Complex Cobordism and Stable Homotopy Groups of Spheres

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To my wife, Michelle

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Preface to the first edition

My initial inclination was to call this book *The Music of the Spheres*, but I was dissuaded from doing so by my diligent publisher, who is ever mindful of the sensibilities of librarians. The purpose of this book is threefold: (i) to make BP-theory and the Adams–Novikov spectral sequence more accessible to nonexperts, (ii) to provide a convenient reference for workers in the field, and (iii) to demonstrate the computational potential of the indicated machinery for determining stable homotopy groups of spheres. The reader is presumed to have a working knowledge of algebraic topology and to be familiar with the basic concepts of homotopy theory. With this assumption the book is almost entirely self-contained, the major exceptions (e.g., Sections 5.4, 5.4, A1.4, and A1.5) being cases in which the proofs are long, technical, and adequately presented elsewhere.

The subject matter is a difficult one and this book will not change that fact. We hope that it will make it possible to learn the subject other than by the only practical method heretofore available, i.e., by numerous exhausting conversations with one of a handful of experts. Much of the material here has been previously published in journal articles too numerous to keep track of. However, a lot of the foundations of the subject, e.g., Chapter 2 and Appendix 1, have not been previously worked out in sufficient generality and the author found it surprisingly difficult to do so.

The reader (especially if she is a graduate student) should be warned that many portions of this volume contain more than he is likely to want or need to know. In view of (ii), results are given (e.g., in Sections 4.3, 6.3, and A1.4) in greater strengh than needed at present. We hope the newcomer to the field will not be discouraged by abundance of material.

The homotopy groups of spheres is a highly computational topic. The serious reader is strongly encouraged to reproduce and extend as many of the computations presented here as possible. There is no substitute for the insight gained by carrying out such calculations oneself.

Despite the large amount of information and techniques currently available, stable homotopy is still very mysterious. Each new computational breakthrough heightens our appreciation of the difficulty of the problem. The subject has a highly experimental character. One computes as many homotopy groups as possible with existing machinery, and the resulting data form the basis for new conjectures and new theorems, which may lead to better methods of computation. In contrast with physics, in this case the experimentalists who gather data and the theoreticians who interpret them are the same individuals.

The core of this volume is Chapters 2–6 while Chapter 1 is a casual nontechnical introduction to this material. Chapter 7 is a more technical description of actual computations of the Adams–Novikov spectral sequence for the stable homotopy groups of spheres through a large range of dimensions. Although it is likely to be read closely by only a few specialists, it is in some sense the justification for the rest of the book, the computational payoff. The results obtained there, along with some similar calculations of Tangora, are tabulated in Appendix 3.

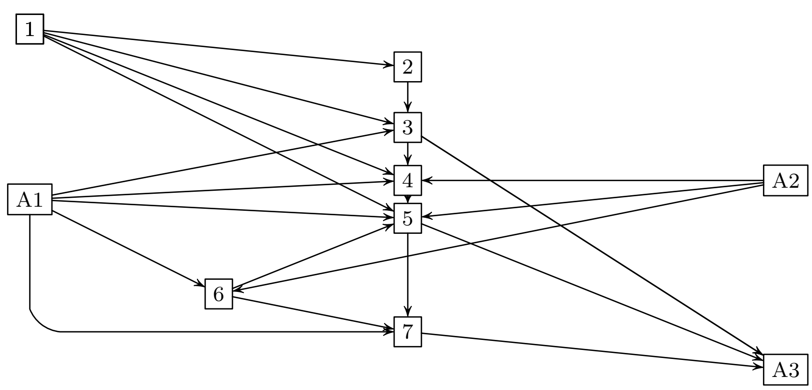
Appendices 1 and 2 are utilitarian in nature and describe technical tools used throughout the book. Appendix 1 develops the theory of Hopf algebroids (of which Hopf algebras are a special case) and useful homological tools such as relative injective resolutions, spectral sequences, Massey products, and algebraic Steenrod operations. It is not entertaining reading; we urge the reader to refer to it only when necessary.

Appendix 2 is a more enjoyable self-contained account of all that is needed from the theory of formal group laws. This material supports a bridge between stable homotopy theory and algebraic number theory. Certain results (e.g., the cohomology of some groups arising in number theory) are carried across this bridge in Chapter 6. The house they inhabit in homotopy theory, the chromatic spectral sequence, is built in Chapter 5.

The logical interdependence of the seven chapters and three appendixes is displayed in the accompanying diagram.

It is a pleasure to acknowledge help received from many sources in preparing this book. The author received invaluable editorial advice from Frank Adams, Peter May, David Pengelley, and Haynes Miller. Steven Mitchell, Austin Pearlman, and Bruce McQuistan made helpful comments on various stages of the manuscript, which owes its very existence to the patient work of innumerable typists at the University of Washington.

Finally, we acknowledge financial help from six sources: the National Science Foundation, the Alfred P. Sloan Foundation, the University of Washington, the Science Research Council of the United Kingdom, the Sonderforschungsbereich of Bonn, West Germany, and the Troisième Cycle of Bern, Switzerland.



Preface to the second edition

The subject of BP-theory has grown dramatically since the appearance of the first edition 17 years ago. One major development was the proof by Devinatz, Hopkins and Smith (see Devinatz, Hopkins and Smith [**2**] and Hopkins and Smith [**3**]) of nearly all the conjectures made in Ravenel [**8**]. An account of this work can be found in our book Ravenel [**13**]. The only conjecture of Ravenel [**8**] that remains is Telescope Conjecture. An account of our unsuccessful attempt to disprove it is given in Mahowald, Ravenel, and Shick [**1**].

Another big development is the emergence of elliptic cohomology and the theory of topological modular forms. There is still no comprehensive introduction to this topic. Some good papers to start with are Ando, Hopkins and Strickland [**1**], Hopkins and Mahowald [**2**], Landweber, Ravenel and Stong [**8**], and Rezk [**1**], which is an account of the still unpublished Hopkins-Miller theorem.

The seventh and final chapter of the book has been completely rewritten and is nearly twice as long as the original. We did this with an eye to carrying out future research in this area.

I am grateful to the many would be readers who urged me to republish this book and to the AMS for its assistance in getting the original manuscript retypeset. Peter Landweber was kind enough to provide me with a copious list of misprints he found in the first edition. Nori Minami and Igor Kriz helped in correcting some errors in § 4.3. Mike Hill and his fellow MIT students provided me with a timely list of typos in the online version of this edition. Hirofumi Nakai was very helpful in motivating me to make the revisions of Chapter 7.

Commonly Used Notations

$\mathbf{Z}$	Integers
$\mathbf{Z}_p$	p-agic integers
$\mathbf{Z}_{(p)}$	Integers localized at p
$\mathbf{Z}/(p)$	Integers $\mod p$
$\mathbf{Q}$	Rationals
$\mathbf{Q}_p$	p-adic numbers
$P(x)$	Polynomial algebra on generators x
$E(x)$	Exterior algebra on generators x
$\square$	Cotensor product (Section A1.1)

Given suitable objects A, B, and C and a map $f\colon A \to B$, the evident map $A \otimes C \to B \otimes C$ is denoted by $f \otimes C$.

CHAPTER 1

An Introduction to the Homotopy Groups of Spheres

This chapter is intended to be an expository introduction to the rest of the book. We will informally describe the spectral sequences of Adams and Novikov, which are the subject of the remaining chapters. Our aim here is to give a conceptual picture, suppressing as many technical details as possible.

In Section 1 we list some theorems which are classical in the sense that they do *not* require any of the machinery described in this book. These include the Hurewicz theorem 1.1.2, the Freudenthal suspension theorem 1.1.4, the Serre finiteness theorem 1.1.8, the Nishida nilpotence theorem 1.1.9, and the Cohen–Moore–Neisendorfer exponent theorem 1.1.10. They all pertain directly to the homotopy groups of spheres and are not treated elsewhere here. The homotopy groups of the stable orthogonal group SO are given by the Bott periodicity theorem 1.1.11. In 1.1.12 we define the J-homomorphism from $\pi_i(SO(n))$ to $\pi_{n+i}(S^n)$. Its image is given in 1.1.13, and in 1.1.14 we give its cokernel in low dimensions. Most of the former is proved in Section 5.3.

In Section 2 we describe Serre's method of computing homotopy groups using cohomological techniques. In particular, we show how to find the first element of order p in $\pi_*(S^3)$ 1.2.4. Then we explain how these methods were streamlined by Adams to give his celebrated spectral sequence 1.2.10. The next four theorems describe the Hopf invariant one problem. A table showing the Adams spectral sequence at the prime 2 through dimension 45 is given in 1.2.15. In Chapter 2 we give a more detailed account of how the spectral sequence is set up, including a convergence theorem. In Chapter 3 we make many calculations with it at the prime 2.

In 1.2.16 we summarize Adams' method for purposes of comparing it with that of Novikov. The basic idea is to use complex cobordism (1.2.17) in place of ordinary mod (p) cohomology. Fig. 1.2.19 is a table of the Adams–Novikov spectral sequence for comparison with Fig. 1.2.15.

In the next two sections we describe the algebra surrounding the E_2-term of the Adams–Novikov spectral sequence. To this end formal group laws are defined in 1.3.1 and a complete account of the relevant theory is given in Appendix 2. Their connection with complex cobordism is the subject of Quillen's theorem (1.3.4) and is described more fully in Section 4.1. The Adams–Novikov E_2-term is described in terms of formal group law theory (1.3.5) and as an Ext group over a certain Hopf algebra (1.3.6).

The rest of Section 3 is concerned with the Greek letter construction, a method of producing infinite periodic families of elements in the E_2-term and (in favorable cases) in the stable homotopy groups of spheres. The basic definitions are given in

1.3.17 and 1.3.19 and the main algebraic fact required is the Morava–Landweber theorem (1.3.16). Applications to homotopy are given in 1.3.11, 1.3.15, and 1.3.18. The section ends with a discussion of the proofs and possible extensions of these results. This material is discussed more fully in Chapter 5.

In Section 4 we describe the deeper algebraic properties of the E_2-term. We start by introducing BP and defining a Hopf algebroid. The former is a minimal wedge summand of MU localized at a prime. A Hopf algebroid is a generalized Hopf algebra needed to describe the Adams–Novikov E_2-term more conveniently in terms of BP (1.4.2). The algebraic and homological properties of such objects are the subject of Appendix 1.

Next we give the Lazard classification theorem for formal group laws (1.4.3) over an algebraically closed field of characteristic p, which is proved in Section A2.2. Then we come to Morava's point of view. Theorem 1.3.5 describes the Adams–Novikov E_2-term as the cohomology of a certain group G with coefficients in a certain polynomial ring L. $\mathrm{Spec}(L)$ (in the sense of abstract algebraic geometry) is an infinite dimensional affine space on which G acts. The points in $\mathrm{Spec}(L)$ can be thought of as formal group laws and the G-orbits as isomorphism classes, as described in 1.4.3. This orbit structure is described in 1.4.4. For each orbit there is a stabilizer or isotropy subgroup of G called S_n. Its cohomology is related to that of G (1.4.5), and its structure is known. The theory of Morava stabilizer algebras is the algebraic machinery needed to exploit this fact and is the subject of Chapter 6. Our next topic, the chromatic spectral sequence (1.4.8, the subject of Chapter 5), connects the theory above to the Adams–Novikov E_2-term. The Greek letter construction fits into this apparatus very neatly.

Section 5 is about unstable homotopy groups of spheres and is not needed for the rest of the book. Its introduction is self-explanatory.

1. Classical Theorems Old and New

We begin by recalling some definitions. The nth homotopy group of a connected space X, $\pi_n(X)$, is the set of homotopy classes of maps from the n-sphere S^n to X. This set has a natural group structure which is abelian for $n \geq 2$.

We now state three classical theorems about homotopy groups of spheres. Proofs can be found, for example, in Spanier [**1**].

1.1.1. THEOREM. *$\pi_1(S^1) = \mathbf{Z}$ and $\pi_m(S^1) = 0$ for $m > 1$.* □

1.1.2. HUREWICZ'S THEOREM. *$\pi_n(S^n) = \mathbf{Z}$ and $\pi_m(S^n) = 0$ for $m < n$. A generator of $\pi_n(S^n)$ is the class of the identity map.* □

For the next theorem we need to define the suspension homomorphism $\sigma\colon \pi_m(S^n) \to \pi_{m+1}(S^{n+1})$.

1.1.3. DEFINITION. *The kth suspension $\Sigma^k X$ of a space X is the quotient of $I^k \times X$ obtained by collapsing $\partial I^k \times X$ onto ∂I^k, ∂I^k being the boundary of I^k, the k-dimensional cube. Note that $\Sigma^i\Sigma^j X = \Sigma^{i+j}X$ and $\Sigma^k f\colon \Sigma^k X \to \Sigma^k Y$ is the quotient of $1 \times f\colon I^k \times X \to I^k \times Y$. In particular, given $f\colon S^m \to S^n$ we have $\Sigma f\colon S^{m+1} \to S^{n+1}$, which induces a homomorphism $\pi_m(S^n) \to \pi_{m+1}(S^{n+1})$.* □

1.1.4. FREUDENTHAL SUSPENSION THEOREM. *The suspension homomorphism $\sigma\colon \pi_{n+k}(S^n) \to \pi_{n+k+1}(S^{n+q})$ defined above is an isomorphism for $k < n-1$ and a surjection for $k = n-1$.* □

1.1.5. Corollary. *The group $\pi_{n+k}(S^n)$ depends only on k if $n > k + 1$.* □

1.1.6. Definition. *The stable k-stem or kth stable homotopy group of spheres π_k^S is $\pi_{n+k}(S^n)$ for $n > k + 1$. The groups $\pi_{n+k}(S^n)$ are called stable if $n > k+1$ and unstable if $n \leq k + 1$. When discussing stable groups we will not make any notational distinction between a map and its suspensions.* □

The subsequent chapters of this book will be concerned with machinery for computing the stable homotopy groups of spheres. Most of the time we will not be concerned with unstable groups. The groups π_k^S are known at least for $k \leq 45$. See the tables in Appendix 3, along with Theorem 1.1.13. Here is a table of π_k^S for $k \leq 15$:

k	0	1	2	3	4	5	6	7	8
π_k^S	$\mathbf{Z}$	$\mathbf{Z}/(2)$	$\mathbf{Z}/(2)$	$\mathbf{Z}/(24)$	0	0	$\mathbf{Z}/(2)$	$\mathbf{Z}/(240)$	$(\mathbf{Z}/(2))^2$

k	9	10	11	12	13	14	15
π_k^S	$(\mathbf{Z}/2)^3$	$\mathbf{Z}/6$	$\mathbf{Z}/(504)$	0	$\mathbf{Z}/(3)$	$(\mathbf{Z}/(2))^2$	$\mathbf{Z}/(480) \oplus \mathbf{Z}/(2)$

This should convince the reader that the groups do not fall into any obvious pattern. Later in the book, however, we will present evidence of some deep patterns not apparent in such a small amount of data. The nature of these patterns will be discussed later in this chapter.

When homotopy groups were first defined by Hurewicz in 1935 it was hoped that $\pi_{n+k}(S^n) = 0$ for $k > 0$, since this was already known to be the case for $n = 1$ (1.1.1). The first counterexample is worth examining in some detail.

1.1.7. Example. $\pi_3(S^2) = \mathbf{Z}$ generated by the class of the Hopf map $\eta\colon S^3 \to S^2$ defined as follows. Regard S^2 (as Riemann did) as the complex numbers $\mathbf{C}$ with a point at infinity. S^3 is by definition the set of unit vectors in $\mathbf{R}^4 = \mathbf{C}^2$. Hence a point in S^3 is specified by two complex coordinates (z_1, z_2). Define η by

$$\eta(z_1, z_2) = \begin{cases} z_1/z_2 & \text{if } z_2 \neq 0 \\ \infty & \text{if } z_2 = 0. \end{cases}$$

It is easy to verify that η is continuous. The inverse image under η of any point in S^2 is a circle, specifically the set of unit vectors in a complex line through the origin in $\mathbf{C}^2$, the set of all such lines being parameterized by S^2. Closer examination will show that any two of these circles in S^3 are linked. One can use quaternions and Cayley numbers in similar ways to obtain maps $\nu\colon S^7 \to S^4$ and $\sigma\colon S^{15} \to S^8$, respectively. Both of these represent generators of infinite cyclic summands. These three maps (η, ν, and σ) were all discovered by Hopf [1] and are therefore known as the Hopf maps.

We will now state some other general theorems of more recent vintage.

1.1.8. Finiteness Theorem (Serre [3]). *$\pi_{n+k}(S^n)$ is finite for $k > 0$ except when $n = 2m$, $k = 2m - 1$, and $\pi_{4m-1}(S^{2m}) = \mathbf{Z} \oplus F_m$, where F_m is finite.* □

The next theorem concerns the ring structure of $\pi_*^S = \bigoplus_{k\geq 0} \pi_k^S$ which is induced by composition as follows. Let $\alpha \in \pi_i^S$ and $\beta \in \pi_j^S$ be represented by $f\colon S^{n+i} \to S^n$ and $g\colon S^{n+i+j} \to S^{n+i}$, respectively, where n is large. Then

$\alpha\beta \in \pi_{i+j}^S$ is defined to be the class represented by $f \cdot g \colon S^{n+i+j} \to S^n$. It can be shown that $\beta\alpha = (-1)^{ij}\alpha\beta$, so π_*^S is an anticommutative graded ring.

1.1.9. Nilpotence Theorem (Nishida [**1**]). *Each element $\alpha \in \pi_k^S$ for $k > 0$ is nilpotent, i.e., $\alpha^t = 0$ for some finite t.* □

For the next result recall that 1.1.8 says $\pi_{2i+1+j}(S^{2i+1})$ is a finite abelian group for all $j > 0$.

1.1.10. Exponent Theorem (Cohen, Moore, and Neisendorfer [**1**]). *For $p \geq 5$ the p-component of $\pi_{2i+1+j}(S^{2i+1})$ has exponent p^i, i.e., each element in it has order $\leq p^i$.* □

This result is also true for $p = 3$ (Neisendorfer [**1**]) as well, but is known to be false for $p = 2$. For example, the 2-component of 3-stem is cyclic of order 4 (see Fig. 3.3.18) on S^3 and of order 8 on S^8 (see Fig. 3.3.10). It is also known (Gray [**1**]) to be the best possible, i.e., $\pi_{2i+1+j}(S^{2i+1})$ is known to contain elements of order p^i for certain j.

We now describe an interesting subgroup of π_*^S, the image of the Hopf–Whitehead J-homomorphism, to be defined below. Let $SO(n)$ be the space of $n \times n$ special orthogonal matrices over $\mathbf{R}$ with the standard topology. $SO(n)$ is a subspace of $SO(n+1)$ and we denote $\bigcup_{n>0} SO(n)$ by SO, known as the stable orthogonal group. It can be shown that $\pi_i(SO) = \pi_i(SO(n))$ if $n > i+1$. The following result of Bott is one of the most remarkable in all of topology.

1.1.11. Bott Periodicity Theorem (Bott [**1**]; see also Milnor [**1**]).

$$\pi_i(SO) = \begin{cases} \mathbf{Z} & \text{if } i \equiv -1 \mod 4 \\ \mathbf{Z}/(2) & \text{if } i = 0 \text{ or } 1 \mod 8 \\ 0 & \text{otherwise.} \end{cases} \qquad \square$$

We will now define a homomorphism $J \colon \pi_i(SO(n)) \to \pi_{n+i}(S^n)$. Let $\alpha \in \pi_i(SO(n))$ be the class of $f \colon S^i \to SO(n)$. Let D^n be the n-dimensional disc, i.e., the unit ball in $\mathbf{R}^n$. A matrix in $SO(n)$ defines a linear homeomorphism of D^n to itself. We define $\hat{f} \colon S^i \times D^n \to D^n$ by $\hat{f}(x,y) = f(x)(y)$, where $x \in S^i$, $y \in D^n$, and $f(x) \in SO(n)$. Next observe that S^n is the quotient of D^n obtained by collapsing its boundary S^{n-1} to a single point, so there is a map $p \colon D^n \to S^n$, which sends the boundary to the base point. Also observe that S^{n+i}, being homeomorphic to the boundary of $D^{i+1} \times D^n$, is the union of $S^i \times D^n$ and $D^{i+1} \times S^{n-1}$ along their common boundary $S^i \times S^{n-1}$. We define $\tilde{f} \colon S^{n+i} \to S^n$ to be the extension of $p\hat{f} \colon S^i \times D^n \to S^n$ to S^{n+i} which sends the rest of S^{n+i} to the base point in S^n.

1.1.12. Definition. *The Hopf–Whitehead J-homomorphism $J \colon \pi_i(SO(n)) \to \pi_{n+i}(S^n)$ sends the class of $f \colon S^i \to SO(n)$ to the class of $\tilde{f} \colon S^{n+i} \to S^n$ as described above.* □

We leave it to the skeptical reader to verify that the above construction actually gives us a homomorphism.

Note that both $\pi_i(SO(n))$ and $\pi_{n+i}(S^n)$ are stable, i.e., independent of n, if $n > i+1$. Hence we have $J \colon \pi_k(SO) \to \pi_k^S$. We will now describe its image.

1.1.13. Theorem (Adams [**1**] and Quillen [**1**]). *$J \colon \pi_k(SO) \to \pi_k^S$ is a monomorphism for $k \equiv 0$ or $1 \mod 8$ and $J(\pi_{4k-1}(SO))$ is a cyclic group whose 2-component* *is* $\mathbf{Z}_{(2)}/(8k)$ *and whose* p*-component for* $p \geq 3$ *is* $\mathbf{Z}_{(p)}/(pk)$ *if* $(p-1) \mid 2k$ *and* 0 *if* $(p-1) \nmid 2k$*, where* $\mathbf{Z}_{(p)}$ *denotes the integers localized at* p*. In dimensions* 1*,* 3*, and* 7*,* $\operatorname{im} J$ *is generated by the Hopf maps* (1.1.7) η*,* ν*, and* σ*, respectively. If we denote by* x_k *the generator in dimension* $4k-1$*, then* ηx_{2k} *and* $\eta^2 x_{2k}$ *are the generators of* $\operatorname{im} J$ *in dimensions* $8k$ *and* $8k+1$*, respectively.* □

The image of J is also known to a direct summand; a proof can be found for example at the end of Chapter 19 of Switzer [**1**]. The order of $J(\pi_{4k-1}(SO))$ was determined by Adams up to a factor of two, and he showed that the remaining ambiguity could be resolved by proving the celebrated Adams conjecture, which Quillen and others did. Denote this number by a_k. Its first few values are tabulated here.

k	1	2	3	4	5	6	7	8	9	10
a_k	24	240	504	480	264	65,520	24	16,320	28,728	13,200

The number a_k has interesting number theoretic properties. It is the denominator of $B_k/4k$, where B_k, is the kth Bernoulli number, and it is the greatest common divisor of numbers $n^{t(n)}(n^{2k}-1)$ for $n \in \mathbf{Z}$ and $t(n)$ sufficiently large. See Adams [**1**] and Milnor and Stasheff [**5**] for details.

Having determined $\operatorname{im} J$, one would like to know something systematic about $\operatorname{coker} J$, i.e., something more than its structure through a finite range of dimensions. For the reader's amusement we record some of that structure now.

1.1.14. Theorem. *In dimensions* ≤ 15*, the* 2*-component of* $\operatorname{coker} J$ *has the following generators, each with order* 2:

$$\eta^2 \in \pi_2^S, \quad \nu^2 \in \pi_6^S, \quad \bar{\nu} \in \pi_8^S, \quad \eta\bar{\nu} = \nu^3 \in \pi_9^S, \quad \mu \in \pi_9^S,$$
$$\eta\mu \in \pi_{10}^S, \quad \sigma^2 \in \pi_{14}^S, \quad \kappa \in \pi_{14}^S \quad \text{and} \quad \eta\kappa \in \pi_{15}^S.$$

(*There are relations* $\eta^3 = 4\nu$ *and* $\eta^2\mu = 4x_3$)*. For* $p \geq 3$ *the* p*-component of* $\operatorname{coker} J$ *has the following generators in dimensions* $\leq 3pq-6$ (*where* $q = 2p-2$)*, each with order* p:

$$\beta_1 \in \pi_{pq-2}^S, \qquad \alpha_1\beta_1 \in \pi_{(p+1)q-3}^S$$

where $\alpha_1 = x_{(p-1)/2} \in \pi_{q-1}^S$ *is the first generator of the* p*-component of* $\operatorname{im} J$*,*

$$\beta_1^2 \in \pi_{2pq-4}^S, \quad \alpha_1\beta_1^2 \in \pi_{(2p+1)q-5}^S, \quad \beta_2 \in \pi_{(2p+1)q-2}^S,$$
$$\alpha_1\beta_2 \in \pi_{(2p+2)q-3}^S, \quad \text{and} \quad \beta_1^3 \in \pi_{3pq-6}^S.$$ □

The proof and the definitions of new elements listed above will be given later in the book, e.g., in Section 4.4.

2. Methods of Computing $\pi_*(S^n)$

In this section we will informally discuss three methods of computing homotopy groups of spheres, the spectral sequences of Serre, Adams, and Novikov. A fourth method, the EHP sequence, will be discussed in Section 5. We will not give any proofs and in some cases we will sacrifice precision for conceptual clarity, e.g., in our identification of the E_2-term of the Adams–Novikov spectral sequence.

The Serre spectral sequence (circa 1951) (Serre [**2**]) is included here mainly for historical interest. It was the first systematic method of computing homotopy groups and was a major computational breakthrough. It has been used as late as the 1970s by various authors (Toda [**1**], Oka [**1, 2, 3**]), but computations made with it were greatly clarified by the introduction of the Adams spectral sequence in 1958 in Adams [**3**]. In the Adams spectral sequence the basic mechanism of the Serre spectral sequence information is organized by homological algebra.

For the 2-component of $\pi_*(S^n)$ the Adams spectral sequence is indispensable to this day, but the odd primary calculations were streamlined by the introduction of the Adams–Novikov spectral sequence (Adams–Novikov spectral sequence) in 1967 by Novikov [**1**]. It is the main subject in this book. Its E_2-term contains more information than that of the Adams spectral sequence; i.e., it is a more accurate approximation of stable homotopy and there are fewer differentials in the spectral sequence. Moreover, it has a very rich algebraic structure, as we shall see, largely due to the theorem of Quillen [**2**], which establishes a deep (and still not satisfactorily explained) connection between complex cobordism (the cohomology theory used to define the Adams–Novikov spectral sequence; see below) and the theory of formal group laws. Every major advance in the subject since 1969, especially the work of Jack Morava, has exploited this connection.

We will now describe these three methods in more detail. The starting point for Serre's method is the following classical result.

1.2.1. Theorem. *Let X be a simply connected space with $H_i(X) = 0$ for $i < n$ for some positive integer $n \geq 2$. Then*

(a) (Hurewicz [**1**]). $\pi_n(X) = H_n(X)$.

(b) (Eilenberg and Mac Lane [**2**]). *There is a space $K(\pi, n)$, characterized up to homotopy equivalence by*

$$\pi_i(K(\pi, n)) = \begin{cases} \pi & \text{if } i = n \\ 0 & \text{if } i \neq n. \end{cases}$$

If X is above and $\pi = \pi_n(X)$ then there is a map $f\colon X \to K(\pi, n)$ such that $H_n(f)$ and $\pi_n(f)$ are isomorphisms. □

1.2.2. Corollary. *Let F be the fiber of the map f above. Then*

$$\pi_i(F) = \begin{cases} \pi_i(X) & \text{for } i \geq n+1 \\ 0 & \text{for } i \leq n. \end{cases} \qquad \square$$

In other words, F has the same homotopy groups as X in dimensions above n, so computing $\pi_*(F)$ is as good as computing $\pi_*(X)$. Moreover, $H_*(K(\pi, n))$ is known, so $H_*(F)$ can be computed with the Serre spectral sequence applied to the fibration $F \to X \to K(\pi, n)$.

Once this has been done the entire process can be repeated: let $n' > n$ be the dimension of the first nontrivial homology group of F and let $H_{n'}(F) = \pi'$. Then $\pi_{n'}(F) = \pi_{n'}(X) = \pi'$ is the next nontrivial homotopy group of X. Theorem 1.2.1 applied to F gives a map $f'\colon F \to K(\pi', n')$ with fiber F', and 1.2.2 says

$$\pi_i(F') = \begin{cases} \pi_i(X) & \text{for } i > n' \\ 0 & \text{for } i \leq n'. \end{cases}$$

Then one computes $H_*(F')$ using the Serre spectral sequence and repeats the process.

As long as one can compute the homology of the fiber at each stage, one can compute the next homotopy group of X. In Serre [**3**] a theory was developed which allows one to ignore torsion of order prime to a fixed prime p throughout the calculation if one is only interested in the p-component of $\pi_*(X)$. For example, if $X = S^3$, one uses 1.2.1 to get a map to $K(\mathbf{Z}, 3)$. Then $H_*(F)$ is described by:

1.2.3. LEMMA. *If F is the fibre of the map $f\colon S^3 \to K(\mathbf{Z}, 3)$ given by 1.2.1, then*

$$H_i(F) = \begin{cases} \mathbf{Z}/(m) & \text{if } i = 2m \text{ and } m > 1 \\ 0 & \text{otherwise.} \end{cases} \qquad \square$$

1.2.4. COROLLARY. *The first p-torsion in $\pi_*(S^3)$ is $\mathbf{Z}/(p)$ in $\pi_{2p}(S^3)$ for any prime p.* $\square$

PROOF OF 1.2.3. (It is so easy we cannot resist giving it.) We have a fibration

$$\Omega K(\mathbf{Z}, 3) = K(\mathbf{Z}, 2) \to F \to S^3$$

and $H^*(K(\mathbf{Z}, 2)) = H^*(\mathbf{C}P^\infty) = \mathbf{Z}[x]$, where $x \in H^2(\mathbf{C}P^\infty)$ and $\mathbf{C}P^\infty$ is an infinite-dimensional complex projective space. We will look at the Serre spectral sequence for $H^*(F)$ and use the universal coefficient theorem to translate this to the desired description of $H_*(F)$. Let u be the generator of $H^3(S^3)$. Then in the Serre spectral sequence we must have $d_3(x) = \pm u$; otherwise F would not be 3-connected, contradicting 1.1.2. Since d_3 is a derivation we have $d_3(x^n) = \pm nux^{n-1}$. It is easily seen that there can be no more differentials and we get

$$H^i(F) = \begin{cases} \mathbf{Z}/(m) & \text{if } i = 2m+1,\ m > 1 \\ 0 & \text{otherwise} \end{cases}$$

which leads to the desired result. $\square$

If we start with $X = S^n$ the Serre spectral sequence calculations will be much easier for $\pi_{k+n}(S^n)$ for $k < n-1$. Then all of the computations are in the stable range, i.e., in dimensions less than twice the connectivity of the spaces involved. This means that for a fibration $F \xrightarrow{i} X \xrightarrow{f} K$, the Serre spectral sequence gives a long exact sequence

$$\cdots \to H_j(F) \xrightarrow{i_*} H_j(X) \xrightarrow{f_*} H_j(K) \xrightarrow{d} H_{j-1}(F) \to \cdots, \tag{1.2.5}$$

where d corresponds to Serre spectral sequence differentials. Even if we know $H_*(X)$, $H_*(K)$, and f_*, we still have to deal with the short exact sequence

$$0 \to \operatorname{coker} f_* \to H_*(F) \to \ker f_* \to 0. \tag{1.2.6}$$

It may lead to some ambiguity in $H_*(F)$, which must be resolved by some other means. For example, when computing $\pi_*(S^n)$ for large n one encounters this problem in the 3-component of $\pi_{n+10}(S^n)$ and the 2-component of $\pi_{n+14}(S^n)$. This difficulty is also present in the Adams spectral sequence, where one has the possibility of a nontrivial differential in these dimensions. These differentials were first calculated by Adams [**12**], Liulevicius [**2**], and Shimada and Yamanoshita [**3**] by methods involving secondary cohomology operations and later by Adams and Atiyah [**13**] by methods involving K-theory

The Adams spectral sequence of Adams [**3**] begins with a variation of Serre's method. One works only in the stable range and only on the p-component. Instead of mapping X to $K(\pi, n)$ as in 1.2.1, one maps to $K = \prod_{j>0} K(H^j(X; \mathbf{Z}/(p)), j)$ by a certain map g which induces a surjection in mod (p) cohomology. Let X_1 be the fiber of g. Define spaces X_i and K_i inductively by $K_i = \prod_{j>0} K(H^j(X_i; \mathbf{Z}/(p)), j)$

and X_{i+1} is the fiber of $g\colon X_i \to K_i$ (this map is defined in Section 2.1, where the Adams spectral sequence is discussed in more detail). Since $H^*(g_i)$ is onto, the analog of 1.2.5 is an short exact sequence in the stable range

$$0 \leftarrow H^*(X_i) \leftarrow H^*(K_i) \leftarrow H^*(\Sigma X_{i+1}) \leftarrow 0, \tag{1.2.7}$$

where all cohomology groups are understood to have coefficients $\mathbf{Z}/(p)$. Moreover, $H^*(K_i)$ is a free module over the mod (p) Steenrod algebra A, so if we splice together the short exact sequences of 1.2.7 we get a free A-resolution of $H^*(X)$

$$0 \leftarrow H^*(X) \leftarrow H^*(K) \leftarrow H^*(\Sigma^1 K_1) \leftarrow H^*(\Sigma^2 K_2) \leftarrow \cdots \tag{1.2.8}$$

Each of the fibration $X_{i+1} \to X_i \to K_i$ gives a long exact sequence of homotopy groups. Together these long exact sequences form an exact couple and the associated spectral sequence is the Adams spectral sequence for the p-component of $\pi_*(X)$. If X has finite type, the diagram

$$K \to \Sigma^{-1} K_1 \to \Sigma^{-2} K_2 \to \cdots \tag{1.2.9}$$

(which gives 1.2.8 in cohomology) gives a cochain complex of homotopy groups whose cohomology is $\operatorname{Ext}_A(H^*(X); \mathbf{Z}/(p))$. Hence one gets

1.2.10. THEOREM (Adams [**3**]). *There is a spectral sequence converging to the p-component of $\pi_{n+k}(S^n)$ for $k < n-1$ with*

$$E_2^{s,t} = \operatorname{Ext}_A^{s,t}(\mathbf{Z}/(p), \mathbf{Z}/(p)) =: H^{s,t}(A)$$

and $d_r\colon E_r^{s,t} \to E_r^{s+r,t+r-1}$. Here the groups $E_\infty^{s,t}$ for $t-s=k$ form the associated graded group to a filtration of the p-component of $\pi_{n+k}(S^n)$. □

Computing this E_2-term is hard work, but it is much easier than making similar computations with Serre spectral sequence. The most widely used method today is the spectral sequence of May [**1, 2**] (see Section 3.2). This is a trigraded spectral sequence converging to $H^{**}(A)$, whose E_2-term is the cohomology of a filtered form of the Steenrod algebra. This method was used by Tangora [**1**] to compute $E_2^{s,t}$ for $p=2$ and $t-s \leq 70$. Most of his table is reproduced here in FIG. A3.1a–C. Computations for odd primes can be found in Nakamura [**2**].

As noted above, the Adams E_2-term is the cohomology of the Steenrod algebra. Hence $E_2^{1,*} = H^1(A)$ is the indecomposables in A. For $p=2$ one knows that A is generated by Sq^{2^i} for $i \geq 0$; the corresponding elements in $E_2^{1,*}$ are denoted by $h_i \in E_2^{1,2^i}$. For $p>2$ the generators are the Bockstein β and $\mathcal{P}^{p^i}$ for $i \geq 0$ and the corresponding elements are $a_0 \in E_2^{1,1}$ and $h_i \in E_2^{1,qp^i}$, where $q = 2p-2$.

For $p=2$ these elements figure in the famous Hopf invariant one problem.

1.2.11. THEOREM (Adams [**12**]). *The following statements are equivalent.*

(a) *S^{2^i-1} is parallelizable, i.e., it has 2^i-1 globally linearly independent tangent vector fields.*

(b) *There is a division algebra (not necessarily associative) over $\mathbf{R}$ of dimension 2^i.*

(c) *There is a map $S^{2\cdot 2^i-1} \to S^{2^i}$ of Hopf invariant one (see 1.5.2).*

(d) *There is a 2-cell complex $X = S^{2^i} \cup e^{2^{i+1}}$ [the cofiber of the map in (c)] in which the generator of $H^{2^{i+1}}(X)$ is the square of the generator of $H^{2^i}(X)$.*

(e) *The element $h_i \in E_2^{1,2^i}$ is a permanent cycle in the Adams spectral sequence.* □

Condition (b) is clearly true for $i = 0$, 1, 2 and 3, the division algebras being the reals $\mathbf{R}$, the complexes $\mathbf{C}$, the quaternions $\mathbf{H}$ and the Cayley numbers, which are nonassotiative. The problem for $i \geq 4$ is solved by

1.2.12. THEOREM (Adams [**12**]). *The conditions of* 1.2.11 *are false for* $i \geq 4$ *and in the Adams spectral sequence one has* $d_2(h_i) = h_0 h_{i-1}^2 \neq 0$ *for* $i \geq 4$. □

For $i = 4$ the above gives the first nontrivial differential in the Adams spectral sequence. Its target has dimension 14 and is related to the difficulty in Serre's method referred to above.

The analogous results for $p > 2$ are

1.2.13. THEOREM (Liulevicius [**2**] and Shimada and Yamanoshita [**3**]). *The following are equivalent.*

(a) *There is a map* $S^{2p^{i+1}-1} \to \widehat{S}^{2p^i}$ *with Hopf invariant one* (*see* 1.5.3 *for the definition of the Hopf invariant and the space* $\widehat{S}^{2m}$).

(b) *There is a p-cell complex* $X = S^{2p^i} \cup e^{4p^i} \cup e^{6p^i} \cup \cdots \cup e^{2p^{i+1}}$ [*the cofiber of the map in* (*a*)] *whose* mod (p) *cohomology is a truncated polynomial algebra on one generator.*

(c) *The element* $h_i \in E_2^{1,qp^i}$ *is a permanent cycle in the Adams spectral sequence.* □

The element h_0 is the first element in the Adams spectral sequence above dimension zero so it is a permanent cycle. The corresponding map in (a) suspends to the element of $\pi_{2p}(S^3)$ given by 1.2.4. For $i \geq 1$ we have

1.2.14. THEOREM (Liulevicius [**2**] and Shimada and Yamanoshita [**3**]). *The conditions of* 1.2.13 *are false for* $i \geq 1$ *and* $d_2(h_i) = a_0 b_{i-1}$, *where* b_{i-1} *is a generator of* E_2^{2,qp^i} (*see Section* 5.2). □

For $i = 1$ the above gives the first nontrivial differential in the Adams spectral sequence for $p > 2$. For $p = 3$ its target is in dimension 10 and was referred to above in our discussion of Serre's method.

FIG. 1.2.15 shows the Adams spectral sequence for $p = 3$ through dimension 45. We present it here mainly for comparison with a similar figure (1.2.19) for the Adams–Novikov spectral sequence. $E_2^{s,t}$ is a $\mathbf{Z}/(p)$ vector space in which each basis element is indicated by a small circle. Fortunately in this range there are just two bigradings [(5,28) and (8,43)] in which there is more than one basis element. The vertical coordinate is s, the cohomological degree, and the horizontal coordinate is $t - s$, the topological dimension. These extra elements appear in the chart to the right of where they should be, and the lines meeting them should be vertical. A d_r is indicated by a line which goes up by r and to the left by 1. The vertical lines represent multiplication by $a_0 \in E_2^{1,1}$ and the vertical arrow in dimension zero indicates that all powers of a_0 are nonzero. This multiplication corresponds to multiplication by p in the corresponding homotopy group. Thus from the figure one can read off $\pi_0 = \mathbf{Z}$, $\pi_{11} = \pi_{45} = \mathbf{Z}/(9)$, $\pi_{23} = \mathbf{Z}/(9) \oplus \mathbf{Z}/(3)$, and $\pi_{35} = \mathbf{Z}/(27)$. Lines that go up 1 and to the right by 3 indicate multiplication by $h_0 \in E_2^{1,4}$, while those that go to the right by 7 indicate the Massey product $\langle h_0, h_0, - \rangle$ (see A1.4.1). The elements a_0 and h_i for $i = 0$, 1, 2 were defined above and the elements $b_0 \in E_2^{2,12}$, $k_0 \in E_2^{2,28}$, and $b_1 \in E_2^{2,36}$ are up to the sign the Massey products $\langle h_0, h_0, h_0 \rangle$, $\langle h_0, h_1, h_1 \rangle$, and $\langle h_1, h_1, h_1 \rangle$, respectively. The unlabeled elements in

$E_2^{i,5i-1}$ for $i \geq 2$ (and $h_0 \in E_2^{1,4}$) are related to each other by the Massey product $\langle h_0, a_0, -\rangle$. This accounts for all of the generators except those in $E_2^{3,26}$, $E_2^{7,45}$ and $E_2^{8,50}$, which are too complicated to describe here.

We suggest that the reader take a colored pencil and mark all of the elements which survive to E_∞, i.e., those which are not the source or target of a differential. There are in this range 31 differentials which eliminate about two-thirds of the elements shown.

Now we consider the spectral sequence of Adams and Novikov, which is the main object of interest in this book. Before describing its construction we review the main ideas behind the Adams spectral sequence. They are the following.

1.2.16. PROCEDURE. (i) Use mod (p)-cohomology as a tool to study the p-component of $\pi_*(X)$. (ii) Map X to an appropriate Eilenberg–Mac Lane space K, whose homotopy groups are known. (iii) Use knowledge of $H^*(K)$, i.e., of the Steenrod algebra, to get at the fiber of the map in (ii). (iv) Iterate the above and codify all information in a spectral sequence as in 1.2.10. □

An analogous set of ideas lies behind the Adams–Novikov spectral sequence, with mod p cohomology being replaced by complex cobordism theory. To elaborate, we first remark that "cohomology" in 1.2.16(i) can be replaced by "homology" and 1.2.10 can be reformulated accordingly; the details of this reformulation need not be discussed here. Recall that singular homology is based on the singular chain complex, which is generated by maps of simplices into the space X. Cycles in the chain complex are linear combinations of such maps that fit together in an appropriate way. Hence $H_*(X)$ can be thought of as the group of equivalence classes of maps of certain kinds of simplicial complexes, sometimes called "geometric cycles," into X.

Our point of departure is to replace these geometric cycles by closed complex manifolds. Here we mean "complex" in a very weak sense; the manifold M must be smooth and come equipped with a complex linear structure on its stable normal bundle, i.e., the normal bundle of some embedding of M into a Euclidean space of even codimension. The manifold M need not be analytic or have a complex structure on its tangent bundle, and it may be odd-dimensional.

The appropriate equivalence relation among maps of such manifolds into X is the following.

1.2.17. DEFINITION. *Maps $f_i\colon M \to X$ $(i = 1, 2)$ of n-dimensional complex (in the above sense) manifolds into X are bordant if there is a map $g\colon W \to X$ where W is a complex mainfold with boundary $\partial W = M_1 \cup M_2$ such that $g|M_i = f_i$. (To be correct we should require the restriction to M_2 to respect the complex structure on M_2 opposite to the given one, but we can ignore such details here.)* □

One can then define a graded group $MU_*(X)$, the *complex bordism* of X, analogous to $H_*(X)$. It satisfies all of the Eilenberg–Steenrod axioms except the dimension axiom, i.e., $MU_*(\mathrm{pt})$, is not concentrated in dimension zero. It is by definition the set of equivalence classes of closed complex manifolds under the relation of 1.2.17 with $X = \mathrm{pt}$, i.e., without any condition on the maps. This set is a ring under disjoint union and Cartesian product and is called the *complex bordism ring.* as are the analogous rings for several other types of manifolds; see Stong [**1**].

1.2.18. THEOREM (Thom [**1**], Milnor [**4**], Novikov [**2**]). *The complex bordism ring, $MU_*(\mathrm{pt})$, is $\mathbf{Z}[x_1, x_2, \ldots]$ where $\dim x_i = 2i$.* □

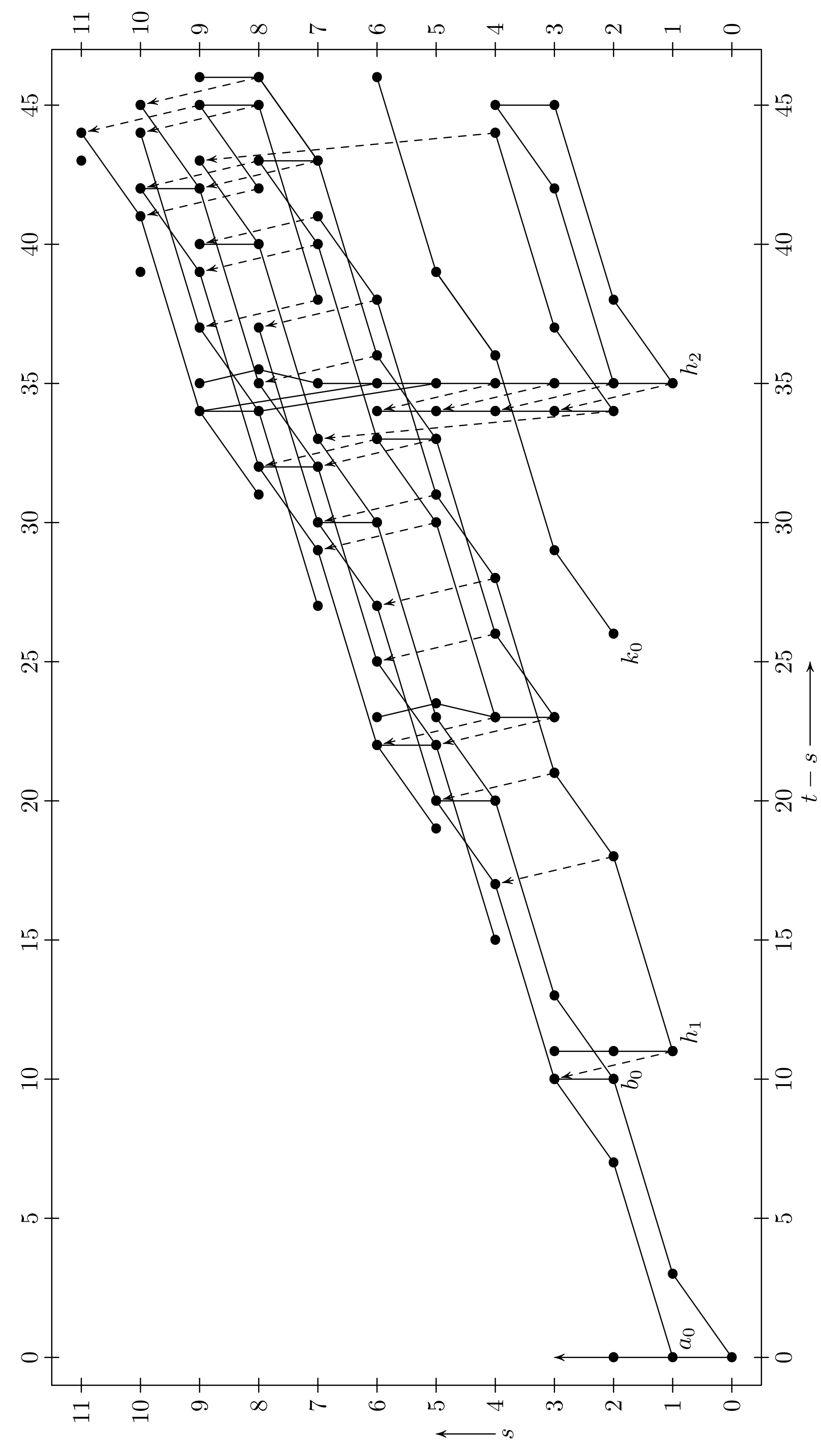


FIGURE 1.2.15. The Adams spectral sequence for $p = 3$, $t - s \leq 45$.

Now recall 1.2.16. We have described an analog of (i), i.e., a functor $MU_*(-)$ replacing $H_*(-)$. Now we need to modify (ii) accordingly, e.g., to define analogs of the Eilenberg–Mac Lane spaces. These spaces (or rather the corresponding spectrum MU) are described in Section 4.1. Here we merely remark that Thom's contribution to 1.2.18 was to equate $MU_i(\mathrm{pt})$ with the homotopy groups of certain spaces and that these spaces are the ones we need.

To carry out the analog of 1.2.16(iii) we need to know the complex bordism of these spaces, which is also described (stably) in Section 4.1. The resulting spectral sequence is formally introduced in Section 4.4, using constructions given in Section 2.2. We will not state the analog of 1.2.10 here as it would be too much trouble to develop the necessary notation. However we will give a figure analogous to 1.2.15.

The notation of Fig. 1.2.19 is similar to that of Fig. 1.2.15 with some minor differences. The E_2-term here is not a $\mathbf{Z}/(3)$-vector space. Elements of order > 3 occur in $E_2^{0,0}$ (an infinite cyclic group indicated by a square), and in $E_2^{1,12t}$ and $E_2^{3,48}$, in which a generator of order 3^{k+1} is indicated by a small circle with k parentheses to the right. The names α_t, β_t, and $\beta_{s/t}$ will be explained in the next section. The names α_{3t} refer to elements of order 3 in, rather than generators of, $E_2^{1,12t}$. In $E_2^{3,48}$ the product $\alpha_1\beta_3$ is divisible by 3.

One sees from these two figures that the Adams–Novikov spectral sequence has far fewer differentials than the Adams spectral sequence. The first nontrivial Adams–Novikov differential originates in dimension 34 and leads to the relation $\alpha_1\beta_1^3$ in $\pi_*(S^0)$. It was first established by Toda [**2, 3**].

3. The Adams–Novikov E_2-term, Formal Group Laws, and the Greek Letter Construction

In this section we will describe the E_2-term of the Adams–Novikov spectral sequence introduced at the end of the previous section. We begin by defining formal group laws (1.3.1) and describing their connection with complex cobordism (1.3.4). Then we characterize the E_2-term in terms of them (1.3.5 and 1.3.6). Next we describe the Greek letter construction, an algebraic method for producing periodic families of elements in the E_2-term. We conclude by commenting on the problem of representing these elements in π_*^S.

Suppose T is a one-dimensional commutative analytic Lie group and we have a local coordinate system in which the identity element is the origin. Then the group operation $T \times T \to T$ can be described locally as a real-valued analytic function of two variables. Let $F(x,y) \in \mathbf{R}[[x,y]]$ be the power series expansion of this function about the origin. Since 0 is the identity element we have $F(x,0) = F(0,x) = x$. Commutativity and associativity give $F(x,y) = F(y,x)$ and $F(F(x,y),z) = F(x,F(y,z))$, respectively.

1.3.1. Definition. *A* formal group law *over a commutative ring with unit R is a power series $F(x,y) \in R[[x,y]]$ satisfying the three conditions above.* □

Several remarks are in order. First, the power series in the Lie group will have a positive radius of convergence, but there is no convergence condition in the definition above. Second, there is no need to require the existence of an inverse because it exists automatically. It is a power series $i(x) \in R[[x]]$ satisfying $F(x, i(x)) = 0$;

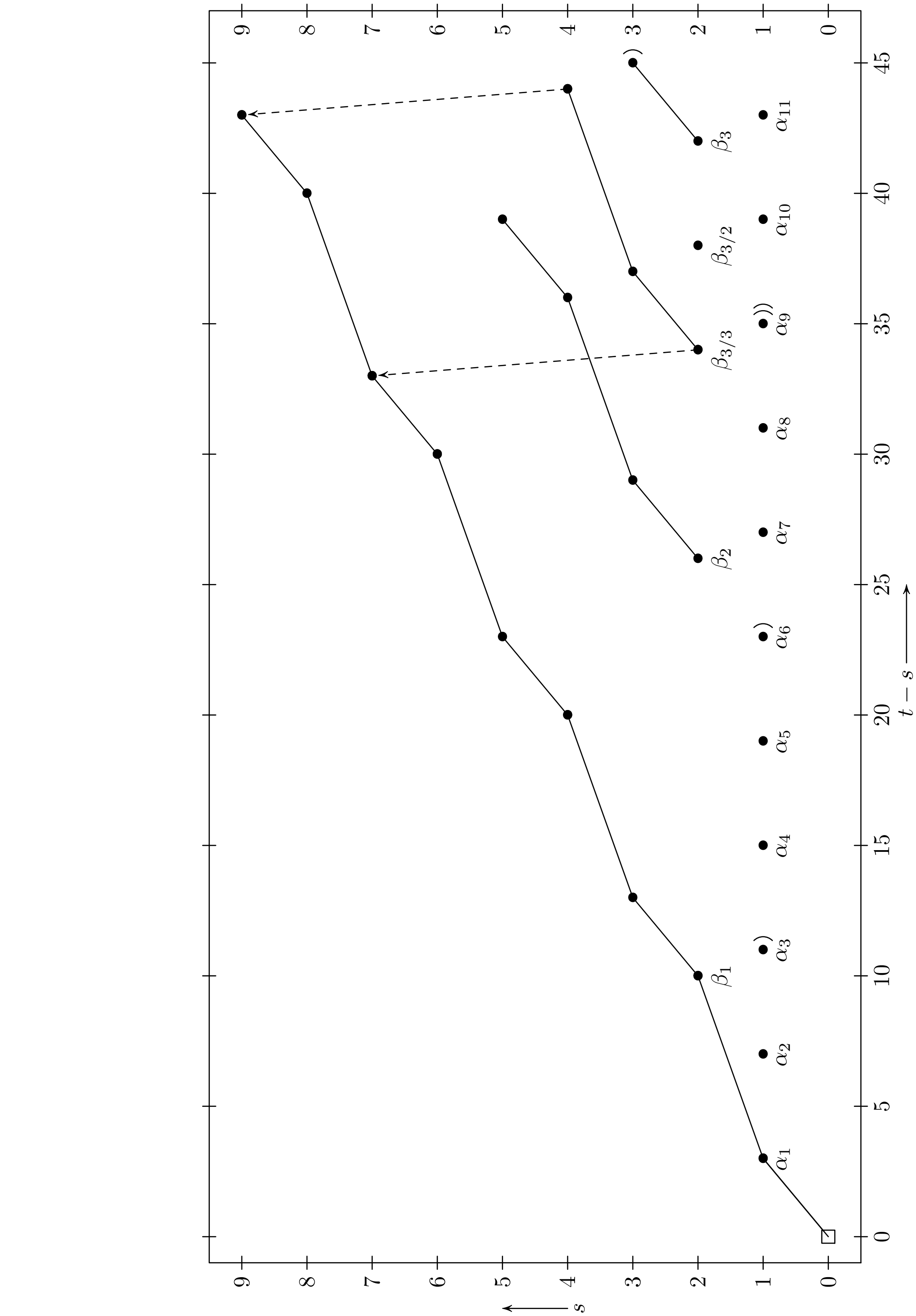


FIGURE 1.2.19. The Adams–Novikov spectral sequence for $p = 3$, $t - s \leq 45$

it is an easy exercise to solve this equation for $i(x)$ given F. Third, a rigorous self-contained treatment of the theory of formal group laws is given in Appendix 2.

Note that $F(x,0) = F(0,x) = x$ implies that $F \equiv x + y \mod (x,y)^2$ and that $x+y$ is therefore the simplest example of a formal group law; it is called the *additive* formal group law and is denoted by F_a. Another easy example is the *multiplicative* formal group law, $F_\mathrm{m} = x+y+rxy$ for $r \in \mathbf{R}$. These two are known to be the only formal group laws which are polynomials. Other examples are given in A2.1.4.

To see what formal group laws have to do with complex cobordism and the Adams–Novikov spectral sequence, consider $MU^*(\mathbf{C}P^\infty)$, the complex cobordism of infinite-dimensional complex projective space. Here $MU^*(-)$ is the cohomology theory dual to the homology theory $MU_*(-)$ (complex bordism) described in Section 2. Like ordinary cohomology it has a cup product and we have

1.3.2. THEOREM. *There is an element $x \in MU^2(\mathbf{C}P^\infty)$ such that*

$$MU^*(\mathbf{C}P^\infty) = MU^*(\mathrm{pt})[[x]]$$

and

$$MU^*(\mathbf{C}P^\infty \times \mathbf{C}P^\infty) = MU^*(\mathrm{pt})[[x \otimes 1, 1 \otimes x]].$$

□

Here $MU^*(\mathrm{pt})$ is the complex cobordism of a point; it differs from $MU_*(\mathrm{pt})$ (described in 1.2.18) only in that its generators are negatively graded. The generator x is closely related to the usual generator of $H^2(\mathbf{C}P^\infty)$, which we also denote by x. The alert reader may have expected $MU^*(\mathbf{C}P^\infty)$ to be a polynomial rather than a power series ring since $H^*(\mathbf{C}P^\infty)$ is traditionally described as $\mathbf{Z}[x]$. However, the latter is really $\mathbf{Z}[[x]]$ since the cohomology of an infinite complex maps onto the inverse limit of the cohomologies of its finite skeleta. [$MU^*(\mathbf{C}P^n)$, like $H^*(\mathbf{C}P^n)$, is a truncated polynomial ring.] Since one usually considers only homogeneous elements in $H^*(\mathbf{C}P^\infty)$, the distinction between $\mathbf{Z}[x]$ and $\mathbf{Z}[[x]]$ is meaningless. However, one can have homogeneous infinite sums in $MU^*(\mathbf{C}P^\infty)$ since the coefficient ring is negatively graded.

Now $\mathbf{C}P^\infty$ is the classifying space for complex line bundles and there is a map $\mu\colon \mathbf{C}P^\infty \times \mathbf{C}P^\infty \to \mathbf{C}P^\infty$ corresponding to the tensor product; in fact, $\mathbf{C}P^\infty$ is known to be a topological abelian group. By 1.3.2 the induced map μ^* in complex cobordism is determined by its behavior on the generator $x \in MU^2(\mathbf{C}P^\infty)$ and one easily proves, using elementary facts about line bundles,

1.3.3. PROPOSITION. *For the tensor product map $\mu\colon \mathbf{C}P^\infty \times \mathbf{C}P^\infty \to \mathbf{C}P^\infty$, $\mu^*(x) = F_U(x \otimes 1, 1 \otimes x) \in MU^*(\mathrm{pt})[[x \otimes 1, 1 \otimes x]]$ is an formal group law over $MU^*(\mathrm{pt})$.* □

A similar statement is true of ordinary cohomology and the formal group law one gets is the additive one; this is a restatement of the fact that the first Chern class of a tensor product of complex line bundles is the sum of the first Chern classes of the factors. One can play the same game with complex K-theory and get a multiplicative formal group law.

$\mathbf{C}P^\infty$ is a good test space for both complex cobordism and K-theory. One can analyze the algebra of operations in both theories by studying their behavior in $\mathbf{C}P^\infty$ (see Adams [**5**]) in the same way that Milnor [**2**] analyzed the mod (2) Steenrod algebra by studying its action on $H^*(\mathbf{R}P^\infty; \mathbf{Z}/(2))$. (See also Steenrod and Epstein [**1**].)

The formal group law of 1.3.3 is not as simple as the ones for ordinary cohomology or K-theory; it is complicated enough to have the following universal property.

1.3.4. THEOREM (Quillen [**2**]). *For any formal group law F over any commutative ring with unit R there is a unique ring homomorphism $\theta\colon MU^*(\mathrm{pt}) \to R$ such that $F(x,y) = \theta F_U(x,y)$.* □

We remark that the existence of such a universal formal group law is a triviality. Simply write $F(x,y) = \sum a_{i,j}x^iy^i$ and let $L = \mathbf{Z}[a_{i,j}]/I$, where I is the ideal generated by the relations among the $a_{i,j}$ imposed by the definition 1.3.1 of an formal group law. Then there is an obvious formal group law over L having the universal property. Determining the explicit structure of L is much harder and was first done by Lazard [**1**]. Quillen's proof of 1.3.4 consisted of showing that Lazard's universal formal group law is isomorphic to the one given by 1.3.3.

