogonal *G*-spectra



projective

stabilization

stable

positive

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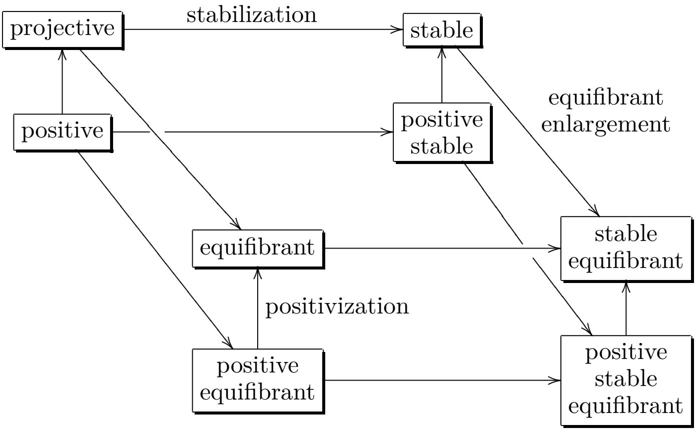

$$\begin{tikzcd}
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Each arrow denotes the identity functor as a left Quillen functor.

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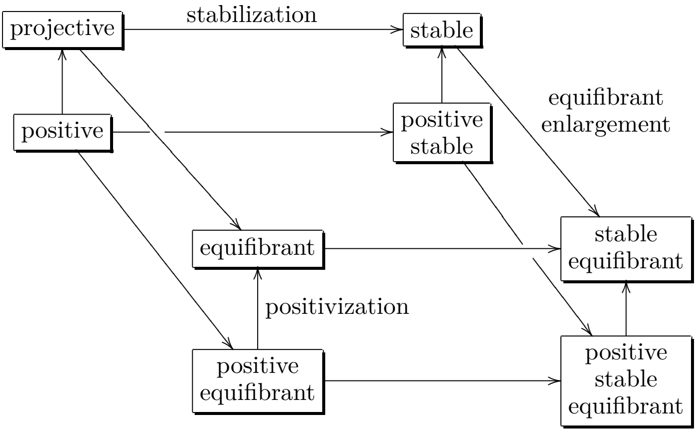

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1. both $F\mathcal{I}$ and $F\mathcal{J}$ permit the small object argument in $\mathcal{N}$ and
2. U takes relative $F\mathcal{J}$-cell complexes in $\mathcal{N}$ to weak equivalences in $\mathcal{M}$.

The Crans-Kan transfer theorem (continued)

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The Crans-Kan transfer theorem (continued)

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$$\begin{tikzcd} \mathcal{M} \arrow[r, shift left=2, "F"] \arrow[r, phantom, "\perp"] & \mathcal{N} \arrow[l, shift left=2, "U"] \end{tikzcd}$$

be a transfer adjunction as above. Then there is a cofibrantly generated model structure on $\mathcal{N}$ (the **transferred model structure**), for which $\mathcal{F}\mathcal{I}$ and $\mathcal{F}\mathcal{J}$ are cofibrant generating sets, and the weak equivalences and fibrations are the maps taken by U to weak equivalences and fibrations in $\mathcal{M}$. Furthermore, with respect to this model structure, (F, U) is a Quillen pair.

This is our main tool for constructing new model structures.

The Crans-Kan transfer theorem (continued)

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This is our main tool for constructing new model structures. Note that $\mathcal{N}$ does not have a model structure to begin with. It gets one though the adjunction.

Enlarging the class of cofibrations in a model category

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$$\mathcal{M} = \mathcal{S}p^G \qquad \text{and} \qquad \mathcal{M}' = \prod_{H\subset G} \mathcal{S}p^H.$$

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We call this process equifibrant enlargement. The resulting model structure plays nicely with the norm and with geometric fixed points.

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Back to functor categories (continued)

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Mike Hill
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Doug Ravenel

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We call this new model structure on $[J, \mathcal{M}]$ a confinement of the projective one.

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The positive model structure on $\mathcal{S}p^G$ is the one induced by the transfer adjunction

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A relevant example of confinement

We want to confine the projective model structure on the category of orthogonal G-spectra

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We call this type of confinement positivization.

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   For $V \neq 0$ this map is a stable equivalence but not a projective one.

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More about stabilization (continued)

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$$(RX)_V = \operatorname*{hocolim}_n \Omega^{n\rho} X_{V+n\rho}.$$

Happy Birthday John!
The Skye is the limit!