Once Quillen's Theorem 1.3.4 is proved, the manifolds used to define complex bordism theory become irrelevant, however pleasant they may be. All of the applications we will consider follow from purely algebraic properties of formal group laws. This leads one to suspect that the spectrum MU can be constructed somehow using formal group law theory and without using complex manifolds or vector bundles. Perhaps the corresponding infinite loop space is the classifying space for some category defined in terms of formal group laws. Infinite loop space theorists, where are you?

We are now just one step away from a description of the Adams–Novikov spectral sequence E_2-term. Let $G = \{f(x) \in \mathbf{Z}[[x]] \mid f(x) \equiv x \mod (x)^2\}$. Here G is a group under composition and acts on the Lazard/complex cobordism ring $L = MU_*(\mathrm{pt})$ as follows. For $g \in G$ define a formal group law F_g over L by $F_g(x,y) = g^{-1}F_U(g(x), g(y))$. By 1.3.4 F_g is induced by a homomorphism $\theta_g\colon L \to L$. Since g is invertible under composition, θ_g is an automorphism and we have a G-action on L.

Note that $g(x)$ defines an isomorphism between F and F_g. In general, isomorphisms between formal group laws are induced by power series $g(x)$ with leading term a unit multiple (not necessarily one) of x. An isomorphism induced by a g in G is said to be *strict*.

1.3.5. THEOREM. *The E_2-term of the Adams–Novikov spectral sequence converging to π_*^S is isomorphic to $H^{**}(G; L)$.* □

There is a difficulty with this statement: since G does not preserve the grading on L, there is no obvious bigrading on $H^*(G; L)$. We need to reformulate in terms of L as a comodule over a certain Hopf algebra B defined as follows.

Let $g \in G$ be written as $g(x) = \sum_{i\geq 0} b_ix^{i+1}$ with $b_0 = 1$. Each b_i for $i > 0$ can be thought of as a $\mathbf{Z}$-valued function on G and they generate a graded algebra of such functions

$$B = \mathbf{Z}[b_1, b_2, \dots] \quad \text{with } \dim b_i = 2i.$$

(Do not confuse this ring with L, to which it happens to be isomorphic.) The group structure on G corresponds to a coproduct $\Delta\colon B \to B \otimes B$ on B given by $\Delta(b) = \sum_{i\geq 0} b^{i+1} \otimes b_i$, where $b = \sum_{i\geq 0} b_i$ and $b_0 = 1$ as before. To see this suppose $g(x) = g^{(1)}(g^{(2)}(x))$ with $g^{(k)}(x) = \sum b_i^{(k)}x^{i+1}$ Then we have

$$\sum b_ix^{i+1} = \sum b_i^{(1)}\left(\sum b_j^{(2)}x^{j+1}\right)^{i+1}$$

from which the formula for Δ follows. This coproduct makes B into a graded connected Hopf algebra over which L is a graded comodule. We can restate 1.3.5 as

1.3.6. THEOREM. *The E_2-term of the Adams–Novikov spectral sequence converging to π_*^S is given by $E_2^{s,t} = \mathrm{Ext}_B^{s,t}(\mathbf{Z}, L)$.* $\square$

The definition of this Ext is given in A1.2.3; all of the relevant homological algebra is discussed in Appendix 1.

Do not be alarmed if the explicit action of G (or coaction of B) on L is not obvious to you. It is hard to get at directly and computing its cohomology is a very devious business.

Next we will describe the *Greek letter construction*, which is a method for producing lots (but by no means all) of elements in the E_2-term, including the α_t's and β_t's seen in 1.2.19. We will use the language suggested by 1.3.5; the interested reader can translate our statements into that of 1.3.6. Our philosophy here is that group cohomology in positive degrees is too hard to comprehend, but $H^0(G;M)$ (the G-module M will vary in the discussion), the submodule of M fixed by G, is relatively straightforward. Hence our starting point is

1.3.7. THEOREM. *$H^0(G;L) = \mathbf{Z}$ concentrated in dimension 0.* $\square$

This corresponds to the 0-stem in stable homotopy. Not a very promising beginning you say? It does give us a toehold on the problem. It tells us that the only principal ideals in L which are G-invariant are those generated by integers and suggests the following. Fix a prime number p and consider the short exact sequence of G-modules

$$0 \to L \xrightarrow{p} L \to L/(p) \to 0. \tag{1.3.8}$$

We have a connecting homomorphism

$$\delta_0 \colon H^i(G; L/(p)) \to H^{i+1}(G;L).$$

1.3.9. THEOREM. *$H^0(G; L/(p)) = \mathbf{Z}/(p)[v_1]$, where $v_1 \in L$ has dimension $q = 2(p-1)$.* $\square$

1.3.10. DEFINITION. *For $t > 0$ let $\alpha_t = \delta_0(v_1^t) \in E_2^{1,qt}$.* $\square$

It is clear from the long exact sequence in cohomology associated with 1.3.8 that $\alpha_t \neq 0$ for all $t > 0$, so we have a collection of nontrivial elements in the Adams–Novikov E_2-term. We will comment below on the problems of constructing corresponding elements in π_*^S; for now we will simply state the result.

1.3.11. THEOREM. (a) (Toda [**4**, IV]) *For $p > 2$ each α_t is represented by an element of order p in π_{qt-1}^S which is in the image of the J-homomophism* (1.1.12).

(b) *For $p = 2$ α_t is so represented provided $t \not\equiv 3 \mod (4)$. If $t \equiv 2 \mod (4)$ then the element has order 4; otherwise it has order 2. It is in $\mathrm{im}\, J$ if t is even.* $\square$

Theorem 1.3.9 tells us that

$$0 \to \Sigma^q L/(p) \xrightarrow{v_1} L/(p) \to L/(p, v_1) \to 0 \tag{1.3.12}$$

is an short exact sequence of G-modules and there is a connecting homomorphism

$$\delta_1 \colon H^i(G; L/(p, v_1)) \to H^{i+1}(G; L/(p)).$$

The analogs of 1.3.9 and 1.3.10 are

1.3.13. THEOREM. *$H^0(G; L/(p, v_1)) = \mathbf{Z}/(p)[v_2]$ where $v_2 \in L$ has dimension $2(p^2 - 1)$.* $\square$

1.3.14. DEFINITION. *For $t > 0$ let $\beta_t = \delta_0\delta_1(v_2^t) \in E_2^{2,t(p+1)q-q}$.* $\square$

More work is required to show that these elements are nontrivial for $p > 2$, and $\beta_1 = 0$ for $p = 2$. The situation in homotopy is

1.3.15. THEOREM (Smith [**1**]). *For $p \geq 5$ β_t is represented by a nontrivial element of order p in $\pi_{(p+1)tq-q-2}(S^0)$.* $\square$

You are probably wondering if we can continue in this way and construct γ_t, δ_t, etc. The following results allow us to do so.

1.3.16. THEOREM (Morava [**3**], Landweber [**4**]). (a) *There are elements $v_n \in L$ of dimension $2(p^n - 1)$ such that $I_n = (p, v_1, v_2, \ldots, v_{n-1}) \subset L$ is a G-invariant prime ideal for all $n > 0$.*

(b) *$0 \to \Sigma^{2(p^n-1)} L/I_n \xrightarrow{v_n} L/I_n \to L/I_{n+1} \to 0$ is an short exact sequence of modules with connecting homorphism*

$$\delta\colon H^i(G; L/I_{n+1}) \to H^{i+1}(G; L/I_n).$$

(c) *$H^0(G; L/I_n) = \mathbf{Z}/(p)[v_n]$.*

(d) *The only G-invariant prime ideals in L are the I_n for $0 < n \leq \infty$ for all primes p.* $\square$

Part (d) above shows how rigid the G-action on L is; there are frightfully many prime ideals in L, but only the I_n for various primes are G-invariant. Using (b) and (c) we can make

1.3.17. DEFINITION. *For $t, n > 0$ let $\alpha_t^{(n)} = \delta_0\delta_1 \ldots \delta_{n-1}(v_n^t) \in E_2^{n,*}$.* $\square$

Here $\alpha^{(n)}$ stands for the nth letter of the Greek alphabet, the length of which is more than adequate given our current state of knowledge. The only other known result comparable to 1.3.11 or 1.3.15 is

1.3.18. THEOREM. (a) (Miller, Ravenel, and Wilson [**1**]) *The element $\gamma_t \in E_2^{3,tq(p^2+p+1)-q(p+2)}$ is nontrivial for all $t > 0$ and $p > 2$.*

(b) (Toda [**1**]) *For $p \geq 7$ each γ_t is represented by a nontrivial element of order p in $\pi_{tq(p^2+p+1)-q(p+2)-3}(S^0)$.* $\square$

It is known that not all γ_t exist in homotopy for $p = 5$ (see 7.6.1). Part (b) above was proved several years before part (a). In the intervening time there was a controversy over the nontriviality of γ_1 which was unresolved for over a year, ending in 1974 (see Thomas and Zahler [**1**]). This unusual state of affairs attracted the attention of the editors of *Science* [**1**] and the *New York Times* [**1**], who erroneously cited it as evidence of the decline of mathematics.

We conclude our discussion of the Greek letter construction by commenting briefly on *generalized* Greek letter elements. Examples are $\beta_{3/3}$ and $\beta_{3/2}$ (and the elements in $E_2^{1,*}$ of order > 3) in 1.2.19. The elements come via connecting homomorphisms from $H^0(G; L/J)$, where J is a G-invariant *regular* (instead of prime) ideal. Recall that a regular ideal $(x_0, x_1, \ldots, x_{n-1}) \subset L$ is one in which each x_i is not a zero divisor modulo $(x_0, \ldots, x_{i-1})$. Hence G-invariant prime ideals are regular as are ideals of the form $(p^{i_0}, v_1^{i_1}, \ldots, v_{n-1}^{i_{n-1}})$. Many but not all G-invariant regular ideals have this form.

1.3.19. DEFINITION. *$\beta_{s/t}$ (for appropriate s and t) is the image of $v_2^s \in H^0(G; L/(p, v_1^t))$ and $\alpha_{s/t}$ is the image of $v_1^s \in H^0(G; L/(p^t))$.* □

Hence $p\alpha_{s/t} = \alpha_{s/t-1}$, $\alpha_{s/1} = \alpha_s$, and $\beta_{t/1} = \beta_t$ by definition.

Now we will comment on the problem of representing these elements in the E_2-term by elements in stable homotopy, e.g., on the proofs of 1.3.11, 1.3.15, and 1.3.18(b). The first thing we must do is show that the elements produced are actually nontrivial in the E_2-term. This has been done only for α's, β's, and γ's. For $p = 2$, β_1 and γ_1 are zero but for $t > 1$ β_t and γ_t are nontrivial; these results are part of the recent computation of $E_2^{2,*}$ at $p = 2$ by Shimomura [**1**], which also tells us which generalized β's are defined and are nontrivial. The corresponding calculation at odd primes was done in Miller, Ravenel, and Wilson [**1**], as was that of $E_2^{1,*}$ for all primes.

The general strategy for representing Greek letter elements geometrically is to realize the relevant short exact sequences [e.g., 1.3.8, 1.3.12, and 1.3.16(b)] by cofiber sequences of finite spectra. For any connective spectrum X there is an Adams–Novikov spectral sequence converging to $\pi_*(X)$. Its E_2-term [denoted by $E_2(X)$] can be described as in 1.3.5 with $L = MU_*(S^0)$ replaced by $MU_*(X)$, which is a G-module. For 1.3.8 we have a cofiber sequence

$$S^0 \xrightarrow{p} S^0 \to V(0),$$

where $V(0)$ is the mod (p) Moore spectrum. It is known (2.3.4) that the long exact sequence of homotopy groups is compatible with the long exact sequence of E_2-terms. Hence the elements v_1^t of 1.3.9 live in $E_2^{0,qt}(V(0))$ and for 1.3.11(a) [which says α_t is represented by an element of order p in $\pi_{qt-1}(S^0)$ for $p > 2$ and $t > 0$] it would suffice to show that these elements are permanent cycles in the Adams–Novikov spectral sequence for $\pi_*(V(0))$ with $p > 0$. For $t = 1$ (even if $p = 2$) one can show this by brute force; one computes $E_2(V(0))$ through dimension q and sees that there is no possible target for a differential coming from $v_1 \in E_2^{0,q}$. Hence v_1 is realized by a map

$$S^q \to V(0)$$

If we can extend it to $\Sigma^q V(0)$, we can iterate and represent all powers of v_1. We can try to do this either directly, using obstruction theory, or by showing that $V(0)$ is a ring spectrum spectrum. In the latter case our extension α would be the composite

$$S^q \wedge V(0) \to V(0) \wedge V(0) \to V(0),$$

where the first map is the original map smashed with the identity on $V(0)$ and the second is the multiplication on $V(0)$. The second method is generally (in similar situation of this sort) easier because it involves obstruction theory in a lower range of dimensions.

In the problem at hand both methods work for $p > 2$ but both fail for $p = 2$. In that case $V(0)$ is not a ring spectrum and our element in $\pi_2(V(0))$ has order 4, so it does not extend to $\Sigma^2 V(0)$. Further calculations show that v_1^2 and v_1^3 both support nontrivial differentials (see 5.3.13) but v_1^4 is a permanent cycle represented by map $S^8 \to V(0)$, which *does* extend to $\Sigma^8 V(0)$. Hence iterates of this map produce the homotopy elements listed in 1.3.11(b) once certain calculation have been made in dimensions ≤ 8.

For $p > 2$ the map $\alpha\colon \Sigma^q V(0) \to V(0)$ gives us a cofibre sequence

$$\Sigma^q V(0) \xrightarrow{\alpha} V(0) \to V(1),$$

realizing the short exact sequence 1.3.12. Hence to arrive at 1.3.15 (which describes the β's in homotopy) we need to show that $v_2 \in E_2^{0,(p+1)q}(V(1))$ is a permanent cycle represented by a map which extends to $\beta\colon \Sigma^{(p+1)q}V(1) \to V(1)$. We can do this for $p \geq 5$ but not for $p = 3$. Some partial results for β's at $p = 3$ and $p = 2$ are described in Section 5.5.

The cofiber of the map β (corresponding to v_2) for $p \geq 5$ is called $V(2)$ by Toda [**1**]. In order to construct the γ's [1.3.18(b)] one needs a map

$$\gamma\colon \Sigma^{2(p^3-1)}V(2) \to V(2)$$

corresponding to v_3. Toda [**1**] produces such a map for $p \geq 7$ but it is known not to exist for $p = 5$ (see 7.6.1).

Toda [**1**] first considered the problem of constructing the spectra $V(n)$ above, and hence of the representation of Greek letter elements in π_*^S, although that terminology (and 1.3.16) was not available at the time. While the results obtained there have not been surpassed, the methods used leave something to be desired. Each positive result is proved by brute force; the relevant obstruction groups are shown to be trivial. This approach can be pushed no further; the obstruction to realizing v_4 lies in a nontrivial group for all primes (5.6.13). Homotopy theorists have yet to learn how to compute obstructions in such situations.

The negative results of Toda [**1**] are proved by ingenious but *ad hoc* methods. The nonexistence of $V(1)$ for $p = 2$ follows easily from the structure of the Steenrod algebra; if it existed its cohomology would contradict the Adem relation $Sq^2Sq^2 = Sq^1Sq^2Sq^1$. For the nonexistence of $V(2)$ at $p = 3$ Toda uses a delicate argument involving the nonassociativity of the mod (3) Moore spectrum, which we will not reproduce here. We will give another proof (5.5.1) which uses the multiplicative structure of the Adams–Novikov E_2-term to show that the nonrealizability of $\beta_4 \in E_2^{2,60}$, and hence of $V(2)$, is a formal consequence of that of $\beta_{3/3} \in E_2^{2,36}$. This was shown by Toda [**2, 3**] using an extended power construction, which will also not be reproduced here. Indeed, all of the differentials in the Adams–Novikov spectral sequence for $p = 3$ in the range we consider are formal consequences of that one in dimension 34. A variant of the second method used for $V(2)$ at $p = 3$ works for $V(3)$ (the cofiber of γ) at $p = 5$.

4. More Formal Group Law Theory, Morava's Point of View, and the Chromatic Spectral Sequence

We begin this section by introducing BP-theory, which is essentially a p-local form of MU-theory. With it many of the explicit calculations behind our results become a lot easier. Most of the current literature on the subject is written in terms of BP rather than MU. On the other hand, BP is not essential for the overall picture of the E_2-term we will give later, so it could be regarded as a technicality to be passed over by the casual reader. Next we will describe the classification of formal group laws over an algebraically closed field of characteristic p. This is needed for Morava's point of view, which is a useful way of understanding the action of G on L (1.3.5). The insights that come out of this approach are made computationally precise in the chromatic spectral sequence , which is the pivotal idea in this book. Technically the chromatic spectral sequence is a trigraded spectral sequence converging to the Adams–Novikov E_2-term; heuristically it is like a spectrum in the astronomical sense in that it resolves the E_2-term into various components each having a different type of periodicity. In particular, it incorporates the Greek letter elements of the previous section into a broader scheme which embraces the entire E_2-term.

BP-theory began with Brown and Peterson [**1**] (after whom it is named), who showed that after localization at any prime p, the MU spectrum splits into an infinite wedge suspension of identical smaller spectra subsequently called BP. One has

$$\pi_*(BP) = \mathbf{Z}_{(p)}[v_1, v_2, \dots], \tag{1.4.1}$$

where $\mathbf{Z}_{(p)}$ denotes the integers localized at p and the v_n's are the same as the generators appearing in the Morava–Landweber theorem 1.3.16. Since $\dim v_n = 2(p^n - 1)$, this coefficient ring, which we will denote by BP_*, is much smaller than $L = \pi_*(MU)$, which has a polynomial generator in every even dimension.

Next Quillen [**2**] observed that there is a good formal group law theoretic reason for this splitting. A theorem of Cartier [**1**] (A2.1.18) says that every formal group law over a $\mathbf{Z}_{(p)}$-algebra is canonically isomorphic to one in a particularly convenient form called a p-typical formal group law (see A2.1.17 and A2.1.22 for the definition, the details of which need not concern us now). This canonical isomorphism is reflected topologically in the above splitting of the localization of MU. This fact is more evidence in support of our belief that MU can somehow be constructed in purely formal group law theoretic terms.

There is a p-typical analog of Quillen's theorem 1.3.4; i.e., $BP^*(\mathbf{C}P^\infty)$ gives us a p-typical formal group law with a similar universal property. Also, there is a BP analog of the Adams–Novikov spectral sequence, which is simply the latter tensored with $\mathbf{Z}_{(p)}$; i.e., its E_2-term is the p-component of $H^*(G; L)$ and it converges to the p-component of π_*^S However, we encounter problems in trying to write an analog of our metaphor 1.3.5 because *there is no p-typical analog of the group G*.

In other words there is no suitable group of power series over $\mathbf{Z}_{(p)}$ which will send any p-typical formal group law into another. Given a p-typical formal group law F over $\mathbf{Z}_{(p)}$ there is a set of power series $g \in \mathbf{Z}_{(p)}[[x]]$ such that $g^{-1}F(g(x), g(y))$ is also p-typical, but this set depends on F. Hence $\operatorname{Hom}(BP_*, K)$ the set of p-typical formal group laws over a $\mathbf{Z}_{(p)}$-algebra K, is acted on not by a group analogous to G, but by a *groupoid*.

Recall that a groupoid is a small category in which every morphism is an equivalence, i.e., it is invertible. A groupoid with a single object is a group. In our case the objects are p-typical formal group laws over K and the morphisms are isomorphisms induced by power series $g(x)$ with leading term x.

Now a Hopf algebra, such as B in 1.3.6, is a *cogroup object* in the category of commutative rings R, which is to say that $\operatorname{Hom}(B, R) = G_R$ is a group-valued functor. In fact G_R is the group (under composition) of power series $f(x)$ over R with leading term x. For a p-typical analog of 1.3.6 we need to replace b by *cogroupoid object* in the category of commutative $\mathbf{Z}_{(p)}$-algebras K. Such an object is called a *Hopf algebroid* (A1.1.1) and consists of a pair (A, Γ) of commutative rings with appropriate structure maps so that $\operatorname{Hom}(A, K)$ and $\operatorname{Hom}(\Gamma, K)$ are the sets of objects and morphisms, respectively, of a groupoid. The groupoid we have in mind, of course, is that of p-typical formal group laws and isomorphisms as above. Hence BP_* is the appropriate choice for A; the choice for Γ turns out to be $BP_*(BP)$, the BP-homology of the spectrum BP. Hence the p-typical analog of 1.3.6 is

1.4.2. THEOREM. *The p-component of the E_2-term of the Adams–Novikov spectral sequence converging to π_*^S is*

$$\operatorname{Ext}_{BP_*(BP)}(BP_*, BP_*). \qquad \square$$

Again this Ext is defined in A1.2.3 and the relevant homological algebra is discussed in Appendix 1.

We will now describe the classification of formal group laws over an algebraically closed field of characteristic p. First we define power series $[m]_F(x)$ associated with a formal group law F and natural numbers m. We have $[0]_F(x) = 0$, $[1]_F(x) = x$, and $[m]_F(x) = F(x, [m-1]_F(x))$. An easy lemma (A2.1.6) says that if F is defined over a field of characteristic p, then $[p]_F(x)$ is in fact a power series over x^{p^n} with leading term ax^{p^n}, $a \neq 0$, for some $n > 0$, provided F is not isomorphic to the additive formal group law, in which case $[p]_F(x) = 0$. This integer n is called the *height* of F, and the height of the additive formal group law is defined to be ∞. Then we have

1.4.3. CLASSIFICATION THEOREM (Lazard [**2**]).
(a) *Two formal group laws defined over the algebraic closure of $\mathbf{F}_p$ are isomorphic iff they have the same height.*
(b) *If F is nonadditive, its height is the smallest n such that $\theta(v_n) \neq 0$, where $\theta\colon L \to K$ is the homomorphism of 1.3.4 and $v_n \in L$ is as in 1.3.16, where K is finite field.* $\square$

Now we come to Morava's point of view. Let $K = \overline{\mathbf{F}}_p$, the algebraic closure of the field with p elements, and let $G_K \subset K[[x]]$ be the group (under composition) of power series with leading term x. We have seen that G_K acts on $\operatorname{Hom}(L, K)$, the set formal group laws defined over K. Since L is a polynomial ring, we can think of $\operatorname{Hom}(L, K)$ as an infinite-dimensional vector space V over K; a set of polynomial generators of L gives a topological basis of V. For a vector $v \in V$, let F_v be the corresponding formal group law.

Two vectors in V are in the same orbit iff the corresponding formal group laws are strictly isomorphic (strict isomorphism was defined just prior to 1.3.5), and the stabilizer group of $v \in V$ (i.e., the subgroup of G_K leaving V fixed) is the strict automorphism group of F_v. This group S_n (where n is the height) can be described explicitly (A2.2.18); it is a profinite group of units in a certain p-adic division algebra, but the details need not concern us here. Theorem 1.4.3 enables us to describe the orbits explicitly.

1.4.4. THEOREM. *There is one G_K-orbit of V for each height as in 1.4.3. The height n orbit V_n is the subset defined by $v_i = 0$ for $i < n$ and $v_n \neq 0$.* $\square$

Now observe that V is the set of closed points in $\operatorname{Spec}(L_n \otimes K)$, and V_n is the set of closed points in $\operatorname{Spec}(L_n \otimes K)$, where $L_n = v_n^{-1}L/I_n$. Here V_n is a homogeneous G_K-space and a standard change-of-rings argument gives

1.4.5. CHANGE-OF-RINGS THEOREM. $H^*(G_K; L_n \otimes K) = H^*(S_n; K)$. $\square$

We will see in Chapter 6 that a form of this isomorphism holds over $\mathbf{F}_p$ as well as over K. In it the right-hand term is the cohomology of a certain Hopf algebra [called the nth Morava stabilizer algebra $\Sigma(n)$] defined over $\mathbf{F}_p$, which, when tensored with $\mathbf{F}_{p^n}$, becomes isomorphic to the dual of $\mathbf{F}_{p^n}[S_n]$, the $\mathbf{F}_{p^n}$-group algebra of S_n.

Now we are ready to describe the central construction of this book, the chromatic spectral sequence, which enables us to use the results above to get more explicit information about the Adams–Novikov E_2-term. We start with a long exact sequence of G-modules, called the chromatic resolution

$$0 \to L \otimes \mathbf{Z}_{(p)} \to M^0 \to M^1 \to \cdots \tag{1.4.6}$$

defined as follows. $M^0 = L \otimes \mathbf{Q}$, and N^1 is the cokernel in the short exact sequence

$$0 \to L \otimes \mathbf{Z}_{(p)} \to M^0 \to N^1 \to 0.$$

M^n and N^n are defined inductively for $n > 0$ by short exact sequences

$$0 \to N^n \to M^n \to N^{n+1} \to 0, \tag{1.4.7}$$

where $M^n = v_n^{-1} N^n$. Hence we have

$$N^1 = L \otimes \mathbf{Q}/\mathbf{Z}_{(p)} = \varinjlim L/(p^i) = L/(p^\infty)$$

and

$$N^{n+1} = \varinjlim L/(p^{i_0}, v_1^{i_1}, \ldots, v_n^{i_n}) = L/(p^\infty, v_1^\infty, \ldots, v_n^\infty).$$

The fact that these are short exact sequences of G-modules is nontrivial. The long exact sequence 1.4.6 is obtained by splicing together the short exact sequences 1.4.7. In Chapter 5, where the chromatic spectral sequence is described in detail, M^n and N^n denote the corresponding objects defined in terms of BP_*. In what follows here $\mathrm{Ext}_B(\mathbf{Z}, M)$ will be abbreviated by $\mathrm{Ext}(M)$ for a B-module (e.g., G-module) M. Standard homological algebra (A1.3.2) gives

1.4.8. PROPOSITION. *There is a spectral sequence converging to* $\mathrm{Ext}(L \otimes \mathbf{Z}_{(p)})$ *with* $E_1^{n,s} = \mathrm{Ext}^s(M^n)$*,* $d_r \colon E_r^{n,s} \to E_r^{n+r,s-r+1}$*, and* $d_1 \colon \mathrm{Ext}(M^n) \to \mathrm{Ext}(M^{n+1})$ *being induced by the maps* $M^n \to M^{n+1}$ *in 1.4.6.* [$E_\infty^{n,s}$ *is a subquotient of* $\mathrm{Ext}^{n+s}(L \otimes \mathbf{Z}_{(p)})$*.*] □

This is the *chromatic spectral sequence*. We can use 1.4.5 to get at its E_1 term as follows. Define G-modules M_i^n for $0 \le i \le n$ by $M_0^n = M^n$, and M_i^n is the kernel in the short exact sequence

$$0 \to M_i^n \to M_{i-1}^n \xrightarrow{v_{i-1}} M_{i-1}^n \to 0, \tag{1.4.9}$$

where $v_0 = p$. This gives $M_n^n = L_n = v_n^{-1} L/I_n$, so the $\mathbf{F}_p$-analog of 1.4.5 describes $\mathrm{Ext}(M_n^n)$ in terms of the cohomology of the stabilizer group S_n. Equation 1.4.9 gives a long exact sequence of Ext groups of a Bockstein spectral sequence computing $\mathrm{Ext}(M_{i-1}^n)$ in terms of $\mathrm{Ext}(M_i^n)$. Hence in principle we can get from $H^*(S_n)$ to $\mathrm{Ext}(M^n)$, although the Bockstein spectral sequences are difficult to handle in practice.

Certain general facts about $H^*(S_n)$ are worth mentioning here. If $(p-1)$ divides n then this cohomology is periodic (6.2.10); i.e., there is an element $c \in H^*(S_n; \mathbf{F}_p)$ such that $H^*(S_n; \mathbf{F}_p)$ is a finitely generated free module over $\mathbf{F}_p[c]$. In this case S_n has a cyclic subgroup of order p to whose cohomology c restricts nontrivially. This cohomology can be used to detect elements in the Adams–Novikov E_2-term of high cohomological degree, e.g., to prove

1.4.10. THEOREM. *For* $p > 2$*, all monomials in the* β_{p^i/p^i} (1.3.19) *are nontrivial.* □

If n is not divisible by $p-1$ then S_n has cohomological dimension n^2; i.e., $H^i(S_n)=0$ if $i>n^2$, and $H^*(S_n)$ has a certain type of Poincaré duality (6.2.10). It is essentially the cohomology of a certain n-stage nilpotent Lie algebra (6.3.5), at least for $n<p-1$. The cohomological dimension implies

1.4.11. MORAVA VANISHING THEOREM. *If $(p-1)\nmid n$, then in the chromatic spectral sequence (1.4.8) $E_1^{n,s}=0$ for $s>n^2$.* □

It is also known (6.3.6) that every sufficiently small open subgroup of S_n has the same cohomology as a free abelian group of rank n^2. This fact can be used to get information about the Adams–Novikov spectral sequence E_2-term for certain Thom spectra (6.5.6).

Now we will explain how the Greek letter elements of 1.3.17 and 1.3.19 appear in the chromatic spectral sequence. If J is a G-invariant regular ideal with n generators [e.g., the invariant prime ideal $I_n=(p,v_1,\dots,v_{n-1})$], then L/J is a submodule of N^n and M^n, so $\mathrm{Ext}^0(L/J)\subset\mathrm{Ext}^0(N^n)\subset\mathrm{Ext}^0(M^n)=E_1^{n,0}$. Recall that the Greek letter elements are images of elements in $\mathrm{Ext}^0(J)$ under the appropriate composition of connecting homomorphisms. This composition corresponds to the edge homomorphism $E_2^{n,0}\to E_\infty^{n,0}$ in the chromatic spectral sequence. [Note that every element in the chromatic $E_2^{n,0}$ is a permanent cycle; i.e., it supports no nontrivial differential although it may be the target of one. Elements in $E_1^{n,0}$ coming from $\mathrm{Ext}(L/J)$ lift to $\mathrm{Ext}(N^n)$ are therefore in $\ker d_1$ and live in $E_2^{n,0}$.] The module N^n is the union of the L/J over all possible invariant regular ideals J with n generators, so $\mathrm{Ext}^0(N^n)$ contains all possible nth Greek letter elements.

To be more specific about the particular elements discussed in Section 3 we must introduce *chromatic notation* for elements in N^n and M^n. Such elements will be written as fractions $\frac{x}{y}$ with $x\in L$ and $y=p^{i_0}v^{i_1}\dots v_{n-1}^{i_{n-1}}$ with all exponent positive, which stands for the image of y in $L/J\subseteq N^n$ where $J=(p^{i_0},v_1^{i_1},\dots,v_{n-1}^{i_{n-1}})$. Hence x/y is annihilated by J and depends only on the mod J reduction of x. The usual rules of addition, subtraction, and cancellation of fractions apply here.

1.4.12. PROPOSITION. *Up to sign the elements $\alpha_t^{(n)}$ (1.3.17), $\alpha_{s/t}$ and $\beta_{s/t}$ (1.3.19) are represented in the chromatic spectral sequence by $v_n^t/pv_1\cdots v_{n-1}\in E_2^{n,0}$, $v_1^s/p^t\in E_2^{1,0}$, and $v_2^s/pv_1^t\in E_2^{2,0}$, respectively.* □

The signs here are a little tricky and come from the double complex used to prove 1.4.8 (see 5.1.18). The result suggests elements of a more complicated nature; e.g., $\beta_{s/i_2,i_1}$ stands for $v_2^s/p^{i_1}v_1^{i_2}$, with the convention that if $i_1=1$ it is omitted from the notation. The first such element with $i_1>1$ is $\beta_{p^2/p,2}$. We also remark that some of these elements require correcting terms in their numerators; e.g., $(v_1^4+8v_1v_2)/2^4$ (but not $v_1^4/2^4$) is in $\mathrm{Ext}^0(N^1)$ and represents $\alpha_{4/4}$, which corresponds to the generator $\sigma\in\pi_*(S^0)$.

We will describe $E_1^{n,*}$ for $n\le 1$ at $p>2$. For all primes $E_1^{0,0}=\mathbf{Q}$ (concentrated in dimension 0) and $E_1^{0,s}=0$ for $s>0$. For $p>2$, $E_1^{1,s}=0$ for $s>1$ and $E_1^{1,1}=\mathbf{Q}/\mathbf{Z}_{(p)}$ concentrated in dimension 0, and $E_1^{1,0}$ is trivial in dimensions not divisible by $q=2(p-1)=\dim v_1$ and is generated by all elements of the form v_1^t/pt for $t\in\mathbf{Z}$. Hence if p^i is the largest power of p dividing t, then $E_1^{1,0}\approx\mathbf{Z}/(p^{i+1})$ in dimension qt, and in dimension 0, $E_1^{1,0}=\mathbf{Q}/\mathbf{Z}_{(p)}$.

The differential $d_1\colon E_1^{0,0} \to E_1^{1,0}$ is the usual map $\mathbf{Q} \to \mathbf{Q}/\mathbf{Z}_{(p)}$. Its kernel $\mathbf{Z}_{(p)}$ is $\mathrm{Ext}^0(L \otimes \mathbf{Z}_{(p)})$. On $E_1^{1,1} = \mathbf{Q}/\mathbf{Z}_{(p)}$ the kernel of d_1 is trivial, so $E_2^{1,1} = E_2^{0,2} = 0$ and $\mathrm{Ext}^2(L \otimes \mathbf{Z}_{(p)}) = E_2^{2,0}$. On $E_1^{1,0}$, the kernel of d_1 consists of all elements in nonnegative dimensions. Since the $\mathbf{Q}/\mathbf{Z}_{(p)}$ in dimension 0 is hit by d_1, $E_2^{1,0}$ consists of the positive dimensional elements in $E_1^{1,0}$ and this group is $\mathrm{Ext}^1(L \otimes \mathbf{Z}_{(p)})$. In $\pi_*(S^0)$ it is represented by the p-component of $\operatorname{im} J$.

Now the chromatic E_1-term is periodic in the following sense. By defintion,

$$M^n = \varinjlim v_n^{-1} L/J,$$

where the direct limit is over all invariant regular ideals J with n generators. For each J, $\mathrm{Ext}^0(v_n^{-1}L/J)$ contains some power of v_n, say v_n^k. Then $\mathrm{Ext}(v_n^{-1}L/J)$ is a module over $\mathbf{Z}_{(p)}[v_n^k, v_n^{-k}]$, i.e., multiplication by v_n^k is an isomorphism, so we say that this Ext is v_n-*periodic*. Hence $E_1^{n,*} = \mathrm{Ext}(M^n)$ is a direct limit of such groups. We may say that an element in the Adams–Novikov spectral sequence E_2-term is v_n-*periodic* if it represents an element in $E_\infty^{n,*}$ of the chromatic spectral sequence.

Hence the chromatic spectral sequence E_∞-term is the trigraded group associated with the filtration of $\mathrm{Ext}(L \otimes \mathbf{Z}_{(p)})$ by v_n-periodicity. This filtration is decreasing and has an infinite number of stages in each cohomological degree. One sees this from the diagram

$$\mathrm{Ext}^s(N^0) \leftarrow \mathrm{Ext}^{s-1}(N^1) \leftarrow \dots \leftarrow \mathrm{Ext}^0(N^s)$$

where $N^0 = L \otimes \mathbf{Z}_{(p)}$; the filtration of $\mathrm{Ext}(N^0)$ is by images of the groups $\mathrm{Ext}(N^n)$. This local finiteness allows us to define an increasing filtration on $\mathrm{Ext}(N^0)$ by $F_i \,\mathrm{Ext}^s(N^0) = \operatorname{im} \mathrm{Ext}^i(N^{s-i})$ for $0 \le i \le s$, and $F_0 \,\mathrm{Ext}(N)$ is the subgroup of Greek letter elements in the most general possible sense.

5. Unstable Homotopy Groups and the EHP Spectral Sequence

In this section we will describe the EHP sequence, which is an inductive method for computing $\pi_{n+k}(S^n)$ beginning with our knowledge of $\pi_*(S^1)$ (1.1.7). We will explain how the Adams vector field theorem, the Kervaire invariant problem, and the Segal conjecture are related to the unstable homotopy groups of spheres. We will not present proofs here or elsewhere in the book, nor will we pursue the topic further except in Section 3.3. We are including this survey here because no comparable exposition exists in the literature and we believe these results should be understood by more than a handful of experts. In particular, this section could serve as an introduction to Mahowald [**4**]. For computations at the prime 3, see Toda [**8**], which extends the known range for unstable 3-primary homotopy groups from 55 to 80.

The EHP sequences are the long exact sequences of homotopy groups associated with certain fibration constructed by James [**1**] and Toda [**6**]. There is a different set of fibrations for each prime p. All spaces and groups are assumed localized at the prime in question. We start with $p = 2$. There we have a fibration

(1.5.1) $$S^n \to \Omega S^{n+1} \to \Omega S^{2n+1},$$

which gives the long exact sequence

(1.5.2)
$$\cdots \to \pi_{n+k}(S^n) \xrightarrow{E} \pi_{n+k+1}(S^{n+1}) \xrightarrow{H} \pi_{n+k+1}(S^{2n+1}) \xrightarrow{P} \pi_{n+k-1}(S^n) \to \cdots .$$

Here E stands for Einhängung (suspension), H for Hopf invariant, and P for Whitehead product. If n is odd the fibration is valid for all primes and it splits at odd primes, so for $p > 2$ we have

$$\pi_{2m+k}(S^{2m}) = \pi_{2m+k-1}(S^{2m-1}) \oplus \pi_{2m+k}(S^{4m-1}).$$

This means that even-dimensional spheres at odd primes are uninteresting. Instead one considers the fibration

$$\widehat{S}^{2m} \to \Omega S^{2m+1} \to \Omega S^{2pm+1}, \tag{1.5.3}$$

where the second map is surjective in $H_*(\ ;\mathbf{Z}_{(p)})$, and $\widehat{S}^{2m}$ is the $(2mp-1)$-skeleton of ΩS^{2m+1}, which is a CW-complex with $p-1$ cells of the form $S^{2m} \cup e^{4m} \cup \cdots \cup e^{2(p-1)m}$. The corresponding long exact sequence is

$$\cdots \to \pi_i(\widehat{S}^{2m}) \xrightarrow{E} \pi_{i+1}(S^{2m+1}) \xrightarrow{H} \pi_{i+1}(S^{2pm+1}) \xrightarrow{P} \pi_{i-1}(\widehat{S}^{2m}) \to \cdots . \tag{1.5.4}$$

There is also a fibration

$$S^{2m-1} \to \Omega\widehat{S}^{2m} \to \Omega S^{2pm-1}, \tag{1.5.5}$$

which gives

$$\cdots \to \pi_{i-1}(S^{2m-1}) \xrightarrow{E} \pi_i(\widehat{S}^{2m}) \xrightarrow{H} \pi_i(S^{2pm-1}) \xrightarrow{P} \pi_{i-2}(S^{2m-1}) \to \cdots . \tag{1.5.6}$$

1.5.4 and 1.5.6 are the EHP sequences for odd primes. Note that for $p = 2$, $\widehat{S}^{2m} = S^{2m}$ and both sequences coincide with (1.5.2).

For each prime these long exact sequences fit together into an exact couple (2.1.6) and we can study the associated spectral sequence, namely

1.5.7. PROPOSITION.

(a) *For $p = 2$ there is a spectral sequence converging to π_*^S (stable homotopy) with*

$$E_1^{k,n} = \pi_{k+n}(S^{2n-1}) \qquad \textit{and} \qquad d_r\colon E_r^{k,n} \to E_r^{k-1,n-r}.$$

$E_\infty^{n,k}$ is the subquotient $\operatorname{im}\pi_{n+k}(S^n)/\operatorname{im}\pi_{n+k-1}(S^{n-1})$ of π_k^S. There is a similar spectral sequence converging to $\pi_*(S^j)$ with $E_1^{k,n}$ as above for $n \le j$ and $E_1^{k,n} = 0$ for $n > j$.

(b) *For $p > 2$ there are similar spectral sequences with*

$$E_1^{k,2m+1} = \pi_{k+2m+1}(S^{2pm+1}) \qquad \textit{and} \qquad E_1^{k,2m} = \pi_{k+2m}(S^{2pm-1}).$$

The analogous spectral sequence with $E_1^{k,n} = 0$ for $n > j$ converges to $\pi_*(S^j)$ if j is odd and to $\pi_*(\widehat{S}^j)$ if j is even. □

This is the *EHP spectral sequence*. We will explain below how it can be used to compute $\pi_{n+k}(S^n)$ [or $\pi_{n+k}(\widehat{S}^n)$ if n is even and p is odd] by double induction on n and k. First we make some easy general observations.

1.5.8. PROPOSITION.

(a) *For all primes $E_1^{k,1} = \pi_{1+k}(S^1)$, which is $\mathbf{Z}_{(p)}$ for $k = 0$ and 0 for $k > 0$.*

(b) *For $p = 2$, $E_1^{k,n} = 0$ for $k < n-1$.*

(c) *For $p = 2$, $E_1^{k,n} = \pi_{k-n+1}^S$ for $k < 3n-3$.*

(d) *For $p > 2$, $E_1^{k,2m+1} = 0$ for $k < qm$ and $E_1^{k,2m} = 0$ for $k < qm-1$, where $q = 2(p-1)$.*

(e) *For $p > 2$, $E_1^{k,2m+1} = \pi_{k-qm}^S$ for $k < q(pm+m+1)-2$, and $E_1^{k,2m} = \pi_{k+1-qm}^S$ for $k < q(pm+m)-3$.* □

Part (b) follows from the connectivity of the $(2n-1)$-sphere and similarly for (d); these give us a vanishing line for the spectral sequence. (c) and (e) follow from the fact that $\pi_{2m-1+k}(S^{2m-1}) = \pi_k^S$ for $k < qm-2$, which is in turn a consequence of 1.5.7. We will refer to the region where $n-1 \leq k$ and $E_1^{k,n}$ is a stable stem as the *stable zone*.

Now we will describe the inductive aspect of the EHP spectral sequence. Assume for the moment that we know how to compute differentials and solve the group extension problems. Also assume inductively that we have computed $E_1^{i,j}$ for all (i,j) with $i < k$ and all (k,j) for $j > n$. For $p = 2$ we have $E_1^{k,n} = \pi_{n+k}(S^{2n-1})$. This group is in the $(k-n+1)$-stem. If $n = 1$, this group is $\pi_{1+k}(S^1)$, which is known, so assume $n > 1$. If $n = 2$ this group is $\pi_{2+k}(S^3)$, which is 0 for $k = 0$, $\mathbf{Z}$ for $k = 1$, and for $k > 1$ is the middle term in the short exact sequence

$$0 \to E_2^{k-1,2} \to \pi_{k+2}(S^3) \to \ker d_1 \subset E_2^{k-1,3} \to 0.$$

Note that $E_2^{k-1,2}$ is the cokernel of the d_1 coming from $E_1^{k,3}$ and is therefore known by induction. Finally, if $n > 2$, $E_1^{k,n} = \pi_{n+k}(S^{2n-1})$ can be read off from the already computed portion of the EHP spectral sequence as follows. As in 1.5.7 one obtains a spectral sequence for $\pi_*(S^{2n-1})$ by truncating the EHP spectral sequence, i.e., by setting all $E_1^{j,m} = 0$ for $m > 2n-1$. The group $\pi_{n+k}(S^{2n-1})$ lies in a stem which is already known, so we have $E_1^{k,n}$. Similar remarks apply to odd primes.

We will illustrate the method in detail for $p = 2$ by describing what happens for $0 \leq k \leq 7$ in Fig. 1.5.9. By 1.5.8(c) we have $E_1^{k,k+1} = \pi_0^S = \mathbf{Z}$. Let x_k denote the standard generator of this group. We will see below (1.5.13) that $d_1(x_k) = 2x_{k-1}$ for even positive k and $d_1(x_k) = 0$ otherwise. Hence $E_2^{1,2} = E_\infty^{1,2} = \pi_1^S = \mathbf{Z}/(2)$, so $E_1^{k,k} = \mathbf{Z}/(2)$ for all $k \geq 2$. We denote the generator of each of these groups by 1 to indicate that, if the generator is a permanent cycle, it corresponds to an element whose Hopf invariant suspends to the element corresponding to x_1. Now the first such generator, that of $E_1^{2,2}$, is not hit by a differential, so we have $E_1^{k,k-1} = \pi_{2k-1}(S^{2k-3}) = \mathbf{Z}/(2)$ for all $k \geq 3$. We denote these generators by 11, to indicate that their Hopf invariants each desuspend to elements with Hopf invariant x_1.

In general we can specify an element $\alpha \in \pi_{n+k}(S^n)$ by a sequence of integers adding up to k as follows. Desuspend α as far as possible, say to S^{m+1}. The first integer is then m (necessarily $\leq k$) and the desuspended α has a Hopf invariant $\beta \in \pi_{m+1+k}(S^{2m+1})$. To get the second integer we desuspend β, and so forth. After a finite number of steps we get an element with Hopf invariant in the zero stem and stop the process. Of course there is some indeterminacy in desuspending but we can ignore it for now. We call this sequence of integers the *serial number* of α. In Fig. 1.5.9 we indicate each element of $E_1^{k,n} = \pi_{n+k}(S^{2n-1})$ by its serial number. In almost all cases if $p\alpha \neq 0$, its serial number differs from that of α itself.

To get back to Fig. 1.5.9, we now have to determine the groups $E_1^{k,k-t} = \pi_{2k-2}(S^{2k-5})$ for $k \geq 4$, which means examining the 3-stem in detail. The groups $E_1^{3,2}$ and $E_1^{3,3}$ are not touched by differentials, so there is an short exact sequence

$$0 \to E_1^{3,2} \to \pi_6(S^3) \to E_1^{3,3} \to 0.$$

The two end terms are $\mathbf{Z}/(2)$ and the group extension can be shown to be nontrivial, so $E_1^{4,2} = \pi_6(S^3) = \mathbf{Z}/(4)$. Using the serial number notation, we denote the generator by 21 and the element of order 2 by 111. Similarly one sees $\pi_5(S^2) =$

	π_k^S	$\mathbf{Z}$	$\mathbf{Z}/(2)$	$\mathbf{Z}/(2)$	$\mathbf{Z}/(8)$	0	0	$\mathbf{Z}/(2)$	$\mathbf{Z}/(16)$
	$n \backslash k$	0	1	2	3	4	5	6	7
S^1	1	x_0							
S^3	2		x_1	1	11	111 21	211	2111	
S^5	3		2	x_2	1	11	111 21 3	31	311
S^7	4				x_3	1	11	111 21 3	
S^9	5				2	x_4	1	11	111 21 3
S^{11}	6						x_5	1	11
S^{13}	7						2	x_6	1
S^{15}	8								x_7

FIGURE 1.5.9. The EHP spectral sequence for $p = 2$ and $k \leq 7$.

$\mathbf{Z}/(2)$, $\pi_7(S^4) = \mathbf{Z} \oplus \mathbf{Z}/(4)$ and there is an short exact sequence

$$0 \to \pi_6(S^3) \to \pi_8(S^5) \to E_2^{3,4} \to 0.$$

Here the subgroup and cokernel are $\mathbf{Z}/(4)$ and $\mathbf{Z}/(2)$, respectively, and the group extension is again nontrivial, so $\pi_8(S^5) = E_1^{k,k-2} = \mathbf{Z}/(8)$ for $k \geq 5$. The generator of this group is the suspension of the Hopf map $\nu\colon S^7 \to S^4$ and is denoted by 3.

To determine $E_1^{k,k-3} = \pi_{2k-3}(S^{2k-7})$ for $k \geq 5$ we need to look at the 4-stem, i.e., at the column $E_r^{4,*}$. The differentials affecting those groups are indicated on the chart. Hence we have $E_2^{4,2} = 0$ so $\pi_7(S^3) = E_1^{5,2} = \mathbf{Z}/(2)$; the d_2 hitting $E_1^{4,3}$ means that the corresponding element *dies* (i.e., becomes null homotopic) when suspended to $\pi_9(S^5)$; since it first appears on S^3 we say it is *born* there. Similarly, the generator of $E_1^{4,4}$ corresponds to an element that is born on S^4 and dies on S^6 and hence shows up in $E_1^{6,3} = \pi_9(S^5)$. We leave it to the reader to determine the remaining groups shown in the chart, assuming the differentials are as shown.

We now turn to the problem of computing differentials and group extensions in the EHP spectral sequence. For the moment we will concentrate on the prime 2. The fibration 1.5.1 can be looped n times to give

$$\Omega^n S^n \to \Omega^{n+1} S^{n+1} \to \Omega^{n+1} S^{2n+1}.$$

In Snaith [**1**] a map is constructed from $\Omega^n S^n$ to $Q\mathbf{R}P^{n-1}$ which is compatible with the suspension map $\Omega^n S^n \to \Omega^{n+1} S^{n+1}$. (Here QX denotes $\varinjlim \Omega^k \Sigma^k X$.) Hence we get a commutative diagram

$$\begin{tikzcd} \Omega^n S^n \arrow[r] \arrow[d] & \Omega^{n+1} S^{n+1} \arrow[r] \arrow[d] & \Omega^{n+1} S^{2n+1} \arrow[d] \\ Q\mathbf{R}P^{n-1} \arrow[r] & Q\mathbf{R}P^n \arrow[r] & QS^n \end{tikzcd} \tag{1.5.10}$$

where both rows are fibre sequences and the right-hand vertical map is the standard inclusion. The long exact sequence in homotopy for the bottom row leads to an exact couple and a spectral sequence as in 1.5.7. We call it the *stable EHP spectral sequence.*

There is an odd primary analog of 1.5.10 in which $\mathbf{R}P^n$ is replaced by an appropriate skeleton of $B\Sigma_p$, the classifying space for the symmetric group on p letters. Recall that its mod (p) homology is given by

$$H_i(B\Sigma_p; \mathbf{Z}/(p)) = \begin{cases} \mathbf{Z}/(p) & \text{if } i \equiv 0 \text{ or } -1 \mod (q) \\ 0 & \text{otherwise.} \end{cases} \tag{1.5.11}$$

1.5.12. Proposition. (a) *For $p = 2$ there is a spectral sequence converging to $\pi_*^S(\mathbf{R}P^\infty)$ (stable homotopy of $\mathbf{R}P^\infty$) with $E_1^{k,n} = \pi_{k-n+1}^S$ for $n \geq 2$ and $d_r\colon E_r^{k,n} \to E_r^{k-1,n-r}$. Here $E_\infty^{k,n}$ is the subquotient $\operatorname{im} \pi_k^S(\mathbf{R}P^{n-1})/\operatorname{im} \pi_k^S(\mathbf{R}P^{n-2})$ of $\pi_k^S(\mathbf{R}P^\infty)$. There is a similar spectral sequence converging to $\pi_*^S(\mathbf{R}P^{j-1})$ with $E_1^{k,n}$ as above for $n \leq j$ and $E_1^{k,n} = 0$ for $n > j$.*

(b) *For $p > 2$ there is a similar spectral sequence converging to $\pi_*^S(B\Sigma_p)$ with $E_1^{k,2m+1} = \pi_k^S$ and $E_1^{k,2m} = \pi_{k+1-mq}$. There is a similar spectral sequence with $E_1^{k,n} = 0$ for $n > j$ converging to $\pi_*^S(B\Sigma_p^{(q)j-1})$ if j is even and to $\pi_*(B\Sigma_p^{(q)(j-1)})$ if j is odd.*

(c) *There are homomorphisms to these from the corresponding EHP spectral sequences of 1.5.7 induced by suspension on the E_1 level, e.g., at $p = 2$ by the suspension map $\pi_{k+n}(S^{2n-1}) \to \pi^S_{k-n+1}$. Hence the E_1-terms are isomorphic in the stable zone.*

We remark that this stable EHP spectral sequence is nothing but a reindexed form of the Atiyah–Hirzebruch spectral sequence (see Adams [**4**], Section 7) for $\pi^S_*(B\Sigma_p)$. In the latter one has $E_2^{s,t} = H_s(B\Sigma_p; \pi^S_t)$ and this group is easily seen to be $E_2^{s+t, f(s)}$ in the EHP spectral sequence where

$$f(s) = \begin{cases} s/(p-1)+1 & \text{if } s \equiv 0 \mod (2p-2) \\ (s+1)/(p-1) & \text{if } s \equiv -1 \mod (2p-2). \end{cases}$$

Since everything in 1.5.12 is stable one can use stable homotopy theoretic methods, such as the Adams spectral sequence and K-theory, to compute differentials and group extensions. This is a major theme in Mahowald [**1**]. Differentials originating $E_r^{k,k+1}$ for $p = 2$ correspond to attaching maps in the cellular structure of $\mathbf{R}P^\infty$, and similarly for $p > 2$. For example, we have

1.5.13. Proposition. *In the stable EHP spectral sequence* (1.5.12)*, the differential $d_1\colon E_1^{k,n} \to E_1^{k-1,n-1}$ is multiplication by p if k is even and trivial if k is odd.* □

Another useful feature of this spectral sequence is *James periodicity*: for each r there is a finite i and an isomorphism $E_r^{k,n} \approx E_r^{k+qp^i, n+2p^i}$ which commutes with differentials (note that $q = 2$ when $p = 2$). This fact is a consequence of the vector field theorem and will be explained more fully below (1.5.18).

For $p = 2$, the diagram 1.5.10 can be enlarged as follows. An element in the orthogonal group $O(n)$ gives a homeomorphism $S^{n-1} \to S^{n-1}$. Suspension gives a basepoint-preserving map $S^n \to S^n$ and therefore an element in $\Omega^n S^n$. Hence we have a map $J\colon O(n) \to \Omega^n S^n$ (compare 1.1.12). We also have the reflection map $r\colon \mathbf{R}P^{n-1} \to O(n)$ sending a line through the origin in $\mathbf{R}^n$ to the orthogonal matrix corresponding to reflection through the orthogonal hyperplane. Combining these we get

$$\begin{tikzcd}
\mathbf{R}P^{n-1} \arrow[r] \arrow[d] & \mathbf{R}P^n \arrow[r] \arrow[d] & S^n \arrow[d, equal] \\
O(n) \arrow[r] \arrow[d] & O(n+1) \arrow[r] \arrow[d] & S^n \arrow[d] \\
\Omega^n S^n \arrow[r] \arrow[d] & \Omega^{n+1} S^{n+1} \arrow[r] \arrow[d] & \Omega^{n+1} S^{2n+1} \arrow[d] \\
Q\mathbf{R}P^{n-1} \arrow[r] & Q\mathbf{R}P^n \arrow[r] & QS^n.
\end{tikzcd} \tag{1.5.14}$$

Here the top row is a cofiber sequence while the others are fiber sequences. The right-hand vertical maps are all suspensions, as is the composite $\mathbf{R}P^n \to Q\mathbf{R}P^n$. The second row leads to a spectral sequence (which we call the *orthogonal spectral sequence*) converging to $\pi_*(O)$ which maps to the EHP spectral sequence. The map on

$E_1^{k,n} = \pi_k(S^{n-1})$ is an isomorphism for $k < 2n-3$ by the Freudenthal suspension theorem 1.1.10. The middle right square of this diagram only commmutes after a single looping. This blemish does not affect calculations of homotopy groups.

Hence we have three spectral sequences corresponding to the three lower rows of 1.5.14 and converging to $\pi_*(O)$, the 2-component of π_*^S, and $\pi_*^S(\mathbf{R}P^\infty)$. In all three we have generators $x_k \in E_1^{k,k+1} = \mathbf{Z}$ and we need to determine the first nontrivial differential (if any exists) on it for k odd. We will see that this differential always lands in the zone where all three spectral sequences are isomorphic. In the orthogonal spectral sequence x_k survives to E_r iff the projection $O(k+1)/O(k+1-r) \to S^k$ admits a cross section. It is well known (and easy to prove) that such a cross section exists iff S^k admits $r-1$ linearly independent tangent vector fields. The question of how many such vector fields exist is the *vector field problem*, which was solved by Adams [**16**] (see 1.5.16). We can give equivalent formulations of the problem in terms of the other two spectral sequences.

1.5.15. THEOREM (James [**2, 3**]). *The following three statements are equivalent*:

(a) S^{k-1} *admits* $r-1$ *linearly independent tangent vector fields.*

(b) *Let* ι *be the generator of* $\pi_{2k-1}(S^{2k-1}) = \mathbf{Z}$. *Then* $P(\iota) \in \pi_{2k-3}(S^{k-1})$ (*see* 1.5.2) *desuspend to* $\pi_{2k-r-2}(S^{k-r})$.

(c) *The stable map* $\mathbf{R}P^{k-1}/\mathbf{R}P^{k-r} \to S^{k-1}$ *admits a cross section.* □

The largest possible r above depends on the largest powers of 2 dividing $k+1$. Let $k = 2^j(2s+1)$,

$$\phi(j) = \begin{cases} 2j & \text{if } j \equiv 1 \text{ or } 2 \mod (4) \\ 2j+1 & \text{if } j \equiv 0 \mod (4) \\ 2j+2 & \text{if } j \equiv 3 \mod (4) \end{cases}$$

and $\rho(k) = \phi(j)$.

1.5.16. THEOREM (Adams [**16**]).

(a) *With notation as above,* S^{k-1} *admits* $\rho(k)-1$ *linearly independent tangent vector fields and no more.*

(b) *Let* $\bar\alpha_0 = 2 \in \pi_0^S$ *and for* $j > 0$ *let* $\bar\alpha_j$ *denote the generator of* $\operatorname{im} J$ *in* $\pi_{\rho(j)-1}^S$ (*see* 1.5.15 (c)). *Then in the* 2*-primary EHP spectral sequence* (1.5.7) $d_{\phi(j)}(x_{k-1})$ *is the* (*nontrivial*) *image of* $\bar\alpha_j$ *in* $E_{\phi(j)}^{k-2,k-j}$. □

We remark that the $\rho(k)-1$ vector fields on S^k were constructed long ago by Hurwitz and Radon (see Eckmann [**1**]). Adams [**16**] showed that no more exist by using real K-theory to solve the problem as formulated in 1.5.15(c).

Now we turn to the odd primary analog of this problem, i.e., finding differentials on the generators x_{qk-1} of $E_1^{qk-1,2k} = \mathbf{Z}$. We know of no odd primary analog of the enlarged diagram 1.5.14, so we have no analogs of 1.5.15(a) or 1.5.16(a), but we still call this the *odd primary vector field problem*. The solution is

1.5.17. THEOREM (Kambe, Matsunaga and Toda [**1**]). *Let* $\bar\alpha_j$ *generate* $\operatorname{im} J \subset \pi_{qj-1}^S$ (1.1.12), *let* x_{qk-1} *generate* $E_1^{qk-1,2k}$ *in the EHP spectral sequence* (1.5.7) *for an odd prime* p (*here* $q = 2p-2$), *and let* $k = p^j s$ *with* s *not divisible by* p. *Then* x_{qk-1} *lives to* E_{2j+2} *and* $d_{2j+2}(x_{qk-1})$ *is the* (*nontrivial*) *image of* $\bar\alpha_{j+1}$ *in* $E_{2j}^{qk-2,2k-2j-2}$. □

Now we will explain the James periodicity referred to above. For $p = 2$ let $\mathbf{R}P^n_m = \mathbf{R}P^n/\mathbf{R}P^{m-1}$ for $m \le n$. There is an i depending only on $n-m$ such that $\mathbf{R}P^{n+2^{i+1}}_{m+2^{i+1}} \simeq \Sigma^{2^{i+1}}\mathbf{R}P^n_m$, a fact first proved by James [**3**]. To prove this, let λ be the canonical real line bundle over $\mathbf{R}P^{n-m}$. Then $\mathbf{R}P^n_m$, is the Thom space for $m\lambda$. The reduced bundle $\lambda - 1$ is an element of finite order $2i+1$ in $KO^*(\mathbf{R}P^{n-m})$, so $(2^{i+1}+m)\lambda = m\lambda + 2^{i+1}$ and the respective Thom spaces $\mathbf{R}P^{n+2^{i+1}}_{m+2^{i+1}}$ and $\Sigma^{2^{i+1}}\mathbf{R}P^n_m$ are equivalent. The relevant computations in $KO^*(\mathbf{R}P^{n-m})$ are also central to the proof of the vector field theorem 1.5.16. Similar statements can be made about the odd primary case. Here one replaces λ by the $\mathbf{C}^{p-1}$ bundle obtained by letting Σ_p act via permutation matrices on $\mathbf{C}^p$ and splitting off the diagonal subspace on which Σ_p acts trivially.

For $p = 2$ one can modify the stable EHP spectral sequence to get a spectral sequence converging to $\pi_*(\mathbf{R}P^n_m)$ by setting $E_1^{k,j} = 0$ for $j < m-1$ and $j > n-1$. Clearly the $d_r\colon E_r^{k,n} \to E_r^{k-1,n-r}$ in the stable EHP spectral sequence is the same as that in the spectral sequence for $\pi_*(\mathbf{R}P^{n-1}_{n-r-1})$ and similar statements can be made for $p > 2$, giving us

1.5.18. James Periodicity Theorem. *In the stable EHP spectral sequence* (1.5.12) *there is an isomorphism $E_r^{k,n} \to E_r^{k+qp^i,n+2p^i}$ commuting with d_r, where $i = [r/2]$.* □

Note that 1.5.17 is simpler than its 2-primary analog 1.5.16(b). The same is true of the next question we shall consider, that of the general behavior of elements in $\operatorname{im} J$ in the EHP spectral sequence. It is ironic that most of the published work in this area, e.g., Mahowald [**2, 4**], is concerned exclusively with the prime 2, where the problem appears to be more difficult.

Theorem 1.5.17 describes the behavior of the elements x_{qk-1} in the odd primary EHP spectral sequence and indicates the need to consider the behavior of $\operatorname{im} J$. The elements $\bar\alpha_j$ and their multiples occur in the stable EHP spectral sequence in the groups $E_1^{qk-2,2m}$ and $E_1^{qk-1,2m+1}$ for all $k > m$. To get at this question we use the spectrum J, which is the fibre of a certain map $bu \to \Sigma^2 bu$, where bu is the spectrum representing connective complex K-theory, i.e., the spectrum obtained by delooping the space $\mathbf{Z} \times BU$. There is a stable map $S^0 \to J$ which maps $\operatorname{im} J \subset \pi_*^S$ isomorphically onto $\pi_*(J)$. The stable EHP spectral sequence, which converges to $\pi_*^S(B\Sigma_p)$, maps to a similar spectral sequence converging to $J_*(B\Sigma_p) = \pi_*(J \wedge B\Sigma_p)$. This latter spectral sequence is completely understood and gives information about the former and about the EHP spectral sequence itself.

1.5.19. Theorem.

(a) *For each odd prime p there is a connective spectrum J and a map $S^0 \to J$ sending the p-component of* $\operatorname{im} J$ (1.1.12) *isomorphically onto $\pi_*(J)$, i.e.,*

$$\pi_i(J) = \begin{cases} \mathbf{Z}_{(p)} & \text{if } i = 0 \\ \mathbf{Z}/(p^{j+1}) & \text{if } i = qk-1,\ k > 0,\ k = sp^j \text{ with } p \nmid s \\ 0 & \text{otherwise.} \end{cases}$$

(b) *There is a spectral sequence converging to $J_*(B\Sigma_p)$ with*

$$E_1^{k,2m+1} = \pi_{k-mq}(J) \qquad \textit{and} \qquad E_1^{k,2m} = \pi_{k+1-mq}(J);$$

the map $S^0 \to J$ induces a map to this spectral sequence from the stable EHP spectral sequence of 1.5.12.

(c) *The d_1 in this spectral sequence is determined by 1.5.13. The resulting E_2-term has the following nontrivial groups and no other*:

$$E_2^{qk-1,2k} = \mathbf{Z}/(p) \quad \text{generated by } x_{qk-1} \text{ for } k > 0,$$
$$E_2^{q(k+j)-2,2k} = \mathbf{Z}/(p) \quad \text{generated by } \bar{\alpha}_j \text{ for } k, j > 0,$$

and

$$E_2^{q(k+j)-1,2k+1} = \mathbf{Z}/(p) \quad \text{generated by } \alpha_j \text{ for } k, j > 0,$$

where α_j is an element of order p in $\pi_{qj-1}(J)$.

(d) *The higher differentials are determined by 1.5.17 and the fact that all group extensions in sight are nontrivial, i.e., with k and j as in 1.5.17,*

$$d_{2j+2}(x_{qk-1}) = \bar{\alpha}_{j+1} \in E_{wj+2}^{qk-2,2(k-j-1)}$$

and d_{2j+3} is nontrivial on $E_{2j+3}^{qk-1,2m+1}$ for $j+2 < m < k$.

(e) *The resulting E_∞-term has the following nontrivial groups and no others*: *$E_\infty^{qk-2,2m}$ for $k > m \geq k-j$ and $E_\infty^{qk-1,2m+1}$ for $1 \leq m \leq j+1$. The group extensions are all nontrivial and we have for $i > 0$*

$$J_i(B\Sigma_p) = \pi_i(J) \oplus \begin{cases} \mathbf{Z}/(p^j) & \text{for } i = qsp^j - 2 \text{ with } p \nmid s \\ 0 & \text{otherwise.} \end{cases} \qquad \square$$

We will sketch the proof of this theorem. We have the fibration $J \to bu \to \Sigma^2 bu$ for which the long exact sequence of homotopy groups is known; actually bu (when localized at the odd prime p) splits into $p-1$ summands each equivalent to an even suspension of $BP\langle 1\rangle$, where $\pi_*(BP\langle 1\rangle) = \mathbf{Z}_{(p)}[v_1]$ with $\dim v_1 = q$. It is convenient to replace the above fibration by $J \to BP\langle 1\rangle \to \Sigma^q BP\langle 1\rangle$. We also have a transfer map $B\Sigma_p \to S^0_{(p)}$, which is the map which Kahn and Priddy [**2**] show induces a surjection of homotopy groups in positive dimensions (see also Adams [**15**]); the same holds for J-homology groups. Let R be the cofiber of this map. One can show that $S^0_{(p)} \to R$ induces a monomorphism in $BP\langle 1\rangle$-homology (or equivalently in bu-homology) and that $BP\langle 1\rangle \wedge R \simeq \bigvee_{j\geq 0} \Sigma^{qj} H\mathbf{Z}_{(p)}$, i.e., a wedge of suspensions of integral Eilenberg–Mac Lane spectra localized at p. Smashing these two fibrations together gives us a diagram

(1.5.20)
$$\begin{tikzcd} J \wedge R \arrow[r] & BP\langle 1\rangle \wedge R \arrow[r] & \Sigma^q BP\langle 1\rangle \wedge R \\ J \arrow[r] \arrow[u] & BP\langle 1\rangle \arrow[r, "f"] \arrow[u] & \Sigma^q BP\langle 1\rangle \arrow[u] \\ J \wedge B\Sigma_p \arrow[r] \arrow[u] & BP\langle 1\rangle \wedge B\Sigma_p \arrow[r] \arrow[u] & \Sigma^q BP\langle 1\rangle \wedge B\Sigma_p \arrow[u] \end{tikzcd}$$

in which each row and column is a cofiber sequence. The known behavior of $\pi_*(f)$ determines that of $\pi_*(f \wedge R)$ and enables one to compute $\pi_*(J \wedge B\Sigma_p) = J_*(B\Sigma_p)$. The answer, described in 1.5.19(c), essentially forces the spectral sequence of 1.5.19

to behave in the way it does. The E_2-term [1.5.19(c)] is a filtered form of $\pi_*(BP\langle 1\rangle\wedge B\Sigma_p)\oplus\pi_*(\Sigma^{q-1}BP\langle 1\rangle\wedge B\Sigma_p)$.

Corresponding statements about the EHP spectral sequence are not yet known but can most likely be proven by using methods of Mahowald [**4**]. We surmise they can be derived from the following.

1.5.21. CONJECTURE.

(a) *The composite* $\pi_k(\Omega^{2n+1}S^{2n+1})\to\pi_k(QB\Sigma_p^{qn})\to J_k(B\Sigma_p^{qn})$ *is onto unless* $k=qsp^j-2$ (*with* $j>0$, $sp^j>p$ *and* $p\nmid s$) *and* $n=sp^j-i$ *for* $1\le i\le j$.

(b) *The groups* $E_\infty^{qk-1,2m+1}$ *of* 1.5.19(e) *pull back to the* E_∞*-term of the EHP spectral sequence and correspond to the element* $\alpha_{k/m}$ (1.3.19) *of order* p^m *in* $\operatorname{im}J\in\pi_{qk-1}^S$. *Hence* $\alpha_{k/m}$ *is born in* S^{2m+1} *and has Hopf invariant* α_{k-m} *except for* α_1, *which is born on* $\widehat{S}^2$ *with Hopf invariant one.* (*This was not suspected when the notation was invented*!) $\square$

We will give an example of an exception to 1.5.21(a) for $p=3$. One has age $\alpha_8\in E_3^{39,5}$, which should support a d_3 hitting $\bar\alpha_9\in E_3^{38,2}$, but $E_1^{38,2}=\pi_{40}(S^5)$ and α_9 is only born on S^7, so the proposed d_3 cannot exist (this problem does not occur in the stable EHP spectral sequence). In fact, $\alpha_1\alpha_8\neq 0\in\pi_{41}(S^7)=E_1^{38,3}$ and this element is hit by a d_2 supported by the $\alpha_8\in E_2^{39,5}$.

The other groups in 1.5.19(e), $J_{pqi-2}(B\Sigma_p)$, are harder to analyze. $E_\infty^{pq-2,q}$ pulls back to the EHP spectral sequence and corresponds to $\beta_1\in\pi_{pq-2}^S$ (1.3.14), the first stable element in coker J (1.1.12), so β_1 is born on $\widehat{S}^q$ and has Hopf invariant α_1. Presumably the corresponding generators of $E_r^{piq-2,2pi-2}$ for $i>1$ each supports a nontrivial d_q hitting a β_1 in the appropriate group. The behavior of the remaining elements of this sort is probably determined by that of the generators of $E_2^{p^jq-2,wp^j-2j}$ for $j\ge 2$, which we now denote by $\tilde\theta_j$. These appear to be closely related to the Arf invariant elements $\theta_j=\beta_{p^{j-1}/p^{j-1}}$ (1.4.10) in E_2^{2,p^jq} of the Adams–Novikov spectral sequence. The latter are known not to survive (6.4.1), so presumably the $\tilde\theta_j$ do not survive either. In particular we know $d_{2p^2-6}(\tilde\theta_2)=\beta_1^p$ in the appropriate group. There are similar elements at $p=1$ as we shall see below. In that case the θ_j are presumed but certainly not known (for $j>5$) to exist in $\pi_{2^{j+1}-2}^S$. Hence any program to prove their existence at $p=2$ is doomed to fail if it would also lead to a proof for $p>2$.

We now consider the 2-primary analog of 1.5.19 and 1.5.21. The situation is more complicated for four reasons.

(1) $\operatorname{im}J$ (1.5.15) is more complicated at $p=2$ than at odd primes.

(2) The homotopy of J (which is the fiber of a certain map $bo\to\Sigma^4 bsp$, where bo and bsp are the spectra representing connective real and symplectic K-theory, respectively) contains more than just $\operatorname{im}J$.

(3) Certain additional exceptions have to be made in the analog 1.5.21.

(4) The groups corresponding to the $J_{piq-2}(B\Sigma_p)$ are more complicated and lead us to the elements $\eta_j\in\pi_{2^j}^S$ of Mahowald [**6**] in addition to the hypothetical $\theta_j\in\pi_{2^{j+1}-2}^S$.

Our first job then is to describe $\pi_*(J)$ and how it differs from $\operatorname{im}J$ as described in 1.1.12. We have $\pi_i(bo)=\pi_{i+7}(O)$ and $\pi_i(bsp)=\pi_{i+3}(O)$ for $i\ge 0$ and $\pi_*(O)$ is described in 1.1.11, i.e.,

$$\pi_i(O) = \begin{cases} \mathbf{Z} & \text{if } i = 3 \mod (4) \\ \mathbf{Z}/(2) & \text{if } i = 0 \text{ or } 1 \mod (8) \\ 0 & \text{otherwise.} \end{cases}$$

The map $bo \to \Sigma^4 bsp$ used to define J is trivial on the torsion in $\pi_*(bo)$, so these groups pull back to $\pi_*(J)$. Hence $\pi_{8i+1}(J)$ and $\pi_{8i+2}(J)$ for $i \geq 1$ contain summands of order 2 not coming from $\operatorname{im} J$.

1.5.22. PROPOSITION. *At $p = 2$*

$$\pi_i(J) = \begin{cases} \mathbf{Z}_{(2)} & \text{if } i = 0 \\ \mathbf{Z}/(2) & \text{if } i = 1 \text{ or } 2 \\ \mathbf{Z}/(8) & \text{if } i \equiv 3 \mod (8) \text{ and } i > 0 \\ \mathbf{Z}/(2) & \text{if } i \equiv 0 \text{ or } 2 \mod 8 \text{ and } i \geq 8 \\ \mathbf{Z}/(2) \oplus \mathbf{Z}/(2) & \text{if } i \equiv 1 \mod (8) \text{ and } i \geq 9 \\ \mathbf{Z}/(2^{j+1}) & \text{if } i = 8m-1,\ m \geq 1 \text{ and } 8m = 2^j(2s+1). \end{cases}$$

Here, $\operatorname{im} J \subset \pi_*(J)$ consists of cyclic summands in $\pi_i(J)$ for $i > 0$ and $i \equiv 7$, 0, 1 or 3 $\mod (8)$. □

Now we need to name certain elements in $\pi_*(J)$. As in 1.5.16 let $\bar{\alpha}_j$ denote the generator of $\operatorname{im} J$ in dimension $\phi(j) - 1$, where

$$\phi(j) - 1 = \begin{cases} 2j-1 & \text{if } j \equiv 1 \text{ or } 2 \mod (4) \\ 2j & \text{if } j \equiv 0 \mod (4) \\ 2j+1 & \text{if } j \equiv 3 \mod (4). \end{cases}$$

We also define elements α_j in $\pi_*(J)$ of order 2 as follows. $\alpha_1 = \eta \in \pi_1(J)$ and $\alpha_{4k+1} \in \pi_{8k+1}(J)$ is a certain element not in $\operatorname{im} J$ for $k \geq 1$. $\alpha_{4k+2} = \eta\alpha_{4k+1}$, $\alpha_{4k+3} = \eta^2\alpha_{4k+1} = 4\bar{\alpha}_{4k+2}$, and $\alpha_{4k} \in \pi_{8k-1}(J)$ is an element of order 2 in that cyclic group.

1.5.23. THEOREM (Mahowald [4]). (a) *There is a spectral sequence converging to $J_*(\mathbf{R}P^\infty)$ with $E_1^{k,n} = \pi_{k-n+1}(J)$; the map $S^0 \to J$ induces a homomorphism to this spectral sequence from the stable EHP spectral sequence of 1.5.12. (We will denote the generator of $E_1^{k,k+1}$ by x_k and the generator of $E_1^{k,k+1+m}$ for $m > 0$ by the name of the corresponding element in $\pi_m(J)$.)*

(b) *The d_1 in this spectral sequence is determined by 1.5.13. The following is a complete list of nontrivial d_2's and d_3's.*

For $k \geq 1$ and $t \geq 0$, d_2 sends

$$\begin{array}{rll} x_{4k+1} \in E_2^{4k+1,4k+2} & \text{to} & \alpha_1 \\ \bar{\alpha}_{4t+3+i} \in E_2^{4k+8+i+8t,4k+2} & \text{to} & \bar{\alpha}_{4t+i} \quad \text{for } i = 0,\ 1 \\ \alpha_{4t+1} \in E_2^{4k+2+8t,4k+2} & \text{to} & \alpha_{4t+2} \\ \bar{\alpha}_{4t+4} \in E_2^{4k+1+8t+7,4k+1} & \text{to} & \bar{\alpha}_{4t+5} \end{array}$$

and

$$\alpha_{4t+i} \in E_2^{4k+i+8t,4k+1} \quad \text{to} \quad \alpha_{4t+i+1} \quad \text{for } i = 1,\ 2.$$

For $k \geq 1$ and $t \geq 1$, d_3 sends

$$\alpha_{4t} \in E_2^{4k+1+8t,4k+3} \quad \text{to} \quad \alpha_{4t+1}$$

and

$$\bar{\alpha}_{4t+1} \in E_2^{4k+8t+1,4k+1} \quad \text{to} \quad \bar{\alpha}_{4t+2}.$$

See FIG. 1.5.24.

(c) *The resulting E_4-term is a $\mathbf{Z}/(2)$-vector space on the following generators for $k \geq 1$, $t \geq 0$.*

$$x_1 \in E_4^{1,2}; \quad \bar{\alpha} \in E_4^{4,2}; \quad \alpha_{4t+i} \in E_4^{8t+i+1,2} \quad \text{for } i = 1,\, 2;$$
$$\bar{\alpha}_{4t+i} \in E_4^{8t+i+5,2} \quad \text{for } i = 3,\, 4,\, 5; \quad \alpha_{4t+1} \in E_4^{8t+3,3}; \quad \alpha_{4t+4} \in E_4^{8t+9,3};$$
$$\bar{\alpha}_{4t+4} \in E_4^{8t+10,3}; \quad x_{4k-1} \in E_4^{4k-1,4k}; \quad \bar{\alpha}_{4t+2} \in E_4^{4k+8t+2,4k};$$
$$\bar{\alpha}_{4t+3} \in E_4^{4k+8t+6,4k}; \quad \alpha_{4t+3} \in E_4^{4k+8t+3,4k+1}; \quad \alpha_{4t+4} \in E_4^{4k+8t+7,4k+1};$$
$$\alpha_{4t+2} \in E_4^{4k+8t+3,4k+2}; \quad \bar{\alpha}_2 \in E_4^{4k+4,4k+2}; \quad \bar{\alpha}_{4t+5} \in E_4^{4k+8t+10,4k+2};$$
$$\alpha_{4t+1} \in E_4^{4k+8t+3,4k+3}; \quad \text{and} \quad \bar{\alpha}_{4t+4} \in E_4^{4k+8t+10,4k+3}.$$

(d) *The higher differentials are determined by* 1.5.15 *and the fact that most group extensions in sight are nontrivial. The resulting E_∞-term has the following additive generators and no others for $t \geq 0$.*

$$x_1 \in E_\infty^{1,2}; \quad \alpha_{4t+4} \in E_\infty^{8t+9,3}; \quad \alpha_{4t+i} \in E_\infty^{8t+i+1,2} \quad \text{for } i = 1,\, 2;$$
$$\alpha_{4t+1} \in E_\infty^{8t+3,3}; \quad x_3 \in E_\infty^{3,4}; \quad \alpha_{4t+4} \in E_\infty^{8t+11,5};$$
$$\bar{\alpha}_{4t+i} \in E_\infty^{8t+i+5,2} \quad \text{for } i = 3,\, 4; \quad x_7 \in E_\infty^{7,8};$$
$$\alpha_{4t+4} \in E_\infty^{8t+15,9}; \quad \alpha_{4t+i} \in E_\infty^{8t+7,8-i} \quad \text{for } i = 1,2,3;$$
$$\alpha_{2^j t+2^j-j-2} \in E_\infty^{2^{j+1}(1+t)-1,*} \quad \text{for } j \geq 3;$$
$$\bar{\alpha}_2 \in E_\infty^{4t+4,4t+2}; \quad \text{and} \quad \bar{\alpha}_j \in E_\infty^{2^{j+1}(t+1)-2,*} \quad \text{for } j \geq 2.$$

(e) *For $i > 0$*

$$J_i(\mathbf{R}P^\infty) = \pi_i(J) \oplus \begin{cases} \mathbf{Z}/(2) & \text{if } i \equiv 0 \mod (4) \\ \mathbf{Z}/(2^j) & \text{if } i = 2^{j+2}s - 2 \text{ for } s \text{ odd} \\ 0 & \text{otherwise.} \end{cases} \qquad \square$$

Note that the portion of the E_∞-term corresponding to the summand $\pi_*(J)$ in 1.5.23(e) [i.e., all but the last two families of elements listed in 1.5.23(d)] is near the line $n = 0$, while that corresponding to the second summand is near the line $n = k$.

The proof of 1.5.23 is similar to that of 1.5.19 although the details are messier. One has fibrations $J \to bo \to \Sigma^4 bsp$ and $\mathbf{R}P^\infty \to S^0_{(2)} \to R$. We have $R \wedge bo \simeq \bigvee_{j \geq 0} \Sigma^{4j} H\mathbf{Z}_{(2)}$ and we can get a description of $R \wedge bsp$ from the fibration $\Sigma^4 bo \to bsp \to H\mathbf{Z}_{(2)}$. The E_4-term in 1.5.22 is a filtered form of $\pi_*(\Sigma^3 bsp \wedge \mathbf{R}P^\infty) \oplus \pi_*(bo \wedge \mathbf{R}P^\infty)$; elements with Hopf invariants of the form $\bar{\alpha}_j$ are in the first summand while the other generators make up the second summand. By studying the analog of 1.5.20 we can compute $J_*(\mathbf{R}P^\infty)$ and again the answer [1.5.23(e)] forces the spectral sequence to behave the way it does.

Now we come to the analog of 1.5.21.

$n \backslash k$	1	2	3	4	5	6	7	8	9	10	11	12	13	14
2	x_1	α_1	α_2	$\bar{\alpha}_2$				$\bar{\alpha}_3$	$\bar{\alpha}_4$	$\bar{\alpha}_5$ α_5	α_6	$\bar{\alpha}_6$		
3			α_1	α_2	α_3				α_4		$\bar{\alpha}_5$ α_5	α_6	α_7	
4			x_3	α_1	α_2					$\bar{\alpha}_3$	$\bar{\alpha}_4$	α_5 $\bar{\alpha}_5$	α_6	$\bar{\alpha}_6$
5					α_1	α_2	α_3				α_4	$\bar{\alpha}_4$	$\bar{\alpha}_5$ α_5	α_6
6					x_5	α_1	α_2	$\bar{\alpha}_2$				$\bar{\alpha}_3$	$\bar{\alpha}_4$	$\bar{\alpha}_5$ α_5
7							α_1	α_2	α_3				α_4	$\bar{\alpha}_4$

FIGURE 1.5.24. A portion of the E_2-term of the spectral sequence of Theorem 1.5.23 converging to $J_*(\mathbf{R}P^\infty)$ and showing the d_2's and d_3's listed in Theorem 1.5.23, part (c).

1.5.25. THEOREM (Mahowald [4]). (a) *The composite*

$$\pi_k(\Omega^{2n+k}S^{2n+1}) \to \pi_k(Q\mathbf{R}P^{2n}) \to J_k(\mathbf{R}P^{2n})$$

is onto unless $k \equiv 0 \mod (4)$ and $k \leq 2n$, or $k \equiv 6 \mod (8)$. It is also onto if $k = 2^j$ for $j \geq 3$ or if $k \equiv 2^j - 2 \mod (2^{j+1})$ and $k \geq 2n + 8 + 2j$. When $k \leq 2n$ is a multiple of 4 and not a power of 2 at least 8, then the cokernel is $\mathbf{Z}/(2)$; when $k \leq 2n$ is 2 less than a multiple of 8 but not 2 less than a power of 2, then the cokernel is $J_k(\mathbf{R}P^{2n}) = J_k(\mathbf{R}P^\infty)$.

(b) *All elements in the E_∞-term corresponding to elements in $\pi_*(J)$ pull back to the EHP spectral sequence except some of the $\bar{\alpha}_{4t+i} \in E_\infty^{8t+i+5,2}$ for $i = 3, 4$ and $t \geq 0$. Hence $H(\alpha_1) = H(\bar{\alpha}_2) = H(\bar{\alpha}_3) = 1$, $H(\alpha_{t+1}) = \alpha_t$, and if $2^i x = \alpha_{t+1}$ for $x \in \operatorname{im} J$ then $H(x) = \alpha_{t-i}$.* $\square$

Theorem 1.5.23 leads one to believe that $H(\bar{\alpha}_{4t+i}) = \bar{\alpha}_{4t+i-1}$ for $i = 4$, 5 and $t \geq 0$, and that these elements are born on S^2, but this cannot be true in all cases. If $\bar{\alpha}_4$ were born on S^2, its Hopf invariant would be in $\pi_{10}(S^3)$, but this group does not contain $\bar{\alpha}_3$, which is born on S^4. In fact we find $H(\bar{\alpha}_4) = \bar{\alpha}_2$, $H(\bar{\alpha}_5) = \bar{\alpha}_2^2$, and $H(\bar{\alpha}_8)$ is an unstable element.

1.5.26. REMARK. Theorem 1.5.25(b) shows that the portion of $\operatorname{im} J$ generated by $\bar{\alpha}_{4t+2}$ and $\bar{\alpha}_{4t+3}$, i.e., the cyclic summands of order ≥ 8 in dimensions $4k-1$, are born on low-dimensional spheres, e.g., $\bar{\alpha}_{4t+2}$ is born on S^5. However, simple calculations with 1.5.14 show that the generator of $\pi_{4k-1}(O)$ pulls back to $\pi_{4k-1}(O(2k+1))$ and no further. Hence $\bar{\alpha}_{4t+2} \in \pi_{qt+8}(S^5)$ is not actually in the image of the unstable J-homomorphism until it is suspended to S^{4t+3}.

Now we consider the second summand of $J_*(\mathbf{R}P^\infty)$ of 1.5.23(e). The elements $\bar{\alpha}_2 \in E_\infty^{4k,4k-2}$ for $k \geq 1$ have no odd primary analog and we treat them first. The main result of Mahowald [6] says there are elements $\eta_j \in \pi_{2^j}(S^0)$ for $j \geq 3$ with Hopf invariant $v = \bar{\alpha}_2$. This takes care of the case $k = 2^{j-2}$ above.

1.5.27. THEOREM. *In the EHP spectral sequence the element $\nu = \bar{\alpha}_2 \in E_1^{4k,4k-2}$ for $k \geq 2$ behaves as follows (there is no such element for $k = 1$).*

(a) *If $k = 2^{j-2}$, $j \geq 3$ then the element is a permanent cycle corresponding to η_j; this is proved by Mahowald [6].*

(b) *If $k = 2s + 1$ then $d_4(\nu) = \nu^2$.* $\square$

1.5.28. CONJECTURE. *If $k = (2s+1)2^{j-2}$ with $s > 0$ then $d_{2^j-2}(\nu) = \eta_j$.* $\square$

The remaining elements in 1.5.23(e) appear to be related to the famous Kervaire invariant problem (Mahowald [7], Browder [1]).

1.5.29. CONJECTURE. *In the EHP spectral sequence the elements*

$$\bar{\alpha}_j \in E_2^{2^{j+1}(t+1)-2,*} \qquad \text{for } j \geq 2,\ t \geq 0$$

behave as follows:

(a) *If there is a framed $(2^{j+1}-2)$-manifold with Kervaire invariant one then $\bar{\alpha}_j \in E_2^{2^{j+1}-2,*}$ is a nontrivial permanent cycle corresponding to an element $\theta_j \in \pi_{2^{j+1}-2}(S^0)$ (These elements are known (Barratt, Jones, and Mahowald [2]) to exist for $j \geq 0$.)*

(b) *If (a) is true then the element $\bar{\alpha}_j \in E_2^{2^{j+1}(2s+1)-2,*}$ satisfies $d_r(\bar{\alpha}_j) = \theta_j$ where $r = 2^{j+1} - 1 - \dim(\bar{\alpha}_j)$.* $\square$

The converse of 1.5.29(a) is proved by Mahowald [**4**] 7.11.

Now we will describe the connection of the EHP spectral sequence with the Segal conjecture. For simplicity we will limit our remarks to the 2-primary case, although everything we say has an odd primary analog. As remarked above, the stable EHP spectral sequence (1.5.12) can be modified so as to converge to the stable homotopy of a stunted projective space. Let $\mathbf{R}P_j = \mathbf{R}P^\infty/\mathbf{R}P_{j-1}$ for $j > 0$; i.e., $\mathbf{R}P^j$ is the infinite-dimensional stunted projective space whose first cell is in dimension j. It is easily seen to be the Thom spectrum of the j-fold Whitney sum of the canonical line bundle over $\mathbf{R}P^\infty$. This bundle can be defined stably for $j \le 0$, so we get Thom spectra $\mathbf{R}P_j$ having one cell in each dimension $\ge j$ for any integer j.

1.5.30. Proposition. *For each $j \in \mathbf{Z}$ there is a spectral sequence converging to $\pi_*(\mathbf{R}P_j)$ with*

$$E_1^{k,n} = \begin{cases} \pi_{k-n+1}(S^0) & \text{if } n-1 \ge j \\ 0 & \text{if } n-1 < j \end{cases}$$

and $d_r\colon E_r^{k,n} \to E_r^{k-1,n-r}$. For $j = 1$ this is the stable EHP spectral sequence of 1.5.12. If $j < 1$ this spectral sequence maps to the stable EHP spectral sequence, the map being an isomorphism on $E_1^{k,n}$ for $n \ge 2$. □

The Segal conjecture for $\mathbf{Z}/(2)$, first proved by Lin [**1**], has the following consequence.

1.5.31. Theorem. *For each $j < 0$ there is a map $S^{-1} \to \mathbf{R}P_j$ such that the map $S^{-1} \to \mathbf{R}P_{-\infty} = \varprojlim \mathbf{R}P_j$ is a homotopy equivalent after 2-adic completion of the source (the target is already 2-adically complete since $\mathbf{R}P_j$ is for j odd). Consequently the inverse limit over j of the spectral sequences of 1.5.30 converges to the 2-component of $\pi_*(S^{-1})$. We will call this limit spectral sequence the* superstable EHP spectral sequence. □

Nothing like this is stated in Lin [**1**] even though it is an easy consequence of his results. A proof and some generalizations are given in Ravenel [**4**]. Notice that $H_*(\mathbf{R}P_{-\infty}) \ne \varprojlim H_*(\mathbf{R}P_j)$; this is a spectacular example of the failure of homology to commute with inverse limits. Theorem 1.5.31 was first conjectured by Mahowald and was discussed by Adams [**14**].

Now consider the spectrum $\mathbf{R}P_0$. It is the Thom spectrum of the trivial bundle and is therefore $S^0 \vee \mathbf{R}P_1$. Hence for each $j < 0$ there is a map $\mathbf{R}P_j \to S^0$ which is nontrivial in mod (2) homology. The cofiber of this map for $j = -1$ can be shown to be R, the cofiber of the map $\mathbf{R}P_1 \to S^0$ of Kahn and Priddy [**2**]. The Kahn–Priddy theorem says this map is surjective in homotopy in positive dimensions. Using these facts we get

1.5.32. Theorem. *In the spectral sequence of 1.5.30 for $j < 0$,*

(a) *no element in $E_r^{0,k}$ supports a nontrivial differential;*

(b) *no element in $E_r^{1,k}$ is the target of a nontrivial differential;*

(c) *every element of $E_1^{0,k} = \pi_{k+1}(S^0)$ that is divisible by 2 is the target of a nontrivial d_1 and every element of $E_2^{0,k}$ for $k > -1$ is the target of some d_r for $r \ge 2$; and*

(d) *every element in $E_1^{1,k} = \pi_k(S^0)$ not of order 2 supports a nontrivial d_1 and every element of $E_2^{1,k}$ supports a nontrivial d_r for some $r \ge 2$.* □

PROOF. Parts (a) and (b) follow from the existence of maps $S^{-1} \to \mathbf{R}P_j \to S^0$, (c) follows from the Kahn–Priddy theorem, and (d) follows from the fact that the map $\varprojlim \mathbf{R}P_j \to S^0$ is trivial. $\square$

Now the spectral sequence converges to $\pi_*(S^{-1})$, yet 1.5.32(c) indicates that the map $S^{-1} \to \mathbf{R}P_{-\infty}$ induces a trivial map of E_∞-terms, except for $E_\infty^{-1,0}$, where it is the projection of $\mathbf{Z}$ onto $\mathbf{Z}/(2)$. [Here we are using a suitably indexed, collapsing AHSS for $\pi_*(S^{-1})$.] This raises the following question: what element in $E_\infty^{k,-n}$ (for some $n > 0$) corresponds to a given element $x \in \pi_k(S^{-1})$? The determination of n is equivalent to finding the smallest n such that the composite $S^k \xrightarrow{x} S^{-1} \to \mathbf{R}P_{-n-1}$ is nontrivial. The Kahn–Priddy theorem tells us this composite is trivial for $n = 0$ if $k \geq 0$ or $k = -1$ and x is divisible by 2; and the Segal conjecture (via 1.5.31) says the map is nontrivial for some $n > 0$. Now consider the cofiber sequence $S^{-n-1} \to \mathbf{R}P_{-n-1} \to \mathbf{R}P_{-n}$. The map from S^k to $\mathbf{R}P_{-n}$ is trivial by assumption so we get a map from S^k to S^{-1-n}, defined modulo some indeterminacy. Hence $x \in \pi_{k+1}(S^0)$ gives us a coset $M(x) \subset \pi_{k+1+n}(S^0)$ which does not contain zero. We call $M(x)$ the Mahowald invariant of x, and note that n, as well as the coset, depends on x. The invariant can be computed in some cases and appears to be very interesting. For example, we have

1.5.33. THEOREM. *Let $\imath$ be a generator of $\pi_0(S^0)$. Then for each $j > 0$, $M(2^j\imath)$ contains α_j, a preimage in $\pi_*(S^0)$ of the $\alpha_j \in \pi_*(J)$ of 1.5.23.* $\square$

A similar result holds for odd primes. In 1.5.31 we replace the $\mathbf{R}P_j$ by Thom spectra of certain bundles over $B\Sigma_p$, and $M(p^j\imath) \ni \alpha_j$ for α_j, as in 1.5.19. We also have

1.5.34. CONJECTURE. *$M(\theta_j)$ contains θ_{j+1} for θ_j as in 1.5.29.* $\square$

1.5.35. CONJECTURE. *Whenever the Greek letter elements (1.3.17) $\alpha_j^{(n)}$ and $\alpha_j^{(n+1)}$ exist in homotopy, $\alpha_j^{(n+1)} \in M(\alpha_j^{(n)})$.* $\square$

One can mimic the definition of the Mahowald invariant in terms of the Adams spectral sequence or Adams–Novikov E_2-terms and in the latter case prove an analog of these conjectures. At $p = 2$ one can show (in homotopy) that $M(\alpha_1) \ni \bar{\alpha}_2$, $M(\bar{\alpha}_2) \ni \bar{\alpha}_3$, and $M(\bar{\alpha}_3) \ni \bar{\alpha}_3^2 = \theta_3$. This suggests using the iterated Mahowald invariant to *define* (up to indeterminacy) Greek letter elements in homotopy, and that θ_j is a special case (namely $\alpha_1^{(j+1)}$) of this definition.

CHAPTER 2

Setting up the Adams Spectral Sequence

In this chapter we introduce the spectral sequence that will be our main object of study. We do not intend to give a definitive account of the underlying theory, but merely to make the rest of the book intelligible. Nearly all of this material is due to Adams. The classical Adams spectral sequence [i.e., the one based on ordinary mod (p) cohomology] was first introduced in Adams [**3**] and a most enjoyable exposition of it can be found in Adams [**7**]. In Section 1 we give a fairly self-contained account of it, referring to Adams [**4**] only for standard facts about Moore spectra and inverse limits. We include a detailed discussion of how one extracts differentials from an exact couple and a proof of convergence.

In Section 2 we describe the Adams spectral sequence based on a generalized homology theory E_* satisfying certain assumptions (2.2.5). We rely heavily on Adams [**4**], referring to it for the more difficult proofs. The E_*-Adams resolutions (2.2.1) and spectral sequences (2.2.4) are defined, the E_2-term is identified, and the convergence question is settled (2.2.3). We do not give the spectral sequence in its full generality; we are only concerned with computing $\pi_*(Y)$, not $[X, Y]$ for spectra X and Y. Most of the relevant algebraic theory, i.e., the study of Hopf algebroids, is developed in Appendix 1.

In Section 3 we study the pairing of Adams spectral sequences induced by a map $\alpha\colon X' \wedge X'' \to X$ and the connecting homomorphism associated with a cofibration realizing a short exact sequence in E-homology. Our smash product result implies that for a ring spectrum the Adams spectral sequence is one of differential algebras. To our knowledge these are the first published proofs of these results in such generality.

Throughout this chapter and the rest of the book we assume a working knowledge of spectra and the stable homotopy category as described, for example, in the first few sections of Adams [**4**].

1. The Classical Adams Spectral Sequence

In this section we will set up the Adams spectral sequence based on ordinary mod (p) cohomology for the homotopy groups of a spectrum X. Unless otherwise stated all homology and cohomology groups will have coefficients in $\mathbf{Z}/(p)$ for a prime number p, and X will be a connective spectrum such that $H^*(X)$ (but not necessarily X itself) has finite type.

Recall that $H^*(X)$ is a module over the mod (p) Steenrod algebra A, to be described explicitly in the next chapter. Our object is to prove

2.1.1. Theorem (Adams [**3**]). *Let X be a spectrum as above. There is a spectral sequence*

$$E_*^{**}(X) \quad \text{with} \quad d_r\colon E_r^{s,t} \to E_r^{s+r,t+r-1}$$

such that

(a) $E_2^{s,t} = \mathrm{Ext}_A^{s,t}(H^*(X), \mathbf{Z}/(p))$.

(b) *if X is of finite type, E_∞^{**} is the bigraded group associated with a certain filtration of $\pi_*(X) \otimes \mathbf{Z}_p$, where $\mathbf{Z}_p$ denotes the ring of p-adic integers.* $\square$

Let $E = H\mathbf{Z}/(p)$, the mod (p) Eilenberg–Mac Lane spectrum. We recall some of its elementary properties.

2.1.2. Proposition.

(a) $H_*(X) = \pi_*(E \wedge X)$.

(b) $H^*(X) = [X, E]$.

(c) $H^*(E) = A$.

(d) *If K is a locally finite wedge of suspensions of E, i.e., a generalized* mod (p) *Eilenberg–Mac Lane spectrum, then $\pi_*(K)$ is a graded $\mathbf{Z}/(p)$-vector space with one generator for each wedge summand of K. More precisely, $\pi_*(K) = \mathrm{Hom}_A(H^*(K), \mathbf{Z}/(p))$.*

(e) *A map from X to K is equivalent to a locally finite collection of elements in $H^*(X)$ in the appropriate dimensions. Conversely, any locally finite collection of elements in $H^*(X)$ determines a map to such a K.*

(f) *If a locally finite collection of elements in $H^*(X)$ generate it as an A-module, then the corresponding map $f\colon X \to K$ induces a surjection in cohomology.*

(g) *$E \wedge X$ is a wedge of suspensions of E with one wedge summand for each $\mathbf{Z}/(p)$ generator of $H^*(X)$. $H^*(E \wedge X) = A \otimes H^*(X)$ and the map $f\colon X \to E \wedge X$ (obtained by smashing X with the map $S^0 \to E$) induces the A-module structure map $A \otimes H^*(X) \to H^*(X)$ in cohomology. In particular $H^*(F)$ is a surjection.* $\square$

The idea behind the Adams spectral sequence is to use maps such as those of (f) or (g) and our knowledge of $\pi_*(K)$ or $\pi_*(E \wedge X)$ to get information about $\pi_*(X)$. We enlist the aid of homological algebra to make the necessary calculations.

More specifically, we have

2.1.3. Definition. *A* mod (p) *Adams resolution (X_s, g_s) for X is a diagram*

$$\begin{tikzcd} X = X_0 \arrow[d, "f_0"] & X_1 \arrow[l, "g_0"'] \arrow[d, "f_1"] & X_2 \arrow[l, "g_1"'] \arrow[d, "f_2"] & \cdots \arrow[l, "g_2"'] \\ K_0 & K_1 & K_2 & \end{tikzcd}$$

where each K_s, is a wedge of suspensions of E, $H^*(f_s)$ is onto and X_{s+1} is the fiber of f_s. $\square$

Proposition 2.1.2(f) and (g) enable us to construct such resolutions for any X, e.g., by setting $K_s = E \wedge X_s$. Since $H^*(f_s)$ is onto we have short exact sequences

$$0 \leftarrow H^*(X_s) \leftarrow H^*(K_s) \leftarrow H^*(\Sigma X_{s+1}) \leftarrow 0.$$

We can splice these together to obtain a long exact sequence

$$0 \leftarrow H^*(X) \leftarrow H^*(K_0) \leftarrow H^*(\Sigma K_1) \leftarrow H^*(\Sigma^2 K_2) \leftarrow \cdots . \tag{2.1.4}$$

Since the maps are A-module homomorphisms and each $H^*(K_s)$ is free over A_p, 2.1.4 is a free A-resolution of $H^*(X)$.

Unfortunately, the relation of $\pi_*(K_s)$ to $\pi_*(X)$ is not as simple as that between the corresponding cohomology groups. Life would be very simple if we knew $\pi_*(f_s)$

was onto, but in general it is not. We have instead long exact sequences

(2.1.5)
$$\begin{tikzcd} \pi_*(X_{s+1}) \arrow[r, "\pi_*(g_s)"] & \pi_*(X_s) \arrow[r, "\pi_*(f_s)"] & \pi_*(K_s) \arrow[ll, bend left, "\partial_{s,*}"] \end{tikzcd}$$

arising from the fibrations
$$X_{s+1} \xrightarrow{g_s} X_s \xrightarrow{f_s} K_s.$$
If we regard $\pi_*(X_s)$ and $\pi_*(K_s)$ for all s as bigraded abelian groups D_1 and E_1, respectively [i.e., $D_1^{s,t} = \pi_{t-s}(X_s)$ and $E_1^{s,t} = \pi_{t-s}(K_s)$] then 2.1.5 becomes

(2.1.6)
$$\begin{tikzcd} D_1 \arrow[rr, "i_1"] & & D_1 \arrow[dl, "j_1"] \\ & E_1 \arrow[ul, "k_1"] & \end{tikzcd}$$

where
$$i_1 = \pi_{t-s}(g_s)\colon D_1^{s+1,t+1} \to D_1^{s,t},$$
$$j_1 = \pi_{t-s}(f_s)\colon D_1^{s,t} \to E_1^{s,t},$$

and
$$k_1 = \partial_{s,t-s}\colon E_1^{s,t} \to D_1^{s+1,t}.$$
The exactness of 2.1.5 translates to $\ker i_1 = \operatorname{im} k_1$, $\ker j_1 = \operatorname{im} i_1$, and $\ker k_1 = \operatorname{im} j_1$. A diagram such as 2.1.6 is known as an *exact couple*. It is standard homological algebra that an exact couple leads one to a spectral sequence; accounts of this theory can be found in Cartan and Eilenberg [**1**, Section XV.7], Mac Lane [**1**, Section XI.5], and Hilton and Stammbach [**1**, Chapter 8] as well as Massey [**2**].

Briefly, $d_1 = j_1 k_1\colon E_1^{s,t} \to E_1^{s+1,t}$ has $(d_1)^2 = j_1 k_1 j_1 k_1 = 0$ so (E_1, d_1) is a complex and we define $E_2 = H(E_1, d_1)$. We get another exact couple, called the derived couple,

(2.1.7)
$$\begin{tikzcd} D_2 \arrow[rr, "i_2"] & & D_2 \arrow[dl, "j_2"] \\ & E_2 \arrow[ul, "k_2"] & \end{tikzcd}$$

where $D_2^{s,t} = i_1 D_1^{s,t}$, i_2 is induced by i_1, $j_2(i_1 d) = j_1 d$ for $d \in D_1$, and $k_2(e) = k_1(e)$ for $e \in \ker d, \subset E_1$. Since 2.1.7 is also an exact couple (this is provable by a diagram chase), we can take its derived couple, and iterating the procedure gives a sequence of exact couples

$$\begin{tikzcd} D_r \arrow[rr, "i_r"] & & D_r \arrow[dl, "j_r"] \\ & E_r \arrow[ul, "k_r"] & \end{tikzcd}$$

where $D_{r+1} = i_r D_r$, $d_r = j_r k_r$, and $E_{r+1} = H(E_r, d_r)$. The sequences of complexes $\{(E_r, d_r)\}$ constitutes a spectral sequence. A close examination of the indices will reveal that $d_r\colon E_r^{s,t} \to E_r^{s+r,t+r-1}$. It follows that for $s < r$, the image of d_r in $E_r^{s,t}$ is trivial so $E_{r+1}^{s,t}$ is a subgroup of $E_r^{s,t}$, hence we can define

$$E_\infty^{s,t} = \bigcap_{r>s} E_r^{s,t}.$$

This group will be identified (2.1.12) in certain cases with a subquotient of $\pi_{t-s}(X)$, namely, $\operatorname{im}\pi_{t-s}(X_s)/\operatorname{im}\pi_{t-s}(X_{s+1})$. The subgroups $\operatorname{im}\pi_*(X_s) = F^s\pi_*(X)$ form a *decreasing filtration* of $\pi_*(X)$ and E_∞ is the *associated bigraded group.*

2.1.8. DEFINITION. *The* mod (p) *Adams spectral sequence for X is the spectral sequence associated to the exact couple 2.1.6.* □

We will verify that $d_r\colon E_r^{s,t} \to E_r^{s+r,t+r-1}$ by chasing diagram 2.1.9, where we write $\pi_*(X_*)$ and $\pi_*(K_*)$ instead of D_1 and E_1, with $u = t - s$.

(2.1.9)

$$\begin{tikzcd}
 & {} \arrow[d] & & {} \arrow[d] & & \\
{} \arrow[r] & \pi_u(X_{s+2}) \arrow[r, "\pi_u(f_{s+2})"] \arrow[d, "\pi_u(g_{s+1})"'] & \pi_u(K_{s+2}) \arrow[r, "\partial_{s+2,u}"] & \pi_{u-1}(X_{s+3}) \arrow[r, "\pi_{u-1}(f_{s+3})"] \arrow[d, "\pi_{u-1}(g_{s+2})"'] & \pi_{u-1}(K_{s+3}) \arrow[r] & {} \\
{} \arrow[r] & \pi_u(X_{s+1}) \arrow[r, "\pi_u(f_{s+1})"] \arrow[d, "\pi_u(g_s)"'] & \pi_u(K_{s+1}) \arrow[r, "\partial_{s+1,u}"] & \pi_{u-1}(X_{s+2}) \arrow[r, "\pi_{u-1}(f_{s+2})"] \arrow[d, "\pi_{u-1}(g_{s+1})"'] & \pi_{u-1}(K_{s+2}) \arrow[r] & {} \\
{} \arrow[r] & \pi_u(X_s) \arrow[r, "\pi_u(f_s)"] \arrow[d] & \pi_u(K_s) \arrow[r, "\partial_{s,u}"] & \pi_{u-1}(X_{s+1}) \arrow[r, "\pi_{u-1}(f_{s+1})"] \arrow[d] & \pi_{u-1}(K_{s+1}) \arrow[r] & {} \\
 & {} & & {} & &
\end{tikzcd}$$

The long exact sequences 2.1.5 are embedded in this diagram; each consists of a vertical step $\pi_*(g_*)$ followed by horizontal steps $\pi_*(f_*)$ and ∂_{**} and so on. We have $E_1^{s,t} = \pi_u(K_s)$ and $d_1^{s,t} = (\pi_{u-1}(f_{s+1}))(\partial_{s,u})$. We have $E_2^{s,t} = \ker d_1^{s,t}/\operatorname{im} d_1^{s-1,t}$. Suppose an element in $E_2^{s,t}$ is represented by $x \in \pi_u(K_s)$. We will now explain how $d_2[x]$ (where $[x]$ is the class represented by x) is defined. x is a d_1 cycle, i.e., $d_1 x = 0$, so exactness in 2.1.4 implies that $\partial_{s,u}x = (\pi_{u+1}(g_{s+1}))(y)$ for some $y \in \pi_{u-1}(X_{s+2})$. Then $(\pi_{u-1}(f_{s+2}))(y)$ is a d_1 cycle which represents $d_2[x] \in E_2^{s+2,t-1}$. If $d_2[x] = 0$ then $[x]$ represents an element in $E_3^{s,t}$ which we also denote by $[x]$. To define $d_3[x]$ it can be shown that y can be chosen so that $y = (\pi_{u-1}(g_{s+2}))(y')$ for some $y' \in \pi_{u-1}(X_{s+3})$ and that $(\pi_{u-1}(f_{s+3}))(y')$ is a d_1 cycle representing a d_2 cycle which represents an element in $E^{s+3,t+2}$ which we define to be $d_3[x]$. These assertions may be verified by drawing another diagram which is related to the derived couple 2.1.7 in the same way that 2.1.9 is related to the original exact couple 2.1.6. The higher differentials are defined in a similar fashion. In practice, even the calculation of d_2 is a delicate business.

Before identifying $E_\infty^{s,t}$ we need to define the homotopy inverse limit of spectra.

2.1.10. DEFINITION. *Given a sequence of spectra and maps*

$$X_0 \xleftarrow{f_1} X_1 \xleftarrow{f_2} X_2 \xleftarrow{f_3} X_3 \longleftarrow \cdots,$$

$\varprojlim X_i$, *is the fiber of the map*

$$g\colon \prod X_i \to \prod X_i$$

whose ith component is the difference between the projection $p_i\colon \prod X_j \to X_i$ *and the composite*

$$\prod X_j \xrightarrow{p_{i+1}} X_{i+1} \xrightarrow{f_{i+1}} X_i. \qquad \square$$

For the existence of products in the stable category see 3.13 of Adams [4]. This $\varprojlim$ is not a categorical inverse limit (Mac Lane [1, Section III.4] because a compatible collection of maps to the X_i, does not give a unique map to $\varprojlim X_i$. For this reason some authors (e.g., Bousfield and Kan [1]) denote it instead by $\underleftarrow{\mathrm{holim}}$. The same can be said of the direct limit, which can be defined as the cofiber of the appropriate self-map of the coproduct of the spectra in question. However this $\varprojlim$ has most of the properties one would like, such as the following.

2.1.11. LEMMA. *Given spectra* $X_{i,j}$ *for* $i, j \geq 0$ *and maps* $f\colon X_{i,j} \to X_{i-1,j}$ *and* $g\colon X_{i,j} \to X_{i,j-1}$ *such that* fg *is homotopic to* gf*,*

$$\varprojlim_i \varprojlim_j X_{i,j} = \varprojlim_j \varprojlim_i X_{i,j}.$$

PROOF. We have for each i a cofibre sequence

$$\varprojlim_j X_{i,j} \to \prod_j X_{i,j} \to \prod_j X_{i,j}.$$

Next we need to know that products preserve cofiber sequences. For this fact, recall that the product of spectra $\prod Y_i$, is defined via Brown's representability theorem (Adams [4], Theorem 3.12) as the spectrum representing the functor $\prod[-, Y_i]$. Hence the statement follows from the fact that a product (although not the inverse limit) of exact sequences is again exact.

Hence we get the following homotopy commutative diagram in which both rows and columns are cofiber sequences.

$$\begin{tikzcd}
\varprojlim_i \varprojlim_j X_{i,j} \arrow[r] \arrow[d] & \varprojlim_i \prod_j X_{i,j} \arrow[r] \arrow[d] & \varprojlim_i \prod X_{i,j} \arrow[d] \\
\prod_i \varprojlim_j X_{i,j} \arrow[r] \arrow[d] & \prod_i \prod_j X_{i,j} \arrow[r] \arrow[d] & \prod_i \prod_j X_{i,j} \arrow[d] \\
\prod_i \varprojlim_j X_{i,j} \arrow[r] & \prod_i \prod_j X_{i,j} \arrow[r] & \prod_i \prod_j X_{i,j}
\end{tikzcd}$$

Everything in sight is determined by the two self-maps of $\prod_i \prod_j X_{i,j}$ and the homotopy that makes them commute. Since the product is categorical we have $\prod_i \prod_j X_{i,j} = \prod_j \prod_i X_{i,j}$. It follows that $\prod_i \varprojlim_j X_{i,j} = \varprojlim_j \prod_i X_{i,j}$ because they are each the fiber of the same map.

Similarly

$$\prod_j \varprojlim_i X_{i,j} = \varprojlim_i \prod_j X_{i,j}$$

so one gets an equivalent diagram with $\varprojlim_j \varprojlim_i X_{i,j}$ in the upper left corner. $\square$

Now we will show that for suitable X, $E_\infty^{s,t}$ is a certain subquotient of $\pi_u(X)$.

2.1.12. LEMMA. *Let X be a spectrum with an Adams resolution (X_s, g_s) such that $\varprojlim X_s = \mathrm{pt}$. Then $E_\infty^{s,t}$ is the subquotient $\operatorname{im} \pi_u(X_s)/\operatorname{im} \pi_u(X_{s+1})$ of $\pi_u(X)$ and $\bigcap \operatorname{im} \pi_*(X_s) = 0$.*

PROOF. For the triviality of the intersection we have $\varprojlim \pi_*(X_s) = 0$ since $\varprojlim X_s = \mathrm{pt}$. Let $G_s = \pi_*(X_s)$ and

$$G_s^t = \begin{cases} G_s & \text{if } s \geq t \\ \operatorname{im} G_t < G_s & \text{if } t \geq s. \end{cases}$$

We have injections $G_s^t \to G_s^{t-1}$ and surjections $G_s^t \to G_{s-1}^t$, so $\varprojlim_t G_s^t = \bigcap_t G_s^t$ and $\varprojlim_s G_s^t = G_t$. We are trying to show $\varprojlim_t G_0^t = 0$. $\varprojlim_t G_s^t$ maps onto $\varprojlim_t G_{s-1}^t$, so $\varprojlim_s \varprojlim_t G_s^t$ maps onto $\varprojlim_t G_0^t$. But $\varprojlim_s \varprojlim_t G_s^t = \varprojlim_t \varprojlim_s G_s^t = \varprojlim_t G_t = 0$.

For the identification of $E_\infty^{s,t}$, let $0 \neq [x] \in E_\infty^{s,t}$.

First we show $\partial_{s,u}(x) = 0$. Since $d_r[x] = 0$, $\partial_{s,u}(x)$ can be lifted to $\pi_{u-1}(X_{s+r+1})$ for each r. It follows that $\partial_{s,u}(x) \in \operatorname{im} \varprojlim \pi_{u-1}(X_{s+r}) = 0$, so $\partial_{s,u}(x) = 0$.

Hence we have $x = \pi_u(f_s)(y)$ for $y \in \pi_u(X_s)$. It suffices to show that y has a nontrivial image in $\pi_u(X)$. If not, let r be the largest integer such that y has a nontrivial image $z \in \pi_u(X_{s-r+1})$. Then $z = \partial_{s-r,u}(w)$ for $w \in \pi_u(K_{s-r})$ and $d_r[w] = [x]$, contradicting the nontriviality of $[x]$. $\square$

Now we prove 2.1.1(a), the identification of the E_2-term.

By 2.1.2(d), $E_1^{s,t} = \operatorname{Hom}_A(H^{t-s}(K_s), \mathbf{Z}/(p))$. Hence applying $\operatorname{Hom}_A(-, \mathbf{Z}/(p))$ to 2.1.4 gives a complex

$$E_1^{0,t} \xrightarrow{\delta} E_1^{1,t} \xrightarrow{\delta} E_1^{2,t} \to \cdots .$$

The cohomology of this complex is by definition the indicated Ext group. It is straightforward to identify the coboundary δ with the d_1 in the spectral sequence and 2.1.1(a) follows.

2.1.13. COROLLARY. *If $f\colon X \to Y$ induces an isomorphism in* mod (p) *homology then it induces an isomorphism (from E_2 onward) in the* mod (p) *Adams spectral sequence.* $\square$

2.1.14. DEFINITION. *Let G be an abelian group and X a spectrum. Then $XG = X \wedge SG$, where SG is the Moore spectrum associated with G (Adams* [**4**], p. 200]. *Let $\widehat{X} = X\mathbf{Z}_p$ (the p-aidc completion of X), where $\mathbf{Z}_p$ is the p-adic integers, and $X^m = X\mathbf{Z}/(p^m)$.* $\square$

2.1.15. LEMMA. (a) *The map $X \to \widehat{X}$ induces an isomorphism of* mod (p) *Adams spectral sequences.*

(b) $\pi_*(\widehat{X}) = \pi_*(X) \otimes \mathbf{Z}_p$.

(c) $\widehat{X} = \varprojlim X^m$, *if x has finite type.*

PROOF. For (a) it suffices by 2.1.11 to show that the map induces an isomorphism in mod (p) homology. For this see Adams [**4**], proposition 6.7, which also shows (b).

Part (c) does not follow immediately from the fact that $S\mathbf{Z}_p = \varprojlim S\mathbf{Z}/(p^m)$ because inverse limits do not in general commute with smash products. Indeed our assertion would be false for $X = S\mathbf{Q}$, but we are assuming that X has finite type.

By 2.1.10 there is a cofibration

$$S\mathbf{Z}_p \to \prod_m S\mathbf{Z}/(p^m) \to \prod_m S\mathbf{Z}/(p^m),$$

so it suffices to show that

$$X \wedge \prod S\mathbf{Z}/(p^m) \simeq \prod X\mathbf{Z}/(p^m).$$

This is a special case (with $X = E$ and $R = \mathbf{Z}$) of Theorem 15.2 of Adams [4]. $\square$

2.1.16. LEMMA. *If X is a connective spectrum with each $\pi_i(X)$ a finite p-group, then for any* mod (p) *Adams resolution (X_s, g_s) of X, $\varprojlim X_s = \mathrm{pt}$.*

PROOF. Construct a diagram

$$X = X'_0 \leftarrow X'_1 \leftarrow X'_2 \leftarrow \cdots$$

(not an Adams resolution) by letting X'_{s+1} be the fiber in

$$X'_{s+1} \to X'_s \to K_s,$$

where the right-hand map corresponds [2.1.2(e)] to a basis for the bottom cohomology group of X_s. Then the finiteness of $\pi_i(X)$ implies that for each i, $\pi_i(X'_s) = 0$ for large s. Moreover, $\pi_*(X'_{s+1}) \to \pi_*(X'_s)$ is monomorphic so $\varprojlim X'_s = \mathrm{pt}$.

Now if (X_s, g_s) is an Adams resolution, the triviality of g_s in cohomology enables us to construct compatible maps $X_s \to X'_s$. It follows that the map $\varprojlim \pi_*(X_s) \to \pi_*(X)$ is trivial. Each X_s also satisfies the hypotheses of the lemma, so we conclude that $\varprojlim \pi_*(X_s)$ has trivial image in each $\pi_*(X_s)$ and is therefore trivial. Since $\pi_i(X_s)$ is finite for all i and s, $\varprojlim^1 \pi_*(X_s) = 0$ so $\varprojlim X_s = \mathrm{pt}$. $\square$

We are now ready to prove 2.1.1(b), i.e., to identify the E_∞-term. By 2.1.15(a) it suffices to replace X by $\widehat{X}$. Note that since $S\mathbf{Z}_p \wedge S\mathbf{Z}/(p^m) = S\mathbf{Z}/(p^m)$, $X^m = \widehat{X}^m$. It follows that given a mod (p) Adams resolution (X_s, g_s) for X, smashing with $S\mathbf{Z}_p$ and $S\mathbf{Z}/(p^m)$ gives resolutions $(\widehat{X}_s, \hat{g}_s)$ and (X_s^m, g_s^m) for $\widehat{X}$ and X^m, respectively. Moreover, X^m satisfies 2.1.16 so $\varprojlim_s X_s^m = \mathrm{pt}$. Applying 2.1.15(c) to each X_s, we get $\widehat{X}_s = \varprojlim_m X_s^m$, so

$$\begin{aligned}
\varprojlim_s X_s &= \varprojlim_s \varprojlim_m X_s^m \\
&= \varprojlim_m \varprojlim_s X_s^m \quad \text{by 2.1.11} \\
&= \mathrm{pt}.
\end{aligned}$$

Hence the result follows from 2.1.12. $\square$

2.1.17. REMARK. The E_∞ term only gives us a series of subquotients of $\pi_*(X) \otimes \mathbf{Z}_p$, not the group itself. After computing E_∞ one may have to use other methods to solve the extension problem and recover the group.

We close this section with some examples.

2.1.18. EXAMPLE. Let $X = H\mathbf{Z}$, the integral Eilenberg–Mac Lane spectrum. The fundamental cohomology class gives a map $f\colon X \to E$ with $H^*(f)$ surjective. The fiber of f is also X, the inclusion map $g\colon X \to X$ having degree p. Hence we get an Adams resolution (2.1.3) with $X_s = X$ and $K_s = E$ for all s, the map $X = X_s \to X_0 = X$, having degree p^s. We have then

$$E_1^{s,t} = \begin{cases} \mathbf{Z}/(p) & \text{if } t = s \\ 0 & \text{if } t \neq s. \end{cases}$$

There is no room for nontrivial differentials so the spectral sequence collapses, i.e., $E_\infty = E_1$. We have $E_\infty^{s,s} = \mathbf{Z}/(p) = \operatorname{im}\pi_0(X_s)/\operatorname{im}\pi_0(X_{s+1})$. In this case $\widehat{X} = H\mathbf{Z}_p$, the Eilenberg–Mac Lane spectrum for $\mathbf{Z}_p$.

2.1.19. EXAMPLE. Let $X = H\mathbf{Z}/(p^i)$ with $i > 1$. It is known that $H^*(X) = H^*(Y) \oplus \Sigma H^*(Y)$ as A-modules, where $Y = H\mathbf{Z}$. This splitting arises from the two right-hand maps in the cofiber sequence

$$Y \to Y \to X \to \Sigma Y,$$

where the left-hand map has degree p^i. Since the E_2-term of the Adams spectral sequence depends only on $H^*(X)$ as an A-module, the former will enjoy a similar splitting. In the previous example we effectively showed that

$$\operatorname{Ext}_A^{s,t}(H^*(Y), \mathbf{Z}/(p)) = \begin{cases} \mathbf{Z}/(p) & \text{if } t = s \\ 0 & \text{if } t \neq s. \end{cases}$$

It follows that in the spectral sequence for X we have

$$E_2^{s,t} = \begin{cases} \mathbf{Z}/(p) & \text{if } t - s = 0 \text{ or } 1 \\ 0 & \text{otherwise} \end{cases}$$

In order to give the correct answer we must have $E_\infty^{s,t} = 0$ if $t - s = 1$ and $E_\infty^{s,t} = 0$ if $t = s$ for all but i values of s. Multiplicative properties of the spectral sequence to be discussed in Section 3 imply that the only way we can arrive at a suitable E_∞ term is to have $d_i\colon E_i^{s,s+1} \to E_i^{s+i,s+i}$ nontrivial for all $s \geq 0$. A similar conclusion can be drawn by chasing the relevant diagrams.

2.1.20. EXAMPLE. Let X be the fiber in $X \to \widehat{S}^0 \to H\mathbf{Z}_p$ where the right-hand map is the fundamental integral cohomology class on S^0. Smashing the above fibration with X we get

$$X \wedge X \xrightarrow{g_0} X \xrightarrow{f_0} X \wedge H\mathbf{Z}$$

It is known that the integral homology of X has exponent p, so $X \wedge H\mathbf{Z}$ is a wedge of E and $H^*(f_0)$ is surjective. Similar statements hold after smashing with X any number of times, so we get an Adams resolution (2.1.3) with $K_s = X_s \wedge H\mathbf{Z}$ and $X_s = X^{(s+1)}$, the $(s+1)$-fold smash product of X with itself, i.e., one of the form

$$\begin{tikzcd} X \arrow[d] & X \wedge X \arrow[l] \arrow[d] & X \wedge X \wedge X \arrow[l] \arrow[d] & \cdots \arrow[l] \\ X \wedge H\mathbf{Z} & X \wedge X \wedge H\mathbf{Z} & X \wedge X \wedge X \wedge H\mathbf{Z}. & \end{tikzcd}$$

Since X is $(2p-4)$-connected X_s, is $((s+1)(2p-3)-1)$-connected, so $\varprojlim X_s$, is contractible.

2. The Adams Spectral Sequence Based on a Generalized Homology Theory

In this section we will define a spectral sequence similar to that of 2.1.1 (the classical Adams spectral sequence) in which the mod (p) Eilenberg–Mac Lane spectrum is replaced by some more general spectrum E. The main example we have in mind is of course $E = BP$, the Brown–Peterson spectrum, to be defined in 4.1.12. The basic reference for this material is Adams [**4**] (especially Section 15, which includes the requisite preliminaries on the stable homotopy category.

Our spectral sequence should have the two essential properties of the classical one: it converges to $\pi_*(X)$ localized or completed at p and its E_2-term is a functor of $E^*(X)$ (the generalized cohomology of X) as a module over the algebra of cohomology operations $E^*(E)$; i.e., the E_2-term should be computable in some homological way, as in 2.1.1. Experience has shown that with regard to the second property we should dualize and consider instead $E_*(X)$ (the generalized homology of X) as a comodule over $E_*(E)$ (sometimes referred to as the coalgebra of cooperations). In the classical case, i.e., when $E = H\mathbf{Z}/(p)$, $E_*(E)$ is the dual Steenrod algebra A_*.

Theorem 2.1.1(a) can be reformulated as $E_2 = \operatorname{Ext}_{A_*}(\mathbf{Z}/(p), H_*(X))$ using the definition of Ext in the category of comodules given in A1.2.3. In the case $E = BP$ substantial technical problems can be avoided by using homology instead of cohomology. Further discussion of this point can be found in Adams [**6**, pp. 51–55].

Let us assume for the moment that we have known enough about E and $E_*(E)$ to say that $E_*(X)$ is a comodule over $E_*(E)$ and we have a suitable definition of $\operatorname{Ext}_{E_*(E)}(E_*(S^0), E_*(X))$, which we abbreviate as $\operatorname{Ext}(E_*(X))$. Then we might proceed as follows.

2.2.1. DEFINITION. *An* E_*-Adams resolution *for* X *is a diagram*

$$\begin{tikzcd} X = X_0 \arrow[d, "f_0"'] & X_1 \arrow[l, "g_0"'] \arrow[d, "f_1"'] & X_2 \arrow[l, "g_1"'] \arrow[d, "f_2"'] & \cdots \arrow[l] \\ K_0 & K_1 & K_2 & \end{tikzcd}$$

such that for all $s \geq 0$ *the following conditions hold.*

(a) X_{s+1} *is the fiber of* f_s.

(b) $E \wedge X_s$ *is a retract of* $E \wedge K_s$*, i.e., there is a map* $h_s \colon E \wedge K_s \to E \wedge X_s$ *such that* $h_s(E \wedge f_s)$ *is an identity map of* $E \wedge X_s$*. particular* $E_*(f_s)$ *is a monomorphism.*

(c) K_s *is a retract of* $E \wedge K_s$.

(d)

$$\operatorname{Ext}^{t,u}(E_*(K_s)) = \begin{cases} \pi_u(K_s) & \text{if } t = 0 \\ 0 & \text{if } t > 0. \end{cases} \qquad \square$$

As we will see below, conditions (b) and (c) are necessary to insure that the spectral sequence is natural, while (d) is needed to give the desired E_2-term. As before it is convenient to consider a spectrum with the following properties.

2.2.2. DEFINITION. *An E-completion* $\widehat{X}$ *of* X *is a spectrum such that*

(a) *There is a map* $X \to \widehat{X}$ *inducing an isomorphism in* E_**-homology.*

(b) $\widehat{X}$ *has an* E_**-Adams resolution* $\{\widehat{X}_s\}$ *with* $\varprojlim \widehat{X}_s = \mathrm{pt}$. $\square$

This is not necessarily the same as the $\widehat{X}$ of 2.1.14, which will be denoted in this section by X_p (2.2.12). Of course, the existence of such a spectrum (2.2.13) is not obvious and we will not give a proof here. Assuming it, we can state the main result of this section.

2.2.3. Theorem (Adams [**4**]). *An E_*-Adams resolution for X (2.2.1) leads to a natural spectral sequence $E_*^{**}(X)$ with $d_r\colon E_r^{s,t} \to E_r^{s+r,t+r-1}$ such that*

(a) $E_2^{s,t} = \operatorname{Ext}(E_*(X))$.

(b) E_∞^{**} *is the bigraded group associated with a certain filtration of $\pi_*(\widehat{X})$, in other words, the spectral sequence converges to the latter.* (*This filtration will be described in* 2.2.14.)

2.2.4. Definition. *The spectral sequence of 2.2.3 is the* Adams spectral sequence for X based on E-homology. □

2.2.5. Assumption. We now list the assumptions on E which will enable us to define Ext and $\widehat{X}$.

(a) E is a commutative associative ring spectrum.

(b) E is *connective*, i.e., $\pi_r(E) = 0$ for $r < 0$.

(c) The map $\mu_*\colon \pi_0(E) \otimes \pi_0(E) \to \pi_0(E)$ induced by the multiplication $\mu\colon E \wedge E \to E$ is an isomorphism.

(d) E is *flat*, i.e., $E_*(E)$ is flat as a left module over $\pi_*(E)$.

(e) Let $\theta\colon \mathbf{Z} \to \pi_0(E)$ be the unique ring homomorphism, and let $R \subset \mathbf{Q}$ be the largest subring to which θ extends. Then $H_r(E; R)$ is finitely generated over R for all r.

2.2.6. Proposition. *$H\mathbf{Z}/(p)$ and BP satisfy* 2.2.5(a)–(e) □

The flatness condition 2.2.5(d) is only necessary for identifying E_2^{**} as an Ext. Without it one still has a spectral sequence with the specified convergence properties. Some well-known spectra which satisfy the remaining conditions are $H\mathbf{Z}$, bo, bu, and MSU. In these cases $E \wedge E$ is not a wedge of suspensions of E as it is when $E = H\mathbf{Z}/(p)$, BP, or MU. $H\mathbf{Z} \wedge H\mathbf{Z}$ is known to be a certain wedge of suspensions of $H\mathbf{Z}/(p)$ and $H\mathbf{Z}$, $bo \wedge bo$ is described by Milgram [**1**], $bu \wedge bu$ by Adams [**4**], Section 17, and $MSU \wedge MSU$ by Pengelley [**1**].

We now turn to the definition of Ext. It follows from our assumptions 2.2.5 that $E_*(E)$ is a ring which is flat as a left $\pi_*(E)$ module. Moreover, $E_*(E)$ is a $\pi_*(E)$ bimodule, the right and left module structures being induced by the maps

$$E = S^0 \wedge E \to E \wedge E \quad \text{and} \quad E = E \wedge S^0 \to E \wedge E,$$

respectively. In the case $E = H\mathbf{Z}/(p)$ these two module structures are identical, but not when $E = BP$. Following Adams [**4**], Section 12, let $\mu\colon E \wedge E$ be the multiplication on E and consider the map

$$(E \wedge E) \wedge (E \wedge X) \xrightarrow{1\wedge\mu\wedge 1} E \wedge E \wedge X.$$

2.2.7. Lemma. *The above map induces an isomorphism*

$$E_*(E) \otimes_{\pi_*(E)} E_*(X) \to \pi_*(E \wedge E \wedge X).$$

Proof. The result is trivial for $X = S^n$. It follows for X finite by induction on the number of cells using the 5-lemma, and for arbitrary X by passing to direct limits. □

Now the map
$$E \wedge X = E \wedge S^0 \wedge X \to E \wedge E \wedge X$$
induces
$$\psi\colon E_*(X) \to \pi_*(E \wedge E \wedge X) = E_*(E) \otimes_{\pi_*(E)} E_*(X).$$
In particular, if $X = E$ we get
$$\Delta\colon E_*(E) \to E_*(E) \otimes_{\pi^*(E)} E_*(E).$$
Thus $E_*(E)$ is a coalgebra over $\pi_*(E)$ as well as an algebra, and $E_*(X)$ is a comodule over $E_*(E)$. One would like to say that $E_*(E)$, like the dual Steenrod algebra, is a commutative Hopf algebra, but that would be incorrect since one uses the bimodule structure in the tensor product $E_*(E) \otimes_{\pi_*(E)} E_*(E)$ (i.e., the product is with respect to the right module structure on the first factor and the left module structure on the second). In addition to the coproduct Δ and algebra structure, it has a right and left unit $\eta_R, \eta_L\colon \pi_*(E) \to E_*(E)$ corresponding to the two module structures, a counit $\varepsilon\colon E_*(E) \to \pi_*(E)$ induced by $\mu\colon E \wedge E \to E$, and a conjugation $c\colon E_*(E) \to E_*(E)$ induced by interchange the factors in $E \wedge E$.

2.2.8. PROPOSITION. *With the above structure maps $(\pi_*(E), E_*(E))$ is a Hopf algebroid* (A1.1.1)*, and E-homology is a functor to the category of left $E_*(E)$-comodules* (A1.1.2)*, which is abelian* (A1.1.3). □

The problem of computing the relevant Ext groups is discussed in Appendix 1, where an explicit complex (the cobar complex A1.2.11) for doing so is given. This complex can be realized geometrically by the canonical E_*-Adams resolution defined below.

2.2.9. LEMMA. *Let $K_s = E \wedge X_s$, and let X_{s+1} be the fiber of $f_s\colon X_s \to K_s$. Then the resulting diagram* (2.2.1) *is an E_*-Adams resolution for X.*

PROOF. Since E is a ring spectrum it is a retract of $E\wedge E$, so $E\wedge X_s$, is a retract of $E \wedge K_s = E \wedge E \wedge X_s$ and 2.2.1(b) is satisfied. $E \wedge X_s$ is an E-module spectrum so 2.2.1(c) is satisfied. For 2.2.1(d) we have $E_*(K_s) = E_*(E) \otimes_{\pi_*(E)} E_*(X_s)$ by 2.2.7 and $\mathrm{Ext}(E_*(K_s))$ has the desired properties by A1.2.1 and A1.2.4. □

2.2.10. DEFINITION. *The* canonical *E_*-Adams resolution for X is the one given by 2.2.9.*

Note that if E is not a ring spectrum then the above f_s need not induce a monomorphism in E-homology, in which case the above would not be an Adams resolution.

Note also that the canonical resolution for X can be obtained by smashing X with the canonical resolution for S^0.

2.2.11. PROPOSITION. *The E_1-term of the Adams spectral sequence associated with the resolution of* 2.2.9 *is the cobar complex $C^*(E_*(X))$* (A1.2.11). □

Next we describe an E-completion $\widehat{X}$ (2.2.2). First we need some more terminology.

2.2.12. DEFINITION. *$X_{(p)} = X\mathbf{Z}_{(p)}$, where $\mathbf{Z}_{(p)}$ denotes the integers localized at p, and $X_p = X\mathbf{Z}_p$ (see 2.1.14).*

2.2.13. THEOREM. *If X is connective and E satisfies 2.2.5(a)–(e) then an E-completion (2.2.2) of X is given by*

$$\widehat{X} = \begin{cases} X\mathbf{Q} & \text{if } \pi_0(E) = \mathbf{Q} \\ X_{(p)} & \text{if } \pi_0(E) = \mathbf{Z}_{(p)} \\ X & \text{if } \pi_0(E) = \mathbf{Z} \\ X_p & \text{if } \pi_0(E) = \mathbf{Z}/(p) \text{ and } \pi_n(X) \\ & \quad \text{is finitely generated for all } n. \end{cases} \qquad \square$$

These are not the only possible values of $\pi_0(E)$, but the others will not concern us. A proof is given by Adams [4], Theorem 14.6 and Section 15. We will sketch a proof using the additional hypothesis that $\pi_1(E) = 0$, which is true in all of the cases we will consider in this book.

For simplicity assume that $\pi_0(X)$ is the first nonzero homotopy group. Then in the cases where $\pi_0(E)$ is a subring of $\mathbf{Q}$ we have $\pi_i(\widehat{X} \wedge \overline{E}^{(s)}) = 0$ for $i < s$, so by setting $\widehat{X}_s = \widehat{X} \wedge \overline{E}^{(s)}$ we get $\varprojlim \widehat{X}_s = \mathrm{pt}$.

The remaining case, $\pi_0(E) = \mathbf{Z}/(p)$ can be handled by an argument similar to that of the classical case. We show $X\mathbf{Z}/(p^m)$ is its own E-completion by modifying the proof of 2.1.16 appropriately. Then X_p can be shown to be E-complete just as in the proof of 2.1.1(b) (following 2.1.16).

Now we are ready to prove 2.2.3(a). As in Section 1 the diagram 2.2.1 leads to an exact couple which gives the desired spectral sequence. To identify the E_2-term, observe that 2.2.1(a) implies that each fibration in the resolution gives a short (as opposed to long) exact sequence in E-homology. These splice together to give a long exact sequence replacing 2.1.3,

$$0 \to E_*(X) \to E_*(K_0) \to E_*(\Sigma K_1) \to \cdots.$$

Condition 2.2.1(c) implies that the E_2-term of the spectral sequence is the cohomology of the complex

$$\mathrm{Ext}^0(E_*(K_0)) \to \mathrm{Ext}^0(E_*(\Sigma K_1)) \to \cdots.$$

By A1.2.4 this is $\mathrm{Ext}(E_*(X))$.

For 2.2.3(b) we know that the map $X \to \widehat{X}$ induces a spectral sequence isomorphism since it induces an E-homology isomorphism. We also know that $\varprojlim \widehat{X}_s = \mathrm{pt}$, so we can identify E_∞^{**} as in 2.1.12.

We still need to show that the spectral sequence is natural and independent (from E_2 onward) of the choice of resolution. The former implies the latter as the identity map on X induces a map between any two resolutions and standard homological arguments show that such a map induces an isomorphism in E_2 and hence in E_r for $r \geq 2$. The canonical resolution is clearly natural so it suffices to show that any other resolution admits maps to and from the canonical one.

We do this in stages as follows. Let $\{f_s \colon X_s \to K_s\}$ be an arbitrary resolution and let R^0 be the canonical one. Let $R^n = \{f_s^n \colon X_s^n \to K_s^n\}$ be defined by $X_s^n = X_s$, and $K_s^n = K_n$ for $s < n$ and $K_s^n = E \wedge X_s^n$; for $s \geq n$. Then R^∞ is the arbitrary resolution and we construct maps $R^0 \leftrightarrow R^\infty$ by constructing maps $R^n \leftrightarrow R^{n+1}$, for which it suffices to construct maps between K_s and $E \wedge X_s$ compatible with the map from X_s. By 2.2.1(b) and (c), K_s and $E \wedge X_s$ are both retracts of $E \wedge K_s$, so we have a commutative diagram

$$\begin{tikzcd}
X_s \arrow[r] \arrow[d] & K_s \arrow[d] \arrow[dr] & \\
E \wedge X_s \arrow[r] \arrow[dr] & E \wedge K_s \arrow[r] \arrow[d] & E \wedge X_s \\
& K_s &
\end{tikzcd}$$

in which the horizontal and vertical composite maps are identities. It follows that the diagonal maps are the ones we want.

The Adams spectral sequence of 2.2.3 is useful for computing $\pi_*(X)$, i.e., $[S^0, X]$. With additional assumptions on E one can generalize to a spectral sequence for computing $[W, X]$. This is done in Adams [**4**] for the case when $E_*(W)$ is projective over $\pi_*(E)$. We omit this material as we have no need for it.

Now we describe the filtration of 2.2.3(b), which will be referred to as the *E_*-Adams filtration* on $\pi_*(\widehat{X})$.

2.2.14. Filtration Theorem. *The filtration on $\pi_*(\widehat{X})$ of* 2.2.3(b) *is as follows. A map $f\colon S^n \to X$ has filtration $\geq s$ if f can be factored into s maps each of which becomes trivial after smashing the target with E.*

Proof. We have seen above that $F^s\pi_*(\widehat{X}) = \operatorname{im}\pi_*(X_s)$. We will use the canonical resolution (2.2.10). Let $\overline{E}$ be the fiber of the unit map $S^0 \to E$. Then $X_2 = \overline{E}^{(s)} \wedge X$, where $\overline{E}^{(s)}$ is the s-fold smash product of $\overline{E}$. $X_{i+1} \to X_i \to X_i \wedge E$ is a fiber sequence so each such composition is trivial and a map $S^n \to X$ which lifts to X_s clearly satisfies the stated condition. It remains to show the converse, i.e., that if a map $f\colon S^n \to X$ factors as

$$S^n \to Y_s \xrightarrow{g_s} Y_{s-1} \xrightarrow{g_{s-1}} \cdots \to Y_0 = X,$$

where each composite $Y_i \xrightarrow{g_i} Y_{i-1} \to Y_{i-1} \wedge E$ is trivial, then it lifts to X_s. We argue by induction on i. Suppose $Y_{i-1} \to X$ lifts to X_{i-1} (a trivial statement for $i = 1$). Since Y_i maps trivially to $Y_{i-1} \wedge E$, it does so to $X_{i-1} \wedge E$ and therefore lifts to X_i. □

3. The Smash Product Pairing and the Generalized Connecting Homomorphism

In this section we derive two properties of the Adams spectral sequence which will prove usefull in the sequel. The first concerns the structure induced by a map

$$\alpha\colon X' \wedge X'' \to X, \tag{2.3.1}$$

e.g., the multiplication on a ring spectrum. The second concerns a generalized connecting homomorphism arising from a cofiber sequence

$$W \xrightarrow{f} X \xrightarrow{g} Y \xrightarrow{h} \Sigma W \tag{2.3.2}$$

when $E_*(h) = 0$. Both of these results are folk theorems long known to experts in the field but to our knowledge never before published in full generality. The first property in the classical case was proved in Adams [**3**], while a weaker form of the second property was proved by Johnson, Miller, Wilson, and Zahler [**1**].

Throughout this section the assumptions 2.2.5 on E will apply. However, the flatness condition [2.2.5(d)] is only necessary for statements explicitly involving Ext, i.e., 2.3.3(e) and 2.3.4(a). For each spectrum X let $E_*^{**}(X)$ be the Adams spectral sequence for X based on E-homology (2.2.3). Our first result is

2.3.3. THEOREM. *Let $2 \le r \le \infty$. Then the map a above induces a natural pairing*

$$E_r^{**}(X') \otimes E_r^{**}(X'') \to E_r^{**}(X)$$

such that

(a) *for $a' \in E_r^{s',t'}(X')$, $a'' \in E_r^{s'',t''}(X'')$,*

$$d_r(a', a'') = d_r(a')a'' + (-1)^{t'-s'} a' d_r(a'');$$

(b) *the pairing on E_{r+1}, is induced by that on E_r;*

(c) *the pairing on E_∞, corresponds to $a_*\colon \pi_*(X') \otimes \pi_*(X'') \to \pi_*(X)$;*

(d) *if $X' = X'' = X$ and $E_*(\alpha)\colon E_*(X) \otimes E_*(X) \to E_*(X)$ is commutative or associative, then so is the pairing [modulo the usual sign conventions, i.e., $a'a'' = (-1)^{(t'-s')(t''-s'')}a''a'$];*

(e) *for $r = 2$ the pairing is the external cup product* (A1.2.13)

$$\operatorname{Ext}(E_*(X')) \otimes \operatorname{Ext}(E_*(X'')) \to \operatorname{Ext}(E_*(X') \otimes_{\pi_*(E)} E_*(X''))$$

composed with the map in Ext *induced by the composition of canonical maps*

$$E_*(X') \otimes_{\pi_*(E)} E_*(X'') \to E_*(X' \wedge X'') \xrightarrow{\alpha_*} E_*(X).$$

In particular, by setting $X' = S^0$ and $X'' = X$ we find that the spectral sequence for X is a module (in the appropriate sense) over that for the sphere S^0. □

The second result is

2.3.4. THEOREM. *Let $E_*(h) = 0$ in 2.3.2. Then for $2 \le r \le \infty$ there are maps $\delta_r\colon E_r^{s,*}(Y) \to E_r^{s+1,*}(W)$ such that*

(a) *δ_2 is the connecting homomorphism associated with the short exact sequence*

$$0 \to E_*(W) \to E_*(X) \to E_*(Y) \to 0,$$

(b) *$\delta_r d_r = d_r \delta_r$ and δ_{r+1} induced by δ_r,*

(c) *δ_∞ is a filtered form of the map $\pi_*(h)$.*

The connecting homomorphism in Ext *can be described as the Yoneda product* (Hilton and Stammbach [1, p. 155] *with the element of* $\operatorname{Ext}^1_{E_*(E)}(E_*(Y), E_*(W))$ *corresponding to the short exact sequence*

$$0 \to E_*(W) \to E_*(X) \to E_*(Y) \to 0.$$

Similarly, given a sequence of maps

$$X_0 \xrightarrow{f_0} \Sigma X_1 \xrightarrow{f_1} \Sigma^2 X_2 \to \cdots \to \Sigma^n X_n$$

with $E_*(f_i) = 0$ one gets maps

$$\delta_r\colon E_r^{s,*}(X_0) \to E_r^{s+n*}(X_n)$$

commuting with differentials where δ_2 can be identified as the Yoneda product with the appropriate element in

$$\operatorname{Ext}^n_{E(E_*)}(E_*(X_0), E_*(X_n)).$$ □

If one generalizes the spectral sequence to source spectra other than the sphere one is led to a pairing induced by composition of maps. This has been studied by Moss [**1**], where it is assumed that one has Adams resolutions satisfying much stronger conditions than 2.2.1. In the spectral sequence for the sphere it can be shown that the composition and smash product pairings coincide, but we will not need this fact.

To prove 2.3.3 we will use the canonical resolutions (2.2.9) for X', X'' and X. Recall that these can be obtained by smashing the respective spectra with the canonical resolution for S^0. Let $K_{s,s+r}$ be the cofiber in

$$\overline{E}^{(s+r)} \to \overline{E}^{(s)} \to K_{s,s+r}, \tag{2.3.5}$$

where E is the fiber of $S^0 \to E$.

These spectra have the following properties.

2.3.6. LEMMA.
(a) *There are canonical fibrations*

$$K_{s+i,s+i+j} \to K_{s,s+i+j} \to K_{s,s+i}.$$

(b) $E_1^{s,*}(X) = \pi_*(X \wedge K_{s,s+1})$.
Let $Z_r^{s,*}(X)$, $B_r^{s,*}(X) \subset E_1^{s,*}(X)$ *be the images of* $\pi_*(X \wedge K_{s,s+r})$ *and* $\pi_*(X \wedge \Sigma^{-1}K_{s-r+1,s})$, *respectively. Then* $E_r^{s,*}(X) = Z_r^{s,*}(X)/B_r^{s,*}(X)$ *and* d_r *is induced by the map*

$$X \wedge K_{s,s+r} \to X \wedge \Sigma K_{s+r,s+2r}.$$

(c) α *induces map* $X'_s \wedge X''_t \to X_{s+1}$ *(where these are the spectra in the canonical resolutions) compatible with the maps* g'_s, g''_t, *and* g_{s+t} *of* 2.2.1.
(d) *The map*

$$K_{s,s+1} \wedge K_{t,t+1} \to K_{s+t,s+t+1},$$

given by the equivalence

$$K_{n,n+1} = E \wedge \overline{E}^{(n)}$$

and the multiplication on E, *lifts to maps*

$$K_{s,s+r} \wedge K_{t,t+r} \to K_{s+t,s+t+r}$$

for $r > 1$ *such that the following diagram commutes*

$$\begin{tikzcd}
K_{s,s+r+1} \wedge K_{t,t+r+1} \arrow[r] \arrow[d] & K_{s+t,s+t+r+1} \arrow[d] \\
K_{s,s+r} \wedge K_{t,t+r} \arrow[r] & K_{s+t,s+t+r}
\end{tikzcd}$$

where the vertical maps come from (a).
(e) *The following diagram commutes*

$$\begin{tikzcd}
K_{s,s+r} \wedge K_{t,t+r} \arrow[r] \arrow[d] & (\Sigma K_{s+r,s+2r} \wedge K_{t,t+r}) \vee (K_{s,s+r} \wedge \Sigma K_{t+r,t+2r}) \arrow[d] \\
K_{s+t,s+t+r} \arrow[r] & \Sigma K_{s+t+r,s+t+2r}
\end{tikzcd}$$

where the vertical maps are those of (d) *and the horizontal maps come from* (a), *the maps to and from the wedge being the sums of the maps to and from the summands.*

PROOF. Part (a) is elementary. For (b) we refer the reader to Cartan and Eilenberg [**1**], Section XV.7, where a spectral sequence is derived from a set of abelian groups $H(p,q)$ satisfying certain axioms. Their $H(p,q)$ in this case is our $\pi_*(K_{p,q})$, and (a) guarantees that these groups have the appropriate properties. For (c) we use the fact that $X'_s = X' \wedge \overline{E}^{(s)}$, $X''_t = X'' \wedge \overline{E}^{(t)}$, and $X_{s+t} = X \wedge \overline{E}^{(s+t)}$.

For (d) we can assume the maps $\overline{E}^{(s+1)} \to \overline{E}^{(s)}$ are all inclusions with $K_{s,s+r} = \overline{E}^{(s)}/\overline{E}^{(s+r)}$. Hence we have

$$K_{s,s+r} \wedge K_{t,t+r} = E^{(s)} \wedge \overline{E}^{(t)}/(\overline{E}^{(s+r)} \wedge \overline{E}^{(t)} \cup \overline{E}^{(s)} \wedge \overline{E}^{(t+r)})$$

and this maps naturally to

$$\overline{E}^{(s+t)}/\overline{E}^{(s+t+r)} = K_{s+t,s+t+r}.$$

For (e) if $\overline{E}^{(s+2r)} \to \overline{E}^{(s+r)} \to \overline{E}^{(s)}$ are inclusions then so is $K_{s+r,s+2r} \to K_{s,s+2r}$ so we have $K_{s,s+r} = K_{s,s+2r}/K_{s+r,s+2r}$ and $K_{t,t+r} = K_{t,t+2r}/K_{t+r,t+2r}$. With this in mind we get a commutative diagram

$$\begin{tikzcd}
K_{s,s+r} \wedge K_{t+r,t+2r} \cup K_{s+r,s+2r} \wedge K_{t,t+r} \arrow[r] \arrow[d] & K_{s+t+r,s+t+2r} \arrow[d] \\
K_{s,s+2r} \wedge K_{t,t+2r} \arrow[r] \arrow[d] & K_{s+t,s+t+2r} \arrow[d] \\
K_{s,s+r} \wedge K_{t,t+r} \arrow[r] \arrow[d] & K_{s+t,s+t+r} \arrow[d] \\
\Sigma(K_{s,s+r} \wedge K_{t+r,t+2r} \cup K_{s+r,s+2r} \wedge K_{t,t+r}) \arrow[r] & \Sigma K_{s+t+r,s+t+2r}
\end{tikzcd}$$

where the horizontal maps come from (d) and the upper vertical maps are inclusions. The lower left-hand map factors through the wedge giving the desired diagram. $\square$

We are now ready to prove 2.3.3. In light of 2.3.6(b), the pairing is induced by the maps of 2.3.6(d). Part 2.3.3(a) then follows from 2.3.6(e) as the differential on $E_r^{**}(X') \otimes E_r^{**}(X'')$ is induced by the top map of 2.3.6(e). Part 2.3.3(b) follows from the commutative diagram in 2.3.6(d). Part 2.3.3(c) follows from the compatibility of the maps in 2.3.6(c) and (d).

Assuming 2.3.3(e), (d) is proved as follows. The pairing on Ext is functorial, so if $E_*(X)$ has a product which is associative or commutative, so will $E_2^{**}(X)$. Now suppose inductively that the product on $E_r^{**}(X)$ has the desired property. Since the product on E_{r+1} is induced by that on E_r the inductive step follows.

It remains then to prove 2.3.3(e). We have $E_*(X' \wedge K_{s,s+1}) = D^s(E_*(X'))$ (A1.2.11) and similarly for X'', so our pairing is induced by a map

$$E_*(X' \wedge K_{s,s+1}) \otimes_{\pi_*(E)} E_*(X'' \wedge K_{t,t+1}) \to E_*(X \wedge K_{s+t,s+t+1}),$$

i.e., by a pairing of resolutions. Hence the pairing on E_2 coincides with the specified algebraic pairing by the uniqueness of the latter (A1.2.14).

We prove 2.3.4 by reducing it to the following special case.

2.3.7. LEMMA. *Theorem 2.3.4 holds when X is such that $\operatorname{Ext}^s(E_*(X)) = 0$ for $s > 0$ and $\pi_*(X) = \operatorname{Ext}^0(E_*(X))$.* □

PROOF OF 2.3.4. Let W' be the fiber of the composite

$$W \xrightarrow{f} X \to X \wedge E.$$

Since Σfh is trivial, h lifts to a map $h'\colon Y \to \Sigma W'$. Now consider the cofiber sequence

$$W \to X \wedge E \to \Sigma W' \to \Sigma W.$$

Lemma 2.3.7 applies here and gives maps

$$\delta_r\colon E_r^{s,*}(\Sigma W') \to E_r^{s+1,*}(\Sigma W).$$

Composing this with the maps induced by h' gives the desired result. □

PROOF OF 2.3.7. Disregarding the notation used in the above proof, let $W' = \Sigma^{-1}Y$, $X' = \Sigma^{-1}Y \wedge E$, and $Y' = Y \wedge \overline{E}$. Then we have a commutative diagram in which both rows and columns are cofiber sequences

$$\begin{tikzcd}
X \arrow[d] & W \arrow[l] \arrow[d] & W' \arrow[l] \arrow[d] \\
X \vee (Y \wedge E) \arrow[d] & X \arrow[l] \arrow[d] & X' \arrow[l] \arrow[d] \\
Y \wedge E & Y \arrow[l] & Y' \arrow[l]
\end{tikzcd}$$

Each row is the beginning of an Adams resolution (possibly noncanonical for W and X) which we continue using the canonical resolutions (2.2.9) for W', X', and Y'. Thus we get a commutative diagram

$$\begin{tikzcd}
W \arrow[d] & W' \arrow[l] \arrow[d] & W' \wedge \overline{E} \arrow[l] \arrow[d] & W' \wedge \overline{E}^{(2)} \arrow[l] \arrow[d] & \cdots \arrow[l] \\
X \arrow[d] & X' \arrow[l] \arrow[d] & X' \wedge \overline{E} \arrow[l] \arrow[d] & X' \wedge \overline{E}^{(2)} \arrow[l] \arrow[d] & \cdots \arrow[l] \\
Y & Y' \arrow[l] & Y' \wedge \overline{E} \arrow[l] & Y' \wedge \overline{E}^{(2)} \arrow[l] & \cdots \arrow[l]
\end{tikzcd} \tag{2.3.8}$$

in which each column is a cofiber sequence. The map $Y \xrightarrow{\simeq} \Sigma W'$ induces maps $\delta_r\colon E_r^{s,*}(Y) \to E_r^{s+1,*}(W)$ which clearly satisfy 2.3.4(a) and (b), so we need only to verify that δ_2 is the connecting homomorphism. The resolutions displayed in 2.3.8 make this verification easy because they yield a short exact sequence of E_1-terms which is additively (though not differentially) split. For $s = 0$ we have

$$\begin{aligned}
E_1^{0,*}(W) &= \pi_*(X), \quad E_1^{0,*}(X) = \pi_*(X \vee (Y \wedge E)),\\
E_1^{0,*}(Y) &= \pi_*(Y \wedge E), \quad E_1^{1,*}(W) = \pi_*(Y \wedge E),\\
E_1^{1,*}(X) &= \pi_*(Y \wedge E \wedge E) \quad \text{and} \quad E_1^{1,*}(Y) = \pi_*(\Sigma Y \wedge \overline{E} \wedge E),
\end{aligned}$$

so the relevant diagram for the connecting homomorphism is

$$\begin{tikzcd}
X \arrow[r] \arrow[d] & X \vee (Y \wedge E) \arrow[r, shift right] \arrow[d, "d"'] & Y \wedge E \arrow[l, shift right, "b"'] \arrow[d] \\
Y \wedge E \arrow[r, shift right] & Y \wedge E \wedge E \arrow[l, shift right, "a"'] \arrow[r] & \Sigma Y \wedge \overline{E} \wedge \overline{E}
\end{tikzcd}$$

where a and b are splitting maps. The connecting homomorphism is induced by adb, which is the identity on $Y \wedge E$, which also induces δ_2.

For $s > 0$ we have

$$E_1^{s,*}(W) = \pi_*(\Sigma^{s-1} Y \wedge E \wedge \overline{E}^{(s-1)}),$$
$$E_1^{s,*}(X) = \pi_*(\Sigma^{s-1} Y \wedge E^{(2)} \wedge \overline{E}^{(s-1)}),$$

and

$$E_1^{s,*}(Y) = \pi_*(\Sigma^{s} Y \wedge E \wedge \overline{E}^{(s)}),$$

so the relevant diagram is

$$\begin{tikzcd}
E \arrow[r] \arrow[d] & E \wedge E \arrow[r, shift right] \arrow[d] & E \wedge \overline{E} \arrow[l, shift right] \arrow[d] \\
\Sigma E \wedge \overline{E} \arrow[r, shift right] & \Sigma E \wedge E \wedge \overline{E} \arrow[l, shift right] \arrow[r] & \Sigma^2 E \wedge \overline{E}^2
\end{tikzcd}$$

and again the connecting homomorphism is induced by the identity on $\Sigma^s Y \wedge E \wedge \overline{E}^s$. $\square$

CHAPTER 3

The Classical Adams Spectral Sequence

In Section 1 we make some simple calculations with the Adams spectral sequence which will be useful later. In particular, we use it to compute $\pi_*(MU)$ (3.1.5), which will be needed in the next chapter. The computations are described in some detail in order to acquaint the reader with the methods involved.

In Sections 2 and 3 we describe the two best methods of computing the Adams spectral sequence for the sphere, i.e., the May spectral sequence and the lambda algebra. In both cases a table is given showing the result in low dimensions (3.2.9 and 3.3.10). Far more extensive charts are given in Tangora [**1, 4**]. The main table in the former is reproduced in Appendix 3.

In Section 4 we survey some general properties of the Adams spectral sequence. We give $E_2^{s,*}$ for $s \le 3$ (3.4.1 and 3.4.2) and then say what is known about differentials on these elements (3.4.3 and 3.4.4). Then we outline the proof of the Adams vanishing and periodicity theorems (3.4.5 and 3.4.6). For $p = 2$ they say that $E_s^{s,t}$ vanishes roughly for $0 < t - s < 2s$ and has a very regular structure for $t - s < 5s$. The E_∞-term in this region is given in 3.4.16. An elementary proof of the nontriviality of most of these elements is given in 3.4.21.

In Section 5 we survey some other past and current research and suggest further reading.

1. The Steenrod Algebra and Some Easy Calculations

In this section we begin calculating with the classical mod (p) Adams spectral sequence of 2.1.1. We start by describing the dual Steenrod algebra A_*, referring the reader to Milnor [**2**] or Steenrod and Epstein [**1**] for the proof. Throughout this book, $P(x)$ will denote a polynomial algebra (over a field which will be clear from the context) on one or more generators x, and $E(x)$ will denote the exterior algebra on same.

3.1.1. Theorem (Milnor [**2**]). *A_* is a graded commutative, noncocommutative Hopf algebra.*

(a) *For $p = 2$, $A_* = P(\xi_1, \xi_2, \dots)$ as an algebra where $|\xi_n| = 2^n - 1$. The coproduct $\Delta\colon A_* \to A_* \otimes A_*$ is given by $\Delta\xi_n = \sum_{0 \le i \le n} \xi_{n-i}^{2^i} \otimes \xi_i$, where $\xi_0 = 1$.*

(b) *For $p > 2$, $A_* = P(\xi_1, \xi_2, \dots) \otimes E(\tau_0, \tau_1, \dots)$ as an algebra, where $|\xi_n| = 2(p^n - 1)$, and $|\tau_n| = 2p^n - 1$. The coproduct $\Delta\colon A_* \to A_* \otimes A_*$ is given by $\Delta\xi_n = \sum_{0 \le i \le n} \xi_{n-i}^{p^i} \otimes \xi_i$, where $\xi_0 = 1$ and $\Delta\tau_n = \tau_n \otimes 1 + \sum_{0 \le i \le n} \xi_{n-i}^{p^i} \otimes \tau_i$.*

(c) *For each prime p, there is a unit $\eta\colon \mathbf{Z}/(p) \to A_*$, a counit $\varepsilon\colon A_* \to \mathbf{Z}/(p)$ (both of which are isomorphisms in dimension 0), and a conjugation (canonical anti-automorphism) $c\colon A_* \to A_*$ which is an algebra map given recursively by $c(\xi_0) = 1$, $\sum_{0 \le i \le n} \xi_{n-i}^{p^i} c(\xi_i) = 0$ for $n > 0$ and $\tau_n + \sum_{0 \le i \le n} \xi_{n-i}^{p^i} c(\tau_i) = 0$ for $n \ge 0$. $\bar{A}_*$ will* *denote* $\ker\varepsilon$; *i.e.,* $\bar{A}_*$ *is isomorphic to* A_* *in positive dimensions, and is trivial in dimension* 0. □

A_* is a commutative Hopf algebra and hence a Hopf algebroid. The homological properties of such objects are discussed in Appendix 1.

We will consider the classical Adams spectral sequence formulated in terms of homology (2.2.3) rather than cohomology (2.1.1). The most obvious way of computing the E_2-term is to use the cobar complex. The following description of it is a special case of 2.2.10 and A1.2.11.

3.1.2. Proposition. *The* E_2*-term for the classical Adams spectral sequence for* $\pi_*(X)$ *is the cohomology of the cobar complex* $C^*_{A_*}(H_*(X))$ *defined by*

$$C^s_{A_*}(H_*(X)) = \bar{A}_* \otimes \cdots \otimes \bar{A}_* \otimes H_*(X)$$

(*with* s *tensor factors of* $\bar{A}_*$). *For* $a_i \in A_*$ *and* $x \in H_*(X)$, *the element* $a_1 \otimes \cdots a_s \otimes x$ *will be denoted by* $[a_1|a_2|\cdots|a_s]x$. *The coboundary operator* $d_s\colon C^s_{A_*}(H_*(X)) \to C^{s+1}_{A_*}(H_*(X))$ *is given by*

$$\begin{aligned} d_s[a_1|\cdots|a_s]x = [1|a_1|\cdots|a_s]x + \sum_{i=1}^{s}(-1)^i[a_1|\cdots|a_{i-1}|a_i'|a_i''|a_{i+1}|\cdots|a_s]x \\ + (-1)^{s+1}[a_1|\cdots|a_s|x']x'', \end{aligned}$$

where $\Delta a_i = a_i' \otimes a_i''$ *and* $\psi(x) = x' \otimes x'' \in A_* \otimes H_*(X)$. [*A priori this expression lies in* $A_*^{\otimes s+1} \otimes H_*(X)$. *The diligent reader can verify that it actually lies in* $\bar{A}_*^{\otimes s+1} \otimes H_*(X)$.] □

This E_2-term will be abbreviated by $\operatorname{Ext}(H_*(X))$.

Whenever possible we will omit the subscript A_*.

The following result will be helpful in solving group extension problems in the Adams spectral sequence. For $p > 2$ let $a_0 \in \operatorname{Ext}^{1,1}_{A_*}(\mathbf{Z}/(p), \mathbf{Z}/(p))$ be the class represented by $[\tau_0] \in C(\mathbf{Z}/(p))$. The analogous element for $p = 2$ is represented by $[\xi_1]$ and is denoted by a_0, $h_{1,0}$, or h_0.

3.1.3. Lemma.

(a) *For* $s \geq 0$, $\operatorname{Ext}^{s,s}(H_*(S^0))$ *is generated by* a_0^s.

(b) *If* $x \in \operatorname{Ext}(H_*(X))$ *is a permanent cycle in the Adams spectral sequence represented by* $\alpha \in \pi_*(X)$, *then* $a_0 x$ *is a permanent cycle represented by* $p\alpha$. [*The pairing* $\operatorname{Ext}(H_*(S^0)) \otimes \operatorname{Ext}(H_*(X)) \to \operatorname{Ext}(H_*(X))$ *is given by* 2.3.3.] □

Proof. Part (a) follows from inspection of $C^*(\mathbf{Z}/(p))$; there are no other elements in the indicated bidegrees. For (b) the naturality of the smash product pairing (2.3.3) reduces the problem to the case $x = 1 \in \operatorname{Ext}(H_*(S^0))$, where it follows from the fact that $\pi_0(S^0) = \mathbf{Z}$. □

The cobar complex is so large that one wants to avoid using it directly at all costs. In this section we will consider four spectra (MO, MU, bo, and bu) in which the change-of-rings isomorphism of A1.1.18 can be used to great advantage. The most important of these for our purposes is MU, so we treat it first. The others are not used in the sequel. Much of this material is covered in chapter 20 of Switzer [**1**].

The computation of $\pi_*(MU)$ is due independently to Milnor [**4**] and Novikov [**2, 3**]. For the definition and basic properties of MU, including the following lemma, we refer the reader to Milnor [**4**] or Stong [**1**] or to Section 4.1.

3.1.4. LEMMA.
(a) $H_*(MU;\mathbf{Z}) = \mathbf{Z}[b_1, b_2, \dots]$, *where* $b_i \in H_{2i}$.
(b) *Let* $H/(p)$ *denote the* mod (p) *Eilenberg–Mac Lane spectrum for a prime* p *and let* $u\colon MU \to H/(p)$ *be the Thom class, i.e., the generator of* $H^0(MU;\mathbf{Z}/(p))$. *Then* $H_*(u)$ *is an algebra map and its image in* $H_*(H/(p)) = A_*$ *is* $P(\xi_1^2, \xi_2^2, \dots)$ *for* $p = 2$ *and* $P(\xi_1, \xi_2, \dots)$ *for* $p > 2$. □

The main result concerning MU is the following.

3.1.5. THEOREM (Milnor [**4**], Novikov [**2, 3**]).
(a) $\pi_*(MU) = \mathbf{Z}[x_1, x_2, \dots]$ *with* $x_i \in \pi_{2i}(MU)$.
(b) *Let* $h\colon \pi_*(MU) \to H_*(MU;\mathbf{Z})$ *be the Hurewicz map. Then modulo decomposables in* $H_*(MU;\mathbf{Z})$,

$$h(x_i) = \begin{cases} -pb_i & \text{if } i = p^k - 1 \text{ for some prime } p \\ -b_i & \text{otherwise.} \end{cases} \qquad \square$$

We will prove this in essentially the same way that Milnor and Novikov did. After some preliminaries on the Steenrod algebra we will use the change-of-rings isomorphisms A1.1.18 and A1.3.13 to compute the E_2-term (3.1.10). It will follow easily that the spectral sequence collapses; i.e., it has no nontrivial differentials.

To compute the E_2-term we need to know $H_*(MU;\mathbf{Z}/(p))$ as an A_*-comodule algebra. Since it is concentrated in even dimensions, the following result is useful.

3.1.6. LEMMA. *Let* M *be a left* A_**-comodule which is concentrated in even dimensions. Then* M *is a comodule over* $P_* \subset A_*$ *defined as follows. For* $p > 2$, $P_* = P(\xi_1, \xi_2, \dots)$ *and for* $p = 2$, $P_* = P(\xi_1^2, \xi_2^2, \dots)$.

PROOF. For $m \in M$, let $\psi(m) = \Sigma m' \otimes m''$. Then each $m' \in A_s$ must be even-dimensional and by coassociativity its coproduct expansion must consist entirely of even-dimensional factors, which means it must lie in P_*. □

3.1.7. LEMMA. *As a left* A_**-comodule,* $H_*(MU) = P_* \otimes C$, *where* $C = P(u_1, u_2, \dots)$ *with* $\dim u_i = 2i$ *and* i *is any positive integer not of the form* $p^k - 1$.

PROOF. $H_*(MU;\mathbf{Z}/(P))$ is a P_*-comodule algebra by 3.1.4 and 3.1.6. It maps onto P_* by 3.1.4(b), so by A1.1.18 it is $P_* \otimes C$, where $C = \mathbf{Z}/(p) \,\square_{P_*}\, H_*(MU)$. An easy counting argument shows that C must have the indicated form. □

3.1.8. LEMMA. *Let* M *be a comodule algebra over* A_* *having the form* $P_* \otimes N$ *for some* A_**-comodule algebra* N. *Then*

$$\operatorname{Ext}_{A_*}(\mathbf{Z}/(p), M) = \operatorname{Ext}_E(\mathbf{Z}/(p), N)$$

where

$$E = A_* \otimes_{P_*} \mathbf{Z}/(p) = \begin{cases} E(\xi_1, \xi_2, \dots) & \text{for } p = 2 \\ E(\tau_0, \tau_1, \dots) & \text{for } p > 2. \end{cases}$$

In particular,

$$\operatorname{Ext}_{A_*}(\mathbf{Z}/(p), H_*(MU)) = \operatorname{Ext}_E(\mathbf{Z}/(p), \mathbf{Z}/(p)) \otimes C.$$

PROOF. The statement about $H_*(MU)$ follows from the general one by 3.1.7. For the latter we claim that $M = A_* \square_E N$. We have $A_* = P_* \otimes E$ as vector spaces and hence as E-comodules by A1.1.20, so

$$A_* \square_E N = P_* \otimes E \square_E N = P_* \otimes N = M,$$

and the result follows from A1.3.13. $\square$

Hence we have reduced the problem of computing the Adams E_2-term for MU to that of computing $\mathrm{Ext}_E(\mathbf{Z}/(p), \mathbf{Z}/(p))$. This is quite easy since E is dual to an exterior algebra of finite type.

3.1.9. LEMMA. *Let Γ be a commutative, graded connected Hopf algebra of finite type over a field K which is an exterior algebra on primitive generators $x_1, x_2, \ldots$, each having odd degree if K has characteristic other than 2 (e.g., let $\Gamma = E$). Then*

$$\mathrm{Ext}_\Gamma(K, K) = P(y_1, y_2, \ldots),$$

where $y_i \in \mathrm{Ext}^{1,|x_i|}$ is represented by $[x_i]$ in $C_\Gamma(K)$ (the cobar complex of A1.2.11).

PROOF. Let $\Gamma_i \subset \Gamma$ be the exterior algebra on x_i. Then an injective Γ_i-resolution of K is given by

$$0 \to K \to \Gamma_i \xrightarrow{d} \Gamma_i \to \Gamma_i \to \cdots$$

where $d(x_i) = 1$ and $d(1) = 0$ applying $\mathrm{Hom}_{\Gamma_i}(K, \ )$ gives a complex with trivial boundary operator and shows $\mathrm{Ext}_{\Gamma_i}(K, K) = P(Y_i)$. Tensoring all the R_i together gives an injective Γ-resolution of K and the result follows from the Kunneth theorem. $\square$

Combining the last three lemmas gives

3.1.10. COROLLARY.

$$\mathrm{Ext}_{A_*}(\mathbf{Z}/(p), H_*(MU)) = C \otimes P(a_0, a_1, \ldots),$$

where C is as in 3.1.7 and $a_i \in \mathrm{Ext}^{1,2p^i-1}$ is represented by $[\tau_i]$ for $p > 2$ and $[\xi_i]$ for $p = 2$ in $C_{A_*}(H_*(MU))$. $\square$

Thus we have computed the E_2-term of the classical Adams spectral sequence for $\pi_*(MU)$. Since it is generated by even-dimensional classes, i.e., elements in $E_2^{s,t}$ with $t - s$ even, there can be no nontrivial differentials, i.e., $E_2 = E_\infty$.

The group extension problems are solved by 3.1.3; i.e., all multiples of a_0^s are represented in $\pi_*(MU)$ by multiples of p^s. It follows that $\pi_*(MU) \otimes \mathbf{Z}_{(p)}$ is as claimed for each p; i.e., 3.1.5(a) is true locally. Since $\pi_i(MU)$ is finitely generated for each i, we can conclude that it is a free abelian group of the appropriate rank.

To get at the global ring structure note that the mod (p) indecomposable quotient in dimension $2i$, $Q_{2i}\pi_*(MU) \otimes \mathbf{Z}/(p)$ is $\mathbf{Z}/(p)$ for each $i > 0$, so $Q_{2i}\pi_*(MU) = \mathbf{Z}$. Pick a generator x_i in each even dimension and let $R = \mathbf{Z}[x_1, x_2, \ldots]$. The map $R \to \pi_*(MU)$ gives an isomorphism after tensoring with $\mathbf{Z}_{(p)}$ for each prime p, so it is isomorphism globally.

To study the Hurewicz map

$$h\colon \pi_*(MU) \to H_*(MU; \mathbf{Z}),$$

recall $H_*(X; \mathbf{Z}) = \pi_*(X \wedge H)$, where H is the integral Eilenberg–Mac Lane spectrum. We will prove 3.1.5(b) by determining the map of Adams spectral sequences induced by $i\colon MU \to MU \wedge H$. We will assume $p > 2$, leaving the obvious changes for $p = 2$ to the reader. The following result on $H_*(H)$ is standard.

3.1.11. LEMMA. *The* mod (p) *homology of the integer Eilenberg–Mac Lane spectrum*

$$H_*(H) = P_* \otimes E(\bar{\tau}_1, \bar{\tau}_2, \dots)$$

as an A_* comodule, where $\bar{\tau}_i$ denotes the conjugate τ_i, i.e., its image under the conjugation c. $\square$

Hence we have

$$H_*(H) = A_* \,\square_{E(\tau_0)}\, \mathbf{Z}/(p)$$

and an argument similar to that of 3.1.8 shows

$$\operatorname{Ext}_{A_*}(\mathbf{Z}/(p), H_*(X \wedge H)) = \operatorname{Ext}_{E(\tau_0)}(\mathbf{Z}/(p), H_*(X)). \tag{3.1.12}$$

In the case $X = MU$ the comodule structure is trivial, so by 3.1.11,

$$\operatorname{Ext}_{A_*}(\mathbf{Z}/(p), H_*(MU \wedge H)) = H_*(MU) \otimes P(a_0).$$

To determine the map of Ext groups induced by i, we consider three cobar complexes, $C_{A_*}(H_*(MU))$, $C_E(C)$, and $C_{E(\tau_0)}(H_*(MU))$. The cohomologies of the first two are both $\operatorname{Ext}_{A_*}(\mathbf{Z}/(p), H_*(MU))$, by 3.1.2 and 3.1.8, respectively, while that of the third is $\operatorname{Ext}_{A_*}(\mathbf{Z}/(p), H_*(MU \wedge H))$ by 3.1.12. There are maps from $C_{A_*}(H_*(MU))$ to each of the other two.

The class $A_n \in \operatorname{Ext}_{A_*}^{1,2p^n-1}(\mathbf{Z}/(p), H_*(MU))$ is represented by $[\tau_n] \in C_E(C)$. The element $-\sum_i [\bar{\tau}_i]\bar{\xi}_{n-i}^{p^i} \in C_{A_*}(H_*(MU))$ [using the decomposition of $H_*(MU)$ given by 3.1.7] is a cycle which maps to $[\tau_n]$ and therefore it also represents a_n. Its image in $C_{E(\tau_0)}(H_*(MU))$ is $[\tau_0]\bar{\xi}_n$, so we have $i_*(a_n) = a_0\xi_n$. Since $\xi_n \in H_*(MU)$ is a generator it is congruent modulo decomposables to a nonzero scalar multiple of b_{p^n-1}, while u_i (3.1.9) can be chosen to be congruent to b_i. It follows that the $x_i \in \pi_{2i}(MU)$ can be chosen to satisfy 3.1.5(b).

We now turn to the other spectra in our list, MO, bu, and bo. The Adams spectral sequence was not used originally to compute the homotopy of these spectra, but we feel these calculations are instructive examples. In each case we will quote without proof a standard theorem on the spectrum's homology as an A_*-comodule and proceed from there.

For similar treatments of MSO, MSU, and MSp see, respectively, Pengelley [**2**], Pengelley [**1**], and Kochman [**1**].

To following result on MO was first proved by Thom [**1**]. Proofs can also be found in Liulevicius [**1**] and Stong [**1**, p. 95].

3.1.13. THEOREM. *For $p = 2$, $H_*(MO) = A_* \otimes N$, where N is a polynomial algebra with one generator in each degree not of the form $2^k - 1$. For $p > 2$, $H_*(MO) = 0$.* $\square$

It follows immediately that

$$\operatorname{Ext}_{A_*}^s(\mathbf{Z}/(2), H_*(MO)) = \begin{cases} N & \text{if } s = 0 \\ 0 & \text{if } s > 0, \end{cases} \tag{3.1.14}$$

the spectral sequence collapses and $\pi_*(MO) = N$.

For bu we have

3.1.15. THEOREM (Adams [8]).

$$H_*(bu) = \bigoplus_{0 \le i < p-1} \Sigma^{2i} M$$

where

$$M = P_* \otimes E(\bar\tau_2, \bar\tau_3, \dots) \qquad \textit{for } p > 2$$
$$M = P_* \otimes E(\bar\xi_3, \bar\xi_4, \dots) \qquad \textit{for } p = 2$$

where $\bar\alpha$ *for* $\alpha \in A_*$ *is the conjugate* $c(\alpha)$. □

Using 3.1.8 we get

$$\mathrm{Ext}_{A_*}(\mathbf{Z}/(p), M) = \mathrm{Ext}_E(\mathbf{Z}/(p), E(\tau_2, \tau_3, \dots))$$

(again we assume for convenience that $p > 2$) and by an easy calculation A1.3.13 gives

$$\mathrm{Ext}_E(\mathbf{Z}/(p), E(\tau_2, \tau_3, \dots)) = \mathrm{Ext}_{E(\tau_0,\tau_1)}(\mathbf{Z}/(p), \mathbf{Z}/(p)) = P(a_0, a_1)$$

by 3.1.11, so we have

3.1.16. THEOREM.

$$\mathrm{Ext}_{A_*}(\mathbf{Z}/(p), H_*(bu)) = \bigoplus_{i=0}^{p-2} \Sigma^{2i} P(a_0, a_1)$$

where $a_0 \in \mathrm{Ext}^{1,1}$ *and* $a_1 \in \mathrm{Ext}^{1,2p-1}$. □

As in the MU case the spectral sequence collapses because the E_2-term is concentrated in even dimensions. The extensions can be handled in the same way, so we recover the fact that

$$\pi_i(bu) = \begin{cases} \mathbf{Z} & \text{if } i \ge 0 \text{ and } i \text{ is even} \\ 0 & \text{otherwise.} \end{cases}$$

The *bo* spectrum is of interest only at the prime 2 because at odd primes it is a summand of *bu* (see Adams [8]). For $p = 2$ we have

3.1.17. THEOREM (Stong [2]). *For* $p = 2$, $H_*(bo) = P(\bar\xi_1^4, \bar\xi_2^2, \bar\xi_3, \bar\xi_4, \dots)$ *where* $\bar\xi_i = c(\xi_i)$. □

Let $A(1)_* = A_*/(\xi_1^4, \bar\xi_2^2, \bar\xi_3, \bar\xi_4, \dots)$. We leave it as an exercise for the reader to show that $A(1)_*$ is dual to the subalgebra $A(1)$ of A generated by Sq^1 and Sq^2, and that

$$H_*(bo) = A_* \,\square_{A(1)_*}\, \mathbf{Z}/(2),$$

so by A1.3.13,

$$\mathrm{Ext}_{A_*}(\mathbf{Z}/(2), H_*(bo)) = \mathrm{Ext}_{A(1)_*}(\mathbf{Z}/(2), \mathbf{Z}/(2)). \tag{3.1.18}$$

$A(1)$ is not an exterior algebra, so 3.1.9 does not apply. We have to use the Cartan–Eilenberg spectral sequence A1.3.15. The reader can verify that the following is an extension (A1.1.15)

$$\Phi \to A(1)_* \to E(\bar\xi_2), \tag{3.1.19}$$

where $\Phi = P(\xi_1)/(\xi_1^4)$. Φ is isomorphic as a coalgebra to an exterior algebra on elements corresponding to ξ_1 and ξ_1^2, so by 3.1.9

$$\mathrm{Ext}_\Phi(\mathbf{Z}/(2), \mathbf{Z}/(2)) = P(h_{10}, h_{11})$$

and

$$\mathrm{Ext}_{E(\bar{\xi}_2)}(\mathbf{Z}/(2), \mathbf{Z}/(2)) = P(h_{20}),$$

where $h_{i,j}$ is represented by $[\bar{\xi}_i^{2^j}]$ in the appropriate cobar complex. Since $P(h_{20})$ has only one basis element in each degree, the coaction of Φ on it is trivial, so by A1.3.15 we have a Cartan–Eilenberg spectral sequence converging to $\mathrm{Ext}_{A(1)_*}(\mathbf{Z}/(2), \mathbf{Z}/(2))$ with

$$E_2 = P(h_{10}, h_{11}, h_{20}) \tag{3.1.20}$$

where $h_{1i} \in E_2^{1,0}$ and $h_{20} \in E_2^{0,1}$. We claim

$$d_2(h_{20}) = h_{10}h_{11}. \tag{3.1.21}$$

This follows from the fact that

$$d(\xi_2) = \xi_1 \otimes \xi_1^2$$

in $C_{A(1)_*}(\mathbf{Z}/(2))$. It follows that

$$E_3 = P(u, h_{10}, h_{11})/(h_{10}h_{11}) \tag{3.1.22}$$

where $u \in E_3^{0,2}$ corresponds to h_{20}^2. Next we claim

$$d_3(u) = h_{11}^3. \tag{3.1.23}$$

We have in $C_{A(1)_*}(\mathbf{Z}/(2))$,

$$d(\bar{\xi}_2 \otimes \bar{\xi}_2) = \bar{\xi}_2 \otimes \xi_1 \otimes \xi_1^2 + \xi_1 \otimes \xi_1^2 \otimes \bar{\xi}_2.$$

In this E_2 this gives

$$d_2 h_{20}^2 = h_{10}h_{11}h_{20} + h_{20}h_{10}h_{11} = 0$$

since E_2 is commutative. However, the cobar complex is not commutative and when we add correcting terms to $\bar{\xi}_2 \otimes \bar{\xi}_2$ in the hope of getting a cycle, we get instead

$$d(\bar{\xi}_2 \otimes \bar{\xi}_2 + \xi_1 \otimes \xi_1^2\bar{\xi}_2 + \xi_1\bar{\xi}_2 \otimes \xi_1^2) = \xi_1^2 \otimes \xi_1^2 \otimes \xi_1^2,$$

which implies 3.1.23. It follows that

$$E_4 = P(h_{10}, h_{11}, v, w)/(h_{10}h_{11}, h_{11}^3, v^2 + h_{10}^2 w, vh_{11}), \tag{3.1.24}$$

where $v \in E_4^{1,2}$ and $w \in E_4^{0,4}$ correspond to $h_{10}h_{20}^2$ and h_{20}^4, respectively.

Finally, we claim that $E_4 = E_\infty$; inspection of E_4 shows that there cannot be any higher differentials because there is no $E_r^{s,t}$ for $r \geq 4$ which is nontrivial and for which $E_r^{s+r,t-r+1}$ is also nontrivial. There is also no room for any nontrivial extensions in the multiplicative structure. Thus we have proved

3.1.25. THEOREM. *The E_2-term for the* mod (2) *Adams spectral sequence for* $\pi_*(bo)$,

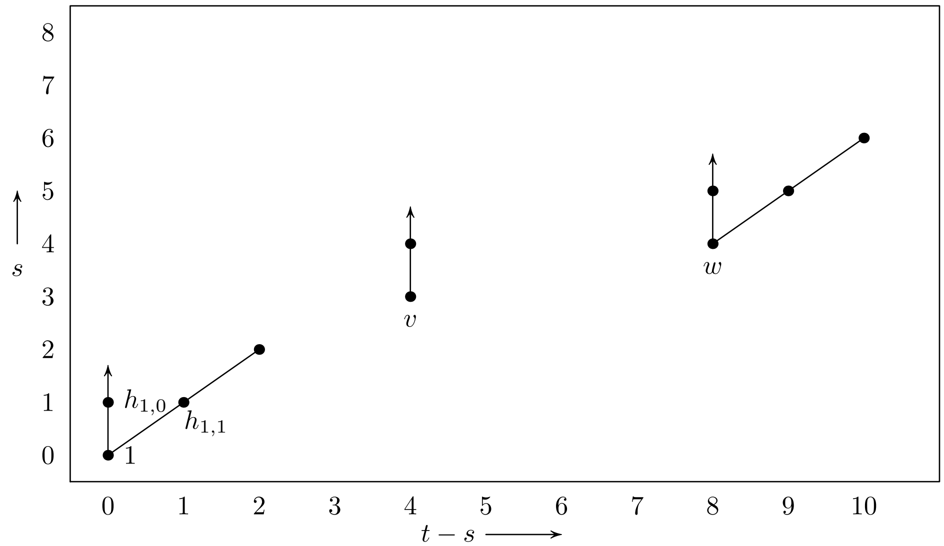
$$\mathrm{Ext}_{A_*}(\mathbf{Z}/(2), H_*(bo)) = \mathrm{Ext}_{A(1)_*}(\mathbf{Z}/(2), \mathbf{Z}/(2))$$

is

$$P(h_{10}, h_{11}, v, w)/(h_{10}h_{11}, h_{11}^3, v^2 + h_{10}^2 w, vh_{11}),$$

where

$$h_{10} \in \mathrm{Ext}^{1,1}, \quad h_{11} \in \mathrm{Ext}^{1,2}, \quad v \in \mathrm{Ext}^{3,7}, \quad \textit{and} \quad w \in \mathrm{Ext}^{4,12}. \qquad \square$$



This E_2-term is displayed in the accompanying figure. A vertical arrow over an element indicates that $h_{10}^s x$ is also present and nontrivial for all $s > 0$.

Now we claim that this Adams spectral sequence also collapses, i.e., $E_2 = E_\infty$. Inspection shows that the only possible nontrivial differential is $d_r(w^n h_{11}) = w^n h_{10}^{n+r}$. However, bo is a ring spectrum so by 2.3.3 the differentials are derivations and we cannot have $d_r(h_{11}) = h_{10}^{r+1}$ because it contradicts the relation $h_{10}h_{11} = 0$. The extension problem is solved by 3.1.3, giving

3.1.26. THEOREM (Bott [1]).

$$\pi_*(bo) = \mathbf{Z}[\eta, \alpha, \beta]/(2\eta, \eta^3, \eta\alpha, \alpha^2 - 4\beta)$$

with $\eta \in \pi_1$, $\alpha \in \pi_4$, $\beta \in \pi_8$, *i.e., for* $i \geq 0$

$$\pi_i(bo) = \begin{cases} \mathbf{Z} & \text{if } i \equiv 0 \mod 4 \\ \mathbf{Z}/2 & \text{if } i \equiv 1 \text{ or } 2 \mod 8 \\ 0 & \text{otherwise.} \end{cases} \qquad \square$$

For future reference we will compute $\mathrm{Ext}_{A(1)}(\mathbf{Z}/(2), M)$ for $M = A(0)_* \equiv E(\xi_1)$ and $M = Y \equiv P(\xi_1)/(\xi_1^4)$. Topologically these are the Adams E_2-terms for the mod (2)-Moore spectrum smashed with bo and bu, respectively. We use the Cartan–Eilenberg spectral sequence as above and our E_2-term is

$$\mathrm{Ext}_\Phi(\mathbf{Z}/(2), \mathrm{Ext}_{E(\bar{\xi}_2)}(\mathbf{Z}/(2), M)).$$

An easy calculation shows that

$$E_2 = P(h_{11}, h_{20}) \quad \text{for } M = A(0)_*$$

and

$$E_2 = P(h_{20}) \quad \text{for } M = Y.$$

In the latter case the Cartan–Eilenberg spectral sequence collapses. In the former case the differentials are not derivations since $A(0)_*$ is not a comodule algebra.

From 3.1.23 we get $d_3(h_{20}^2) = h_{11}^3$, so

$$E_\infty = E_4 = P(w) \otimes \{1, h_{11}, h_{11}^2, h_{20}, h_{20}h_{11}, h_{20}h_{11}^2\}.$$

This Ext is not an algebra but it is a module over $\operatorname{Ext}_{A(1)_*}(\mathbf{Z}/(2), \mathbf{Z}/(2))$. We will show that there is a nontrivial extension in this structure, namely $h_{10}h_{20} = h_{11}^2$. We do this by computing in the cobar complex $C_{A(1)_*}(A(0)_*)$. There the class h_{20} is represented by $[\xi_2]+[\xi_1^2]\xi_1$, so $h_{10}h_{20}$ is represented by $[\xi_1|\xi_2]+[\xi_1|\xi_1^2]\xi_1$. The sum of this and $[\xi_1^2|\xi_1^2]$ (which represents $h_{1,1}^2$) is the coboundary of $[\xi_1\xi_2] + [\xi_1^3 + \xi_2]\xi_1$.

From these considerations we get

3.1.27. THEOREM. *As a module over* $\operatorname{Ext}_{A(1)_*}(\mathbf{Z}/(2), \mathbf{Z}/(2))$ (3.1.25) *we have*

(a) $\operatorname{Ext}_{A(1)_*}, (\mathbf{Z}/(2), A(0)_*)$ *is generated by* $1 \in \operatorname{Ext}^{0,0}$ *and* $h_{20} \in \operatorname{Ext}^{1,3}$ *with* $h_{10} \cdot 1 = 0$, $h_{10}h_{20} = h_{11}^2 \cdot 1$, $v \cdot 1 = 0$, *and* $vh_{20} = 0$.

(b) $\operatorname{Ext}_{A(1)_*}(\mathbf{Z}(2), Y)$ *is generated by* $\{h_{20}^i \colon 0 \le i \le 3\}$ *with* $h_{10}h_{20}^i = h_{11}h_{20}^i = vh_{20}^i = 0$. $\square$

We will also need an odd primary analog of 3.1.27(a). $A(1) = E(\tau_0, \tau_1) \otimes P(\xi_1)/(\xi_1^p)$ is the dual to the subalgebra of A generated by the Bockstein β and the Steenrod reduced power P^1. Instead of generalizing the extension 3.1.19 we use

$$P(0)_* \to A(0)_* \to E(1)_*,$$

where $P(0)_* = P(\xi_1)/(\xi_1^p)$ and $E(1)_* = E(\tau_0, \tau_1)$. The Cartan–Eilenberg spectral sequence E_2-term is therefore

$$\operatorname{Ext}_{p(0)_*}(\mathbf{Z}/(p), \operatorname{Ext}_{E(1)_*}(\mathbf{Z}/(p), A(0)_*)),$$

where $A(0)_* = E(\tau_0)$. An easy calculation gives

3.1.28. THEOREM. *For* $p > 2$

$$\operatorname{Ext}_{A(1)_*}(\mathbf{Z}/(p), A(0)_*) = E(h_0) \otimes P(a_1, b_0),$$

where $h_0 \in \operatorname{Ext}^{1,q}$, $a_1 \in \operatorname{Ext}^{1,q+1}$, *and* $b_0 \in \operatorname{Ext}^{2,pq}$ *are represented by* $[\xi_1]$, $[\xi_1]\tau_0 + [\tau_1]$, *and* $\sum_{0<i<p} p^{-1}\binom{p}{i}[\xi_1^i|\xi_1^{p-i}]$, *respectively.* $\square$

2. The May Spectral Sequence

In this section we discuss a method for computing the classical Adams E_2-term, $\operatorname{Ext}_{A_*}(\mathbf{Z}/(p), \mathbf{Z}/(p))$, which we will refer to simply as Ext. For the reader hoping to understand the classical Adams spectral sequence we offer two pieces of advice. First, do as many explicit calculations as possible yourself. Seeing someone else do it is no substitute for the insight gained by firsthand experience. The computations sketched below should be reproduced in detail and, if possible, extended by the reader. Second, the E_2-term and the various patterns within it should be examined and analyzed from as many viewpoints as possible. For this reason we will describe several methods for computing Ext. For reasons to be given in Section 4.4, we will limit our attention here to the prime 2.

The most successful method for computing Ext through a range of dimensions is the spectral sequence of May [**1**]. Unfortunately, crucial parts of this material have never been published. The general method for computing Ext over a Hopf algebra is described in May [**2**], and the computation of the differentials in the May spectral sequence for the Steenrod algebra through dimension 70 is described by Tangora [**1**]. A revised account of the May E_2-term is given in May [**4**].

In our language May's approach is to filter A_* by copowers of the unit coideal (A1.3.10) and to study the resulting spectral sequence. Its E_2-term is the Ext over the associated graded Hopf algebra E^0A_*. The structure of this Hopf algebra is as follows.

3.2.1. THEOREM (May [1]). (a) *For $p = 2$,*

$$E^0A_* = E(\xi_{i,j}\colon i > 0, j \geq 0)$$

with coproduct

$$\Delta(\xi_{i,j}) = \sum_{0 \leq k \leq i} \xi_{i-k,j+k} \otimes \xi_{k,j},$$

where $\xi_{0,j} = 1$ and $\xi_{i,j} \in E^0_i A_*$ is the projection of $\xi_i^{2^j}$.

(b) *For $p > 2$,*

$$E^0A_* = E(\tau_i\colon i \geq 0) \otimes T(\xi_{i,j}\colon i > 0, j \geq 0)$$

with coproduct given by

$$\Delta(\xi_{i,j}) = \sum_{0 \leq k \leq i} \xi_{i-k,j+k} \otimes \xi_{k,j}$$

and

$$\Delta(\tau_i) = \tau_i \otimes 1 + \sum_{0 \leq k \leq i} \xi_{i-k,k} \otimes \tau_k,$$

where $T(\ )$ denotes the truncated polynomial algebra of height p on the indicated generators, $\tau_i \in E^0_{i+1}A_*$ is the projection of $\tau_i \in A_*$, and $\xi_{i,j} \in E^0_i A^*_p$ is the projection of $\xi_i^{p^j}$. □

May actually filters the Steenrod algebra A rather than its dual, and proves that the associated bigraded Hopf algebra E_0A is primitively generated, which is dual to the statement that each primitive in $E^0A^*_p$ is a generator. A theorem of Milnor and Moore [3] says that every graded primitively generated Hopf algebra is isomorphic to the universal enveloping algebra of a restricted Lie algebra. For $p = 2$ let $x_{i,j} \in E_0A$ be the primitive dual to $\xi_{i,j}$. These form the basis of a Lie algebra under commutation, i.e.,

$$[x_{i,j}, x_{k,m}] \equiv x_{i,j}x_{k,m} - x_{k,m}x_{i,j} = \delta^i_k x_{i,m} - \delta^m_i x_{k,j}$$

where δ^i_j is the Kronecker δ. A *restriction* in a graded Lie algebra L is an endomorphism ξ which increases the grading by a factor of p. In the case at hand this restriction is trivial. The universal enveloping algebra $V(L)$ of a restricted Lie algebra L (often referred to as the restricted enveloping algebra) is the associative algebra L generated by the elements of L subject to the relations $xy - yx = [x, y]$ and $x^p = \xi(x)$ for $x, y \in L$.

May [1] constructs an efficient complex (i.e., one which is much smaller than the cobar complex) for computing Ext over such Hopf algebras. In particular, he proves

3.2.2. THEOREM (May [1]). *For $p = 2$, $\mathrm{Ext}^{***}_{E^0A_*}(\mathbf{Z}/(2), \mathbf{Z}/(2))$ (the third grading being the May filtration) is the cohomology of the complex*

$$V^{***} = P(h_{i,j}\colon i > 0,\ j \geq 0)$$

with $d(h_{i,j}) = \sum_{0<k<i} h_{k,j}h_{i-k,k+j}$, where $h_{i,j} \in V^{1,2^j(2^i-1),i}$ corresponds to $\xi_{i,j} \in A^*_2$. □

Our $h_{i,j}$ is written R_i^j by May [**1**] and R_{ji} by Tangora [**1**], but as $h_{i,j}$ (in a slightly different context) by Adams [**3**]. Notice that in $C^*(\mathbf{Z}/(2))$ one has $d[\xi_i^{2^j}] = \sum_{0<k<i}[\xi_{k-i}^{2^{i+j}}|\xi_k^{2^j}]$, which corresponds to the formula for $d(h_{i,j})$ above. The theorem asserts that $E^0C^*(\mathbf{Z}/(2))$ is chain homotopy equivalent to the polynomial algebra on the $[\xi_{i,j}]$. We will see below (3.2.7) that $C^*(\mathbf{Z}/(2))$ itself does not enjoy the analogous property and that the May differentials are a measure of its failure to do so.

From 3.2.2 May derives a spectral sequence of the following form.

3.2.3. Theorem (May [**1**]). *There is a spectral sequence converging to*

$$\operatorname{Ext}_{A_*}^{**}(\mathbf{Z}/(2), \mathbf{Z}/(2))$$

with $E_1^{***} = V^{***}$ *and* $d_r\colon E_r^{s,t,u} \to E_r^{s+1,t,u+1-r}$.

Proof of 3.2.2 and 3.2.3. The spectral sequence is a reindexed form of that of A1.3.9, so 3.2.3 follows from 3.2.2. We will show that the same spectral, sequence can be obtained more easily by using a different increasing filtration of A_*. An increasing filtration is defined by setting $|\xi_i^{2^j}| = 2i-1$. Then it follows easily that this E^0A_* has the same algebra structure as in 3.2.1 but with each $\xi_{i,j}$ primitive. Hence E^0A_* is dual to an exterior algebra and its Ext is V^{***} (suitably reindexed) by 3.1.9. A1.3.9 gives us a spectral sequence associated to this filtration. In particular, it will have $d_1(h_{i,j}) = \sum h_{k,j}h_{i-k,j+k}$ as in 3.2.2. Since all of the $h_{i,j}$ have odd filtration degree, all of the nontrivial differentials must have odd index. It follows that this spectral sequence can be reindexed in such a way that each d_{2r-1} gets converted to a d_r and the resulting spectral sequence is that of 3.2.3. $\square$

For $p > 2$ the spectral sequence obtained by this method is not equivalent to May's but is perhaps more convenient as the latter has an E_1-term which is nonassociative. In the May filtration one has $|\tau_{i-1}| = |\xi_i^{p^j}| = i$. If we instead set $|\tau_{i-1}| = |\xi_i^{p^j}| = 2i-1$, then the resulting E^0A_* has the same algebra structure (up to indexing) as that of 3.2.1(b), but all of the generators are primitive. Hence it is dual to a product of exterior algebras and truncated polynomial algebras of height p. To compute its Ext we need, in addition to 3.1.11, the following result.

3.2.4. Lemma. *Let* $\Gamma = T(x)$ *with* $\dim x = 2n$ *and* x *primitive. Then*

$$\operatorname{Ext}_\Gamma(\mathbf{Z}/(p), \mathbf{Z}/(p)) = E(h) \otimes P(b),$$

where

$$h \in \operatorname{Ext}^1 \quad \textit{is represented in } C_\Gamma(\mathbf{Z}/(p)) \textit{ by } [x]$$

and

$$b \in \operatorname{Ext}^2 \quad \textit{by} \quad \sum_{0<i<p} \frac{1}{p}\binom{p}{i}[x^i|x^{p-i}]. \qquad \square$$

The proof is a routine calculation and is left to the reader.

To describe the resulting spectral sequence we have

3.2.5. Theorem. *For* $p > 2$ *the dual Steenrod algebra* (3.1.1) A_* *can be given an increasing filtration with* $|\tau_{i-1}| = |\xi_i^{p^j}| = 2i-1$ *for* $i-1,\ j \geq 0$. *The associated bigraded Hopf algebra* E^0A_* *is primitively generated with the algebra structure of* 3.2.1(b). *In the associated spectral sequence* (A1.3.9)

$$E_1^{***} = E(h_{i,j}\colon i>0,\ j\geq 0) \otimes P(b_{i,j}\colon i>0,\ j\geq 0) \otimes P(a_i\colon i\geq 0),$$

where

$$h_{i,j} \in E_1^{1,2(p^i-1)p^j,2i-1},$$
$$b_{i,j} \in E_1^{2,2(p^i-1)p^{1+j},p(2i-1)},$$

and

$$a_i \in E_1^{1,2p^i-1,2i+1}$$

($h_{i,j}$ and a_i correspond respectively to $\xi_i^{p^j}$ and τ_i). One has $d_r\colon E_r^{s,t,u} \to E_r^{s-1,t,u-r}$, and if $x \in E_r^{s,t,u}$ then $d_r(xy) = d_r(x)y + (-1)^{t-s}xd_r(y)$. d_1 is given by

$$d_1(h_{i,j}) = -\sum_{0<k<i} h_{k,j}h_{i-k,k+j},$$
$$d_1(a_i) = -\sum_{0\le k<i} a_k h_{i-k,k},$$
$$d_1(b_{i,j}) = 0. \qquad \square$$

In May's spectral sequence for $p > 2$, indexed as in 3.2.3, the E_1-term has the same additive structure (up to indexing) as 3.2.5 and di is the same on the generators, but it is a derivation with respect to a different multiplication, which is unfortunately nonassociative.

We will illustrate this nonassociativity with a simple example for $p = 3$.

3.2.6. Example. In the spectral sequence of 3.2.5 the class $h_{10}h_{20}$ corresponds to a nontrivial permanent cycle which we call g_0. Clearly $h_{10}g_0$=0 in E_∞, but for $p = 3$ it could be a nonzero multiple of $h_{11}b_{10}$ in Ext. The filtration of $h_{10}g_0$ and $h_{11}b_{10}$ are 5 and 4, respectively. Using Massey products (A1.4), one can show that this extension in the multiplicative structure actually occurs in the following way. Up to nonzero scalar multiplication we have $b_{10} = \langle h_{10}, h_{10}, h_{10}\rangle$ and $g_0 = \langle h_{10}, h_{10}, h_{11}\rangle$ (there is no indeterminacy), so

$$\begin{aligned} h_{10}g_0 &= h_{10}\langle h_{10}, h_{10}, h_{11}\rangle \\ &= \langle h_{10}, h_{10}, h_{10}\rangle h_{11} \\ &= b_{10}h_{11}. \end{aligned}$$

Now in the May filtration, both $h_{10}g_0$ and $b_{10}h_{11}$ have weight 4, so this relation must occur in E_1, i.e., we must have

$$0 \neq h_{10}g_0 = h_{10}(h_{10}g_0) \neq (h_{10}h_{10})g_0 = 0,$$

so the multiplication is nonassociative.

To see a case where this nonassociativity affects the behavior of May's d_1, consider the element $h_{10}h_{20}h_{30}$. It is a d_1 cycle in 3.2.5. In E_2 the Massey product $\langle h_{10}, h_{11}, h_{12}\rangle$ is defined and represented by $\pm(h_{10}h_{21} + h_{20}h_{12}) = \pm d_1(h_{30})$. Hence in Ext we have

$$\begin{aligned} 0 &= g_0\langle h_{10}, h_{11}, h_{12}\rangle \\ &= \langle g_0h_{10}, h_{11}, h_{12}\rangle \\ &= \pm\langle h_{11}b_{10}, h_{11}, h_{12}\rangle \\ &= \pm b_{10}\langle h_{11}, h_{11}, h_{12}\rangle. \end{aligned}$$

The last bracket is represented by $\pm h_{11}h_{21}$, which is a permanent cycle g_1. This implies (A1.4.12) $d_2(h_{10}h_{20}h_{30}) = \pm b_{11}g_1$. In May's grading this differential is a d_1.

Now we return to the prime 2.

3.2.7. EXAMPLE. The computation leading to 3.1.25, the Adams E_2-term for bo, can be done with the May spectral sequence. One filters $A(1)_*$ (see 3.1.18) and gets the sub-Hopf algebra of E^0A_* generated by ξ_{10}, ξ_{11}, and ξ_{20}. The complex analogous to 3.2.2 is $P(h_{10}, h_{11}, h_{20})$ with $d(h_{20}) = h_{10}h_{11}$. Hence the May E_2-term is the Cartan–Eilenberg E_3-term (3.1.22) suitably reindexed, and the d_3 of 3.1.23 corresponds to a May d_2.

We will illustrate the May spectral sequence for the mod (2) Steenrod algebra through the range $t-s \leq 13$. This range is small enough to be manageable, large enough to display some nontrivial phenomena, and is convenient because no May differentials originate at $t-s = 14$. May [**1, 4**] was able to describe his E_2-term (including d_2) through a very large range, $t-s \leq 164$ (for $t-s \leq 80$ this description can be found in Tangora [**1**]). In our small range the E_2-term is as follows.

3.2.8. LEMMA. *In the range $t-s \leq 13$ the E_2-term for the May spectral sequence* (3.2.3) *has generators*

$$h_j = h_{1,j} \in E_2^{1,2^j,1},$$
$$b_{i,j} = h_{i,j}^2 \in E_2^{2,2^{j+1}(2^i-1),2i},$$

and

$$x_7 = h_{20}h_{21} + h_{11}h_{30} \in E_2^{2,9,4}$$

with relations

$$h_j h_{j+1} = 0,$$
$$h_2 b_{20} = h_0 x_7,$$

and

$$h_2 x_7 = h_0 b_{21}. \qquad \square$$

This list of generators is complete through dimension 37 if one adds x_{16} and x_{34}, obtained from x_7 by adding 1 and 2 to the second component of each index. However, there are many more relations in this larger range.

The E_2-term in this range is illustrated in FIG. 3.2.9. Each dot represents an additive generator. If two dots are joined by a vertical line then the top element is h_0 times the lower element; if they are joined by a line of slope $\frac{1}{3}$ then the right-hand element is h_2 times the left-hand element. Vertical and diagonal arrows mean that the element has linearly independent products with all powers of h_0 and h_1, respectively.

3.2.10. LEMMA. *The differentials in* 3.2.3 *in this range are given by*
(a) $d_r(h_j) = 0$ *for all* r,
(b) $d_2(b_{2,j}) = h_j^2 h_{j+2} + h_{j+1}^3$,
(c) $d_2(x_7) = h_0 h_2^2$,
(d) $d_2(b_{30}) = h_1 b_{21} + h_3 b_{20}$, *and*
(e) $d_4(b_{20}^2) = h_0^4 h_3$.

PROOF. In each case we make the relevant calculation in the cobar complex $C_{A_*}(\mathbf{Z}/(2))$ of 3.1.2. For (a), $[\xi_i^{2^j}]$ is a cycle. For (b) we have

$$d([\xi_2|\xi_2] + [\xi_1^2|\xi_1\xi_2] + [\xi_2\xi_1^2|\xi_1]) = [\xi_1^2|\xi_1^2|\xi_1^2] + [\xi_1^4|\xi_1|\xi_1].$$

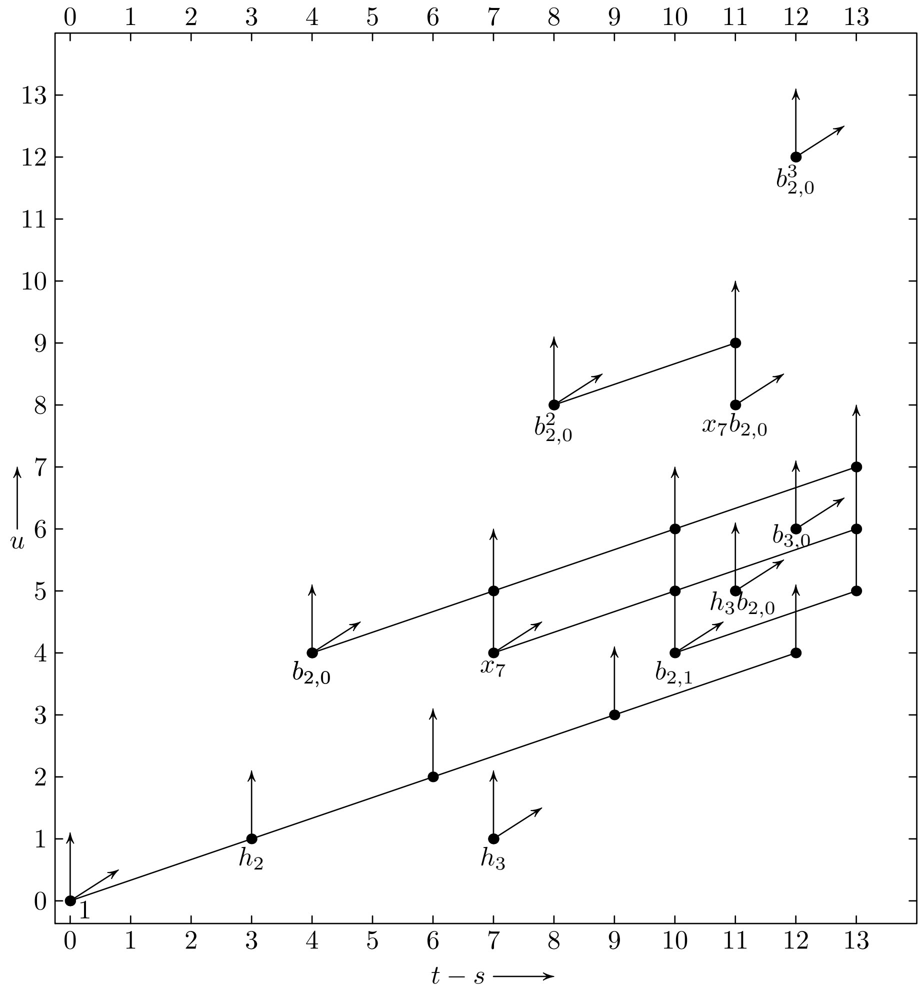


FIGURE 3.2.9. The May E_2-term for $p = 2$ and $t - s \leq 13$

For (c) we have

$$d([(\xi_1^3 + \xi_2)|\xi_2^2] + [(\xi_3 + \xi_1^4\xi_2 + \xi_1\xi_2^2 + \xi_1^7)|\xi_1^2] + [\xi_1|\xi_1^2\xi_2^2]) = [\xi_1|\xi_1^4|\xi_1^4].$$

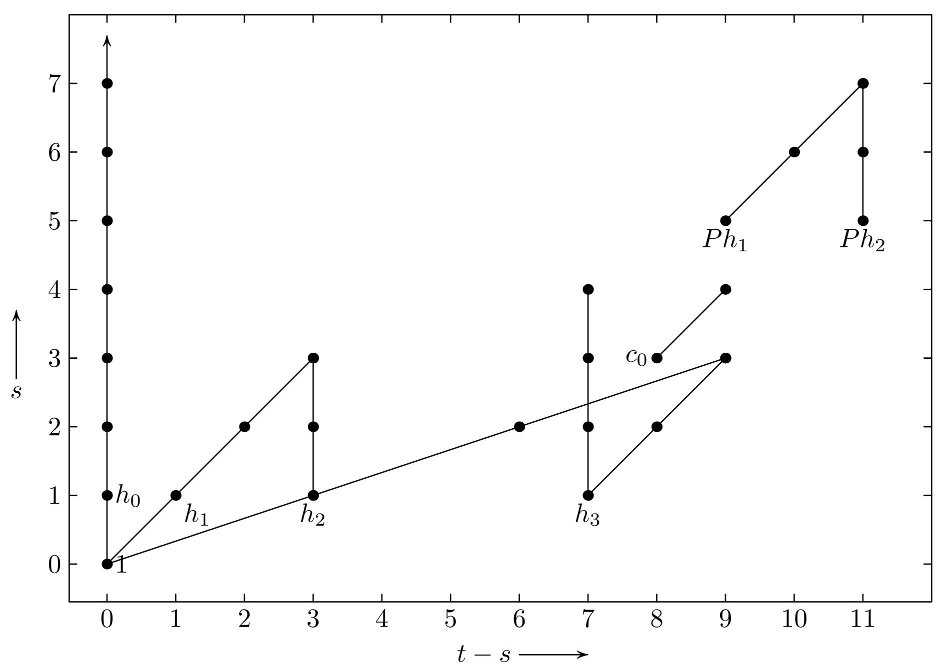
For (d) we use the relation $x_7^2 = h_1^2b_{30} + b_{20}b_{21}$ (which follows from the definition of the elements in question); the right-hand term must be a cycle in E_2 and we can use this fact along with (b) to calculate $d_2(b_{30})$.

Part (e) follows from the fact that $h_0^4h_3 = 0$ in Ext, for which three different proofs will be given below. These are by direct calculation in the Λ-algebra (Section 3.3), by application of a Steenrod squaring operation to the relation $h_0h_1 = 0$, and by the Adams vanishing theorem (3.4.5). $\square$

It follows by inspection that no other differentials can occur in this range. Since no May differentials originate in dimension 14 we get

3.2.11. THEOREM. $\mathrm{Ext}_{A_*}^{s,t}(\mathbf{Z}/(2), \mathbf{Z}(2))$ *for* $t - s \leq 13$ *and* $s \leq 7$ *is generated as a vector space by the elements listed in the accompanying table.* (*There are no* *generators for $t-s=12$ and 13, and the only generators in this range with $s>7$ are powers of h_0.)*

In the table c_0 corresponds to h_1x_7, while Px corresponds to $b_{2,0}^2x$. There are relations $h_1^3=h_0^2h_2$, $h_2^3=h_1^2h_3$, and $Ph_1^3=Ph_0^2h^2=h_0^2Ph_2$. $\square$



Inspecting this table one sees that there are no differentials in the Adams spectral sequence in this range, and all of the group extensions are solved by 3.1.3 and we get

3.2.12. COROLLARY. *For $n \leq 13$ the 2-component of $\pi_n(S^0)$ are given by the following table.*

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13
$\pi_n(S^0)$	$\mathbf{Z}_{(2)}$	$\mathbf{Z}/(2)$	$\mathbf{Z}/(2)$	$\mathbf{Z}/(8)$	0	0	$\mathbf{Z}/(2)$	$\mathbf{Z}/(16)$	$(\mathbf{Z}/(2))^2$	$(\mathbf{Z}/(2))^3$	$\mathbf{Z}/(2)$	$\mathbf{Z}/(8)$	0	0

In general the computation of higher May differentials is greatly simplified by the use of algebraic Steenrod operations (see Section A1.5). For details see Nakamura [**1**].

Now we will use the May spectral sequence to compute $\mathrm{Ext}_{A(2)_*}(\mathbf{Z}/(2), A(0)_*)$, where $A(n)_* = P(\xi_1, \xi_2, \ldots, \xi_{n+1})/(\xi_i^{2^{n+2-1}})$ is dual to the subalgebra $A(n) \subset A$ generated by $Sq^1, Sq^2, \ldots, Sq^{2^n}$. We filter $A(2)_*$ just as we filter A_*. The resulting May E_1-term is $P(h_{11}, h_{12}, h_{20}, h_{21}, h_{30})$ with $d_1(h_{1,i}) = 0 = d_1(h_{20})$, $d_1(h_{21}) = h_{11}h_{12}$, and $d_1(h_{30}) = h_{20}h_{12}$. This gives

$$E_2 = P(b_{21}, b_{30}) \otimes ((P(h_{11}, h_{20}) \otimes E(x_7)) \oplus \{h_{12}^i \colon i > 0\}), \tag{3.2.13}$$

where $b_{21} = h_{21}^2$, $b_{30} = h_{30}^2$, and $x_7 = h_{11}h_{30} + h_{20}h_{21}$. The d_2's are trivial except for

$$d_2(h_{20}^2) = h_{11}^3, \quad d_2(b_{21}) = h_{12}^3, \quad \text{and} \quad d_2(b_{30}) = h_{11}b_{21}. \tag{3.2.14}$$

Since $A(0)_*$ is not a comodule algebra, this is not a spectral sequence of algebras, but there is a suitable pairing with the May spectral sequence of 3.2.3.

Finding the resulting E_3-term requires a little more ingenuity. In the first place we can factor out $P(b_{30}^2)$, i.e., $E_2 = E_2/(b_{30}^2) \otimes P(b_{30}^2)$ as complexes. We denote $E_2/(b_{30}^2)$ by $\overline{E}_2$ and give it an increasing filtration as a differential algebra by letting $F_0 = P(h_{11}, h_{20}) \otimes E(x_7) \oplus \{h_{12}^i \colon i > 0\}$ and letting $b_{21}, b_{30} \in F_1$. The cohomology of the subcomplex F_0 is essentially determined by 3.1.27(a), which gives $\mathrm{Ext}_{A(1)_*}(\mathbf{Z}/(2), A(0)_*)$. Let B denote this object suitably regraded for the present purpose. Then we have

$$H^*(F_0) = B \otimes E(x_7) \oplus \{h_{12}^i \colon i > 0\}. \tag{3.2.15}$$

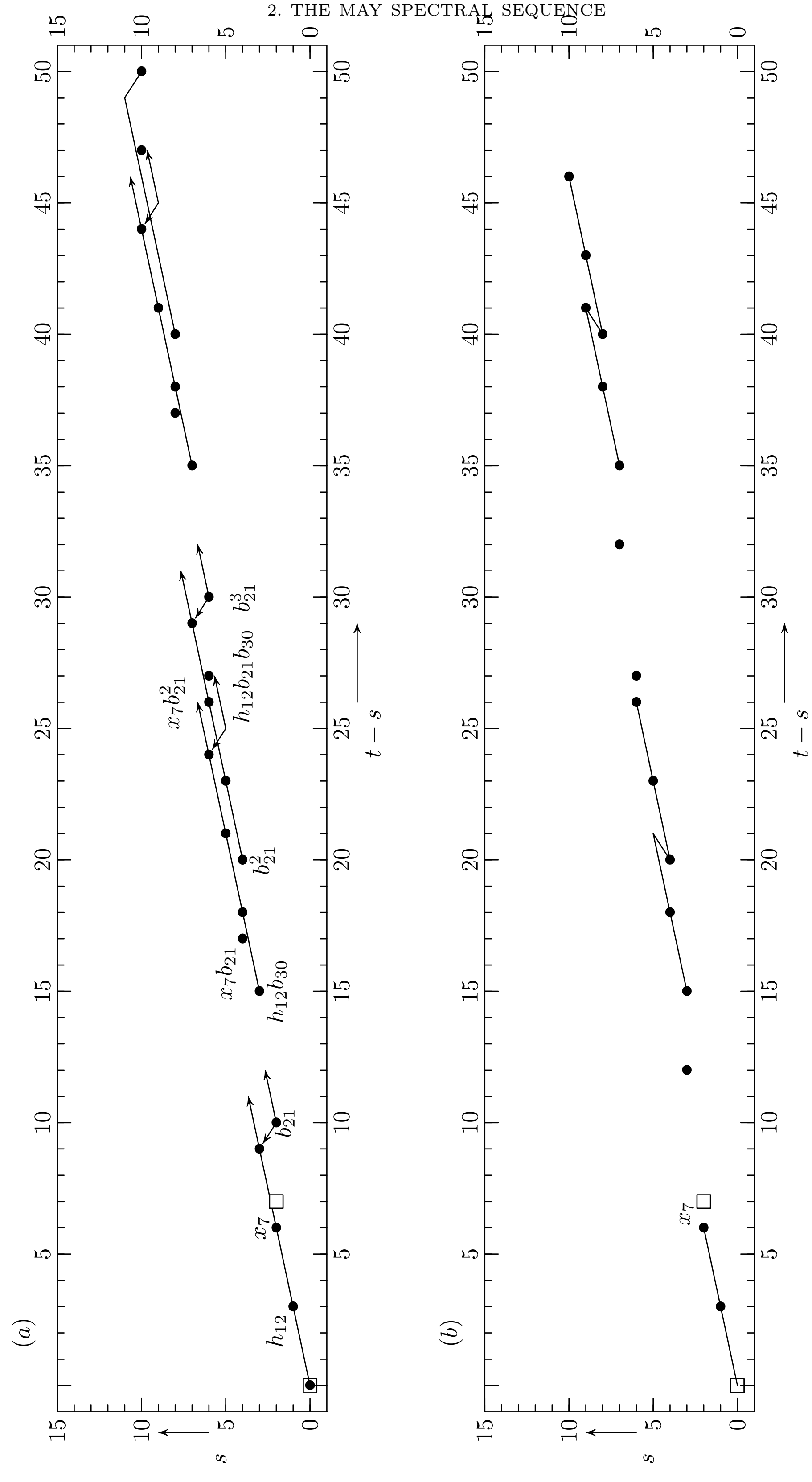
For $k > 0$ we have $F_k/F_{k-1} = \{b_{21}^k, b_{21}^{k-1}b_{30}\} \otimes F_0$ with $d_2(b_{21}^{k-1}b_{30}) = b_{21}^k h_{11}$. Its cohomology is essentially determined by 3.1.27(b), which describes $\mathrm{Ext}_{A(1)_*}(\mathbf{Z}/(2), Y)$. Let C denote this object suitably regraded, i.e., $C = P(h_{20})$. Then we have for $k > 0$

$$H^*(F_k/F_{k-1}) = C\{b_{21}^k\} \otimes E(X_7) \oplus \{b_{21}^k h_{12}^i, b_{30}b_{21}^{k-1}h_{12}^i \colon i > 0\}. \tag{3.2.16}$$

This filtration leads to a spectral sequence converging to $\overline{E}_3$ in which the only nontrivial differential sends

$$b_{21}^k b_{30}^\varepsilon h_{12}^i \quad \text{to} \quad k b_{21}^{k-1} b_{30}^\varepsilon h_{12}^{i+3}$$

for $\varepsilon = 0, 1$, $k > 0$ and $i \geq 1$. This is illustrated in FIG. 3.2.17(a), where a square indicates a copy of B and a large circle indicates a copy of C. Arrows pointing to the left indicate further multiplication by h_{12}, and diagonal lines indicate differentials. Now b_{21} supports a copy of C and a differential. This leads to a copy of C in $\overline{E}_3$ supported by $h_{20}b_{21}$ shown in 3.2.17(b). There is a nontrivial multiplicative extension $h_{20}h)_{12}b_{30} = x_7 b_{21}$ which we indicate by a copy of C in place of $h_{12}b_{30}$ in (b). Fig. 3.2.17(b) also shows the relation $h_{11}b_{21}^2 = h_{12}^3 b_{30}$.


(a)
(b)
h_{12}
x_7
b_{21}
$x_7 b_{21}$
$h_{12} b_{30}$
b_{21}^2
$x_7 b_{21}^2$
$h_{12} b_{21} b_{30}$
b_{21}^3
s
$t - s$

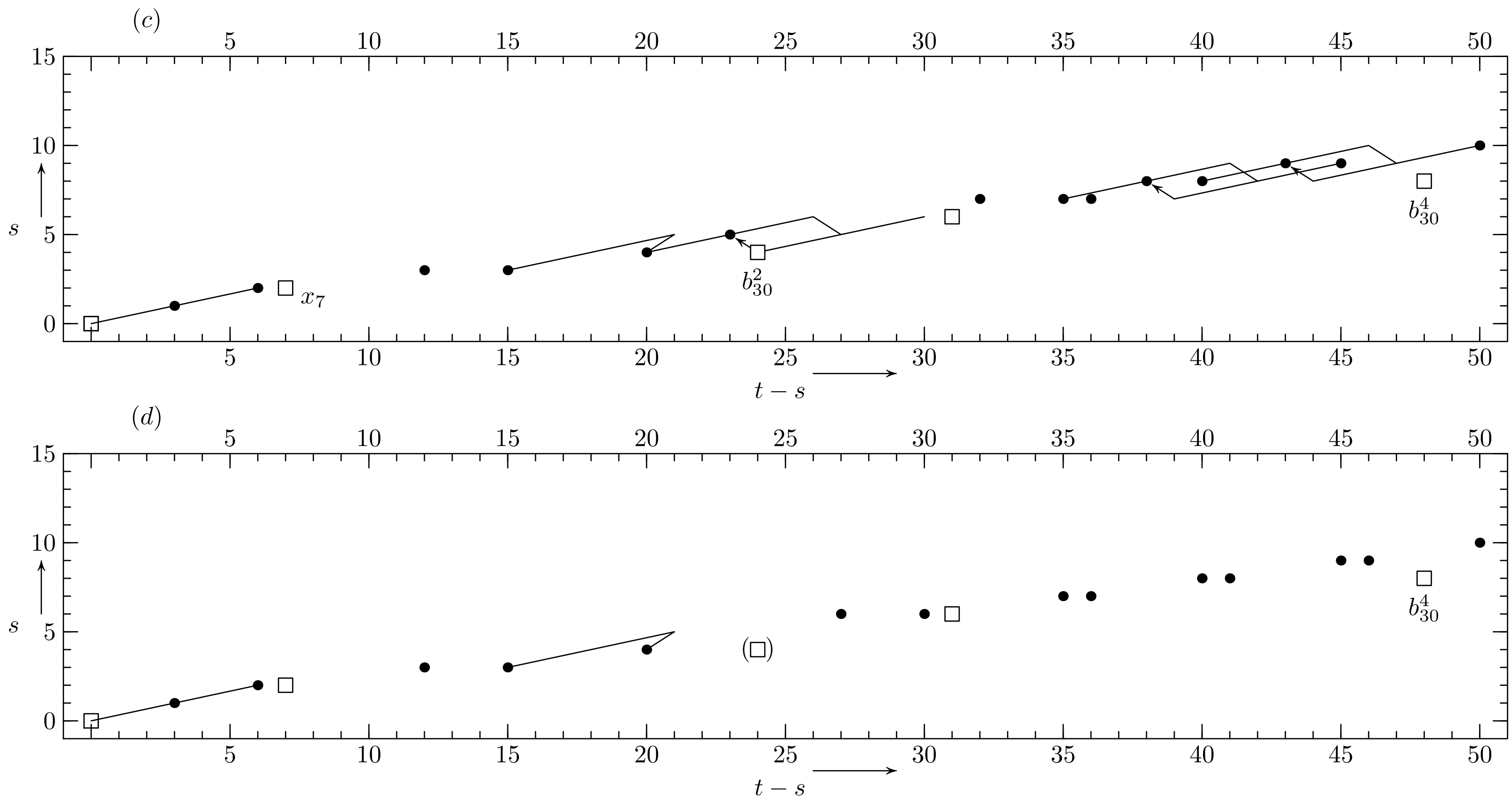


FIGURE 3.2.17. The May spectral sequence for $\mathrm{Ext}_{A(2)_*}(\mathbf{Z}/(2), A(0)_*)$. (a) The spectral sequencex for $\overline{E}_3$; (b) the $\overline{E}_3$-term; (c) differentials in E_3; (d) E_∞

The differentials in E_3 are generated by $d_3(b_{30}^2) = h_{12}b_{21}^2$ and are shown in 3.2.17(c). The resulting $E_4 = E_\infty$ is shown in 3.2.17(d), where the symbol in place of b_{30}^2 indicates a copy of B with the first element missing.

3. The Lambda Algebra

In this section we describe the lambda algebra of Bousfield et al. [**2**] at the prime 2 and the algorithm suggested by it for computing Ext. For more details, including references, see Tangora [**2, 3**] and Richter [**1**] . For most of this material we are indebted to private conversations with E. B. Curtis. It is closely related to that of Section 1.5.

The lambda algebra Λ is an associative differential bigraded algebra whose cohomology, like that of the cobar complex, is Ext. It is much smaller than the cobar complex; it is probably the smallest such algebra generated by elements of cohomological degree one with cohomology isomorphic to Ext. Its greatest attraction, which will not be exploited here, is that it contains for each $n > 0$ a subcomplex $\Lambda(n)$ whose cohomology is the E_2-term of a spectral sequence converging to the 2-component of the *unstable* homotopy groups of S^n. In other words $\Lambda(n)$ is the E_1-term of an unstable Adams spectral sequence.

More precisely, Λ is a bigraded $\mathbf{Z}/(2)$-algebra with generators $\lambda_n \in \Lambda^{1,n+1}$ $(n \geq 0)$ and relations

$$\lambda_i\lambda_{2i+1+n} = \sum_{j\geq 0} \binom{n-j-1}{j} \lambda_{i+n-j}\lambda_{2i+1+j} \quad \text{for } i, n \geq 0 \tag{3.3.1}$$

with differential

$$d(\lambda_n) = \sum_{j\geq 1} \binom{n-j}{j} \lambda_{n-j}\lambda_{j-1}. \tag{3.3.2}$$

Note that d behaves formally like left multiplication by λ_{-1}.

3.3.3. Definition. *A monomial $\lambda_{i_1}\lambda_{i_2}\cdots\lambda_{i_s} \in \Lambda$ is admissible if $2i_r \geq i_{r+1}$ for $1 \leq r < s$. $\Lambda(n) \subset \Lambda$ is the subcomplex spanned by the* admissible *monomials with $i_1 < n$.*

The following is an easy consequence of 3.3.1 and 3.3.2.

3.3.4. Proposition.
(a) *The admissible monomials constitute an additive basis for Λ.*
(b) *There are short exact sequences of complexes*

$$0 \to \Lambda(n) \to \Lambda(n+1) \to \Sigma^n\Lambda(2n+1) \to 0. \qquad \square$$

The significant property of Λ is the following.

3.3.5. Theorem (Bousfield *et al.* [**2**]). (a) *$H(\Lambda) = \mathrm{Ext}_{A_*}(\mathbf{Z}/(2), \mathbf{Z}/(2))$, the classical Adams E_2-term for the sphere.*

(b) *$H(\Lambda(n))$ is the E_2-term of a spectral sequence converging to $\pi_*(S^n)$.*

(c) *The long exact sequence in cohomology* (3.3.6) *given by* 3.3.4(b) *corresponds to the EHP sequence, i.e., to the long exact sequence of homotopy groups of the fiber sequence* (*at the prime* 2)

$$S^n \to \Omega S^{n+1} \to \Omega S^{2n+1} \quad \text{(see 1.5.1)}. \qquad \square$$

The spectral sequence of (b) is the *unstable Adams spectral sequence*. The long exact sequence in (c) above is

$$\text{(3.3.6)} \qquad \to H^{s,t}(\Lambda(n)) \xrightarrow{E} H^{s,t}(\Lambda(n+1)) \xrightarrow{H} H^{s-1,t-n-1}(\Lambda(2n+1)) \xrightarrow{P} H^{s+1,t}(\Lambda(n)) \to .$$

The letters E, H, and P stand respectively for suspension (*Einhängung* in German), Hopf invariant, and Whitehead product. The map H is obtained by dropping the first factor of each monomial. This sequence leads to an inductive method for calculating $H^{s,t}(\Lambda(n))$ which we will refer to as the *Curtis algorithm*.

Calculations with this algorithm up to $t = 51$ (which means up to $t - s = 33$) are recorded in an unpublished table prepared by G. W. Whitehead. Recently, Tangora [**4**] has programmed a computer to find $H^{s,t}(\Lambda)$ at $p = 2$ for $t \leq 48$ and $p = 3$ for $t \leq 99$. Some related machine calculations are described by Wellington [**1**].

For the Curtis algorithm, note that the long exact sequences of 3.3.6 for all n constitute an exact couple (see Section 2.1) which leads to the following spectral sequence, similar to that of 1.5.7.

3.3.7. Proposition (Algebraic EHP spectral sequence).

(a) *There is a trigraded spectral sequence converging to* $H^{s,t}(\Lambda)$ *with*

$$E_1^{s,t,n} = H^{s-1,t-n}(\Lambda(2n-1)) \quad \text{for } s > 0$$

and

$$E_1^{0,t,n} = \begin{cases} \mathbf{Z}/(2) & \text{for } t = n = 0 \\ 0 & \text{otherwise,} \end{cases}$$

and $d_r\colon E_r^{s,t,n} \to E_r^{s+1,t,n-r}$.

(b) *For each* $m > 0$ *there is a similar spectral sequence converging to* $H^{s,t}(\Lambda(m))$ *with*

$$E_1^{s,t,n} = \begin{cases} \text{as above} & \text{for } n \leq m \\ 0 & \text{for } n > m. \end{cases} \qquad \square$$

The EHP sequence in homotopy leads to a similar spectral sequence converging to stable homotopy filtered by sphere of origin which is described in Section 1.5.

At first glance the spectral sequence of 3.3.7 appears to be circular in that the E_1-term consists of the same groups one is trying to compute. However, for $n > 1$ the groups in $E_1^{s,t,n}$ are from the $(t-s-n+1)$-stem, which is known by induction on $t - s$. Hence 3.3.7(b) for odd values of m can be used to compute the E_1-terms. For $n = 1$, we need to know $H^*(\Lambda(1))$ at the outset, but it is easy to compute. $\Lambda(1)$ is generated simply by the powers of λ_0 and it has trivial differential. This corresponds to the homotopy of S^1.

Hence the EHP spectral sequence has the following properties,

3.3.8. Lemma. *In the spectral sequence of* 3.3.7(a),

(a) $E_1^{s,t,n} = 0$ *for* $t - s < n - 1$ (vanishing line);

(b) $E_1^{s,t,n} = \mathbf{Z}/(2)$ *for* $t - s = n - 1$ *and all* $s \geq 0$ *and if in addition* $n - 1$ *is even and positive,* $d_1\colon E_1^{s,t,n} \to E_1^{s+1,t,n-1}$ *is nontrivial for all* $s \geq 0$ (*diagonal groups*);

(c) $E_1^{s,t} = H^{s-1,t-n}(\Lambda)$ *for* $t - s < 3n$ (*stable zone*); *and*

(d) $E_1^{s,t,1} = 0$ *for* $t > s$.

PROOF. The groups in (a) vanish because they come from negative stems in $\Lambda(2n-1)$. The groups in (b) are in the 0-stem of $\Lambda(2n-1)$ and correspond to $\lambda_{n-1}\lambda_0^{s-1} \in \Lambda$. If $n-1$ is even and positive, 3.3.2 gives

$$d(\lambda_{n-1}\lambda_0^{s-1}) \equiv \lambda_{n-2}\lambda_0^s \mod \Lambda(n-2),$$

which means d_1 behaves as claimed. The groups in (c) are independent of n by 3.3.6. The groups in (d) are in $\Lambda(1)$ in positive stems. □

The above result leaves undecided the fate of the generators of $E_1^{0,n-1,n}$ for $n-1$ odd, which correspond to the λ_{n-1}. We use 3.3.2 to compute the differentials on these elements. (See Tangora [**2**] for some helpful advice on dealing with these binomial coefficients.) We find that if n is a power of 2, λ_{n-1} is a cycle, and if $n = k \cdot 2^j$ for odd $k > 1$ then

$$d(\lambda_{n-1}) \equiv \lambda_{n-1-2^j}\lambda_{2^j-1} \mod \Lambda(n-1-2^j).$$

This equation remains valid after multiplying on the right by any cycle in Λ, so we get

3.3.9. PROPOSITION. *In the spectral sequence of 3.3.7(a) every element in $E_1^{s,t,2^j}$ is a permanent cycle. For $n = k2^j$ for $k > 1$ odd, then every element in $E_r^{s,t,k2^j}$ is a d_r-cycle for $r < 2^j$ and*

$$d_{2^j}\colon E_{2^j}^{0,k\cdot 2^j-1,k2^j} \to E_2^{1,k\cdot 2^j-1,(k-1)2^j}$$

is nontrivial, the target corresponding to λ_{2^j-1} under the isomorphism of 3.3.7. The cycle λ_{2^j-1} corresponds to $h_j \in \operatorname{Ext}^{1,2^j}$. □

Before proceeding any further it is convenient to streamline the notation. Instead of $\lambda_{i_1}\lambda_{i_2}\cdots\lambda_{i_s}$ we simply write $i_1 i_2 \ldots i_s$, e.g., we write 411 instead of $\lambda_4\lambda_1\lambda_1$. If an integer ≥ 10 occurs we underline all of it but the first digit, thereby removing the ambiguity; e.g., $\lambda_{15}\lambda_3\lambda_{15}$ is written as $1\underline{5}31\underline{5}$. Sums of monomials are written as sums of integers, e.g., $d(9) = 71 + 53$ means $d(\lambda_9) = \lambda_7\lambda_1 + \lambda_5\lambda_3$; and we write ϕ for zero, e.g., $d(15) = \phi$ means $d(\lambda_{15}) = 0$.

We now study the EHP spectral sequence [3.3.7(a)] for $t - s \leq 14$. It is known that no differentials or unexpected extensions occur in this range in any of the unstable Adams spectral sequences, so we are effectively computing the 2-component of $\pi_{n+k}(S^n)$ for $k \leq 13$ and all n.

For $t-s = 0$ we have $E_1^{s,s,1} = \mathbf{Z}/(2)$ for all $s \geq 0$ and $E_1^{s,s,n} = 0$ for $n > 1$. For $t - s = 1$ we have $E_2^{1,2,2} = \mathbf{Z}/(2)$, corresponding to λ_1 or h_1, while $E_2^{s,1+s,n} = 0$ for all other s and n. From this and 3.3.8(c) we get $E_1^{2,n+2,n} = \mathbf{Z}/(2)$ generated by $\lambda_{n-1}\lambda_1$ for all $n \geq 2$, while $E_1^{s,t,t-s} = 0$ for all other s, t. The element 11 cannot be hit by a differential because 3 is a cycle, so it survives to a generator of the 2-stem, and it gives generators of $E_1^{3,n+4,n}$ (corresponding to elements with Hopf invariant 11) for $n \geq 2$, while $E_1^{s,t,t-s-1} = 0$ for all other s and t.

This brings us to $t - s = 3$. In addition to the diagonal groups $E_1^{s,s+3,4}$ given by 3.3.8(b) we have $E_1^{2,5,3}$ generated by 21 and $E_1^{3,6,2}$ generated by 111, with no other generators in this stem. These two elements are easily seen to be nontrivial permanent cycles, so $H^{s,s+3}(\Lambda)$ has three generators; 3, 21, and 111. Using 3.3.1 one sees that they are connected by left multiplication by 0 (i.e., by λ_0).

Thus for $t-s\leq 3$ we have produced the same value of Ext as given by the May spectral sequence in 3.2.11. The relation $h_0^2h_2=h_1^3$ corresponds to the relation $003=111$ in Λ, the latter being easier to derive. It is also true that 300 is cohomologous in Λ to 111, the difference being the coboundary of $40+22$. So far no differentials have occurred other than those of 3.3.8(b).

These and subsequent calculations are indicated in FIG. 3.3.10, which we now describe. The gradings $t-s$ and n are displayed; we find this more illuminating than the usual practice of displaying $t-s$ and s. All elements in the spectral sequence in the indicated range are displayed except the infinite towers along the diagonal described in 3.3.8(b). Each element (except the diagonal generators) is referred to by listing the leading term of its Hopf invariant with respect to the left lexicographic ordering; e.g., the cycle $4111+221+1123$ is listed in the fifth row as 111. An important feature of the Curtis algorithm is that it *suffices to record the leading term of each element*. We will illustrate this principle with some examples. For more discussion see Tangora [**3**]. The arrows in the figure indicate differentials in the spectral sequence. Nontrivial cycles in Λ for $0<t-s<14$ are listed at the bottom. We do not list them for $t-s=14$ because the table does not indicate which cycles in the 14th column are hit by differentials coming from the 15th column.

3.3.11. EXAMPLE. Suppose we are given the leading term 4111 of the cycle above. We can find the other terms as follows. Using 3.3.1 and 3.3.2 we find $d(4111)=21111$. Refering to Fig. 3.3.10 we find 1111 is hit by the differential from 221, so we add 2221 to 4111 and find that $d(4111+2221)=11121$. The figure shows that 121 is killed by 23, so we add 1123 to our expression and find that $d(4111+2221+1123)=\phi$ i.e., we have found all of the terms in the cycle.

Now suppose the figure has been completed for $t-s<k$. We wish to fill in the column $t-s=k$. The box for $n=1$ is trivial by 3.3.8(d) and the boxes for $n\geq 3$ can be filled in on the basis of previous calculations. (See 3.3.12.) The elements in the box for $n=2$ will come from the *cycles* in the box for $n=3$, $t-s=k-1$, and the elements in the box for $n=2$, $t-s=k-1$ which are *not hit* by d_1's. Hence before we can fill in the box for $b=2$, $t-s=k$, we must find the d_1's originating in the box for $n=3$. The procedure for computing differentials will be described below. Once the column $t-s=k$ has been filled in, one computes the differentials for successively larger values of n.

The above method is adequate for the limited range we will consider, but for more extensive calculations it has a drawback. One could work very hard to show that some element is a cycle only to find at the next stage that it is hit by an easily computed differential. In order to avoid such redundant work one should work by induction on t, then on s and then on n; i.e., one should compute differentials originating in $E_r^{s,t,n}$ only after one has done so for all $E_r^{s',t',n'}$ with $t'<t$, with $t'=t$ and $s'<s$, and with $s'=s$, $t'=t$, and $n'<n$. This triple induction is awkward to display on a sheet of paper but easy to write into a computer program. On the other hand Tangora [**4**, last paragraph starting on page 48] used downward rather than upward induction on s because given knowledge of what happens at all lower values of t, the last group needed for the $(t-s)$-stem is the one with the largest value of s possible under the vanishing line, the unstable analog of 3.4.5. There are advantages to both approaches.

The procedure for finding differentials in the EHP spectral sequence (3.3.7) is the following. We start with some sequence α in the $(n+1)$th row. Suppose

$2n-1$	n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
1	1	*														
3	2		0 *	1	11	111 21	211	2111			233	24111 1233	124111 11233 2233	1124111 224111 21233 2333	2124111 211233	21124111
5	3			*	1	11	111 21 3	31	311	33	4111	233	24111 1233 333	2233 433 124111 11233	1124111 224111 44111 3233 353	324111 31233 3333
7	4				0 *	1	11	111 21 3			33	4111 511 61	233 611 53	24111 1233 6111 333	124111 11233 2233 433	1124111 224111 44111 533
9	5					*	1	11	111 21 3			33	4111 511 61 7	233 53 71	24111 1233 711 333	124111 2233 433 73
11	6						0 *	1	11	111 21 3			33	4111 511 61 7	233 53	24111 1233 333
13	7							*	1	11	111 21 3			33	4111 511 61 7	233 53
15	8								0 *	1	11	111 21 3			33	4111 511 61 7
17	9									*	1	11	111 21 3			33
19	10										0 *	1	11	111 21 3		
21	11											*	1	11	111 21 3	
23	12												0 *	1	11	111 21 3
25	13													*	1	11
27	14														0 *	1
29	15															*
Nontrivial permanent cycles			1	11	111 21 3			33	4111 511 61 7	233 53	1233 24111 333	124111	1124111 224111 44111			

Figure 3.3.10. The EHP spectral sequence (3.3.7) for $t-s \leq 14$

inductively that some correcting terms have already been added to $\lambda_n\alpha$, in the manner about to be described, to give an expression x. We use 3.3.1 and 3.3.2 to find the leading term $i_1 i_2 \dots i_{s+1}$ of $d(x)$. If $d(x) = 0$, then our α is a permanent cycle in the spectral sequence. If not, then beginning with $u = 0$ we look in the table for the sequence $i_{s-u+1} i_{s-u+2} \dots i_{s+1}$ in the $(i_{s-u}+1)$th row until we find one that is hit by a differential from some sequence β in the $(m+1)$th row or until $u = s-1$. In the former event we add $\lambda_{i_1} \dots \lambda_{i_{s-u-1}} \lambda_m \beta$ to x and repeat the process. The coboundary of the new expression will have a smaller leading term since we have added a correcting term to cancel out the original leading coboundary term.

If we get up to $u = s-1$ without finding a target of a differential, then it follows that our original α supports a d_{n-i_1} whose target is $i_2 \cdots i_{s+1}$.

It is not necessary to add all of the correcting terms to x to show that our a is a permanent cycle. The figure will provide a finite list of possible targets for the differential in question. As soon as the leading term of $d(x)$ is smaller (in the left lexicographic ordeninng) than any of these candidates then we are done.

In practice it may happen that one of the sequences $i_{s-u+1} \cdots i_{s+1}$ in the $(i_{s-u}+1)$th row supports a nontrivial differential. This would be a contradiction indicating the presence of an error, which should be found and corrected before proceeding further. Inductive calculations of this sort have the advantage that mistakes usually reveal themselves by producing contradictions a few stems later. Thus one can be fairly certain that a calculation through some range that is free of contradictions is also correct through most of that range. In publishing such computations it is prudent to compute a little beyond the stated range to ensure the accuracy of one's results.

We now describe some sample calculations in 3.2.11.

3.3.12. Example. Filling in the table. Consider the boxes with

$$t - s - (n-1) = 8.$$

To fill them in we need to know the 8-stem of $H(\Lambda(2n-1))$. For convenience the values of $2n-1$ are listed at the extreme left. The first element in the 8-stem is 233, which originates on S^3 and hence appears in all boxes for $n \geq 2$. Next we have the elements 53, 521, and 5111 originating on S^6. The latter two are trivial on S^7 and so do not appear in any of our boxes, while 53 appears in all boxes with $n \geq 4$. The element 611 is born on S^7 and dies on S^9 and hence appears only in the box for $n = 4$. Similarly, 71 appears only in the box for $n = 5$.

3.3.13. Example. Computing differentials We will compute the differentials originating in the box for $t - s = 11$, $n = 11$. To begin we have $d(1\underline{0}1) = (90 + 72 + 63 + 54)1 = 721 + 631 + 541$. The table shows that 721 is hit by 83 and we find

$$d(83) = (70 + 61 + 43)3 = 721 + 433.$$

Hence

$$d(1\underline{0}1 + 83) = 631 + 541 + 433.$$

The figure shows that 31 is hit by 5 so we compute

$$d(65) = 631 + (50 + 32)5 = 631 + 541,$$

so

$$d(101 + 83 + 65) = 433,$$

which is the desired result.

Even in this limited range one can see the beginnings of several systematic phenomena worth commenting on.

3.3.14. REMARK. JAMES PERIODICITY. (Compare 1.5.18.) In a neighborhood of the diagonal one sees a certain in the differentials in addition to that of 3.3.9. For example, the leading term of $d(\lambda_n\lambda_1)$ is $\lambda_{n-2}\lambda_1\lambda_1$ if $n \equiv 0$ or $1 \mod (4)$ and $n \geq 4$, giving a periodic family of d_2's in the spectral sequence. The differential computed in 3.3.13 can be shown to recur every 8 stems; add any positive multiple of 8 to the first integer in each sequence appearing in the calculation and the equation remains valid modulo terms which will not affect the outcome.

More generally, one can show that $\Lambda(n)$ is isomorphic to

$$\Sigma^{-2^m}\Lambda(n+2^m)/\Lambda(2^m)$$

through some range depending on n and m, and a general result on the periodicity of differentials follows. It can be shown that $H^*(\Lambda(n+k)/\Lambda(n))$ is isomorphic in the stable zone [3.3.8(c)] to the Ext for $H^*(RP^{n+k-1}/RP^{n-1})$ and that this periodicity of differentials corresponds to James periodicity. The latter is the fact that the stable homotopy type of RP^{n+k}/RP^n depends (up to suspension) only on the congruence class of n modulo a suitable power of 2. For more on this subject see Mahowald [**1, 2, 3, 4**].

3.3.15. REMARK. THE ADAMS VANISHING LINE. Define a collection of admissible sequences (3.3.3) a_i for $i > 0$ as follows.

$$a_1 = 1, \quad a_2 = 11, \quad a_3 = 111, \quad a_4 = 4111,$$
$$a_5 = 24111, \quad a_6 = 124111, \quad a_7 = 1124111, \quad a_8 = 41124111, \text{ etc.}$$

That is, for $i > 1$

$$a_i = \begin{cases} (1, a_{i-1}) & \text{for } i \equiv 2, 3 \mod (4) \\ (2, a_{i-1}) & \text{for } i \equiv 1 \mod (4) \\ (4, a_{i-1}) & \text{for } i \equiv 0 \mod (4) \end{cases}$$

It can be shown that all of these are nontrivial permanent cycles in the EHP spectral sequence and that they correspond to the elements on the Adams vanishing line (3.4.5). Note that $H(a_{i+1}) = a_i$. All of these elements have order 2 (i.e., are killed by λ_0 multiplication) and half of them, the a_i for $i \equiv 3$ and $0 \mod (4)$, are divisible by 2. The a_{4i+3} are divisible by 4 but not by 8; the sequences obtained are $(2, a_{4i+2})$ and $(4, a_{4i+1})$ except for $i = 1$, when the latter sequence is 3. These little towers correspond to cyclic summands of order 8 in π^S_{8i+3} (see 5.3.7). The a_{4i} are the tops of longer towers whose length depends on i. The sequences in the tower are obtained in a similar manner; i.e., sequences are contracted by adding the first two integers; e.g., in the 7-stem we have 4111, 511, 61, and 7. Whenever i is a power of 2 the tower goes all the way down to filtration 1; i.e., it has $4i$ elements, of which the bottom one is $8i - 1$. The table of Tangora [**1**] shows that the towers in the 23-, 29-, and 55-stems have length 6, while that in the 47-stem has length 12. Presumably this result generalizes in a straightforward manner. These towers are also discussed in 3.4.21 and following 4.4.47.

3.3.16. REMARK. d_1's. It follows from 3.3.9 that all d_1's originate in rows with n odd and that they can be computed by left multiplication by λ_0. In particular, the towers discussed in the above remark will appear repeatedly in the E_1-term and be almost completely cancelled by d_1's, as one can see in Fig. 3.3.10. The elements cancelled by d_1's do not appear in any $H^*(\Lambda(2n-1))$, so if one is not interested in $H^*(\Lambda(2n))$ they can be ignored. This indicates that a lot of repetition could be avoided if one had an algorithm for computing the spectral sequence starting from E_2 instead of E_1.

3.3.17. REMARK. S^3. As indicated in 3.3.5, Λ gives unstable as well as stable Ext groups. From a figure such as 3.3.11 one can extract unstable Adams E_2-terms for each sphere. For the reader's amusement we do this for S^3 for $t-s \leq 28$ in FIG. 3.3.18. One can show that if we remove the infinite tower in the 0-stem, what remains is isomorphic above a certain line of slope $\frac{1}{5}$ to the stable Ext for the mod (2) Moore spectrum. This is no accident but part of a general phenomenon described by Mahowald [**3**].

It is only necessary to label a few of the elements in FIG. 3.3.18 because most of them are part of certain patterns which we now describe. There are clusters of six elements known as *lightning flashes*, the first of which consists of 1, 11, 111, 21, 211, 2111. Vertical and diagonal lines as usual represent right multiplication by λ_0 and λ_1, i.e., by h_0 and h_0 respectively. This point is somewhat delicate. For example the element with in the 9-stem with filtration 4 has leading term (according to 3.3.10) 1233, not 2331. However these elements are cohomologous, their difference being the coboundary of 235.

If the first element of a lightning flash is x, the others are $1x$, $11x$, $2x$, $21x$, and $211x$. In the clusters containing 23577 and 233577, the first elements are missing, but the others behave as if the first ones were 4577 and 43577, respectively. For example, the generator of $E_2^{5,30}$ is 24577. In these two cases the sequences $1x$ and $11x$ are not admissible, but since $14 = 23$ by 3.3.1, we get the indicated values for $1x$.

If $x \in E_2^{s,t}$ is the first element of a lightning flash, there is another one beginning with $Px \in E_2^{s+4,t+12}$. The sequence for Px is obtained from that for x by adding 1 to the last integer and then adjoining 4111 on the right, e.g., $P(233) = 2344111$. This operator P can be iterated any number of times, is related to Bott periodicity, and will be discussed more in the next section.

There are other configurations which we will call *rays* begining with 245333 and 235733. Successive elements in a ray are obtained by left multiplication by λ_2. This operation is related to complex Bott periodicity.

In the range of this figure the only elements in positive stems not part of a ray or lightning flash are 23333 and 2335733. This indicates that the Curtis algorithm would be much faster if it could be modified in some way to incorporate this structure.

Finally, the figure includes Tangora's labels for the stable images of certain elements. This unstable Adams spectral sequence for $\pi_*(S^3)$ is known to have nontrivial d_2's originating on 245333, 222245333, and 2222245333, and d_3's on 2235733 and 22235733. Related to these are some exotic additive and multiplicative extensions: the homotopy element corresponding to $Ph_1d_0 = 243344111$ is twice any representative of $h_0h_2g = 235733$ and η (the generator of the 1-stem) times a representative of 2245333. Hence the permanent cycles 2245333, 24334111, 235733,

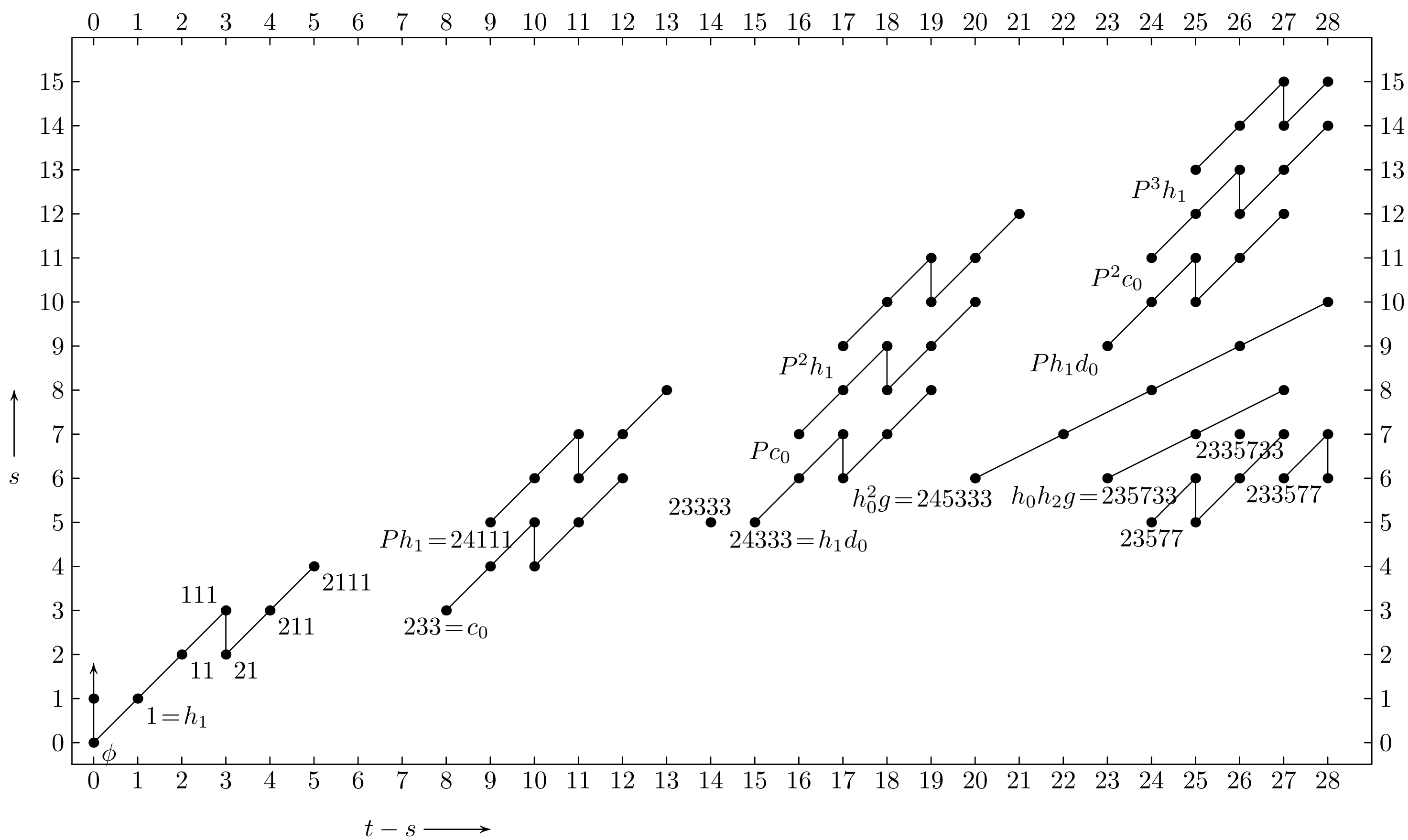


FIGURE 3.3.18. The unstable Adams E_2-term for S^3.

22245333, 224334111, and the missing element 35733 in some sense constitute an exotic lightning flash.

4. Some General Properties of Ext

In this section we abbreviate $\mathrm{Ext}_{A_*}(\mathbf{Z}/(p), \mathbf{Z}/(p))$ by Ext. First we describe Ext^s for small values of s. Then we comment on the status of its generators in homotopy. Next we give a vanishing line, i.e., a function $f(s)$ such that $\mathrm{Ext}^{s,t} = 0$ for $0 < t - s < f(s)$. Then we give some results describing $\mathrm{Ext}^{s,t}$ for t near $f(s)$.

3.4.1. THEOREM. *For $p = 2$*

(a) $\mathrm{Ext}^0 = \mathbf{Z}/(2)$ *generated by* $1 \in \mathrm{Ext}^{0,0}$.

(b) Ext^1 *is spanned by* $\{h_i \colon i \geq 0\}$ *with* $h_i \in \mathrm{Ext}^{1,2^i}$ *represented by* $[\xi_1^{2^i}]$.

(c) (Adams [**12**]) Ext^2 *is spanned by* $\{h_i h_j \colon 0 \leq i \leq j,\ j \neq i+1\}$.

(d) (Wang [**1**]) Ext^3 *is spanned by* $h_i h_j h_k$, *subject to the relations*

$$h_i h_j = h_j h_i, \quad h_{ih_{i+1}} = 0 \quad h_1 h_{i+2}^2 = 0 \quad h_i^2 h_{i+2} = h_{i+1}^3,$$

along with the elements

$$c_i = \langle h_{i+1}, h_i, h_{i+2}^2 \rangle \in \mathrm{Ext}^{3, 11\cdot 2^i}. \qquad \square$$

3.4.2. THEOREM. *For $p > 2$*

(a) $\mathrm{Ext}^0 = \mathbf{Z}/(p)$ *generated by* $1 \in \mathrm{Ext}^{0,0}$.

(b) Ext^1 *is spanned by* a_0 *and* $\{h_i \colon i \geq 0\}$ *where* $a_0 \in \mathrm{Ext}^{1,1}$ *is represented by* $[\tau_0]$ *and* $h_i \in \mathrm{Ext}^{1,qp^i}$ *is represented by* $[\xi_i^{p^i}]$.

(c) (Liulevicius [**2**]) Ext^2 *is spanned by* $\{h_i h_j \colon 0 \leq i < j-1\}$, a_0^2, $\{a_0 h_i \colon i > 0\}$, $\{g_i \colon i \geq 0\}$, $\{k_i \colon i \geq 0\}$, $\{b_i \colon i \geq 0\}$, *and* $\Pi_0 h_0$, *where*

$$g_i = \langle h_i, h_i, h_{i+1} \rangle \in \mathrm{Ext}^{2,(2+p)p^i q}, \quad k_i = \langle h_i, h_{i+1}, h_{i+1} \rangle \in \mathrm{Ext}^{2,(2p+1)p^i q},$$

$$b_i = \langle h_i, h_i, \ldots, h_i \rangle \in \mathrm{Ext}^{2,qp^{i+1}} \quad \text{(with p factors h_i)},$$

and

$$\Pi_0 h_0 = \langle h_0, h_0, a_0 \rangle \in \mathrm{Ext}^{2,1+2q}. \qquad \square$$

Ext^3 for $p > 2$ has recently been computed by Aikawa [**1**].

The behavior of the elements in Ext^1 in the Adams spectral sequence is described in Theorems 1.2.11–1.2.14.

We know that most of the elements in Ext^2 cannot be permanent cycles, i.e.,

3.4.3. THEOREM. (a) (Mahowald and Tangora [**8**]). *With the exceptions $h_0 h_2$, $h_0 h_3$, and $h_2 h_4$ the only elements in* Ext^2 *for $p = 2$ which can possibly be permanent cycles are h_j^2 and $h_1 h_j$.*

(b) (Miller, Ravenel, and Wilson [**1**]). *For $p > 2$ the only elements in* Ext^2 *which can be permanent cycles are a_0^2, $\Pi_0 h_0$, k_0, $h_0 h_i$, and b_i.* $\square$

Part (b) was proved by showing that the elements in question are the only ones with preimages in the Adams–Novikov E_2-term. A similar proof for $p = 2$ is possible using the computation of Shimomura [**1**]. The list in Mahowald and Tangora [**8**] includes $h_2 h_5$ and $h_3 h_6$; the latter is known not to come from the Adams–Novikov spectral sequence and the former is known to support a differential.

The cases $h_0 h_i$ and b_i, for $p > 3$ and $h_1 h_i$ for $p = 2$ are now settled.

3.4.4. THEOREM. (a) (Browder [**1**]). *For* $p = 2$ h_j^2 *is a permanent cycle iff there is a framed manifold of dimension* $2^{j+1} - 2$ *with Kervaire invariant one. Such are known to exist for* $j \leq 5$. *For more discussion see* 1.5.29 *and* 1.5.35.

(b) (Mahowald [**6**]). *For* $p = 2$ $h_1 h_j$ *is a permanent cycle for all* $j \geq 3$.

(c) (Ravenel [**7**]). *For* $p > 3$ *and* $i \geq 1$, b_i *is not a permanent cycle.* (*At* $p = 3$ b_1 *is not permanent but* b_2 *is*; b_0 *is permanent for all odd primes.*)

(d) (R. L. Cohen [**3**]). *For* $p > 2$ $h_0 b_i$ *is a permanent cycle corresponding to an element of order* p *for all* $i \geq 0$. □

The proof of (c) will be given in Section 6.4.

Now we describe a vanishing line. The main result is

3.4.5. VANISHING THEOREM (Adams [**17**]). (a) *For* $p = 2$ $\mathrm{Ext}^{s,t} = 0$ *for* $0 < t - s < f(s)$, *where* $f(s) = 2s - \varepsilon$ *and* $\varepsilon = 1$ *for* $s \equiv 0, 1 \mod (4)$, $\varepsilon = 2$ *for* $s \equiv 2$ *and* $\varepsilon = 3$ *for* $s \equiv 3$.

(b) (May [**6**]). *For* $p > 2$ $\mathrm{Ext}^{s,t} = 0$ *for* $0 < t - s < sq - \varepsilon$, *where* $\varepsilon = 1$ *if* $s \not\equiv 0 \mod (p)$ *and* $\varepsilon = 2$ *if* $s \equiv 0$. □

Hence in the usual picture of the Adams spectral sequence, where the x and y coordinates are $t - s$ and s, the E_2-term vanishes above a certain line of slope $1/q$ (e.g., $\frac{1}{2}$ for $p = 2$). Below this line there are certain periodicity operators Π_n which raise the bigrading so as to move elements in a direction parallel to the vanishing line. In a certain region these operators induce isomorphisms.

3.4.6. PERIODICITY THEOREM (Adams [**17**], May [**6**]).

(a) *For* $p = 2$ *and* $n \geq 1$ $\mathrm{Ext}^{s,t} \simeq \mathrm{Ext}^{s+2^{n+1}, t+3\cdot 2^{n+1}}$ *for*

$$0 < t - s < \min(g(s) + 2^{n+2}, h(s)),$$

where $g(s) = 2s - 4 - \tau$ *with* $\tau = 2$ *if* $s \equiv 0, 1 \mod (4)$, $\tau = 1$ *if* $s \equiv 3$, *and* $\tau = 0$ *if* $s \equiv 2$, *and* $h(s)$ *is defined by the following table:*

s	1	2	3	4	5	6	7	8	≥ 9
$h(s)$	1	1	7	10	17	22	25	32	$5s - 7$

(b) *For* $p > 2$ *and* $n \geq 0$ $\mathrm{Ext}^{s,t} \simeq \mathrm{Ext}^{s+p^n, s+(q+1)/p^n}$ *for*

$$0 < t - s < \min(g(s) + p^n q, h(s)),$$

where $g(s) = qs - 2p - 1$ *and* $h(s) = 0$ *for* $s = 1$ *and* $h(s) = (p^2 - p - 1)s - \tau$ *with* $\tau = 2p^2 - 2p + 1$ *for even* $s > 1$ *and* $\tau = p^2 + p - 2$ *for odd* $s > 1$. □

These two theorems are also discussed in Adams [**7**].

For $p = 2$ these isomorphisms are induced by Massey products (A1.4) sending x to $\langle x, h_0^{2^{n+1}}, h_{n+2}\rangle$. For $n = 1$ this operator is denoted in Tangora [**1**] and elsewhere in this book by P. The elements x are such that $h_0^{2^{n+1}} x$ is above the vanishing line of 3.4.5, so the Massey product is always defined. The indeterminacy of the product has the form $xy + h_{n+2} z$ with $y \in \mathrm{Ext}^{2^{n+1}, 3\cdot 2^{n+1}}$ and $z \in \mathrm{Ext}^{s-1+2^{n+1}, t+2^{n+1}}$. The group containing y is just below the vanishing line and we will see below that it is always trivial. The group containing z is above the vanishing line so the indeterminacy is zero.

Hence the theorem says that any group close enough to the vanishing line [i.e., satisfying $t - s < 2^{n+2} + g(s)$] and above a certain line with slope $\frac{1}{5}$ [$t - s < h(s)$] is acted on isomorphically by the periodicity operator. In Adams [**17**] this line had slope $\frac{1}{3}$. It is known that $\frac{1}{5}$ is the best possible slope, but the intercept could probably be improved by pushing the same methods further. The odd primary case is due entirely to May [**6**]. We are grateful to him for permission to include this unpublished material here.

Hence for $p = 2$ $\mathrm{Ext}^{s,t}$ has a fairly regular structure in the wedge-shaped region described roughly by $2s < t - s < 5s$. Some of this (partially below the line of slope $\frac{1}{5}$ given above) is described by Mahowald and Tangora [**14**] and an attempt to describe the entire structure for $p = 2$ is made by Mahowald [**13**].

However, this structure is of limited interest because we know that almost all of it is wiped out by differentials. All that is left in the E_∞-term are certain few elements near the vanishing line related to the J-homomorphism (1.1.12). We will not formulate a precise statement or proof of this fact, but offer the following explanation. In the language of Section 1.4, the periodicity operators Π_n in the Adams spectral sequence correspond to v_1-periodicity in the Adams–Novikov spectral sequence. More precisely, Π_n corresponds to multiplication by $v_1^{p^n}$. The behavior of the v_1-periodic part of the Adams–Novikov spectral sequence is analyzed completely in Section 5.3. The v_1-periodic part of the Adams–Novikov E_∞-term must correspond to the portion of the Adams spectral sequence E_∞-term lying above (for $p = 2$) a suitable line of slope $\frac{1}{5}$. Once the Adams–Novikov spectral sequence calculation has been made it is not difficult to identify the corresponding elements in the Adams spectral sequence. The elements in the Adams–Novikov spectral sequence all have low filtrations, so it is easy to establish that they cannot be hit by differentials. The elements in the Adams spectral sequence are up near the vanishing line so it is easy to show that they cannot support a nontrivial differential. We list these elements in 3.4.16 and in 3.4.21 give an easy direct proof (i.e., one that does not use BP-theory or K-theory) that most (all for $p > 2$) of them cannot be hit by differentials.

The proof of 3.4.5 involves the comodule M given by the short exact sequence

(3.4.7) $$0 \to \mathbf{Z}/(p) \to A_* \,\square_{A(0)_*}\, \mathbf{Z}/(p) \to M \to 0,$$

where $A(0)_* = E(\tau_0)$ for $p > 2$ and $E(\xi_1)$ for $p = 2$. M is the homology of the cofiber of the map from S^0 to H, the integral Eilenberg–Mac Lane spectrum. The E_2-term for H was computed in 2.1.18 and it gives us the tower in the 0-stem. Hence the connecting homomorphism of 3.4.7 gives an isomorphism

(3.4.8) $$\mathrm{Ext}_{A_*}^{s-1,t}(\mathbf{Z}/(p), M) \simeq \mathrm{Ext}^{s,t}$$

for $t - s > 0$.

We will consider the subalgebras $A(n) \subset A$ generated by $\{Sq^1, Sq^2, \ldots, Sq^{2^n}\}$ for $p = 2$ and $\{\beta, P^1, P^p, \ldots, P^{p^{n-1}}\}$ for $p > 2$. Their duals $A(n)_*$ are $P(\xi_1, \xi_2, \ldots, \xi_{n+1})/(\xi_i^{2^{n+2-i}})$ for $p = 2$ and

$$E(\tau_0, \ldots, \tau_n) \otimes P(\xi_1, \ldots, \xi_n)/(\xi_i^{p^{n+1-i}})$$

for $p > 2$.

We will be considering A_*-comodules N which are free over $A(0)_*$ and (-1)-connexted. $\Sigma^{-1}M$ is an example. Unless stated otherwise N *will be assumed to have these properties for the rest of the section.*

Closely related to the questions of vanishing and periodicity is that of approximation. For what (s,t) does $\mathrm{Ext}_{A_*}^{s,t}(\mathbf{Z}/(p), N) = \mathrm{Ext}_{A(n)_*}^{s,t}(\mathbf{Z}/(p), N)$? This relation is illustrated by

3.4.9. APPROXIMATION LEMMA. *Suppose that there is a nondecreasing function $f_n(s)$ defined such that for any N as above,* $\mathrm{Ext}^{s,t}_{A(n)_*}(\mathbf{Z}/(p), N) = 0$ *for* $t - s < f_n(s)$. *Then for* $r \geq n$ *this group is isomorphic to* $\mathrm{Ext}^{s,t}_{A(r)_*}(\mathbf{Z}/(p), N)$ *for* $t - s < p^n q + f_n(s-1)$, *and the map from the former to the latter is onto for* $t - s = p^n q + f_n(s)$. □

Hence if $f_n(s)$ describes a vanishing line for $A(n)$-cohomology then there is a parallel line below it, above which it is isomorphic to A-cohomology. For $n = 1$ such a vanishing line follows easily from 3.1.27(a) and 3.1.28, and it has the same slope as that of 3.4.5.

PROOF OF 3.4.9. The comodule structure map $N \to A(r)_* \otimes N$ gives a monomorphism $N \to A(r)_* \,\square_{A(n)_*}\, N$ with cokernel C. Then C is $A(0)_*$-free and $(p^n q - 1)$-connected. Then we have

$$\begin{tikzcd}
\mathrm{Ext}^{s-1}_{A(r)_*}(C) \arrow[r] & \mathrm{Ext}^{s}_{A(r)_*}(N) \arrow[r] \arrow[dr] & \mathrm{Ext}^{s}_{A(r)_*}(A(r)_* \,\square_{A(n)_*}\, N) \arrow[r] \arrow[d, "\simeq"] & \mathrm{Ext}^{s}_{A(r)_*(C)} \\
 & & \mathrm{Ext}^{s}_{A(n)_*}(N) &
\end{tikzcd}$$

where $\mathrm{Ext}_{A(r)_*}(-)$ is an abbreviation for $\mathrm{Ext}_{A(r)_*}(\mathbf{Z}/(p), -)$. The isomorphism is given by A1.1.18 and the diagonal map is the one we are considering. The high connectivity of C and the exactness of the top row give the desired result. □

PROOF OF 3.4.5. We use 3.4.9 with $N = M$ as in 3.4.7. An appropriate vanishing line for M will give 3.4.5 by 3.4.8. By 3.4.9 it suffices to get a vanishing line for $\mathrm{Ext}_{A(1)_*}(\mathbf{Z}/(p), M)$. We calculate this by filtering M skeletally as an $A(0)_*$-comodule. Then $E^0 M$ is an extended $A(0)_*$-determined by 3.1.27(a) or 3.1.28 and the additive structure of M. Considering the first two (three for $p = 2$) subquotients is enough to get the vanishing line. We leave the details to the reader. □

The periodicity operators in 3.4.6 which raise s by p^n correspond in $A(n)$-cohomology to multiplication by an element $\omega_n \in \mathrm{Ext}^{p^n,(q+1)p^n}$. In view of 3.4.9, 3.4.6 can be proved by showing that this multiplication induces an isomorphism in the appropriate range. For $p = 2$ our calculation of $\mathrm{Ext}_{A(2)_*}(\mathbf{Z}/(2), A(0)_*)$ (3.2.17) is nessesary to establish periodicity above a line of slope $\frac{1}{5}$. To get these ω_n we need

3.4.10. LEMMA. *There exist cochains $c_n \in C_{A_*}$ satisfying the following.*

(a) *For $p = 2$ $c_n \equiv [\xi_2|\cdots|\xi_2]$ with 2^n factors modulo terms involving ξ_1, and for $p > 2$ $c_n \equiv [\tau_1|\cdots|\tau_1]$ with p^n factors.*

(b) *For $p=2$ $d(c_1) = [\xi_1|\xi_1|\xi_1^4] + [\xi_1^2|\xi_1^2|\xi_1^2]$ and for $n>1$ $d(c_n) = [\xi_1|\cdots|\xi_1|\xi_1^{2^{n+1}}]$ factors ξ_1; and for $p > 2$ $d(c_n) = -[\tau_0|\cdots|\tau_0|\xi_1^{p^n}]$.*

(c) *c_n is uniquely determined up to a coboundary by* (a) *and* (b).

(d) *For $n \geq 1$ ($p > 2$) or $n \geq 2$ ($p = 2$) c_n projects to a cocycle in $C_{A(n)_*}$ representing a nontrivial element $\omega_n \in \mathrm{Ext}^{p^n,(q+1)p^n}_{A(n)_*}(\mathbf{Z}/(p), \mathbf{Z}/(p))$.*

(e) *For $p = 2$, ω_2 maps to ω as in 3.1.27, and in general ω_{n+1} maps to ω_n^p.*

PROOF. We will rely on the algebraic Steenrod operations in Ext described in Section A1.5. We treat only the case $p = 2$. By A1.5.2 there are operations

$Sq^i\colon \mathrm{Ext}^{s,t} \to \mathrm{Ext}^{s+i,2t}$ satisfying a Cartan formula with $Sq^0(h_i) = h_{i+1}$ (A1.5.3) and $Sq^1(h_i) = h_1^2$. Applying Sq^1 to the relation $h_0h_1 = 0$ we have

$$\begin{aligned} 0 &= Sq^1(h_0h_1) = Sq^0(h_0)Sq^1(h_1) + Sq^1(h_0)Sq^0(h_1) \\ &= h_1^3 + h_0^2h_2. \end{aligned}$$

Applying Sq^2 to this gives $h_1^4h_2 + h_0^4h_3 = 0$. Since $h_1h_2 = 0$ this implies $h_0^4h_3 = 0$. Applying Sq^4 to this gives $h_0^8h_4 = 0$. Similarly, we get $h_0^{2^i}h_{i+1} = 0$ for all $i \geq 2$. Hence there must be cochains c_n satisfying (b) above.

To show that these cochains can be chosen to satisfy (a) we will use the Kudo transgression theorem A1.5.7. Consider the cocentral extension of Hopf algebras (A1.1.15)

$$P(\xi_1) \to P(\xi_1, \xi_2) \to P(\xi_2).$$

In the Cartan–Eilenberg spectral sequence (A1.3.14 and A1.3.17) for

$$\mathrm{Ext}_{P(\xi_1,\xi_2)}(\mathbf{Z}/(2), \mathbf{Z}/(2))$$

one has $E_2 = P(h_{1j}, h_{2j}\colon y \geq 0)$ with $h_{1j} \in E_2^{1,0}$ and $h_{2j} \in E_2^{0,1}$. By direct calculation one has $d_2(h_{20}) = h_{10}h_{11}$. Applying Sq^2Sq^1 one gets $d_5(h_{20}^4) = h_{10}^4h_{13} + h_{11}^4h_{12}$. The second term was killed by $d_2(h_{11}^3h_{21})$ so we have $d_5(h_{20}^4) = h_{10}h_{13}^4$. Applying appropriate Steenrod operations gives $d_{2^n+1}(h_{20}^{2^n}) = h_{10}^{2^n}h_{1n+1}$. Hence our cochain c_n can be chosen in $C_{P(\xi_1,\xi_2)}$ so that its image in $C_{P(\xi_2)}$ is $[\xi_2|\cdots|\xi_2]$ representing $h_{20}^{2^n}$, so (a) is verified.

For (c), note that (b) determines c_n up to a cocycle, so it suffices to show that each cocycle in that bidegree is a coboundary, i.e., that $\mathrm{Ext}^{2^n,3\cdot 2^n} = 0$. This group is very close to the vanishing line and can be computed directly by what we already know.

For (d), (a) implies that c_n projects to a cocycle in $C_{A(n)_*}$ which is nontrivial by (b); (e) follows easily from the above considerations. □

For $p = 2$ suppose $x \in \mathrm{Ext}$ satisfies $h_0^{2^n}x = 0$. Let $\hat{x} \in C_{A_*}$ be a cocycle representing x and let y be a cochain with $d(y) = \hat{x}[\xi_1|\cdots|\xi_1]$ with 2^n factors. Then $\hat{x}c_n + y[\xi_1^{2^{n+1}}]$ is a cocycle representing the Massey product $\langle x, h_0^{2^n}, h_{n+1}\rangle$, which we define to be the nth periodicity operator Π_n. This cocycle maps to $\hat{x}c_n$ in $c_{A(n)_*}$, so Π_n corresponds to multiplication by ω_n as claimed. The argument for $p > 2$ is similar.

Now we need to examine ω_1 multiplication in $\mathrm{Ext}(A(1)_*)(\mathbf{Z}/(p), A(0)_*)$ for $p > 2$ using 3.1.28 and ω_2 multiplication in $\mathrm{Ext}_{A(2)_*}(\mathbf{Z}/(2), A(0)_*)$ using 3.2.17. The result is

3.4.11. Lemma.

(a) *For $p = 2$, multiplication by ω_2 in $\mathrm{Ext}_{A(2)_*}^{s,t}(\mathbf{Z}/(2), A(0)_*)$ is an isomorphism for $t - s < v(s)$ and an epimorphism for $t - s < w(s)$, where $v(s)$ and $w(s)$ are given in the following table.*

s	0	1	2	3	4	5	≥ 6
$v(s)$	1	8	6	18	18	21	$5s+3$
$w(s)$	1	8	10	18	23	25	$5s+3$

(b) *For $p > 2$ multiplication by ω_1 in $\operatorname{Ext}^{s,t}_{A(1)_*}(\mathbf{Z}/(p), A(0)_*)$ is a monomorphism for all $s \geq 0$ and an epimorphism for $t - s < w(s)$ where*

$$w(s) = \begin{cases} (p^2 - p - 1)s - 1 & \text{for } s \text{ even} \\ (p^2 - p - 1)s + p^2 - 3p & \text{for } s \text{ odd} \end{cases} \qquad \square$$

Next we need an analogous result where $A(0)_*$ is replaced by a (-1)-connected comodule N free over $A(0)_*$. Let $N^0 \subset N$ be the smallest free $A(0)_*$-subcomodule such that N/N^0 is 1-connected. Then

$$0 \to N^0 \to N \to N/N^0 \to 0$$

is an short exact sequence of $A(0)_*$-free comodules inducing an long exact sequence of $A(n)$–Ext groups on which ω_n acts. Hence one can use induction and the 5-lemma to get

3.4.12. LEMMA. *Let N be a connective $A(n)_*$-comodule free over $A(0)_*$.*

(a) *For $p = 2$ multiplication by ω_2 in $\operatorname{Ext}^{s,t}_{A(2)_*}(\mathbf{Z}/(2), M)$ is an isomorphism for $t - s < \tilde{v}(s)$ and an epimorphism for $t - s < \widetilde{w}(s)$, where these functions are given by the following table*

s	0	1	2	3	4	5	6	≥ 7
$\tilde{v}(x)$	-4	1	6	10	18	21	25	$5s - 2$
$\widetilde{w}(s)$	1	7	10	18	22	25	33	$5s + 3$

(b) *For $p > 2$ a similar result holds for ω_1-multiplication where*

$$\tilde{v}(s) = \begin{cases} (p^2 - p - 1)s - 2p + 1 & \textit{for } s \textit{ even} \\ (p^2 - p - 1)s - p^2 + p & \textit{for } s \textit{ odd} \end{cases}$$

and

$$\widetilde{w}(s) = \begin{cases} (p^2 - p - 1)s - 1 & \textit{for } s \textit{ even} \\ (p^2 - p - 1)s - p^2 + 2p - 1 & \textit{for } s \textit{ odd.} \end{cases} \qquad \square$$

3.4.13. REMARK. If N/N^0 is $(q-1)$-connected, as it is when $N = \Sigma^{-q}M$ (3.4.7), then the function $\tilde{v}(s)$ can be improved slightly. This is reflected in 3.4.6 and we leave the details to the reader.

The next step is to prove an analogous result for ω_n-multiplication. We sketch the proof for $p = 2$. Let N be as above and define $\overline{N} = A(n)_* \square_{A(2)_*} N$, and let $C = \overline{N}/N$. Then C is 7-connected if N is (-1)-connected, and $\operatorname{Ext}_{A(n)_*}(\mathbf{Z}/(2), \overline{N}) = \operatorname{Ext}_{A(2)_*}(\mathbf{Z}/(2), N)$. Hence in this group $\omega_n = \omega_2^{2^{n-2}}$ and we know its behavior by 3.4.12. We know the behavior of ω_n on C by induction, since C is highly connected, so we can argue in the usual way by the 5-lemma on the long exact sequence of Ext groups. If N satisfies the condition of 3.4.13, so will $\overline{N}$ and $\overline{C}$, so we can use the improved form of 3.4.12 to start the induction. The result is

3.4.14. LEMMA. *Let N be as above and satisfy the condition of 3.4.13. Then multiplication by ω_n (3.4.10) in $\operatorname{Ext}^{s,t}_{A(n)_*}(\mathbf{Z}/(p), N)$ is an isomorphism for $t - s < h(s+1) - 1$ and an epimorphism for $t - s < h(s) - 1$, where $h(s)$ is as in 3.4.6.* $\square$

Now the periodicity operators Π_n, defined above as Massey products, can be described in terms of the cochains c_n of 3.4.10 as follows. Let x represent a class in Ext (also denoted by x) which is annihilated by $h_0^{2^n}$ and let y be a cochain whose coboundary is $x[\xi_1|\xi_1|\cdots|\xi_1]$ with 2^n factors ξ_1. Then $y[\xi_1^{2^{n+1}}]+xc_n$ is a cochain representing $\Pi_n(x)$.

Hence it is evident that the action of Π_n in Ext corresponds to multiplication by ω_n in $A(n)_*$-cohomology. Hence 3.4.14 gives a result about the behavior of Π_n in $\mathrm{Ext}_{A_*}(\mathbf{Z}/(p),M)$ with M as in 3.4.7, so 3.4.6 follows from the isomorphism 3.4.8.

Having proved 3.4.6 we will list the periodic elements in Ext which survive to E_∞ and correspond to nontrivial homotopy elements. First we have

3.4.15. Lemma. *For $p=2$ and $n\geq 2$, $\Pi_n(h_0^{2^n-1}h_{n+1})=h_0^{2^{n+1}-1}h_{n+2}$. For $p>2$ and $n\geq 1$, $\Pi_n(a_0^{p^n-1}h_n)=a_0^{p^{n+1}-1}h_{n+1}$ up to a nonzero scalar. [It is not true that $\Pi_0(h_0)=a_0^{p-1}h_1$.]*

Proof. We do not know how to make this computation directly. However, 3.4.6 says the indicated operators act isomorphically on the indicated elements, and 3.4.21 below shows that the indicated image elements are nontrivial. Since the groups in question all have rank one the result follows. (3.4.6 does not apply to Π_0 acting on h_0 for $p>2$.) $\square$

3.4.16. Theorem.

(a) *For $p>2$ the set of elements in the Adams E_∞-term on which all iterates of some periodicity operator Π_n are nontrivial is spanned by $\Pi_n^i(a_0^{p^n-j}h_n)$ with $n\geq 0$, $0<j\leq n+1$ and $i\not\equiv -1 \mod (p)$. (For $i\equiv -1$ these elements vanish for $n=0$ and are determined by 3.4.15 for $n>0$.) The corresponding subgroup of $\pi_*(S^0)$ is the image of the J-homomorphism (1.1.12). (Compare 1.5.19.)*

(b) *For $p=2$ the set is generated by all iterates of Π_2 on h_1, h_1^2, $h_1^3=h_0^2h_2$, h_0h_2, h_2, c_0, and h_1c_0 (where $c_0=\langle h_1,h_0,h_2^2\rangle\in\mathrm{Ext}^{3,11}$) and by $\Pi_n^ih_nh_0^{2^{n-1}-j}$ with $n\geq 3$, i odd, and $0<j\leq n+1$. (For even i these elements are determined by 3.4.15.) The corresponding subgroup of $\pi_*(S^0)$ is $\pi_*(J)$ (1.5.22). In particular, $\operatorname{im}J$ corresponds to the subgroup of E_∞ spanned by all of the above except $\Pi_2^ih_1$ for $i>0$ and $\Pi_2^ih_1^2$ for $i\geq 0$.* $\square$

This can be proved in several ways. The cited results in Section 1.5 are very similar and their proofs are sketched there; use is made of K-theory. The first proof of an essentially equivalent theorem is the one of Adams [**1**], which also uses K-theory. For $p=2$ see also Mahowald [**15**] and Davis and Mahowald [**1**]. The computations of Section 5.3 can be adapted to give a BP-theoretic proof.

The following result is included because it shows that most (all if $p>2$) of the elements listed above are not hit by differentials, and the proof makes no use of any extraordinary homology theory. We will sketch the construction for $p=2$. It is a strengthened version of a result of Maunder [**1**]. Recall (3.1.9) the spectrum bo (representing real connective K-theory) with $H_*(bo)=A_*\,\square_{A(1)_*}\,\mathbf{Z}/(2)=P(\xi_1^4,\bar\xi_2^2,\bar\xi_3,\dots)$. For each $i\geq 0$ there is a map to $\Sigma^{4i}H$ (where H is the integral Eilenberg–Mac Lane spectrum) under which ξ_1^{4i} has a nontrivial image. Together these define a map f from bo to $W=\bigvee_{i\geq 0}\Sigma^{4i}H$. We denote its cofiber by $\overline{W}$. There is a map of cofiber sequences

$$\begin{tikzcd} S^0 \arrow[r] \arrow[d] & H \arrow[r] \arrow[d] & \overline{H} \arrow[d] \\ bo \arrow[r, "f"] & W \arrow[r] & \overline{W} \end{tikzcd} \tag{3.4.17}$$

in which each row induces an short exact sequence in homology and therefore an long exact sequence of Ext groups. Recall (3.1.26) that the Ext group for bo has a tower in every fourth dimension, as does the Ext group for W. One can show that the former map injectively to the latter. Then it is easy to work out the Adams E_2-term for $\overline{W}$, namely

(3.4.18)

$$\operatorname{Ext}^{s,t}(H_*(\overline{W})) = \begin{cases} \operatorname{Ext}^{s+1,t}(H_*(bo)) & \text{if } t-s \not\equiv 0 \mod (4) \\ \mathbf{Z}/(2) & \text{if } t-s \equiv 0 \text{ and } \operatorname{Ext}^{s,t}(H_*(bo)) = 0 \\ 0 & \text{otherwise,} \end{cases}$$

where $\operatorname{Ext}(M)$ is an abbreviation for $\operatorname{Ext}_{A_*}(\mathbf{Z}/(2), M)$. See FIG. 3.4.20. Combining 3.4.17 and 3.4.8 gives us a map

$$\operatorname{Ext}^{s,t}(\mathbf{Z}/(2)) \to \operatorname{Ext}^{s-1,t}(H_*(\overline{W})) \quad \text{for } t-s>0 \tag{3.4.19}$$

Since this map is topologically induced it commutes with Adams differentials. Hence any element in Ext with a nontrivial image in 3.4.19 cannot be the target of a differential.

One can show that each h_n for $n>0$ is mapped monomorphically in 3.4.19, so each h_n supports a tower going all the way up to the vanishing line as is required in the proof of 3.4.15. Note that the vanishing here coincides with that for Ext given in 3.4.5.

A similar construction at odd primes detects a tower going up to the vanishing line in every dimension $\equiv -1 \mod (2p-2)$.

To summarize

3.4.21. THEOREM.

(a) *For $p=2$ there is a spectrum $\overline{W}$ with Adams E_2-term described in 3.4.18 and 3.4.20. The resulting map 3.4.19 commutes with Adams differentials and is nontrivial on h_n for all $n>0$ and all Π_2 iterates of h_1, h_1^2, $h_1^3 = h_0^2h_2$, h_2, and $h_0^3h_3$. Hence none of these elements is hit by Adams differentials.*

(b) *A similar construction for $p>2$ gives a map as above which is nontrivial on h_n for all $n \geq 0$ and on all the elements listed in 3.4.16*(a). $\square$

The argument above does not show that the elements in question are permanent cycles. For example, all but a few elements at the top of the towers built on h_n for large n support nontrivial differentials, but map to permanent cycles in the Adams spectral sequence for $\overline{W}$.

We do not know the image of the map in 3.4.19. For $p=2$ it is clearly onto for $t-s = 2^n-1$. For $t-s+1 = (2k+1)2^n$ with $k>0$ the image is at least as big as it is for $k=0$, because the appropriate periodicity operator acts on h_n. However, the actual image appears to be about $\frac{3}{2}$ as large. For example, the towers in Ext in dimensions 23 and 39 have 6 elements instead of the 4 in dimension 7, while the one in dimension 47 has 12. We leave this as a research question for the interested reader.

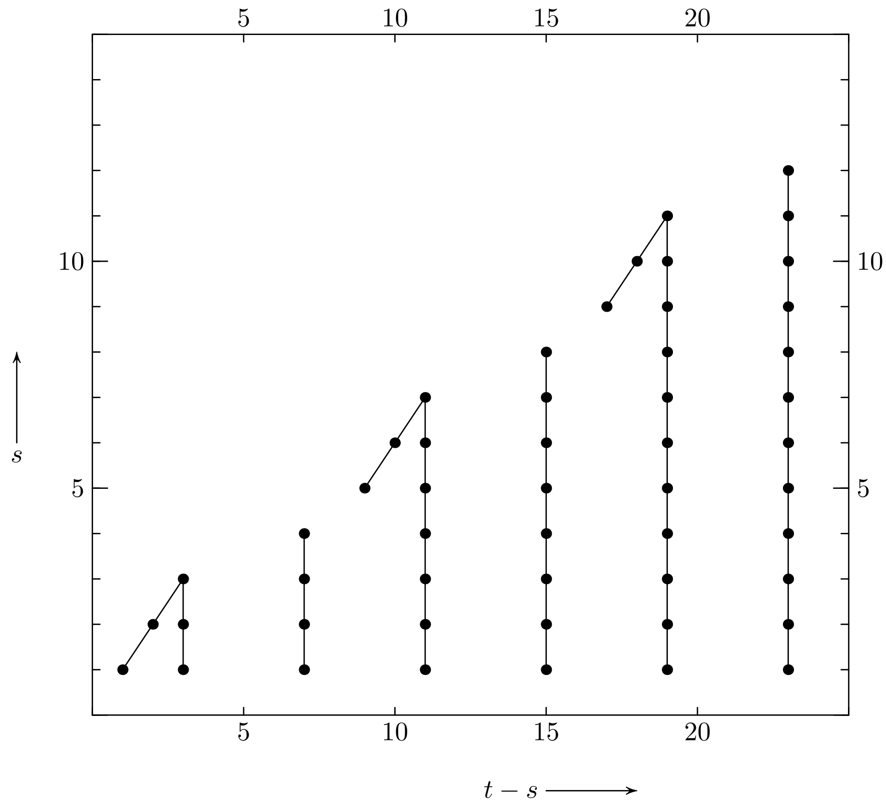


FIGURE 3.4.20. $\mathrm{Ext}^{s-1,t}\, H_*(\overline{W})$.

5. Survey and Further Reading

In this section we survey some other research having to do with the classical Adams spectral sequence, published and unpublished. We will describe in sequence results related to the previous four sections and then indicate some theorems not readily classified by this scheme.

In Section 1 we made some easy Ext calculations and thereby computed the homotopy groups of such spectra as MU and bo. The latter involved the cohomology of $A(1)$, the subalgebra of the mod (2) Steenrod algebra generated by Sq^1 and Sq^2. A pleasant partial classification of $A(1)$-modules is given in section 3 of Adams and Priddy [**10**]. They compute the Ext groups of all of these modules and show that many of them can be realized as bo-module spectra. For example, they use this result to analyze the homotopy type of $bo \wedge bo$.

The cohomology of the subalgebra $A(2)$ was computed by Shimada and Iwai [**2**]. Recently, Davis and Mahowald [**4**] have shown that $A//A(2)$ is *not* the cohomology of any connective spectrum. In Davis and Mahowald [**5**] they compute $A(2)$–Ext groups for the cohomology of stunted real projective spaces.

More general results on subalgebras of A can be found in Adams and Margolis [**11**] and Moore and Peterson [**1**].

The use of the Adams spectral sequence in computing cobordism rings is becoming more popular. The spectra MO, MSO, MSU, and $MSpin$ were originally analyzed by other methods (see Stong [1] for references) but in theory could be analyzed with the Adams spectral sequence; see Pengelley [1, 2] and Giambalvo and Pengelley [1].

The spectrum $MO\langle 8\rangle$ (the Thom spectrum associated with the 7-connected cover of BO) has been investigated by Adams spectral sequence methods in Giambalvo [2], Bahri [1], Davis [3, 6], and Bahri and Mahowald [1].

In Johnson and Wilson [5] the Adams spectral sequence is used to compute the bordism ring of manifolds with free G-action for an elementary abelian p-group G.

The most prodigious Adams spectral sequence calculation to date is that for the symplectic cobordism ring by Kochman [1, 2, 3]. He uses a change-of-rings isomorphism to reduce the computation of the E_2-term to finding Ext over the coalgebra

$$B = P(\xi_1, \xi_2, \dots)/(\xi_i^4) \tag{3.5.1}$$

for which he uses the May spectral sequence. The E_2-term for MSp is a direct sum of many copies of this Ext and these summands are connected to each other by higher Adams differentials. He shows that MSp is indecomposable as a ring spectrum and that the Adams spectral sequence has nontrivial d_r's for arbitrarily large r.

In Section 2 we described the May spectral sequence. The work of Nakamura [1] enables one to use algebraic Steenrod operations (A1.5) to compute May differentials.

The May spectral sequence is obtained from an increasing filtration of the dual Steenrod algebra A_*. We will describe some decreasing filtrations of A_* for $p = 2$ and the spectral sequences they lead to. The method of calculation these results suggest is conceptually more complicated than May's but it may have some practical advantages. The E_2-term (3.5.2) can be computed by another spectral sequence (3.5.4) whose E_2-term is the $A(n)$ cohomology (for some fixed n) of a certain trigraded comodule T. The structure of T is given by a third spectral sequence (3.5.10) whose input is essentially the cohomology of the Steenrod algebra through a range of dimensions equal to 2^{-n-1} times the range one wishes to compute.

This method is in practice very similar to Mahowald's unpublished work on "Koszul resolutions".

3.5.2. PROPOSITION. *For each $n \geq 0$, A_* has a decreasing filtration* (A1.3.5) $\{F^s A_*\}$ *where F^s is the smallest possible subgroup satisfying $\bar\xi_i^{2^j} \in F^{2^{i+j-n-1}-1}$ for $j \leq n+1$.*

$\square$

In particular, $F^0/F^1 = A(n)_*$, so $A(n)_* \subset E_0 A_*$ where

$$A(n)_* = A_*/(\bar\xi_1^{2^{n+1}}, \bar\xi_2^{2^n}, \dots, \bar\xi_n^2, \bar\xi_{n+1}\bar\xi_{n+2}, \dots).$$

We also have $\bar\xi_i^{2^j} \in F^{2^{j-n-1}(2^i-1)}$ for $j \geq n+1$. Hence there is a spectral sequence (A1.3.9) converging to $\mathrm{Ext}_{A_*}(\mathbf{Z}/c(2), M)$ with $E_1^{s,t,u} = \mathrm{Ext}_{E_0A_*}^{s,t}(\mathbf{Z}/(2), E_0M)$ and $d_r\colon E_r^{s,t,u} \to E_r^{s+1,t,u+r}$, where the third grading is that given by the filtration, M is any A_*-comodule, and E_0M is the associated E_0A_*-comodule (A1.3.7).

Now let $G(n)_* = E_0A_* \,\square_{A(n)_*}\, \mathbf{Z}/(2)$. It inherits a Hopf algebra structure from E_0A_* and

$$A(n)_* \to E_0A_* \to G(n)_* \tag{3.5.3}$$

is an extension of Hopf algebras (A1.1.15). Hence we have a Cartan–Eilenberg spectral sequence (A1.3.14), i.e.,

3.5.4. LEMMA. *Associated with the extension 3.5.3 there is a spectral sequence with*

$$E_2^{s_1,s_2,t,u} = \operatorname{Ext}^{s_1}_{A(n)_*}(\mathbf{Z}/(2), \operatorname{Ext}^{s_2,t,u}_{G(n)_*}(\mathbf{Z}/(2), M))$$

with $d_r\colon E_r^{s_1,s_2,t,u} \to E_r^{s_1+r,s_2-r+1,t,u}$ *converging to* $\operatorname{Ext}^{s_1+s_2,t,u}_{E_0A_*}$ *for any* E_0A_* *comodule* M. [$\operatorname{Ext}_{G(n)_*}(\mathbf{Z}/(2), M)$ *is the* T *referred to above.*] □

3.5.5. REMARK. According to A1.3.11(a) the cochain complex W used to compute Ext over $G(n)_*$ can be taken to be one of $A(n)_*$-comodules. The E_2-term of the spectral sequence is the $A(n)_*$ Ext of the cohomology of W, and the E_∞-term is the cohomology of the double complex obtained by applying $C^*_{A(n)_*}(\ )$ (A1.2.11) to W. This W is the direct sum [as a complex of $A(n)_*$-comodules] of its components for various u (the filtration grading). The differentials are computed by analyzing this W.

Next observe that E_0A_* and $G(n)_*$ contain a sub-Hopf algebra $A_*^{(n+1)}$ isomorphic up to regrading to A_*; i.e., $A_*^{(n+1)} \subset E_0A_*$ is the image of $P(\bar\xi_i^{2^{n+1}}) \subset A_*$. The isomorphism follows from the fact that the filtration degree 2^i-1 of $\bar\xi_i^{2^{n+1}}$ coincides with the topological degree of $\bar\xi_i$. Hence we have

$$\operatorname{Ext}^{s,t}_{A_*}(\mathbf{Z}/(2), \mathbf{Z}/(2)) = \operatorname{Ext}^{s,2^{n+1}t,t}_{A_*^{(n+1)}}(\mathbf{Z}/(2), \mathbf{Z}/(2)) \tag{3.5.6}$$

and we can take these groups as known inductively.

Let $L(n)_* = G(n)_* \otimes_{A_*^{(n+1)}} \mathbf{Z}/(2)$ and get an extension

$$A_*^{(n+1)} \to G(n)_* \to L(n)_*. \tag{3.5.7}$$

$L(n)_*$ is easily seen to be cocommutative with

$$\operatorname{Ext}^{s,t,u}_{L(n)_*}(\mathbf{Z}/(2), \mathbf{Z}/(2)) = P(h_{i,j}\colon 0 \le j \le n,\ i \ge n+2-j), \tag{3.5.8}$$

where $h_{i,j} \in \operatorname{Ext}^{1,2^j(2^i-1),2^{i+j-n-1}-1}$ corresponds as usual to $\bar\xi_i^{2^j}$. This Ext is a comodule algebra over $A_*^{(n+1)}$ (A1.3.14) with coaction given by

$$\psi(h_{i,j}) = \sum_{k>0} \xi_k^{2^{i+j-k}} \otimes h_{i-k,j} \tag{3.5.9}$$

Hence by A1.3.14 we have

3.5.10. LEMMA. *The extension 3.5.7 leads to a spectral sequence as in 3.5.4 with*

$$E_2^{s_1,s_2,t,u} = \operatorname{Ext}^{s_1}_{A_*^{(n+1)}}(\mathbf{Z}/(2), \operatorname{Ext}^{s_2,t,u}_{L(n)_*}(\mathbf{Z}/(2), M))$$

converging to $\operatorname{Ext}^{s_1+s_2,t,u}_{G(n)_*}(\mathbf{Z}/(2), M)$ *for any* $G(n)_*$*-comodule* M. *For* $M = \mathbf{Z}/(2)$, *the* Ext *over* $L(n)_*$ *and its comodule algebra structure are given by 3.5.8 and 3.5.9. Moreover, this spectral sequence collapses from* E_2.

PROOF. All is clear but the last statement, which we prove by showing that $G(n)_*$ possesses an extra grading which corresponds to s_2 in the spectral sequence.

It will follow that differentials must respect this grading so $d_r = 0$ for $r \geq 2$. Let $\bar{\xi}_{i,j} \in G(n)_*$ be the element corresponding to $\bar{\xi}_i^{2^j}$. The extra grading is defined by

$$|\bar{\xi}_{i,j}| = \begin{cases} 1 & \text{if } j \leq n \\ 0 & \text{if } j > n. \end{cases}$$

Since the $\bar{\xi}_{i,j}$ for $j \leq n$ are all exterior generators, the multiplication in $G(n)_*$ respects this grading. The coproduct is given by

$$\Delta(\bar{\xi}_{i,j}) = \sum_k \bar{\xi}_{k,j} \otimes \bar{\xi}_{i=k,k+j}.$$

If $j \geq n+1$ then all terms have degree 0, and if $j \leq n$, we have $k+j \geq n+2$ so all terms have degree 1, so Δ also respects the extra grading. $\square$

We now describe how to use these results to compute Ext. If one wants to compute through a fixed range of dimensions, the isomorphism 3.5.6 reduces the calculation of the spectral sequence of 3.5.10 to a much smaller range, so we assume inductively that this has been done. The next step is to compute in the spectral sequence of 3.5.4. The input here is the trigraded $A(n)_*$-comodule $\operatorname{Ext}_{G(n)_*}^{s,t,u}(\mathbf{Z}/(2), \mathbf{Z}/(2))$. We began this discussion by assuming we could compute Ext over $A(n)_*$, but in practice we cannot do this directly if $n > 1$. However, for $0 \leq m < n$ we can reduce an $A(n)_*$ calculation to an $A(m)_*$ calculation by proceeding as above, starting with the mth filtration of $A(n)_*$ instead of A_*. We leave the precise formulation to the reader. Thus we can compute the $A(n)_*$ Ext of $\operatorname{Ext}_{G(n)_*}^{s,t,u}(\mathbf{Z}/(2), \mathbf{Z}/(2))$ separately for each u; the slogan here is divide and conquer.

This method can be used to compute the cohomology of the Hopf algebra B (3.5.1) relevant to MSp. Filtering with $n = 1$, the SS analogous to 3.5.4 has

$$E_2 = \operatorname{Ext}_{A(1)_*}(\mathbf{Z}/(2), P(h_{21}, h_{30}, h_{31}, h_{40}, \dots))$$

with $\psi(h_{i+1,0}) = \xi_1 \otimes h_{i,1} + 1 \otimes h_{i+1,0}$ and $\psi(h_{i,1}) = 1 \otimes h_{i,1}$ for $i \geq 2$. This Ext is easy to compute. Both this spectral sequence and the analog of the one in 3.5.2 collapse from E_2. Hence we get a description of the cohomology of B which is more concise though less explicit than that of Kochman [**1**].

In Section 3 we described Λ and hinted at an unstable Adams spectral sequence. For more on this theory see Bousfield and Kan [**3**], Bousfield and Curtis [**4**], Bendersky, Curtis, and Miller [**1**], Curtis [**1**], and Singer [**3, 4, 5**]. A particularly interesting point of view is developed by Singer [**2**].

In Mahowald [**3**] the double suspension homomorphism

$$\Lambda(2n-1) \to \Lambda(2n+1)$$

is studied. He shows that the cohomology of its cokernel $W(n)$ is isomorphic to $\operatorname{Ext}_{A_*}^{s,t}(\mathbf{Z}/(2), \Sigma^{2n-1}A(0)_*)$ for $t - s < 5s + k$ for some constant k, i.e., above a line with slope $\frac{1}{5}$. This leads to a similar isomorphism between $H^*(\Lambda(2n+1)/\Lambda(1))$ and $\operatorname{Ext}_{A_*}(\mathbf{Z}/(2), \widetilde{H}_*(RP^{2n}))$. In Mahowald [**4**] he proves a geometric analog, showing that a certain subquotient of $\pi_*(S^{2n+1})$ is isomorphic to that of $\pi_*^S(RP^{2n})$. The odd primary analog of the algebraic result has been demonstrated by Harper and Miller [**1**]. The geometric result is very likely to be true but is still an open question. This point was also discussed in Section 1.5.

Now we will describe some unpublished work of Mahowald concerning generalizations of Λ. In 3.3.3 we defined subcomplexes $\Lambda(n) \subset \Lambda$ by saying that an admissible monomial $\lambda_{i_1} \cdots \lambda_{i_k}$ is in $\Lambda(n)$ if $i_1 < n$. The short exact sequence

$$\Lambda(n-1) \to \Lambda(n) \to \Sigma^n \Lambda(2n-1)$$

led to the algebraic EHP spectral sequence of 3.3.7. Now we define quotient complexes $\Lambda(n)$ by $\Lambda\langle n\rangle = \Lambda/\Lambda(\lambda_0, \dots, \lambda_{n-1})$, so $\Lambda(0) = \Lambda$ and $\varinjlim M\Lambda\langle n\rangle = \mathbf{Z}/(2)$. Then there are short exact sequences

$$0 \to \Sigma^n \Lambda\langle (n+1)/2\rangle \to \Lambda\langle n\rangle \to \Lambda\langle n+1\rangle \to 0 \tag{3.5.11}$$

where the fraction $(n+1)/2$ is taken to be the integer part thereof. This leads to a spectral sequence similar to that of 3.3.7 and an inductive procedure for computing $H_*(\Lambda)$.

Next we define A_*-comodules B_n as follows. Define an increasing filtration on A_* (different from those of 3.5.2) by $\xi_i \in F_{2^i}$ and let $B_n = F_n$. The B_n is realized by the spectra of Brown and Gitler [**3**]. They figure critically in the construction of the η_j's in Mahowald [**6**] and in the Brown–Peterson–Cohen program to prove that every closed smooth n-manifold immerses in $\mathbf{R}^{2n-\alpha(n)}$, where $\alpha(n)$ is the number of ones in the dyadic expansion of n. Brown and Gitler [**3**] show that $\mathrm{Ext}_{A_*}(\mathbf{Z}/(2), B_n) = H^*(\Lambda\langle n\rangle)$ and that the short exact sequence 3.5.11 is realized by a cofibration. It is remarkable that the Brown–Gitler spectra and the unstable spheres both lead in this way to Λ.

Now let $N = (n_1, n_2, \dots)$ be a nonincreasing sequence of nonnegative integers. Let $A(N) = A_*/(\xi_1^{2^{n_1}}, \xi_2^{2^{n_2}}, \dots)$. This is a Hopf algebra. Let $M(N) = A_* \square_{A(N)} \mathbf{Z}/(2)$, so $M(N) = P(\xi_1^{2^{n_1}}, \xi_2^{2^{n_2}}, \dots)$. The filtration of A_* defined above induces one on $M(N)$ and we have

$$F_i M(N)/F_{i-1}M(N) = \begin{cases} \Sigma^i F_{[i/2]} M(N^1) & \text{if } 2^{n_1} \mid i \\ 0 & \text{otherwise} \end{cases} \tag{3.5.12}$$

where N^k is the sequence $(n_{k+1}, n_{k+2}, \dots)$. For $N = (0, 0, \dots)$ $A(N) = A_*$ and this is equivalent to 3.5.11.

3.5.13. Proposition. *The short exact sequence*

$$0 \to F_{i-1}M(N) \to F_i M(N) \to F_i/F_{i-1} \to 0$$

is split over $A(N)$. □

This result can be used to construct an long exact sequence of A_*-comodules

$$0 \to \mathbf{Z}/(2) \to C_N^0 \to C_N^1 \to C_N^2 \to \cdots \tag{3.5.14}$$

such that C_N^k is a direct sum of suspensions of $M(N^k)$ indexed by sequences $(i_1, i_2, \dots, i_k)$ satisfying $1 + i_j \equiv 0 \mod 2^{n_{1+k-j}}$ and $i_j \le 2i_{j-1}$. Equation 3.5.14 leads to a spectral sequence (A1.3.2) converging to Ext with

$$E_1^{k,s} = \mathrm{Ext}^s_{A_*}(\mathbf{Z}/(2), C_N^k). \tag{3.5.15}$$

The splitting of C_N^k and the change-of-rings isomorphism A1.3.13 show that $E_1^{k,*}$ is a direct sum of suspensions $\mathrm{Ext}_{A(N^k)}(\mathbf{Z}/(2), \mathbf{Z}/(2))$.

The E_1-term of this spectral sequence is a "generalized Λ" in that it consists of copies of $A(N^k)$ Ext groups indexed by certain monomials in Λ. The d_1 is closely related to the differential in Λ.

We will describe the construction of 3.5.14 in more detail and then discuss some examples. Let $\overline{M}(N)$ be the quotient in

$$0 \to \mathbf{Z}/(2) \to M(N) \to \overline{M}(N) \to 0.$$

In 3.5.14 we want $C_N^0 = M(N)$ and $C_N^1 = \bigoplus_{i>0} \Sigma^{i2^{n_1}} M(N^1)$, so we need to embed $\overline{M}(N)$ in this putative C_N^1. The filtration on $M(N)$ induces ones on $M(N)$ and C_N^s; in the latter F_i should be a direct sum of suspensions of $M(N^1)$. Consider the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & F_{i-1}\overline{M}(N) \arrow[r] \arrow[d] & F_i\overline{M}(N) \arrow[r] \arrow[d] \arrow[dl, dashed] & F_i/F_{i-1}\overline{M}(M) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & F_{i-1}C_N^1 \arrow[r] \arrow[d, equal] & X_i \arrow[r] \arrow[d] & F_i/F_{i-1}\overline{M}(N) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & F_{i-1}C_N^1 \arrow[r] & F_iC_N^i \arrow[r] & \Sigma^? N(N^1) \arrow[r] & 0
\end{tikzcd}$$

with exact rows. The upper short exact sequence splits over $A(N)$ (3.5.13) and hence over $A(N^1)$. Since $F_{i-1}C_N^1$ splits as above, the change-of-rings isomorphism A1.3.13 implies that the map

$$\operatorname{Hom}_{A_*}(F_i\overline{M}(N), F_{i-1}C_N^1) \to \operatorname{Hom}_{A_*}(F_{i-1}\overline{M}(N), F_{i-1}C_N^1)$$

is onto, so the diagonal map exists. It can be used to split the middle short exact sequence, so the lower short exact sequence can be taken to be split and C_N^1 is as claimed.

The rest of 3.5.14 can be similarly constructed.

Now we consider some examples. If $N = (0, 0, \cdots)$ the spectral sequence collapses and we have the Λ-algebra. If $N = (1, 1, \dots)$ we have $\operatorname{Ext}_{A(N)} = P(a_0, a_1, \dots)$ as computed in 3.1.9, and the E_1-term is this ring tensored with the subalgebra of Λ generated by λ_i with i odd, which is isomorphic up to regrading with Λ itself. This is also the E_1-term of a spectral sequence converging to the Adams–Novikov E_2-term to be discussed in Section 4.4. The SS of 3.5.15 in this case can be identified with the one obtained by filtering Λ by the number of λ_i, with i odd occurring in each monomial.

For $N = (2, 2, \cdots)$ we have $A(N) = B$ as in 3.5.1, so the E_1-term is Ext_B tensored with a regraded Λ.

Finally, consider the case $N = (2, 1, 0, 0, \dots)$. We have $E_1^{0,s} = \operatorname{Ext}^s_{A(1)_*}$ and $E_1^{1,s} = \bigoplus_{i>0} \Sigma^{4i} \operatorname{Ext}^s_{A(0)_*}$. One can study the quotient spectral sequence obtained by setting $E_1^{k,s} = 0$ for $k > 1$. The resulting $E_2 = E_\infty$ is the target of a map from Ext, and this map is essentially the one given in 3.4.19. More generally, the first few columns of the spectral sequence of 3.5.15 can be used to detect elements in Ext.

In Section 4 we gave some results concerning vanishing and periodicity. In particular we got a vanishing line of slope $\frac{1}{2}$ (for $p = 2$) for any connective comodule free over $A(0)_*$. This result can be improved if the comodule is free over $A(n)_*$ for some $n > 0$; e.g., one gets a vanishing line of slope $\frac{1}{5}$ for $n = 1$, $p = 2$. See Anderson and Davis [**1**] and Miller and Wilkerson [**8**].

The periodicity in Section 4 is based on multiplication by powers of h_{20} ($p = 2$) or a_1 ($p > 2$) and these operators act on classes annihilated by some power of h_{10} or a_0. As remarked above, this corresponds to v_1-periodicity in the Adams–Novikov spectral sequence (see Section 1.4). Therefore one would expect to find other operators based on multiplication by powers of $h_{n+1,0}$ or a_n corresponding to v_n-periodicity for $n > 1$. A v_n-periodicity operator should be a Massey product defined on elements annihilated by some v_{n-1}-periodicity operator. For $n = 2$, $p = 2$ this phenomenon is investigated by Davis and Mahowald [**1**] and Mahowald [**10, 11, 12**].

More generally one can ask if there is an Adams spectral sequence version of the chromatic SS (1.4.8). For this one would need an analog of the chromatic resolution (1.4.6), which means inverting periodicity operators. This problem is addressed by Miller [**4, 7**].

A v_n-periodicity operator in the Adams spectral sequence for $p = 2$ moves an element along a line of slope $1/(2^{n+1} - 2)$. Thus v_n-periodic families of stable homotopy elements would correspond to families of elements in the Adams spectral sequence lying near the line through the origin with this slope. We expect that elements in the E_∞-term cluster around such lines.

Now we will survey some other research with the Adams spectral sequence not directly related to the previous four sections. For $p = 2$ and $t - s \leq 45$, differentials and extensions are analyzed by Mahowald and Tangora [**9**], Barratt, Mahowald, and Tangora [**1**], Tangora [**5**], and Bruner [**2**]. Some systematic phenomena in the E_2-term are described in Davis [**2**], Mahowald and Tangora [**14**], and Margolis, Priddy, and Tangora [**1**]. Some machinery useful for computing Adams spectral sequence differentials involving Massey products is developed by Kochman [**4**] and Section 12 of Kochman [**2**]. See also Milgram [**2**] and Kahn [**2**] and Bruner *et al* [**1**], and Makinen [**1**].

The Adams spectral sequence was used in the proof of the Segal conjecture for $\mathbf{Z}/(2)$ by Lin [**1**] and Lin *et al.* [**2**]. Computationally, the heart of the proof is the startling isomorphism

$$\operatorname{Ext}^{s,t}_{A_*}(\mathbf{Z}/(2), M) = \operatorname{Ext}^{s,t+1}_{A_*}(\mathbf{Z}/(2), \mathbf{Z}/(2)),$$

where M is dual to the A-module $\mathbf{Z}/(2)[x, x^{-1}]$ with $\dim x = 1$ and $Sq^k x^i = \binom{i}{k} x^{i+k}$ (this binomial coefficient makes sense for any integer i). This isomorphism was originally conjectured by Mahowald (see Adams [**14**]). The analogous odd primary result was proved by Gunawardena [**1**]. The calculation is streamlined and generalized to elementary abelian p-groups by Adams, Gunawardena, and Miller [**18**]. This work makes essential use of ideas due to Singer [**1**] and Li and Singer [**1**].

In Ravenel [**4**] we proved the Segal conjecture for cyclic groups by means of a modified form of the Adams spectral sequence in which the filtration is altered. This method was used by Miller and Wilkerson [**9**] to prove the Segal conjecture for periodic groups.

The general Segal conjecture, which is a statement about the stable homotopy type of the classifying space of a finite group, has been proved by Gunnar Carlsson [**1**]. A related result is the Sullivan conjecture, which concerns says among other things that there are no nontrivial maps to a finite complex from such a classifying space. It was proved by Haynes Miller in [**10**] . New insight into both proofs was provided by work of Jean Lannes on unstable modules over the Steenrod algebra, in particular his T-functor, which is an adjoint to a certain tensor product. See Lannes [**1**], Lannes [**2**] and Lannes and Schwartz [**3**]. An account of this theory is given in the book by Lionel Schwartz [**1**].

Recent work of Palmieir (Palmieri [**1**] and Palmieri [**2**]) gives a global description of Ext over the Steenrod algebra modulo nilpotent elements.

Finally, we must mention the Whitehead conjecture. The n-fold symmetric product $Sp^n(X)$ of a space X is the quotient of the n-fold Cartesian product by the action of the symmetric group Σ_n. Dold and Thom [**1**] showed that $Sp^\infty(X) = \varinjlim Sp^n(X)$ is a product of Eilenberg–Mac Lane spaces whosw homotopy is the homotopy of X. Symmetric products can be defined on spectra and we have $Sp^\infty(S^0) = HJ$, the integer Eilenbergh–Mac Lane spectrum. After localizing at the prime p one considers

$$S^0 \to Sp^p(S^0) \to Sp^{p^2}(S^0) \to \cdots$$

and

$$H \leftarrow S^0 \leftarrow \Sigma^{-1} Sp^p(S^0)/S^0 \leftarrow \Sigma^{-2} Sp^{p^2}(S^0)/Sp^p(S^0) \leftarrow \cdots . \tag{3.5.16}$$

Whitehead conjectured that this diagram induces an long exact sequence of homotopy groups. In particular, the map $\Sigma^{-1} Sp^p(S^0)/S^0 \to S^0$ shouls induce a surjection in homotopy in positive dimensions; this is the famous theorem of Kahn and Priddy [**2**]. The analogous statement about Ext groups was proved by Lin [**3**]. Miller [**6**] generalized this to show that 3.5.16 induces an long exact sequence of Ext groups. The long exact sequence of homotopy groups was established by Kuhn [**1**]. The spectra in 3.5.16 were studied by Welcher [**1, 2**]. He showed that $H_*(Sp^{p^{n+1}}(S^0)/Sp^{p^n}(S^0))$ is free over $A(n)_*$, so its Ext groups has a vanishing line given by Anderson and Davis [**1**] and Miller and Wilkerson [**8**] and the long exact sequence of 3.5.16 is finite in each bigrading.

CHAPTER 4

BP-Theory and the Adams–Novikov Spectral Sequence

In this chapter we turn to the main topic of this book, the Adams–Novikov spectral sequence. In Section 1 we develop the basic properties of MU and the Brown–Peterson spectrum BP, using the calculation of $\pi_*(MU)$ (3.1.5) and the algebraic theory of formal group laws as given in Appendix 2. The main result is 4.1.19, which describes $BP_*(BP)$, the BP-theoretic analog of the dual Steenrod algebra.

Section 2 is a survey of other aspects of BP-theory not directly related to this book.

In Section 3 we study $BP_*(BP)$ more closely and obtain some formulas, notably 4.3.13, 4.3.18, 4.3.22, and 4.3.33, which will be useful in subsequent calculations.

In Section 4 we set up the Adams–Novikov spectral sequence and use it to compute the stable homotopy groups of spheres through a middling range of dimensions, namely ≤ 24 for $p = 2$ and $\leq 2p^3 - 2p - 1$ for $p > 2$.

1. Quillen's Theorem and the Structure of $BP_*(BP)$

In this section we will construct the Brown–Peterson spectrum BP and determine the structure of its Hopf algebroid of cooperations, $BP_*(BP)$, i.e., the analog of the dual Steenrod algebra. This will enable us to begin computing with the Adams–Novikov spectral sequence (ANSS) in Section 4. The main results are Quillen's theorem 4.1.6, which identifies $\pi_*(MU)$ with the Lazard ring L (A2.1.8); the Landweber–Novikov theorem 4.1.11, which describes $MU_*(MU)$; the Brown–Peterson theorem 4.1.12, which gives the spectrum BP; and the Quillen–Adams theorem 4.1.19, which describes $BP_*(BP)$.

We begin by informally defining the spectrum MU. For more details see Milnor and Stasheff [**5**]. Recall that for each $n \geq 0$ the group of complex unitary $n \times n$ matrices $U(n)$ has a classifying space $BU(n)$. It has a complex n-plane bundle γ_n over it which is universal in the sense that any such bundle ξ over a paracompact space X is the pullback of γ_n, induced by a map $f\colon X \to BU(n)$. Isomorphism classes of such bundles ξ are in one-to-one correspondence with homotopy classes of maps from X to $BU(n)$. Any $\mathbf{C}^n$-bundle ξ has an associated disc bundle $D(\xi)$ and sphere bundle $S(\xi)$. The *Thom space* $T(\xi)$ is the quotient $D(\xi)/S(\xi)$. Alternatively, for compact X, $T(\xi)$ is the one-point compactification of the total space of ξ.

$MU(n)$ is $T(\gamma_n)$, the Thom space of the universal n-plane bundle γ_n over $BU(n)$. The inclusion $U(n) \to U(n+1)$ induces a map $BU(n) \to BU(n+1)$. The pullback of γ_{n+1} under this map has Thom space $\Sigma^2 MU(n)$. Thom spaces are functorial so we have a map $\Sigma^2 MU(n) \to MU(n+1)$. Together these maps give the spectrum MU.

It follows from the celebrated theorem of Thom [**1**] that $\pi_*(MU)$ is isomorphic to the complex cobordisrn ring (see Milnor [**4**]) which is defined as follows. A *stably complex manifold* is one with a complex structure on its stable normal bundle. (This notion of a complex manifold is weaker than others, e.g., algebraic, analytic, and almost complex.) All such manifolds are oriented. Two closed stably complex manifolds M_1 and M_2 are *cobordant* if there is a stably complex manifold W whose boundary is the disjoint union of M_1 (with the opposite of the given orientation) and M_2. Cobordism, i.e., being cobordant, is an equivalence relation and the set of equivalence classes forms a ring (the complex cobordism ring) under disjoint union and Cartesian product. Milnor and Novikov's calculation of $\pi_*(MU)$ (3.1.5) implies that two such manifolds are cobordant if they have the same Chern numbers. For the definition of these and other details of the above we refer the reader to Milnor and Stasheff [**5**] or Stong [**1**].

This connection between MU and complex manifolds is, however, not relevant to most of the applications we will discuss, nor is the connection between MU and complex vector bundles. On the other hand, the connection with formal group laws (A2.1.1) discovered by Quillen [**2**] (see 4.1.6) is essential to all that follows. This leads one to suspect that there is some unknown formal group theoretic construction of MU or its associated infinite loop space. For example, many well-known infinite loop spaces have been constructed as classifying spaces of certain types of categories (see Adams [**9**], section 2.6), but to our knowledge no such description exists for MU. This infinite loop space has been studied in Ravenel and Wilson [**2**].

In order to construct BP and compute $BP_*(BP)$ we need first to analyze MU. Our starting points are 3.1.4, which describes its homology, and the Milnor–Novikov theorem 3.1.5, which describes its homotopy and the behavior of the Hurewicz map. The relevant algebraic information is provided by A2.1, which describes universal formal group laws and related concepts and which should be read before this section. The results of this section are also derived in Adams [**5**].

Before we can state Quillen's theorem (4.1.6), which establishes the connection between formal group laws and complex cobordism, we need some preliminary discussion.

4.1.1. DEFINITION. *Let E be an associative commutative ring spectrum. A* complex orientation *for E is a class $x_E \in \widetilde{E}^2(\mathbf{C}P^\infty)$ whose restriction to*

$$\widetilde{E}(\mathbf{C}P^1) \simeq \widetilde{E}^2(S^2) \cong \pi_0(E)$$

is 1, where $\mathbf{C}P^n$ denotes n-dimensional complex projective space. □

This definition is more restrictive than that given in Adams [**5**] (2.1), but it is adequate for our purposes.

Of course, not all ring spectra (e.g., bo) are orientable in this sense. Two relevant examples of oriented spectra are the following.

4.1.2. EXAMPLE. Let $E = H$, the integral Eilenberg–Mac Lane spectrum. Then the usual generator of $H^2(\mathbf{C}P^\infty)$ is a complex orientation x_H.

4.1.3. EXAMPLE. Let $E = MU$. Recall that MU is built up out of Thom spaces $MU(n)$ of complex vector bundles over $BU(n)$ and that the map $BU(n) \to MU(n)$ is an equivalence when $n = 1$. The composition

$$\mathbf{C}P^\infty = BU(1) \xrightarrow{\simeq} MU(1) \to MU$$

gives a complex orientation $x_{MU} \in MU^2(\mathbf{C}P^\infty)$. Alternatively, x_{MU} could be defined to be the first Conner–Floyd Chern class of the canonical complex line bundle over $\mathbf{C}P^\infty$ (see Conner and Floyd [**1**]).

4.1.4. LEMMA. *Let E be a complex oriented ring spectrum.*
(a) $E^*(\mathbf{C}P^\infty) = E^*(\mathrm{pt})[[x_E]]$.
(b) $E^*(\mathbf{C}P^\infty \times \mathbf{C}P^\infty) = E^*(\mathrm{pt})[[x_E \otimes 1, 1 \otimes x_E]]$.
(c) *Let $t\colon \mathbf{C}P^\infty \times \mathbf{C}P^\infty \to \mathbf{C}P^\infty$ be the H-space structure map, i.e., the map corresponding to the tensor product of complex line bundles, and let $F_E(x,y) \in E^*(\mathrm{pt})[[x,y]]$ be defined by $t^*(x_E) = F_E(x_E \otimes 1, 1 \otimes_{x_E})$. Then F_E is a formal group law* (A2.1.1) *over $E^*(\mathrm{pt})$.*

PROOF. For (c), the relevant properties of F_E follow from the fact that $\mathbf{C}P^\infty$ is an associative, commutative H-space with unit.

For (a) and (b) one has the Atiyah–Hirzebruch spectral sequence (AHSS) $H^*(X; E^*(\mathrm{pt})) \Rightarrow E^*(X)$ (see section 7 of Adams [**4**]). For $X = \mathbf{C}P^\infty$ the class x_E represents a unit multiple of $x_H \in H^2(\mathbf{C}P^\infty)$. Hence x_H and all of its powers are permanent cycles so the spectral sequence collapses and (a) follows. The argument for (b) is similar. $\square$

Hence a complex orientation x_E leads to a formal group law F_E over $E^*(\mathrm{pt.})$. Lazard's theorem A2.1.8 asserts that F_E is induced by a homomorphism $\theta_E\colon L \to E^*(\mathrm{pt.})$, where L is a certain ring over which a universal formal group law is defined. Recall that $L = \mathbf{Z}[x_1, x_2, \dots]$, where x_i has degree $2i$. There is a power series over $L \otimes \mathbf{Q}$

$$\log(x) = \sum_{i \ge 0} m_i x^{i+1}$$

where $m_0 = 1$ such that

$$L \otimes \mathbf{Q} = \mathbf{Q}[m_1, m_2 \dots]$$

and

$$\log(F(x,y)) = \log(x) + \log(y)$$

This formula determines the formal group law $F(x,y)$.

The following geometric description of θ_{MU}, while interesting, is not relevant to our purposes, so we refer the reader to Adams [**5**, Theorem 9.2] for a proof.

4.1.5. THEOREM (Mischenko [**1**]). *The element $(n+1)\theta_{MU}(m_n) \in \pi_*(MU)$ is represented by the complex manifold $\mathbf{C}P^n$.* $\square$

4.1.6. THEOREM (Quillen [**2**]). *θ_{MU} is an isomorphism.* $\square$

We will prove this with the help of the diagram

$$\begin{tikzcd} L \arrow[r, "j"] \arrow[d, "\theta_{MU}"'] & M \arrow[d, "\phi"] \\ \pi_*(MU) \arrow[r, "h"] & H_*(MU) \end{tikzcd}$$

where $M = \mathbf{Z}[m_1, m_2 \dots]$ is defined in A2.1.9(b) and contains L. The map ϕ will be constructed below. Recall [A2.1.10(b)] that modulo decomposables in M, the image of j is generated by

$$\begin{cases} pm_i & \text{if } i = p^k - 1 \text{ for some prime } p, \\ m_i & \text{otherwise.} \end{cases}$$

Recall also that $H_*(MU) = \mathbf{Z}[b_1, b_2, \dots]$ [3.1.4(a)] and that modulo decomposables in $H_*(MU)$, the image of h is generated by

$$\begin{cases} -pb_i & \text{if } i = p^k - 1 \text{ for some prime } p, \\ -b_i & \text{otherwise} \end{cases}$$

Hence it suffices to construct ϕ and show that it is an isomorphism.

Before doing this we need two lemmas.

First we must compute $E_*(MU)$. It follows easily from 4.1.4(a) that $E_*(\mathbf{C}P^\infty)$ is a free $\pi_*(E)$ module on elements β_i^E dual to x_E^i. We have a stable map $\mathbf{C}P^\infty \to \Sigma^2 MU$ and we denote by b_i^E the image of β_{i+1}^E.

4.1.7. Lemma. *If E is a complex oriented ring spectrum then*

$$E_*(MU) = \pi_*(E)[b_1^E, b_2^E \dots].$$

Proof. We use the Atiyah–Hirzebruch spectral sequence $H_*(MU, \pi_*(E)) \to E_*(MU)$. The b_i^E represent unit multiples of $b_i \in H_{2i}(MU)$ [3.1.4(a)], so the b_i are permanent cycles and the Atiyah–Hirzebruch spectral sequence collapses. □

If E is complex oriented so is $E \wedge MU$. The orientations x_E and x_{MU} both map to orientations for $E \wedge MU$ which we denote by $\hat{x}_E$ and $\hat{x}_{MU}$, respectively. We also know by 4.1.7 that

$$\pi_*(E \wedge MU) = E_*(MU) = \pi_*(E)[b_i^E]$$

4.1.8. Lemma. *Let E be a complex oriented ring spectrum. Then in $(E \wedge MU)^2(\mathbf{C}P^\infty)$.*

$$\hat{x}_{MU} = \sum_{i \geq 0} b_i^E \hat{x}_E^{i+1},$$

where $b_0 = 1$. This power series will be denoted by $g_E(\hat{x}_E)$.

Proof. We will show by induction on n that after restricting to $\mathbf{C}P^n$ we get

$$\hat{x}_{MU} = \sum_{0 \leq i < n} b_i^E \hat{x}_E^{i+1}.$$

For $n = 1$ this is clear since x_E and x_{MU} restrict to the canonical generators of $\widetilde{E}^*(\mathbf{C}P^1)$ and $MU^*(\mathbf{C}P^1)$. Now notice that x_E^n is the composite

$$\mathbf{C}P^n \to S^{2n} \to \Sigma^{2n} E$$

where the first map is collapsing to the top cell and the second map is the unit. Also b_{n-1}^E is by definition the composite

$$S^{2n} \xrightarrow{\beta_n^E} \mathbf{C}P^n \wedge E \xrightarrow{x_{MU} \wedge E} \Sigma^2 MU \wedge E.$$

Hence we have a diagram

$$\begin{tikzcd}
\mathbf{C}P^{n-1} \arrow[r] \arrow[d] & \mathbf{C}P^{n-1} \wedge E \arrow[dr, "x_{MU}\wedge E"] & \\
\mathbf{C}P^{n} \arrow[r] \arrow[ddd] \arrow[dddr, "x_E^n"'] & \mathbf{C}P^{n} \wedge E \arrow[r, "x_{MU}\wedge E"] \arrow[u, "g"] & \Sigma^2 MU \wedge E \\
 & \mathbf{C}P^{n} \wedge E \wedge E \arrow[r, "x_{MU}\wedge E\wedge E"] \arrow[u, "\mathbf{C}P^n\wedge m"] & \Sigma^2 MU \wedge E \wedge E \arrow[u, "\Sigma^2 MU\wedge m"'] \\
 & & \\
S^{2n} \arrow[r] & S^{2n} \wedge E \arrow[uu, "\beta_n^E\wedge E"] \arrow[uur, "b_{n-1}^E\wedge E"'] &
\end{tikzcd}$$

where $m\colon E \wedge E \to E$ is the multiplication and g is the cofiber projection of $(\mathbf{C}P^n \wedge m)\,(\beta_n^E \wedge E)$. is now split as $(\mathbf{C}P^{n-1} \wedge E) \vee (S^{2n} \wedge E)$ and $x_{MU} \wedge E\colon \mathbf{C}P \wedge E \to \Sigma^2 MU \wedge E$ is the sum of $(x_{MU} \wedge E)g$ and the map from $S^{2n} \wedge E$. Since x_{MU} is the composition

$$\mathbf{C}P^n \to \mathbf{C}P^n \wedge E \xrightarrow{X_{MU}\wedge E} \Sigma^2 MU \wedge E$$

and the lower composite map from $\mathbf{C}P^n$ to $\Sigma^2 MU \wedge E$ is $b_{n-1}^E x_E^n$, the inductive step and the result follow. □

4.1.9. Corollary. *In $\pi_*(E \wedge MU)[[x, y]]$,*

$$F_{MU}(x, y) = g_E(F_E(g_E^{-1}(x), g_E^{-1}(y))).$$

Proof. In $(E \wedge MU)^*(\mathbf{C}P^\infty \times \mathbf{C}P^\infty)$,

$$\begin{aligned}
F_{MU}(\hat{x}_{MU} \otimes 1, 1 \otimes \hat{x}_{MU}) &= t^*(x_{MU}) \\
&= g_E(t^*(\hat{x}_E)) \\
&= g_E(F_E(\hat{x}_E \otimes 1, 1 \otimes \hat{x}_E)) \\
&= g_E(F_E(g_E^{-1}(x_{MU}) \otimes 1, 1 \otimes g_E^{-1}(\hat{x}_{MU}))).
\end{aligned}$$

□

Now we are ready to prove 4.1.6. The map ϕ in 4.1.6 exists if the logarithm of the formal group law defined over $H_*(MU)$ by $h\theta_{MU}$ is integral, i.e., if the formal group law is isomorphic to the additive one. For $E = H$, $F_E(x, y) = x + y$, so the formal group law over $H_*(MU) = \pi_*(H \wedge MU)$ is indeed isomorphic to the additive one, so ϕ exists. Moreover, $\log_E(\hat{x}_E) = \hat{x}_E$, so

$$\hat{x}_E = \sum \phi(m_i)\hat{x}_{MU}^{i-1} = g_H^{-1}(\hat{x}_{MU})$$

by 4.1.9. It follows that $\sum \phi(m_i)x^{i+1}$ is the functional inverse of $\sum b_i x^{i+1}$, i.e.,

$$h\theta_{MU} \exp(x) = \sum_{i \geq 0} b_i x^{i+1}, \tag{4.1.10}$$

where exp is the functional inverse of the logarithm (A2.1.5), so $\phi(m_i) \equiv -b_i$, modulo decomposables in $H^*(MU)$ and 4.1.6 follows.

Now we will determine the structure of $MU_*(MU)$. We know it as an algebra by 4.1.7. In particular, it is a free $\pi_*(MU)$ module, so MU is a flat ring spectrum. Hence by 2.2.8 ($\pi_*(MU)$, $MU_*(MU)$) is a Hopf algebroid (A1.1.1). We will show that it is isomorphic to (L, LB) of A2.1.16. We now recall its structure. As an algebra, $LB = L[b_1, b_2 \dots]$ with deg $b_i = 2i$. There are structure maps $\varepsilon\colon LB \to L$ (augmentation), $\eta_L, \eta_R\colon L \to LB$ (left and right units), $\Delta\colon LB \to LB \otimes_L LB$ (coproduct), and $c\colon LB \to LB$ (conjugation) satisfying certain identities listed in A1.1.1.

$\varepsilon\colon LB \to L$ is defined by $\varepsilon(b_i) = 0$; $\eta_L\colon L \to LB$ is the standard inclusion, while $\eta_R\colon L \otimes \mathbf{Q} \to LB \otimes \mathbf{Q}$ is given by

$$\sum_{i \geq 0} \eta_R(m_i) = \sum_{i \geq 0} m_i \left(\sum_{j \geq 0} c(b_j) \right)^{i+1},$$

where $m_0 = b_0 = 1$;

$$\sum_{i \geq 0} \Delta(b_i) = \sum_{j \geq 0} \left(\sum_{i \geq 0} b_i \right)^{j+1} \otimes b_j;$$

and $c\colon LB \to LB$ is determined by $c(m_i) = \eta_R(m_i)$ and

$$\sum_{i \geq 0} c(b_i) \left(\sum_{j \geq 0} b_j \right)^{i+1} = 1.$$

Note that the maps η_L and η_R, along with the identities of A1.1.1, determine the remaining structure maps ε, Δ, and c.

The map θ_{MU} of 4.1.6 is an isomorphism which can be extended to LB by defining $\theta_{MU}(b_i)$ to be $b_i^{MU} \in MU_{2i}(MU)$ (4.1.8).

4.1.11. Theorem (Novikov [**1**], Landweber [**2**]). *The map $\theta_{MU}\colon LB \to MU_*(MU)$ defined above gives a Hopf algebroid isomorphism $(L, LB) \to (\pi_*(MU), MU_*(MU))$.*

Proof. Recall that the Hopf algebroid structure of (L, LB) is determined by the right unit $\eta_R\colon L \to LB$. Hence it suffices to show that θ_{MU} respects η_R. Now the left and right units in $MU_*(MU)$ are induced by $MU \wedge S^0 \to MU \wedge MU$ and $S^0 \wedge MU \to MU \wedge MU$, respectively. These give complex orientations x_L and x_R for $MU \wedge MU$ and hence formal group laws (4.1.4) F_R and F_L over $MU_*(MU)$. The b_i in LB are the coefficients of the power series of the universal isomorphism between two universal formal group laws. Hence it suffices to show that $x_R = \sum_{i \geq 0} b_i^{MU} x_R^{i+1}$, but this is the special case of 4.1.9 where $E = MU$. $\square$

Our next objective is

4.1.12. Theorem. *[Brown and Peterson [**1**], Quillen [**1**]] For each prime p there is a unique associative commutative ring spectrum BP which is a retract of $MU_{(p)}$ (2.1.12) such that the map $g\colon MU_{(p)} \to BP$ is multiplicative,*

(a) $\pi_*(BP) \otimes \mathbf{Q} = \mathbf{Q}[g_*(m_{p^k-1})\colon k > 0]$ *with $g_*(m_n) = 0$ for $n \neq p^k - 1$;*

(b) $H_*(BP\colon \mathbf{Z}/(p)) = P_*$ (3.1.6) *as comodule algebras over the dual Steenrod algebra A_** (3.1.1); *and*

(c) $\pi_*(BP) = \mathbf{Z}_{(p)}[v_1, v_2 \dots]$ *with $v_n \in \pi_{2(p^n-1)}$ and the composition $\pi_*(g)\theta_{MU_{(p)}}$ factors through the map $L \times \mathbf{Z}_{(p)} \to V$ of* A2.1.25*, giving an isomorphism from V to $\pi_*(BP)$.* $\square$

The spectrum BP is named after Brown and Peterson, who first constructed it via its Postnikov tower. Recall (3.1.9) that $H_*(MU; \mathbf{Z}/(p))$ splits as an A_*-comodule into many copies of P_*. Theorem 4.1.12 implies that there is a corresponding splitting of $MU_{(p)}$. Since P_* is dual to a cyclic A-module, it is clear that

BP cannot be split any further. Brown and Peterson [**1**] also showed that BP can be constructed from H (the integral Eilenberg–Mac Lane spectrum) by killing all of the torsion in its integral homology with Postnikov fibrations. More recently, Priddy [**1**] has shown that BP can be constructed from $S^0_{(p)}$ by adding local cells to kill off all of the torsion in its homotopy.

The generators v_n of $\pi_*(BP)$ will be defined explicitly below.

Quillen [**2**] constructed BP in a more canonical way which enabled him to determine the structure of $BP_*(BP)$. BP bears the same relation to p-typical formal group laws (A2.1.17) that MU bears to formal group laws as seen in 4.1.6. The algebraic basis of Quillen's proof of 4.1.12 is Cartier's theorem A2.1.18, which states that any formal group law over a $\mathbf{Z}_{(p)}$-algebra is canonically isomorphic to p-typical one. Accounts of Quillen's work are given in Adams [**5**] and Araki [**1**].

Following Quillen [**2**], we will construct a multiplicative map $g\colon MU_{(p)} \to MU_{(p)}$ which is idempotent, i.e., $g^2 = g$. This map will induce an idempotent natural transformation or cohomology operation on $MU^*_{(p)}(-)$. The image of this map will be a functor satisfying the conditions of Brown's representability theorem (see Brown [**2**] or, in terms of spectra, 3.12 of Adams [**4**]) and will therefore be represented by a spectrum BP. The multiplicativity of BP and its other properties will follow from the corresponding properties of g.

To construct g we need two lemmas.

4.1.13. LEMMA. *Let E be an oriented ring spectrum. Then orientations of E are in one-to-one correspondence with multiplicative maps from MU to E; i.e., given an orientation $y_E \in E^2(\mathbf{C}P^\infty)$, there is a unique multiplicative map $g\colon MU \to E$ such that $g^*(x_{MU}) = y_E$ and vice versa.*

PROOF. By 4.1.4, $E^*(\mathbf{C}P^\infty) = \pi_*(E)[[x_E]]$ so we have

$$y_E = f(x_E) = \sum_{i\ge 0} f_i x^{i+1}$$

with $f_0 = 1$ and $f_i \in \pi_{2i}(E)$. Using arguments similar to that of 4.1.8 and 4.1.6 one shows

$$E^*(MU) \cong \mathrm{Hom}_{\pi_*(E)}(E_*(MU), \pi_*(E)) \tag{4.1.14}$$

and

$$E^*(\mathbf{C}P^\infty) \cong \mathrm{Hom}_{\pi_*(E)}(E_*(\mathbf{C}P^\infty), \pi_*(E)).$$

A diagram chase shows that a map $MU \to E$ is multiplicative if the corresponding map $E_*(MU) \to \pi_*(E)$ is a $\pi_*(E)$-algebra map. The map y_E corresponds to the map which sends β^E_{i+1} to f_i and β^E_{i+1} by definition maps to $b^E_i \in E_{2i}(MU)$, so we let g be the map which sends b^E_i to f_i. □

4.1.15. LEMMA. *A map $g\colon MU_{(p)} \to MU_{(p)}$ (or $MU \to MU$) is determined up to homotopy by its behavior on π_*.*

PROOF. We do the MU case first. By 4.1.14,

$$MU^*(MU) = \mathrm{Hom}_{\pi_*(MU)}(MU_*(MU), \pi_*(MU)).$$

This object is torsion-free so we lose no information by tensoring with $\mathbf{Q}$. It follows from 4.1.11 that $MU_*(MU) \otimes \mathbf{Q}$ is generated over $\pi_*(MU) \otimes \mathbf{Q}$ by the image of η_R, which is the Hurewicz map. Therefore the map

$$MU^*(MU) \otimes \mathbf{Q} \to \mathrm{Hom}_{\mathbf{Q}}(\pi_*(MU) \otimes \mathbf{Q}, \pi_*(MU) \otimes \mathbf{Q})$$

is an isomorphism, so the result follows for MU.

For the $MU_{(p)}$ case we need to show

$$MU^*_{(p)}(MU_{(p)}) = MU^*(MU) \otimes \mathbf{Z}_{(p)}. \tag{4.1.16}$$

This will follow from 4.1.13 if we can show that the map

$$MU^*_{(p)}(MU) \to MU^*_{(p)}(MU_{(p)}) \tag{4.1.17}$$

is an isomorphism, i.e., that $MU^*_{(p)}(C) = 0$, where C is the cofiber of $MU \to MU_{(p)}$. Now C is trivial when localized at p, so any p-local cohomology theory vanishes on it. Thus 4.1.15 and the $MU_{(p)}$ case follow. $\square$

We are now ready to prove 4.1.12. By 4.1.13 and 4.1.15 a multiplicative map $g\colon MU_{(p)} \to MU_{(p)}$ is determined by a power series $f(x)$ over $\pi_*(MU_{(p)})$. We, take $f(x)$ to be as defined by A2.1.23. By 4.1.15 the corresponding map g is idempotent if $\pi_*(g) \otimes \mathbf{Q}$ is. To compute the latter we need to see the effect of g^* on

$$\log(x_{MU}) = \sum m_i x_{MU}^{i+1} \in MU^2(\mathbf{C}P^\infty) \otimes \mathbf{Q}.$$

Let $F'_{MU_{(p)}}$ be the formal group law associated with the orientation $f(x_{MU})$, and let $\mathrm{mog}(x)$ be its logarithm (A2.1.6). The map g^* preserves formal group laws and hence their logarithms, so we have $g^*(\log(x_{MU})) = \mathrm{mog}(f(x_{MU}))$. By A2.1.24 $\mathrm{mog}(x) = \sum_{k \geq 0} m_{p^k - 1} x^{p^k}$ and it follows that $\pi_*(g)$ has the indicated behavior; i.e., we have proved 4.1.12(a).

For (b), we have $H_*(BP; \mathbf{Q}) = \pi_*(BP) \otimes \mathbf{Q}$, and $H_*(BP; \mathbf{Z}_{(p)})$ is torsionfree, so $H_*(BP; \mathbf{Z}/(p)) = P_*$ as algebras. Since BP is a retract of $MU_{(p)}$ its homology is a direct summand over A_* and (b) follows.

For (c) the structure of $\pi_*(BP)$ follows from (a) and the fact that BP is a retract of $MU_{(p)}$. For the isomorphism from V we need to complete the diagram

$$\begin{tikzcd}
L \otimes \mathbf{Z}_{(p)} \arrow[r] \arrow[d, "\theta_{MU_{(p)}}"'] & V \arrow[d, dashed] \\
\pi_*(MU_{(p)}) \arrow[r, "g_*"] & \pi_*(BP)
\end{tikzcd}$$

The horizontal maps are both onto and the left-hand vertical map is an isomorphism so it suffices to complete the diagram tensored with $\mathbf{Q}$. In this case the result follows from (a) and A2.1.25. This completes the proof of 4.1.12.

Our last objective in this section is the determination of the Hopf algebroid (A1.1.1) $(\pi_*(BP), BP_*(BP))$. (Proposition 2.2.8 says that this object is a Hopf algebroid if BP is flat. It is since $MU_{(p)}$ is flat.) We will show that it is isomorphic to (V, VT) of A2.1.27, which bears the same relation to p-typical formal group laws that (L, LB) (A2.1.16 and 4.1.11) bears to ordinary formal group laws. The ring V (A2.1.25), over which the universal p-typical formal group law is defined, is isomorphic to $\pi_*(BP)$ by 4.1.12(c). $V \otimes \mathbf{Q}$ is generated by $m_{p^i - 1}$ for $i \geq 0$, and we denote this element by λ_i. Then from A2.1.27 we have

4.1.18. THEOREM. *In the Hopf algebroid* (V, VT) (*see* A1.1.1)

(a) $V = Z_{(p)}[v_1, v_2, \dots]$ *with* $|v_n| = 2(p^n - 1)$,

(b) $VT = V[t_1, t_2, \dots]$ *with* $|t_n| = 2(p^n - 1)$, *and*

(c) $\eta_L\colon V \to VT$ *is the standard inclusion and* $\varepsilon\colon VT \to V$ *is defined by* $\varepsilon(t_i) = 0$, $\varepsilon(v_i) = v_i$.

(d) $\eta_R\colon V \to VT$ *is determined by* $\eta_R(\lambda_n) = \sum_{0\le i\le n} \lambda_i t_{n-i}^{p^i}$*, where* $\lambda_0 = t_0 = 1$*,*

(e) Δ *is determined by*

$$\sum_{i,j\ge 0} \lambda_i \Delta(t_j)^{p^i} = \sum_{i,j,k\ge 0} \lambda_i t_j^{p^i} \otimes t_k^{p^{i+j}},$$

and

(f) c *is determined by*

$$\sum_{i,j,k\ge 0} \lambda_i t_j^{p^i} c(t_k)^{p^{i+j}} = \sum_{i\ge 0} \lambda_i.$$

(g) *The forgetful functor from* p*-typical formal group laws to formal group laws induces a surjection of Hopf algebroids* (A1.1.19)

$$(L \otimes \mathbf{Z}_{(p)}, LB \otimes \mathbf{Z}_{(p)}) \to (V, VT). \qquad \square$$

4.1.19. THEOREM (Quillen [**2**], Adams [**5**]). *The Hopf algebroid* $(\pi_*(BP), BP_*(BP))$ *is isomorphic to* (V, VT) *described above.*

PROOF. Consider the diagram

$$\begin{tikzcd}
(L, LB) \otimes \mathbf{Z}_{(p)} \arrow[r] \arrow[d, "\theta_{MU}"] & (V, VT) \arrow[d, dashed] \\
MU_*(MU) \otimes \mathbf{Z}_{(p)} \arrow[r, "g_*"] & (\pi_*(BP), BP_*(BP)).
\end{tikzcd}$$

The left-hand map is an isomorphism by 4.1.11 and the horizontal maps are both onto by (g) above and by 4.1.12. Therefore it suffices to complete the diagram with an isomorphism over $\mathbf{Q}$. One sees easily that $VT \otimes \mathbf{Q}$ and $BP_*(BP) \otimes \mathbf{Q}$ are both isomorphic to $V \otimes V \otimes \mathbf{Q}$. $\square$

2. A Survey of BP-Theory

In this section we will give an informal survey of some aspects of complex cobordism theory not directly related to the Adams–Novikov spectral sequence. (We use the terms complex cobordism and BP interchangeably in light of 4.1.12.) Little or no use of this material will be made in the rest of the book. This survey is by no means exhaustive.

The history of the subject shows a movement from geometry to algebra. The early work was concerned mainly with applications to manifold theory, while more recent work has dealt with the internal algebraic structure of various cohomology theories and their applictions to homotopy theory. The present volume is, of course, an example of the latter. The turning point in this trend was Quillen's theorem 4.1.6, which established a link with the theory of formal groups treated in Appendix 2. The influential but mostly unpublished work of Jack Morava in the early 1970s was concerned with the implications of this link.

Most geometric results in the theory, besides the classification of closed manifolds up to cobordism, rest on the notion of the *bordism groups* $\Omega_*(X)$ of a space X, first defined by Conner and Floyd [**2**]. $\Omega_n(X)$ is the group (under disjoint union) of equivalence classes of maps from closed n-dimensional manifolds (possibly with some additional structure such as an orientation or a stable complex structure) to X. Two such maps $f_i\colon M_i \to X$ $(i = 1, 2)$ are equivalent if there is a map $f\colon W \to X$ from a manifold whose boundary is $M_1 \cup M_2$ with f extending f_1 and f_2. It can be shown (Conner and Floyd [**2**]) that the functor $\Omega_*(-)$ is a generalized homology theory and that the spectrum representing it is the appropriate Thom spectrum for the manifolds in question. For example, if the manifolds are stably complex (see the beginning of Section 1) the bordism theory, denoted by $\Omega_*^U(-)$, coincides with $MU_*(-)$, the generalized homology theory represented by the spectrum MU, i.e., $\Omega_n^U(X) = \pi_n(MU \wedge X)$. The notation $\Omega_*(-)$ with no superscript usually denotes the oriented bordism group, while the unoriented bordism group is usually denoted by $N_*(-)$.

These bordism groups are usually computed by algebraic methods that use properties of the Thom spectra. Thom [**1**] showed that MO, the spectrum representing unoriented bordism, is a wedge of mod (2) Eilenberg–Mac Lane spectra, so $N_*(X)$ is determined by $H_*(X; \mathbf{Z}/(2))$. MSO (which represents oriented bordism) when localized at the prime 2 is known (Stong [**1**, p. 209]) to be a wedge of integral and mod (2) Eilenberg–Mac Lane spectra, so $\Omega_*(X)_{(2)}$ is also determined by ordinary homology. Brown and Peterson [**1**] showed that when localized at any odd prime the spectra MSO, MSU, and MSp as well as MU are wedges of various suspensions of BP, so the corresponding bordism groups are all determined by $BP_*(X)$. Conner and Floyd [**2**] showed effectively that $BP_*(X)$ is determined by $H_*(X; \mathbf{Z}_{(p)})$ when the latter is torsion-free.

For certain spaces the bordism groups have interesting geometric interpretations. For example, $\Omega_*(BO)$ is the cobordism group of vector bundles over oriented manifolds. Since $H_*(BSO)$ has no odd torsion, it determines this group. If X_n is the nth space in the Ω-spectrum for MSO, then $\Omega_*(X_n)$ is the cobordism group of maps of codimension n between oriented manifolds. The unoriented analog was treated by Stong [**3**] and the complex analog by Ravenel and Wilson [**2**].

For a finite group G, $\Omega_*(BG)$ is the cobordism group of oriented manifolds with free G-actions, the manifolds mapped to BG being the orbit spaces. These groups were studied by Conner and Floyd [**2**] and Conner [**4**]. Among other things they computed $\Omega_*(BG)$ for cyclic G. In Landweber [**6**] it was shown that the map $MU_*(BG) \to H_*(BG)$ is onto iff G has periodic cohomology. In Floyd [**1**] and tom Dieck [**1**] it is shown that the ideal of $\pi_*(MU)$ represented by manifolds on which an abelian p-group with n cyclic summands can act without stationary points is the prime ideal I_n defined below. The groups $BP_*(BG)$ for $G = (\mathbf{Z}/(p))^n$ have been computed by Johnson and Wilson [**5**].

We now turn to certain other spectra related to MU and BP. These are constructed by means of either the Landweber exact functor theorem (Landweber [**3**]) or the *Sullivan–Baas construction* (Baas [**1**]), which we now describe. Dennis Sullivan (unpublished, circa 1969) wanted to construct "manifolds with singularities" (admittedly a contradiction in terms) with which any ordinary homology class could be represented; i.e., any element in $H_*(X; \mathbf{Z})$ could be realized as the image of the fundamental homology class of such a "manifold" M under some map $M \to X$. It was long known that not all homology classes were representable in this sense by ordinary manifolds, the question having been originally posed by Steenrod. (I heard Sullivan begin a lecture on the subject by saying that homology was like the weather; everybody talks about it but nobody does anything about it.)

In terms of spectra this nonrepresentability is due to the fact that MU (if we want our manifolds to be stably complex) is *not* a wedge of Eilenberg–Mac Lane spectra. The Sullivan–Baas construction can be regarded as a way to get from MU to H.

Let $y \in \pi_k(MU)$ be represented by a manifold X. A closed n-dimensional manifold with *singularity of type* (y) $(n > k)$ is a space W of the form $A \cup (B \times CM)$, where CM denotes the cone on a manifold M representing y, B is a closed $(n-k-1)$-dimensional manifold, A is an n-dimensional manifold with boundary $B \times M$, and A and $B \times CM$ are glued together along $B \times M$. It can be shown that the bordism group defined using such objects is a homology theory represented by a spectrum $C(y)$ which is the cofiber of

$$\Sigma^k MU \xrightarrow{y} MU, \quad \text{so} \quad \pi_* C(Y) = \pi_*(MU)/y.$$

This construction can be iterated any number of times. Given a sequence $y_1, y_2, \dots$ of elements in $\pi_*(MU)$ we get spectra $C(y_1, y_2, \dots y_n)$ and cofibrations

$$\Sigma^{|y_n|} C(y_1, \dots, y_{n-1}) \to C(y_1, \dots, y_{n-1}) \to C(y_1, \dots, y_n).$$

If the sequence is regular, i.e., if y_n is not a zero divisor in $\pi_*(MU)/(y_1, \dots, y_{n-1})$, then each of the cofibrations will give a short exact sequence in homotopy, so we get

$$\pi_*(C(y_1, \dots, y_n)) = \pi(MU)/(y_1, \dots, y_n).$$

In this way one can kill off any regular ideal in $\pi_*(MU)$. In particular, one can get H by killing $(x_1, x_2, \dots)$. Sullivan's idea was to use this to show that any homology class could be represented by the corresponding type of manifold with singularity. One could also get BP by killing the kernel of the map $\pi_*(MU) \to \pi_*(BP)$ and then localizing at p. This approach to BP does not reflect the splitting of $MU_{(p)}$.

Much more delicate arguments are needed to show that the resulting spectra are multiplicative (Shimada and Yagita [**1**], Morava [**1**], Mironov [**1**]), and the proof only works at odd primes. Once they are multiplicative, it is immediate that they are orientable in the sense of 4.1.1.

The two most important cases of this construction are the *Johnson–Wilson spectra* $BP\langle n\rangle$ (Johnson and Wilson [**2**]) and the Morava K-theories $K(n)$ (Morava's account remains unpublished; see Johnson and Wilson [**3**]).

$BP\langle n\rangle$ is the spectrum obtained from BP (one can start there instead of MU since BP itself is a product of the Sullivan–Baas construction) by killing $(v_{n+1}, v_{n+2}, \dots) \subset \pi_*(BP)$. One gets

$$\pi_*(BP\langle n\rangle) = \mathbf{Z}_{(p)}[v_1, \dots, v_n]$$

and

$$H_*(BP\langle n\rangle, \mathbf{Z}/(p)) = P_* \otimes E(\tau_{n+1}, \tau_{n+2}, \dots).$$

(It is an easy exercise using the methods of Section 3.1 to show that a connective spectrum with that homology must have the indicated homotopy.) One has fibrations

$$\Sigma^{2(p^n-1)} BP\langle n\rangle \xrightarrow{v_n} BP\langle n\rangle \to BP\langle n-1\rangle.$$

$BP\langle 0\rangle$ is $H_{(p)}$ and $BP\langle 1\rangle$ is a summand of $bu_{(p)}$, the localization at p of the spectrum representing connective complex K-theory. One can iterate the map

$$v_n \colon \Sigma^{2(p^n-1)} BP\langle n\rangle \to BP\langle n\rangle$$

and form the direct limit

$$E(n) = \varinjlim_{v_n} \Sigma^{-2i(p^n-1)} BP\langle n\rangle.$$

$E(1)$ is a summand of periodic complex K-theory localized at p. Johnson and Wilson [2] showed that

$$E(n)_*(X) = BP_*(X) \otimes_{BP_*} E(n)_*.$$

$E(n)$ can also be obtained by using the Landweber exact functor theorem below.

The $BP\langle n\rangle$ are interesting for two reasons. First, the fibrations mentioned above split unstably; i.e., if $BP\langle n\rangle_k$ is the kth space in the Q-spectrum for $BP\langle n\rangle$ (i.e., the space whose homotopy starts in dimension k) then

$$BP\langle n\rangle_k \simeq BP\langle n-1\rangle_k \times BP\langle n\rangle_{k+2(p^n-1)}$$

for $k \le 2(p^n-1)/(p-1)$ (Wilson [2]). This means that if X is a finite complex then $BP_*(X)$ is determined by $BP\langle n\rangle_*(X)$ for an appropriate n depending on the dimension of X.

The second application of $BP\langle n\rangle$ concerns $\operatorname{Hom\,dim} BP_*(X)$, the projective dimension of $BP_*(X)$ as a module over $\pi_*(BP)$, known in some circles as the ugliness number. Johnson and Wilson [2] show that the map $BP_*(X) \to BP\langle n\rangle_*(X)$ is onto iff $\operatorname{Hom\,dim} BP_*(X) \le n+1$. The cases $n=0$ and $n=1$ of this were obtained earlier by Conner and Smith [3].

We now turn to the *Morava K-theories* $K(n)$. These spectra are periodic, i.e., $\Sigma^{2(p^n-1)}K(n) = K(n)$. Their connective analogs $k(n)$ are obtained from BP by killing $(p, v_1, \dots, v_{n-1}, v_{n+1}, v_{n+2}, \dots)$. Thus one has $\pi_*(k(n)) = \mathbf{Z}/(p)[v_n]$ and $H_*(k(n), \mathbf{Z}/(p)) = A/(Q_n)_*$. One has fibrations

$$\Sigma^{2(p^n-1)}k(n) \xrightarrow{v_n} k(n) \to H\mathbf{Z}/(p),$$

and one defines

$$K(n) = \varinjlim_{v_n} \Sigma^{-2i(p^n-1)} k(n).$$

$K(1)$ is a summand of mod p complex K-theory and it is consistent to define $K(0)$ to be HQ, rational homology.

The coefficient ring $\pi_*(K(n)) = \mathbf{F}_p[v_n, v_n^{-1}]$ is a graded field in the sense that every graded module over it is free. One has a Künneth isomorphism

$$K(n)_*(X \times Y) = K(n)_*(X) \otimes_{\pi_*(K(n))} K(n)_*(Y).$$

This makes $K(n)_*(-)$ much easier to compute with than any of the other theories mentioned here. In Ravenel and Wilson [3] we compute the Morava K-theories of all the Eilenberg–Mac Lane spaces, the case $n=1$ having been done by Anderson and Hodgkin [2]. We show that for a finite abelian group G, $K(n)_*(K(G,m))$ is finite-dimensional over $\pi_*(K(n))$ for all m and n, and is isomorphic to $K(n)_*(\mathrm{pt})$ if $m > n$. $K(n+1)^*K(\mathbf{Z}, m+2)$ for $m, n \ge 0$ is a power series ring on $\binom{n}{m}$ variables. In all cases the $K(n)$-theory is concentrated in even dimensions. These calculations enabled us to prove the conjecture of Conner and Floyd [2] which concerns $BP_*(B(\mathbf{Z}/p)^n)$.

To illustrate the relation between the $K(n)$'s and BP we must introduce some more theories. Let $I_n = (p, v_1, \dots, v_{n-1}) \subset \pi_*(BP)$ (see 4.3.2) and let $P(n)$ be the spectrum obtained from BP by killing I_n. Then one has fibrations

$$\Sigma^{2(p^n-1)}P(n) \xrightarrow{v_n} P(n) \to P(n+1)$$

and we define

$$B(n) = \varinjlim_{v_n} \Sigma^{-2i(p^n-1)} P(n).$$

$P(n)_*(X)$ is a module over $\mathbf{F}_p[v_n]$ and its torsion-free quotient maps monomorphically to $B(n)_*(X)$. In Johnson and Wilson [**3**] it is shown that $B(n)_*(X)$ is determined by $K(n)_*(X)$. In Würgler [**2**] it is shown that a certain completion of $B(n)$ splits into a wedge of suspensions of $K(n)$.

This splitting has the following algebraic antecedent. The formal group law associated with $K(n)$ (4.1.4) is essentially the standard height n formal group law F_n of A2.2.11, while $\pi_*(B(n)) = \mathbf{F}_p[v_n, v_n^{-1}, v_{n+1}, \dots]$ is the universal ring for all p-typical formal group laws of height n (A2.2.8). In A2.2.12 it is shown that over the algebraic closure of $\mathbf{F}_p$ any height n formal group law is isomorphic to the standard one. Heuristically this is why $B(n)_*(X)$ is determined by $K(n)_*(X)$.

This connection between $K(n)$ and height n formal group laws also leads to a close relation between $K(n)_*(K(n))$ and the endomorphism ring of F_n (A2.2.18). An account of $K(n)_*(K(n))$ is given in Yagita [**1**]. The reader should be warned that $K(n)_*(K(n))$ is not the Hopf algebroid $K(n)_*K(n)$ of Ravenel [**5, 6**], which is denoted herein by $\Sigma(n)$; in fact, $K(n)_*(K(n)) = \Sigma(n) \otimes E(\tau_0, \tau_1, \dots, \tau_{n-1})$, where the τ_i are analogous to the τ_i in A_*.

Most of the above results on $K(n)$ (excluding the results about Eilenberg–Mac Lane spaces) were known to Morava and communicated by him to the author in 1973.

The invariance of the I_n (4.3.2) under the BP-operations makes it possible to construct the spectra $P(n)$, $B(n)$, and $K(n)$ and to show that they are ring spectra for $p > 2$ by more algebraic means, i.e., without using the Sullivan–Baas construction. This is done in Würgler [**1**], where the structure of $P(n)_*(P(n))$ is also obtained. $k(n)_*(k(n))$ is described in Yagita [**2**].

We now turn to the important work of Peter Landweber on the internal algebraic structure of MU- and BP-theories. The starting point is the invariant prime ideal theorem 4.3.2, which first appeared in Landweber [**4**], although it was probably first proved by Morava. It states that the only prime ideals in $\pi_*(BP)$ which are invariant (A1.1.21), or, equivalently, which are subcomodules over $BP_*(BP)$, are the $I_n = (p, v_1, v_2, \dots, v_{n-1})$ for $0 \le n \le \infty$. In Conner and Smith [**3**] it is shown that for a finite complex X, $BP_*(X)$ is finitely presented as a module over $\pi_*(BP)$. [The result there is stated in terms of $MU_*(X)$, but the two statements are equivalent.] From commutative algebra one knows that such a module over such a ring has a finite filtration in which each of the successive subquotients is isomorphic to the quotient of the ring by some prime ideal. Of course, as anyone who has contemplated the prospect of algebraic geometry knows, a ring such as $\pi_*(BP)$ has a very large number of prime ideals. However, Landweber [**3**] shows that the coaction of $BP_*(BP)$ implies that the filtration of $BP_*(X)$ [or of any $BP_*(BP)$-comodule which is finitely presented as a module over $\pi_*(BP)$] can be chosen so that each successive subquotient has the form $\pi_*(BP)/I_n$ for some finite n. {The corresponding statement about $MU_*(X)$ appeared earlier in Landweber [**5**].} The submodules in the filtration can be taken to be subcomodules and n (the number of generators of the prime ideal) never exceeds the projective dimension of the module. This useful result is known as the *Landweber filtration theorem*.

It leads to the *Landweber exact functor theorem*, which addresses the following question. For which $\pi_*(BP)$-modules M is the functor $BP_*(-)\otimes_{\pi_*(BP)} M$ a generalized homology theory? Such a functor must be exact in the sense that it converts cofiber sequences into long exact sequences of modules. This will be the case if M is flat, i.e., if $\mathrm{Tor}_1^{\pi_*(BP)}(M, N) = 0$ for all modules N. However, in view of the filtration theorem it suffices for this Tor group to vanish only for $N = \pi_*(BP)/I_n$ for all n. This weaker (than flatness) condition on M can be made more explicit as follows. For each n, multiplication by v_n in $M \otimes_{\pi_*(BP)} \pi_*(BP)/I_n$ is monic. Thus Landweber [**3**] shows that any M satisfying this condition gives a homology theory.

For example, the spectrum $E(n)$ mentioned above (in connection with Johnson–Wilson spectra) can be so obtained since

$$\pi_*(E(n)) = \mathbf{Z}_{(p)}[v_1, v_2, \ldots, v_n, v_n^{-1}]$$

satisfies Landweber's condition. [Multiplication by v_i is monic in $\pi_*(E(n))$ itself for $i \le n$, while for $i > n$, $\pi_*(E(n)) \otimes_{\pi_*(BP)} \pi_*(BP)/I_i = 0$ so the condition is vacuous.]

As remarked earlier, $E(1)$ is a summand of complex K-theory localized at p. The exact functor theorem can be formulated globally in terms of MU-theory and $\pi_*(K)$ [viewed as a $\pi_*(MU)$-module via the Todd genus $td\colon \pi_*(MU) \to \mathbf{Z}$] satisfies the hypotheses. Thereby one recovers the *Conner–Floyd isomorphism*

$$K_*(X) = MU_*(X) \otimes_{\pi_*(MU)} \pi_*(K)$$

and similarly for cohomology. In other words, complex K-theory is determined by complex cobordism. This result was first obtained by Conner and Floyd [**1**], whose proof relied on an explicit K-theoretic orientation of a complex vector bundle. Using similar methods they were able to show that real K-theory is determined by symplectic cobordism.

Landweber's results have been generalized as follows. Let $J \subset \pi_*(BP)$ be an invariant regular ideal (see Landweber [**7**]), and let BPJ be the spectrum obtained by killing J; e.g., $P(n)$ above is BPI_n. Most of the algebra of BP-theory carries over to these spectra, which are studied systematically in a nice paper by Johnson and Yosimura [**4**]. The case $J = I_n$ was treated earlier by Yagita [**3**] and Yosimura [**1**]. The mod I_n version of the exact functor enables one to get $K(n)$ from $P(n)$.

Johnson and Yosimura [**4**] also prove some important facts about $\pi_*(BP)$ modules M which are comodules over $BP_*(BP)$. They show that if an element $m \in M$ is v_n-torsion (i.e., it is annihilated by some power of v_n) then it is v_{n-1}-torsion. If all of the primitive elements in M [i.e., those with $\psi(m) = 1 \otimes m$] are v_n-torsion, then so is every element, and, if none is, then M is v_n-torsion free. If M is a v_{n-1}-torsion module, then $v_n^{-1}M$ is still a comodule over $BP_*(BP)$. Finally, they show that $v_n^{-1}BP_*(X) = 0$ if $E(n)_*(X) = 0$.

This last result may have been prompted by an erroneous claim by the author that the spectrum $v_n^{-1}BP$ splits as a wedge of suspensions of $E(n)$. It is clear from the methods of Würgler [**2**] that one must complete the spectra in some way before such a splitting can occur. Certain completions of MU are studied in Morava [**2**].

We now turn to the last topic of this section, the applications of BP-theory to unstable homotopy theory. This subject began with Steve Wilson's thesis (Wilson [**1, 2**]) in which he studied the spaces in the Ω-spectra for MU and BP. He obtained the splitting mentioned above (in connection with the Johnson–Wilson spectra) and showed that all of the spaces in question have torsion-free homology.

Both the homology and cohomology of each space are either an exterior algebra on odd-dimensional generators or a polynomial algebra on even-dimensional generators.

These spaces were studied more systematically in Ravenel and Wilson [**2**]. There we found it convenient to consider all of them simultaneously as a graded space. The mod (p) homology of such an object is a bigraded coalgebra. The fact that this graded space represents a multiplicative homology theory implies that its homology is a ring object in the category of bigraded coalgebras; we call such an object a *Hopf ring*. We show that the one in question has a simple set of generators and relations which are determined by the structure of $MU^*(\mathbf{C}P^\infty)$, i.e., by $\pi_*(MU)$ and the associated formal group law. We obtain similar results for the value on this graded space of any complex oriented (4.1.1) generalized homology theory.

As mentioned earlier, the complex bordism of the graded space associated with MU is the cobordism group of maps between stably complex manifolds. We show that it is a Hopf ring generated by maps from a manifold to a point and the linear embeddings of $\mathbf{C}P^n$ in $\mathbf{C}P^{n+1}$.

The Hopf ring point of view is also essential in Ravenel and Wilson [**3**], where we calculate $K(n)_*(K(G,m))$. We show that the Hopf ring $K(n)_*(K(\mathbf{Z}/(p^j),{}^*))$ is a certain type of free object on $K(n)_*(K(\mathbf{Z}/(p^j),1))$. The ordinary homology of $K(\mathbf{Z}/(p^j),{}^*)$ can be described in similar terms and the methods of our paper may lead to simpler proofs of the classical theorems about it (see Wilson [**3**], section II.8).

Knowing the BP homology of the spaces in the BP spectrum is analogous to knowing the mod (p) homology of the mod (p) Eilenberg–Mac Lane spaces. This information, along with some ingenious formal machinery, is needed to construct the unstable Adams spectral sequence, i.e., a spectral sequence for computing the homotopy groups of a space X rather than a spectrum. This was done in the BP case by Bendersky, Curtis, and Miller [**1**]. Their spectral sequence is especially convenient for $X = S^{2n+1}$. In that case they get an E_1-term which is a subcomplex of the usual E_1-term for the sphere spectrum, i.e., of the cobar complex of A1.2.11. Their E_2-term is Ext in an appropriate category. For S^{2n+1} they compute Ext^1, which is a subgroup of the stable Ext^1, and get some corresponding information about $\pi_*(S^{2n+1})$.

In Bendersky [**2**] the spectral sequence is applied to the special unitary groups $SU(n)$. In Bendersky, Curtis, and Ravenel [**3**] the E_2-terms for various spheres are related by an analog of the EHP sequence.

3. Some Calculations in $BP_*(BP)$

In this section we will prove the Morava–Landweber theorem (4.3.2), which classifies invariant prime ideals in $\pi_*(BP)$. Then we will derive several formulas in $BP_*(BP)$ (4.1.18 and 4.1.19). These results are rather technical. Some of them are more detailed than any of the applications in this book require and they are included here only for possible future reference. The reader is advised to refer to this material only when necessary.

Theorem 4.3.3 is a list of invariant regular ideals that will be needed in Chapter 5. Lemma 4.3.8 gives some generalizations of the Witt polynomials. They are used to give more explicit formulas for the formal group law (4.3.9), the coproduct (4.3.13), and the right unit (4.3.18). We define certain elements, $b_{i,J}$ (4.3.14) and $c_{i,J}$ (4.3.19), which are used to give approximations (modulo certain prime ideals) of the coproduct (4.3.15) and right unit (4.3.20). Explicit examples of the right unit are given in 4.3.21. The coboundaries of $b_{i,J}$ and $c_{i,J}$ in the cobar complex are given in 4.3.22.

In 4.3.23 we define a filtration of $BP_*(BP)/I_n$ which leads to a May spectral sequence which will be used in Section 6.3. The structure of the resulting bigraded Hopf algebroid is given in 4.3.32–34.

From now on $\pi_*(BP)$ will be abbreviated by BP_*. Recall (A2.2.3) that we have two sets of generators for the ring BP_* given by Hazewinkel [**2**] (A2.2.1) and Araki [**1**] (A2.2.2). The behavior of the right unit $\eta_R\colon BP_* \to BP_*(BP)$ on the Araki generators is given by A2.2.6, i.e.,

$$\sum_{i,j\geq 0}^{F} t_i\eta_R(v_j)^{p^i} = \sum_{i,j\geq 0}^{F} v_i t_j^{p^i} \tag{4.3.1}$$

For the Hazewinkel generators this formula is true only mod (p).

This formula will enable us to define some invariant ideals in BP_*. In each case it will be easy to show that the ideal in question is independent of the choice of generators used. The most important result of this sort is the following.

4.3.2. Theorem (Morava [**3**], Landweber [**4**]). *Let $I_n = (p, v_1, \ldots v_{n-1}) \subset BP_*$.* (a) *$I_n$ is invariant.*

(b) *For $n > 0$,*

$$\mathrm{Ext}^0_{BP_*(BP)}(BP_*, BP_*/I_n) = \mathbf{Z}/(p)[v_n]$$

and

$$\mathrm{Ext}^0_{BP_*(BP)}(BP_*, BP_*) = \mathbf{Z}_{(p)}.$$

(c) *$0 \to \Sigma^{2(p^n-1)}BP_*/I_n \xrightarrow{v_n} BP_*/I_n \to BP_*/I_{n+1} \to 0$ is a short exact sequence of comodules.*

(d) *The only invariant prime ideals in BP_* are the I_n for $0 \leq n \leq \infty$.*

Proof. Part (a) follows by induction on n, using (c) for the inductive step. Part (c) is equivalent to the statement that

$$v_n \in \mathrm{Ext}^0_{BP_*(BP)}(BP_*, BP_*/I_n)$$

and is therefore a consequence of (b). For (d) suppose J is an invariant prime ideal which properly contains some I_n. Then the smallest dimensional element of J not in I_n must be invariant modulo I_n, i.e., it must be in $\mathrm{Ext}^0_{BP_*(BP)}(BP_*(BP/I_n))$, so by (b) it must be a power of v_n (where $v_0 = p$). Since J is prime this element must be v_n itself, so $J \supset I_{n+1}$. If this containment is proper the argument can be repeated. Hence, if J is finitely generated, it is I_n for some $n < \infty$. If J is infinitely generated we have $J \supset I_\infty$, which is maximal, so (d) follows.

Hence it remains only to prove (b). It is clear from 4.3.1 that $\eta_R(v_n) \equiv v_n \mod I_n$, so it suffices to show that $\mathrm{Ext}^0_{BP_*(BP)}(BP_*, BP_*/I_n)$ is no bigger than indicated. From 4.3.1 we see that in $BP_*(BP)/I_n$,

$$\eta_R(v_{n+j}) \equiv v_{n+j} + v_n t_j^{p^n} - v_n^{p^j} t_j \mod (t_1, t_2, \ldots, t_{j-1}),$$

so the set $\{v_{n+j}, \eta_R(v_{n+j}) \mid j > 0\} \cup \{v_n\}$ is algebraically independent. It follows that if $\eta_R(v) = v$ then v must be a polynomial in v_n. □

Now we will construct some invariant regular ideals in BP_*. Recall that an ideal $(x_0, x_1, \ldots, x_{n-1})$ is regular if x_i is not a zero divisor in $BP_*/(x_0, \ldots, x_{i-1})$ for $0 \le i < n$. This means that the sequence

$$0 \to BP_*/(x_0, \ldots, x_{i-1}) \xrightarrow{x_i} BP_*/(x_0, \ldots, x_{i-1}) \to BP_*/(x_0, \ldots, x_i) \to 0$$

is exact. The regular sequence $(x_0, x_1, \ldots)$ is invariant if the above is a short exact sequence of comodules. Invariant regular ideals have been studied systematically by Landweber [**7**]. He shows that an invariant regular ideal with n generators is primary with radical I_n, and that any invariant ideal with n generators and radical I_n is regular. Invariant ideals in general need not be regular, e.g., I_n^k for $k > 1$.

4.3.3. THEOREM. *Let $i_1, i_2, \ldots$ be a sequence of positive integers such that for each $n > 0$, i_{n+i} is divisible by the smallest power of p not less than i_n, and let $k \ge 0$. Then for each $n > 0$, the regular ideal $(p^{1+k}, v_1^{i_1 p^k}, v_2^{i_2 p^{2k}}, \ldots, v_n^{i_n p^{kn}})$ is invariant.* □

In order to prove this we will need the following.

4.3.4. LEMMA. *Let B, A_1, A_2, ... be ideals in a commutative ring. Then if*

$$x \equiv y \mod pB + \sum_i A_i,$$

then

$$x^{p^n} \equiv y^{p^n} \mod p^{n+1}B + \sum_{k=0}^{n} p^k \sum_i A_i^{p^{n-k}}.$$

PROOF. The case $n > 1$ follows easily by induction on n from the case $n = 1$. For the latter suppose $x = y + pb + \sum a_i$, with $b \in B$ and $a_i \in A_i$. Then

$$x^p = y^p + \sum_{0<j<p} \binom{p}{j} y^{p-j} \left(pb + \sum a_i\right)^j + \left(pb + \sum a_i\right)^p$$

and we have

$$\binom{p}{j} y^{p-j} \left(pb + \sum a_i\right)^j \in p^2 B + p \sum A_i$$

and

$$\left(pb + \sum a_i\right)^p \in p^2 B + p \sum A_i + \sum A_i^p.$$ □

PROOF OF 4.3.3. We have $v_n \equiv \eta_R(v_n) \mod I_n$, so we apply 4.3.4 to the ring $BP_*(BP)$ by setting $B = (1)$, $A_i = (v_i)$. Then we get

$$v_n^{p^m} \equiv \eta_R(v_n)^{p^m} \mod (p^{m+1}) + \sum_{j=0}^{m} \sum_{i=1}^{n-1} (p^j v_i^{m-j}).$$

To prove the theorem we must show that the indicated power of v_n is invariant modulo the ideal generated by the first n elements. It suffices to replace this ideal by the smaller one obtained by replacing each of $i_1, \ldots, i_{n-1}$ by the smallest power of p not less than it, i.e., by an ideal of the form

$$(p^{1+k}, v_1^{j_1}, v_2^{j_2}, \ldots, v_{n-1}^{j_{n-1}}) = I$$

with $j_t = p^{tk+k_t}$ where $0 \le k_1 \le k_2 \cdots \le k_{n-1}$. Then the hypothesis on i_n is that it is divisible by $p^{k_{n-1}}$, so it suffices to assume that $i_n = p^{k_{n-1}}$. Hence we must show

$$v_n^{j_{n-1}p^k} \equiv \eta_R(v_n)^{j_{n-1}p^k} \mod I.$$

We have $v_n \equiv \eta_R(v_n) \bmod I_n$, so we apply 4.3.4 to the ring $BP_*(BP)$ by setting $B = (1)$, $A_i = (v_i)$. Then we get

$$v_n^{p^m} \equiv \eta_R(v_n)^{p_m} \mod (p^{m+1}) + \sum_{j=0}^{m}\sum_{i=0}^{n-1}(p^j v_i^{p^{m-j}}).$$

We are interested in the case $m = kn + k_{n-1}$. Careful inspection shows that the indicated ideal in this case is contained in I. □

Theorem 4.3.3 leads to a list of invariant regular ideals which one might hope is complete. Unfortunately, it is not. For example, it gives $\{(p^{k+1}, v^{ip^k}) \mid k \ge 0, i \ge 0\}$ as a list of I_2-primary regular ideals, and this list can be shown to be a complete for $p > 2$, but at $p = 2$ the ideal $(16, v_1^4 + 8v_1v_2)$ is regular and invariant but not in the list. Similarly, for $p > 2$ the ideal

$$(p, v_1^{p^2+p-1}, v_2^{2p^2} - 2v_1^{p^2} v_2^{p^2-p} v_3^p - 2v_1^{p^2-1} v_2^{2p^2-p+1})$$

is invariant, regular, and not predicted by 4.3.3. This example and others like it were used by Miller and Wilson [**3**] to produce unexpected elements in $\mathrm{Ext}^1_{BP_*(BP)}(BP_*, BP_*/I_n)$ (see Section 5.2).

Now we will make the structure of $BP_*(BP)$ (4.1.19) more explicit. We start with the formal group law.

Recall the lemma of Witt (see, e.g., Lang [**1**, pp. 234–235]) which states that there are symmetric integral polynomials $w_n = w_n(x_1, x_2, \dots)$ of degree p^n in any number of variables such that

$$w_0 = \sum x_t \quad \text{and} \quad \sum_t x_t^{p^n} = \sum_j p^j w_j^{p^{n-j}}. \tag{4.3.5}$$

For example,

$$w_1 = \Big(\sum (x_t^p) - \Big(\sum x_t\Big)^p\Big)\Big/p \tag{4.3.6}$$

and for $p = 2$ with two variables,

$$w_2 = -x_1^3x_2 - 2x_1^2x_2^2 - x_1x_2^3.$$

Witt's lemma can be restated as follows. Let G be the formal group law with logarithm $\sum_{i\ge0} x^{p^i}/p^i$. Then

$$\sum\nolimits^G x_t = \sum\nolimits^G w_n. \tag{4.3.7}$$

This formula is in some sense more explicit than the usual

$$\log\Big(\sum\nolimits^G x_t\Big) = \sum \log x_t.$$

We will derive a similar formula for the universal formal group law.

First we need some notation. Let $I = (i_1, i_2, \dots, i_m)$ be a finite (possibly empty) sequence of positive integers. Let $|I| = m$ and $\|I\| = \sum i_t$. For positive integers n let $\Pi(n) = p - p^{(p^n)}$ and define integers $\Pi(I)$ recursively by $\Pi(\phi) = 1$ and $\Pi(I) = \Pi(\|I\|)\Pi(i_1, \dots, i_{m-1})$. Note that $\Pi(I) \equiv p^{|I|} \bmod p^{|I|+1}$. Given sequences I and J let IJ denote the sequence $(i_1, \dots, i_m, j_1, \dots, j_n)$. Then we have $|IJ| = |I| + |J|$ and $\|IJ\| = \|I\| + \|J\|$. We will need the following analog of Witt's lemma (4.3.5), which we will prove at the end of this section.

4.3.8. LEMMA.
(a) *For each sequence I as above there is a symmetric polynomial of degree $p^{\|I\|}$ in any number of variables with coefficients in $\mathbf{Z}_{(p)}$, $w_I = w_I(x_1, x_2, \dots)$ with $w_\phi = \sum_t x_t$ and*

$$\sum_t x_t^{p^{\|K\|}} = \sum_{IJ=K} \frac{\Pi(K)}{\Pi(I)} w_J^{p^{\|I\|}}.$$

(b) *Let w_i be the polynomial defined by 4.3.5. Then*

$$w_I \equiv w_{|I|}^{p^{\|I\|-|I|}} \mod (p). \qquad \square$$

Now let v_i be Araki's generator and define v_I by $v_\phi = 1$ and $v_I = v_{i_1}(v_{I'})^{(p^{i_1})}$ where $I' = (i_2, i_3 \dots)$. Hence dim $v_I = 2(p^{\|I\|} - 1)$. Then our analog of 4.3.7 is

4.3.9. THEOREM. *With notation as above,*

$$\sum_t^F x_t = \sum_I^F v_I w_I(x_1, x_2, \dots).$$

(*An analogous formula and proof in terms of Hazewinkel's generators can be obtained, by replacing $\Pi(I)$ by $p^{|I|}$ throughout. This requires a different definition of w_J, which is still congruent to $w_{|J|}^{p^{\|J\|-|J|}}$ modulo p.*)

PROOF. Araki's formula (A2.2.2) is

$$p\lambda_n = \sum_{0 \le i \le n} \lambda_i v_{n-i}^{p^i}$$

which can be written as

$$\Pi(n)\lambda_n = \sum_{0 \le i < n} \lambda_i v_{n-i}^{p^i}.$$

By a simple exercise this gives

$$\lambda_n = \sum_{\|I\|=n} \frac{v_I}{\Pi(I)},$$

i.e.,

$$\log(x) = \sum \frac{v_I x^{p^{\|I\|}}}{\Pi(I)}. \tag{4.3.10}$$

Therefore we have

$$\begin{aligned}
\log\left(\sum_J^F v_J w_J\right) &= \sum_J \log v_J w_J \\
&= \sum \frac{v_{IJ}}{\Pi(I)} w_J^{p^{\|I\|}} \\
&= \sum_{I,J} \frac{v_K}{\Pi(K)} \frac{\Pi(K)}{\Pi(I)} w_J^{p^{\|I\|}} \quad \text{(where } K = IJ\text{)} \\
&= \sum_{t,K} \frac{v_K}{\Pi(K)} x_t^{p^{\|K\|}} \quad \text{by 4.3.8} \\
&= \sum_t \log x_t \quad \text{by 4.3.10} \\
&= \log \sum_t^F x_t. \qquad \square
\end{aligned}$$

In the structure formulas for $BP_*(BP)$ we encounter expressions of the form $\sum_{n,i}^F a_{n,i}$, where $a_{n,i}$ is in $BP_*(BP)$ or $BP_*(BP) \otimes_{BP_*} BP_*(BP)$ (or more generally in some commutative graded BP_* algebra D) and has dimension $2(p^n - 1)$. We can use 4.3.9 to simplify such expressions in the following way.

Define subsets A_n and B_n of D as follows. $A_n = B_n = \phi$ for $n \leq 0$ and for $n > 0$, $A_n = \{a_{n,i}\}$ while B_n is defined recursively by

$$B_n = A_n \cup \bigcup_{|J|>0} \{v_J w_J(B_{n-\|J\|})\}.$$

4.3.11. Lemma. *With notation as above,* $\sum_{n,i}^F a_{n,i} = \sum_{n>0}^F w_\phi(B_n)$.

Proof. We will show by induction on m that the statement is true in dimensions $< 2(p^m - 1)$. Our inductive hypothesis is

$$\sum^F a_{n,i} = \sum_{0<n<m}^F w_\phi(B_n) +_F \sum_{\substack{n<m \\ \|J\|+n\geq m}}^F v_J w_J(B_n) +_F \sum_{n\geq m}^F a_{n,i},$$

which is trivial for $m = 1$. The set of formal summands of dimension $2(p^m - 1)$ on the right is B_m. By 4.3.9 the formal sum of these terms is $\sum^F v_J w_J(B_m)$, so we get

$$\begin{aligned}
\sum^F a_{n,i} &= \sum_{0<n<m}^F w_\phi(B_n) +_F \sum_J^F v_J w_J(B_m) +_F \sum_{\substack{n<m \\ \|J\|+n>m}}^F v_J w_J(B_n) +_F \sum_{n>m}^F a_{n,i} \\
&= \sum_{0<n\leq m}^F w_\phi(B_n) +_F \sum_{\substack{n\leq m \\ \|J\|+n>m}}^F v_J w_J(B_n) +_F \sum_{n>m}^F a_{n,i}.
\end{aligned}$$

which completes the inductive step and the proof. $\square$

Recall now the coproduct in $BP_*(BP)$ given by 4.1.18(e), i.e.,

$$\sum_{i\geq 0} \log(\Delta(t_i)) = \sum_{i,j\geq 0} \log(t_i \otimes t_j^{p^i}),$$

which can be rewritten as

$$\sum_{i\geq 0}^{F} \Delta(t_i) = \sum_{i,j\geq 0}^{F} t_i \otimes t_j^{p^i} \tag{4.3.12}$$

To apply 4.3.11, let $M_n = \{t_i \otimes t_{n-i}^{p^i} \mid 0 \leq i \leq n\}$ (M here stands for Milnor since these terms are essentially Milnor's coproduct 3.1.1) and let

$$\Delta_n = M_n \cup \bigcup_{|J|>0} \{v_J w_J(\Delta_{n-\|J\|})\}.$$

Then we get from 4.3.11 and 4.3.12

4.3.13. THEOREM. *With notation as above,*

$$\Delta(t_n) = w_\phi(\Delta_n) \in BP_*(BP) \otimes_{BP_*} BP_*(BP). \qquad \square$$

For future reference we make

4.3.14. DEFINITION. *In $BP_*(BP) \otimes_{BP_*} BP_*(BP)$ let $b_{i,j} = w_{(j+1)}(\Delta_i)$.* $\square$

For example,

$$b_{1,j} = -\frac{1}{p - p^{(p^{j+1})}} \sum_{0<i<p^{j+1}} \binom{p^{j+1}}{i} t_1^i \otimes t_1^{p^{j+1}-i}.$$

This $b_{i,j}$ can be regarded as an element of degree 2 in the cobar complex (A1.2.11) $C(BP_*)$. It will figure in subsequent calculations and we will give a formula for its coboundary (4.3.22) below.

If we reduce modulo I_n, 4.3.13 simplifies as follows.

4.3.15. COROLLARY. *In $BP_*(BP) \otimes_{BP_*} BP_*(BP)/I_n$ for $k \leq 2n$*

$$\Delta(t_k) = \sum_{0\leq i\leq k} t_i \otimes t_{k-i}^{p^i} + \sum_{0\leq j\leq k-n-1} v_{n+j} b_{k-n-j,n+j-1}. \qquad \square$$

Now we will simplify the right unit formula 4.3.1. First we need a lemma. Define t_I as wedefines v_I, replacing v_i by t_i.

4.3.16. LEMMA. *In $BP_*(BP)$, for each $k \geq 0$*

$$\sum_{i,|I|\geq 0}^{F} [(-1)^{|I|}] \left(t_I t_i^{p^{\|I\|}}\right)^{p^k} = \sum_{i,|I|\geq 0}^{F} [(-1)^{|I|}](t_i(t_I)^{p^i})^{p^k} = 1.$$

It is shown in A2.2.5 that for $p > 2$, $[-1](x) = -x$ for any p-typical formal group law, and a formulas for it are given for $p = 2$ in terms of both the Hazewinkel generators of (A2.2.1) and the Araki generators of (A2.2.2). $[n](x)$ is defined in A2.1.19.

PROOF. In the first expression, for each $I = (i_1, i_2, \ldots, i_n)$ with $n > 0$, the expression $(t_I)^{p^k}$ appears twice: once as $(t_I t_0)^{p^k}$ and once as $(t_{I'}(t_{i_n})^{p^{\|I'\|}})^{p^k}$ where $I' = (i_1, \ldots, i_{n-1})$. These two terms have opposite formal sign and hence cancel, leaving 1 as the value of the first expression. The argument for the second expression is similar. $\square$

Now we need to use the conjugate formal group law $c(F)$ over $BP_*(BP)$, defined by the homomorphism $\eta_R\colon BP_* \to BP_*(BP)$. Its logarithm is

$$\log_{c(F)}(x) = \sum_{i\geq 0} \eta_R(\lambda_i)x^{p^i} = \sum_{i,j\geq 0} \lambda_i t_j^{p^i} x^{p^{i+j}}.$$

An analog of 4.3.9 holds for $c(F)$ with v_I replaced by $\eta_R(v_I)$.

The last equation in the proof of A2.2.6 reads

$$\sum \lambda_i v_j^{p^i} t_k^{p^{i+j}} = \sum \lambda_i t_j^{p^i} \eta_R(v_k)^{p^{i+j}} = \sum \eta_R(\lambda_i)\eta_R(v_j)^{p^i}$$

while 4.3.16 gives

$$\sum \lambda_i = \sum (-1)^{|K|} \lambda_i t_j^{p^i} t_K^{p^{i+j}}.$$

Combining these and reindexing gives

$$\sum (-1)^{|J|} \eta_R(\lambda_i)(t_J(v_k t_l^{p^k})^{p^{\|J\|}})^{p^i} = \sum \eta_R(\lambda_i)\eta_R(v_j)^{p^i},$$

which is equivalent to

$$\sum_{i\geq 0}^{c(F)} \eta_R(v_i) = \sum_{|I|,j,k\geq 0}^{c(F)} [(-1)^{|I|}]_{c(F)}(t_I(v_j t_k^{p^j})^{p^{\|I\|}}). \tag{4.3.17}$$

In A2.2.5 we show that for $p > 2$, $[-1]_{c(F)}(x) = -x$, and for $p = 2$,

$$[-1]_{c(F)}(x) = x -_{c(F)} \sum_{i\geq 0}^{c(F)} \eta_R(v_i)x^{2^i}.$$

For $p > 2$ we define finite subsets of $BP_*(BP)$ for $n > 0$

$$N_n = \bigcup_{\|I\|+i+j=n} \left\{(-1)^{|I|} t^I (v_j t_k^{p^i})^{p^{\|I\|}}\right\}$$
$$R_n = N_n \cup \bigcup_{\substack{\|J\|=i\\ 0<i<n}} \{\eta_R(v_J)w_J(R_{n-i})\}.$$

Then we get

4.3.18. Theorem. *In $BP_*(BP)$ for $p > 2$, we have $\eta_R(v_n) = w_\phi(R_n)$.* ☐

For $p = 2$ we have to rewrite (4.3.17) as

$$\sum_{i\geq 0}^{c(F)} \eta_R(v_i) +_{c(F)} \sum_{\substack{|I|,j,k\geq 0\\ |I| \text{ odd}}}^{c(F)} t_I(v_j t_k^{p^j})^{p^{\|I\|}} = \sum_{\substack{|I|,j,k\geq 0\\ |I| \text{ even}}}^{c(F)} t_I(v_j t_k^{p^j})^{p^{\|I\|}}.$$

4.3.19. Definition. *In $BP_*(BP)$, $c_{i,J} = w_J(R_i)$. For $J = (j)$ this will be written as $c_{i,j}$.*

Again we can simplify further by reducing modulo I_n.

4.3.20. Corollary. *In $BP_*(BP)/I_n$ for $0 < k \leq 2n$,*

$$\sum_{0\leq i\leq k} v_{n+i} t_{k-i}^{p^{n+i}} - \eta_R(v_{n+k-i})^{p^i} t_i = \sum_{0\leq j\leq k-n-1} v_{n+j} c_{k-j,n+j}.$$

(Note that the right-hand side vanishes if $k \leq n$.) ☐

4.3.21. COROLLARY. *In $BP_*(BP)/I_n$,*

$$\begin{aligned}
\eta_R(v_{n+1}) &= v_{n+1} + v_n t_1^{p^n} - v_n^p t_1 \\
&\qquad \text{for } n \geq 1; \\
\eta_R(v_{n+2}) &= v_{n+2} + v_{n+1} t_1^{p^{n+1}} + v_n t_2^{p^n} - v_{n+1}^p t_1 - v_n^{p^2} t_2 \\
&\quad + v_n^{p^2} t_1^{1+p} - v_n^p t_1^{1+p^{n+1}} \\
&\qquad \text{for } n \geq 2; \\
\eta_R(v_{n+3}) &= v_{n+3} + v_{n+2} t_1^{p^{n+2}} + v_{n+1} t_2^{p^{n+1}} + v_n t_3^{p^n} - v_{n+2}^p t_1 \\
&\quad - v_{n+1}^{p^2} t_2 - v_n^{p^3} t_3 - v_{n+1}^p t_1^{1+p^{n+2}} - v_n^p t_1 t_2^{p^{n+1}} \\
&\quad - v_n^{p^2} t_2 t_1^{p^{n+2}} + v_{n+1}^{p^2} t_1^{1+p} + v_n^{p^3} t_1 t_2^p + v_n^{p^3} t_2 t_1^{p^2} \\
&\quad + v_n^{p^2} t_1^{1+p+p^{n+2}} - v_n^{p^3} t_1^{1+p+p^2} \\
&\qquad \text{for } n \geq 3; \\
\eta_R(v_3) &= v_3 + v_2 t_1^{p^2} + v_1 t_2^p - v_2^p t_1 - v_1^{p^2} t_2 - v_1^p t_1^{1+p^2} \\
&\quad + v_1^{p^2} t_1^{1+p} + v_1 w_1(v_2, v_1 t_1^p, -v_1^p t_1) \\
&\qquad \text{for } n = 1,\ p > 2\ (\text{add } v_1^5 t_1^2 \text{ for } p = 2) \\
\text{and} \qquad \eta_R(v_5) &= v_5 + v_4 t_1^{p^4} + v_3 t_2^{p^3} + v_2 t_3^{p^2} - v_4^p t_1 - v_3^{p^2} t_2 - v_2^{p^3} t_3 \\
&\quad - v_3^p t_1^{1+p^4} - v_2^p t_1 t_2^{p^3} - v_2^{p^2} t_1^{p^4} t_2 + v_3^{p^2} t_1^{1+p} + v_2^{p^3} t_1 t_2^p \\
&\quad + v_2^{p^3} t_1^{p^2} t_2 + v_2^{p^2} t_1^{1+p+p^4} - v_2^{p^3} t_1^{1+p+p^2} \\
&\quad + v^2 w_1(v_3, v_2 t_1^{p^2}, -v_2^p t_1^p)^p \\
&\qquad \text{for } n = 2,\ p > 2\ (\text{add } v_2^9 t_1^4 \text{ for } p = 2). \qquad \square
\end{aligned}$$

Now we will calculate the coboundaries of $b_{i,j}$ (4.3.14) and $c_{i,j}$ (4.3.19) in the cobar complex $C(BP_*/I_n)$ (A1.2.11).

4.3.22. THEOREM. *In $C(BP_*/I_n)$ for $0 < i \leq n$ and $0 \leq j$*
(a) $d(b_{i,j}) = \sum_{0<k<i} b_{k,j} \otimes t_{i-k}^{p^{k+j+1}} - t_k^{p^{1+j}} \otimes b_{i-k,k+j}$ *and* (b) $d(c_{n+i,j+1}) = \sum_{0\leq k<i} v_{n+k}^{p^{1+j}} b_{i-k,n+k+j} - v_{n+k}^{p^{i+1+j-k}} b_{i-k,j}$.

PROOF. (a) It suffices to assume $i = n$. Recall that in $C(BP_*/I_n)$, $d(t_i) = t_i \otimes 1 + 1 \otimes t_i - \Delta(t_i)$ and $d(v_{n+i}) = n_R(v_{n+i}) - v_{n+i}$. $\Delta(t_{2n}) - 1 \otimes t_{2n} - t_{2n} \otimes 1$, given by 4.3.13, is a coboundary and hence a cocycle. Calculating its coboundary term by term using 4.3.13 and 4.3.17 will give the desired formula for $d(b_{n,n-1})$ and the result will follow. The details are straightforward and left to the reader.

For (b) we assume $i = n$ if $i + n$ is even and $i = n - 1$ if $i + n$ is odd. Then we use the fact that $d(v_{2n+i})$ is a cocycle to get the desired formula, as in the proof of (a). $\square$

Now we will construct an increasing filtration on the Hopf algebroid $BP_*(BP)/I_n$. We will use it in Section 6.3.

To do this we first define integers $d_{n,i}$ by

$$d_{n,i} = \begin{cases} 0 & \text{for } i \leq 0 \\ \max(i, p d_{n,i-n}) & \text{for } i > 0. \end{cases}$$

We then set $\deg t_i^{p^j} = \deg v_{n+i}^{p^j} = d_{n,i}$ for $i,\ j \geq 0$. The subgroups $F_r \subset BP_*(BP)/I_n$ are defined to be the smallest possible subgroups satisfying the above conditions.

The associated graded algebra $E_0BP_*(BP)/I_n$ is defined by $E_0^iBP_*(BP)/I_n = F_i/F_{i-1}$. Its structure is given by

4.3.23. PROPOSITION.

$$E_0BP_*(BP)/I_n = T(t_{i,j}, v_{n+i,j} \colon i > 0,\ j \geq 0),$$

where $t_{i,j}$ and $v_{n+1,j}$ are elements corresponding to $t_i^{p^j}$ and $v_{n+1}^{p^j}$, respectively, $T(x) = R[x]/(x^p)$ and $R = \mathbf{Z}(p)[v_n]$. □

4.3.24. THEOREM. *With the above filtration, $BP_*(BP)/I_n$ is a filtered Hopf algebroid, and $E_0BP_*(BP)/I_n$ is a Hopf algebroid.*

PROOF. For a set of elements X in $B_*(BP)/I_n$ or $BP_*(BP)\otimes_{BP_*}BP_*(BP)/I_n$, let $\deg X$ be the smallest integer i such that $X \subset F_i$. It suffices to show then that $\deg \Delta_i = \deg R_{n+i} = d_{n,i}$. We do this by induction on i, the assertion being obvious for $i = 1$.

First note that

$$d_{n,a+b} \geq d_{n,a} + d_{n,b} \tag{4.3.25}$$

and

$$d_{n,a+bn} \geq p^b d_{n,a}. \tag{4.3.26}$$

It follows from 4.3.25 that $\deg M_i = degN_{n+i} = d_{n,i}$. It remains then to show that for $\|J\| < i$

$$\deg(v_J w_J(\Delta_{i-\|J\|})) \leq d_{n,i} \tag{4.3.27}$$

and

$$\deg(v_J w_J(R_{n+i-\|J\|})) \leq d_{n,i}. \tag{4.3.28}$$

Since

$$\deg w_J(X) \leq p^{|J|} \deg X, \tag{4.3.29}$$

both 4.3.27 and 4.3.28 reduce to showing

$$d_{n,i} \geq \deg v_J + p^{|J|} d_{n,i-\|J\|}. \tag{4.3.30}$$

Now if $v_J \not\equiv 0 \bmod I_n$ we can write

$$J = (n + j_1', m + j_2', \ldots, n + j_l')$$

with $j_t' \geq 0$, so

$$|J| = l, \quad \|J\| = \ln + \sum_{t=1}^{l} j_t', \quad \text{and} \quad \deg v_J = \sum_{t=1}^{l} d_{n,j_t'}.$$

If we set $k = \|J\| - n|J|$, then 4.3.25 implies

$$d_{n,k} \geq \deg v_J. \tag{4.3.31}$$

However, by 4.3.25 and 4.3.26

$$\begin{aligned} d_{n,i} &\geq d_{n,k} + d_{n,i-\|J\|+n|J|} \\ &\geq d_{n,k} + p^{|J|} d_{n,i-\|J\|} \end{aligned}$$

so 4.3.20 follows from 4.3.31. □

We now turn to the Hopf algebroid structure of $E_0BP_*(BP)/I_n$. Let $\overline{M}_i$, $\bar{\Delta}_i$, $\overline{N}_{n+i}$, and $\overline{R}_{n+i}$ denote the associated graded analogs of M_i, Δ_i, N_{n+i}, and R_{n+i}, respectively, with trivial elements deleted. (An element in one of the latter sets will correspond to a trivial element if its degree is less than $d_{n,i}$.) All we have to do is describe these subsets. Let $\bar{t}_I$, $\bar{v}_I$, and $\overline{w}_I(x)$ denote the associated graded elements corresponding to t_I, v_I, and $w_I(x)$, respectively.

4.3.32. LEMMA.

$$\overline{M}_i = \begin{cases} \bigcup_{0\le j\le i} \{t_{j,0}\otimes t_{i-j,j}\} & \text{for } i \le m \\ \{t_{i,0}\otimes 1, 1\otimes t_{i,0}\} & \text{for } i > m \end{cases}$$

$$\overline{N}_{n+i} = \begin{cases} \bigcup_{\|I\|+j+k=i} \{(-1)^{|I|}\bar{t}_I v_{j,\|I\|} t_{k,j+\|I\|}\} & \text{for } i \le m \\ \{v_{n+i,0}, v_n t_{i,n}, -v_n^{p^i} t_{i,0}\} & \text{for } i > m \end{cases}$$

where $m = pn/(p-1)$.

PROOF. This follows from the fact that equality holds in 4.3.25 if $a+b \le m$. $\square$

4.3.33. LEMMA.

$$\bar{\Delta}_i = \begin{cases} \overline{M}_i & \text{for } i < m \\ \overline{M}_i \cup \{v_n \overline{w_1^{p^{n-1}}(M_{i-n})}\} & \text{for } i = m \\ \overline{M}_i \cup \bigcup_{\substack{\|J\|=n|J| \\ 0<\|J\|<i \\ i-\|J\|\ge m-n}} \{v_J \overline{w_J(\Delta_{i-n|J|})}\} & \text{for } i > m \end{cases}$$

$$\overline{R}_{n+i} = \begin{cases} \overline{N}_{n+i} & \text{for } i < m \\ \overline{N}_{n+i} \cup \{v_n \overline{w_1^{p^{n-1}}(R_{i-n})}\} & \text{for } i = m \\ \overline{N}_{n+i} \cup \bigcup_{\substack{\|J\|=n|J| \\ 0<\|J\|<i \\ i-\|J\|\ge m-n}} \overline{\{v_J w_J(R_{i-n|J|})\}} & \text{for } i > m \end{cases}$$

[*Note that the case* $i = m$ *occurs only if* $(p-1)|n$, *and that the only* J*'s we need to consider for* $i > m$ *are those of the form* $(n, n, \ldots, n)$.]

PROOF. We use the observation made in the proof of 4.3.32 along with the fact that equality holds in 4.3.26 if $a \ge m = n$.

Now both $\overline{R}_{n+i}$, and $\bar{\Delta}$, will consist only of the terms associated with those J for which equality holds in 4.3.30. For $i > m$ this can occur only if $\deg v_J = 0$, i.e., if $J = (n, n, \ldots, n)$; the condition $i - \|J\| \ge m - n$ is necessary to ensure that $d_{n,i} = p^{|J|} d_{n,i-n|J|}$. For $i \le m$ we still need $i - \|J\| \ge m - n$. Since $\|J\| \ge n$ in all nontrivial terms, the only possibility is $J = (n)$ when $i = m$. $\square$

Now let $\Delta_{i,j}$ and $R_{n+i,j}$ be the subsets obtained from Δ_i and R_{n+i}, respectively, by raising each element to the p^jth power. The corresponding subsets $\bar{\Delta}_{i,j}$ and $\overline{R}_{n+i,j}$ of the appropriate associated graded objects are related to $\bar{\Delta}_i$ and $\overline{R}_{n+i}$ in an obvious way. Note that

$$\begin{aligned} \overline{w_J(\Delta_i)} &= \overline{w_{|J|}(\Delta_{i,\|J\|-|J|})} \\ &= w_{|J|}(\bar{\Delta}_{i,\|J\|-|J|}). \end{aligned}$$

4.3.34. THEOREM. *With $\bar{\Delta}_{i,j}$ and $\overline{R}_{n+i,j}$ as above, the Hopf algebroid structure of $E_0BP_*(BP)/I_n$, is given by*

$$\Delta(t_{i,j}) = w_0(\bar{\Delta}_{i,j})$$
$$\eta_R(v_{n+i,j}) = w_0(\overline{R}_{n+i,j}). \qquad \square$$

None of the $t_{i,j}$ for $i > 1$ are primitive, so we could not get a Hopf algebroid with deg $t_{i,j} < d_{n,i}$ once we have set deg $t_{1,j} = 1$.

Note finally that the structure of $E_0BP_*(BP)/I_n$ depends in a very essential way on the prime p.

Theorem 4.3.34 implies that $E_0BP_*(BP)/I_n$ is cocommutative for $n = 1$ and $p > 2$. For any n and p we can use this filtration to construct a spectral sequence as in A1.3.9. The cocommutativity in the case above permits a complete, explicit determination of the E_2-term, and hence a very promising beginning for a computation of $\mathrm{Ext}_{BP_*(BP)}(BP_*, BP/I_1)$. However, after investigating this method thoroughly we found the E_2-term to be inconveniently large and devised more efficient strategies for computing Ext, which will be described in Chapter 7. Conceivably the approach at hand could be more useful if one used a machine to do the bookkeeping. We leave the details to the interested reader.

PROOF OF 4.3.8. We will prove (a) and (b) simultaneously by induction on $m = |K|$. If $K' = (1 + k_1, k_2, \ldots, k_m)$ then it follows from (b) that

$$w_{K'} \equiv w_K^p \qquad \mod (p).$$

Let $K'' = (k_1 + k_2, k_3, \ldots, k_m)$ and $K''' = (k_2, k_3, \ldots, k_m)$. Then by the inductive hypothesis $w_{K''}$ and $w_{K'''}$ exist with

$$w_{K''} \equiv w_{K'''}^{a_1} \qquad \mod (p),$$

where $a_1 = p^{k_1}$. Since $\|K\| = \|K''\|$ we have

$$\sum_{IJ=K} \frac{\Pi(K)}{\Pi(I)} w_J^{p^{\|I\|}} = \sum_{IJ=K''} \frac{\Pi(K'')}{\Pi(I)} w_J^{p^{\|I\|}}.$$

Expanding both sides partly we get

$$\begin{aligned} \Pi(K)w_K + \frac{\Pi(K)}{\Pi(k_1)} w_{K'''}^{a_1} + \sum_{\substack{|I|\geq 2\\ IJ=K}} \frac{\Pi(K)}{\Pi(I)} w_J^{p^{\|I\|}} \\ = \Pi(K'')w_{K''} + \sum_{\substack{|I|\geq 1\\ IJ=K''}} \frac{\Pi(K'')}{\Pi(I)} w_J^{p^{\|I\|}}. \end{aligned}$$

Note that the same $w_J^{p^{\|I\|}}$ occur on both sides, and one can use the definition of $\Pi(k)$ to show that they have the same coefficients so the sums cancel. The remaining terms give

$$\Pi(K)\left(w_K + \frac{w_{K'''}^{a_1}}{\Pi(k_1)}\right) = \Pi(K'')w_{K''}.$$

Since $\Pi(k_1) \equiv p \mod (p^2)$ and $w_{K''} = w_{k'''}^{a_1} \mod (p)$, we get an integral expression when we solve for w_K.

This completes the proof of (a).

For (b) we have

$$\sum x_t^{p^{\|K\|}} = \Pi(K)w_K + \sum_{\substack{IJ=K\\|I|>0}} \frac{\Pi(K)}{\Pi(I)} w_J^{p^{\|I\|}}.$$

Since $\Pi(K) \equiv p^{|K|} \mod (p^{1+|K|})$ and $\Pi(I) \equiv p^{|I|} \mod (p^{1+|I|})$, we get $\Pi(K)/\Pi(I) \equiv p^{|J|} \mod (p^{1+|J|})$. By definition

$$\begin{aligned}
\frac{\Pi(K)}{\Pi(I)} &= \Pi(\|K\|)\Pi(\|K\| - j_{|J|}) \cdots \Pi(\|I\| + j_1)\\
&= (p - p^{p^{\|K\|}})(p - p^{p^{\|K\|-j_{|J|}}}) \cdots (p - p^{p^{\|I\|+j_1}})\\
&\equiv p^{|J|} \qquad \mod (p^{|J|-1+p^{\|I\|+j_1}})\\
&\equiv p^{|J|} \qquad \mod (p^{|K|+1}) \quad \text{since}\\
& \qquad |J| - 1 + p^{\|I\|+j_1} \geq |J| - 1 + \|I\| + 2\\
& \qquad\qquad \geq |K| + 1.
\end{aligned}$$

By the inductive hypothesis

$$w_J \equiv w_{|J|}^{p^{\|J\|-|J|}} \mod (p)$$

so $w_J^{p^{\|I\|}} \equiv w_{|J|}^{p^{\|K\|-|J|}} \mod (p^{1+\|I\|})$. Combining these two statements gives

$$\frac{\Pi(K)}{\Pi(I)} w_J^{p^{\|I\|}} \equiv w_{|J|}^{\|K\|-|J|} \mod (p^{1+|K|}).$$

Hence the defining equation for w_K becomes

$$\sum x_t^{p^{\|K\|}} \equiv p^{|K|} w_K + \sum_{\substack{IJ=K\\|I|>0}} p^{|J|} w_{|J|}^{p^{\|K\|-|J|}} \mod (p^{1+|K|}).$$

Let $n = \|K\| - |K|$. Substituting $x_t^{p^n}$ for x_t in 4.3.5 gives

$$\sum x_t^{p^{\|K\|}} = p^{\|K\|} w_{|K|}(x_t^{p^n}) + \sum_{0 \leq j < |K|} p^j w_j^{p^{|K|-j}}(x_t^{p^n}).$$

Since $w_j(x_t^{p^n}) \equiv w_j^{p^n} \mod (p)$,

$$w_j^{p^{|K|-j}}(x_t^{p^n}) \equiv w_j^{p^{n+|K|-j}} \mod (p^{1+|K|-j}),$$

so we get

$$\sum x_t^{p^{\|K\|}} \equiv p^{|K|} w_{|K|}(x_t^{p^n}) + \sum_{0 \leq j < |K|} p^j w_j^{p^{\|K\|-j}} \mod (p^{1+|K|}).$$

Comparing this with the defining equation above gives

$$w_K \equiv w_{|K|}(x_t^{p^n}) \equiv w_{|K|}^{p^{\|K\|-|K|}} \mod (p)$$

as claimed. □

4. Beginning Calculations with the Adams–Novikov Spectral Sequence

In this section we introduce the main object of interest in this book, the Adams–Novikov spectral sequence, i.e., the BP_*-Adams spectral sequence (2.2.4). There is a different BP_*-theory and hence a different Adams–Novikov spectral sequence for each prime p. One could consider the MU_*-Adams spectral sequence (as Novikov [**1**] did originally) and capture all primes at once, but there is no apparent advantage in doing so. Stable homotopy theory is a very local (in the arithmetic sense) subject. Even though the structure formulas for $BP_*(BP)$ are more complicated than those of $MU_*(MU)$ (both are given in Section 1) the former are easier to work with once one gets used to them. (Admittedly this adjustment has been difficult. We hope this book, in particular the results of Section 3, will make it easier.)

The Adams–Novikov spectral sequence was first constructed by Novikov [**1**] and the first systematic calculations at the primes 2 and 3 were done by Zahler [**1**]. In this section we will calculate the E_2-term for $t-s\le 25$ at $p=2$ and for $t-s\le(p^2+p)q$ for $p>2$, where $q=2p-2$. In each case we will compute all the differentials and extensions and thereby find $\pi(S^0)$ through the indicated range. At $p=2$ this will be done by purely algebraic methods based on a comparison of the Adams–Novikov spectral sequence and Adams spectral sequence E_2-terms. At odd primes we will see that the Adams spectral sequence E_2-term sheds no light on the Adams–Novikov spectral sequence and one must compute differentials by other means. Fortunately, there is only one differential in this range and it is given by Toda [**2, 3**]. The more extensive calculations of later chapters will show that in a much larger range all nontrivial differentials follow formally from the first one.

In Section 2.2 we developed the machinery necessary to set up the Adams–Novikov spectral sequence and we have

4.4.1. Adams–Novikov spectral sequence Theorem (Novikov [**1**]). *For any spectrum X there is a natural spectral sequence $E_*^{**}(X)$ with $d_r\colon E_r^{s,t}\to E_r^{s+r,t+r-1}$ such that*

(a) $E_2=\operatorname{Ext}_{BP_*(BP)}(BP_*,BP_*(X))$ *and*

(b) *if X is connective and p-local then E_∞^{**} is the bigraded group associated with the following filtration of $\pi_*(X)$: a map $f\colon S^n\to X$ has filtration $\ge s$ if it can be factored with s maps each of which becomes trivial after smashing the target with BP.* □

The fact that $BP_*(BP)$, unlike the Steenrod algebra, is concentrated in dimensions divisible by $q=2p-2$ has the following consequence.

4.4.2. Proposition: Sparseness. Suppose $BP_*(X)$ is concentrated in dimensions divisible by $q=2p-2$ (e.g., $X=S^0$). Then in the Adams–Novikov spectral sequence for X, $E_r^{s,t}=0$ for all r and s except when t is divisible by q. Consequently d_r is nontrivial only if $r\equiv 1 \mod (q)$ and $E_{mq+2}^{**}=E_{mq+q+1}^{**}$ for all $m\ge 0$. □

For $p=2$ this leads to the "checkerboard phenomenon": $E_r^{s,t}=0$ if $t-s$ and s do not have the same parity.

To compare the Adams spectral sequence and Adams–Novikov spectral sequence we will construct two trigraded spectral sequences converging to the Adams spectral sequence and Adams–Novikov spectral sequence E_2-terms. The former is a Cartan–Eilenberg spectral sequence (A1.3.15) for a certain Hopf algebra extension involving the Steenrod algebra, while the latter arises from a filtration of $BP_*(BP)$ (A1.3.9). The point is that up to reindexing these two spectral sequences have the same E_2-term. Moreover, at odd primes (but not at $p = 2$) the former spectral sequence collapses, which means that the Adams spectral sequence E_2-term when suitably reindexed is a trigraded E_2-term of a spectral sequence converging to the Adams–Novikov spectral sequence E_2-term. It is reasonable to expect there to be a close relation between differentials in the trigraded filtration spectral sequence, which Miller [**2**] calls the "algebraic Novikov spectral sequence," and the differentials in the Adams spectral sequence. Miller [**4**] has shown that many Adams d_2's can be accounted for in this way. At any rate this indicates that at odd primes the Adams spectral sequence E_2-term has less information than the Adams–Novikov spectral sequence E_2-term.

To be more specific, recall (3.1.1) that the dual Steenrod algebra A_* as an algebra is

$$A_* = \begin{cases} P(\xi_1, \xi_2, \dots) & \text{with} \quad \dim \xi_i = 2^i - 1 \text{ for } p = 2 \\ E(\tau_0, \tau_1, \dots) \otimes P(\xi_1, \xi_2, \dots) & \text{with} \quad \dim \tau_i = 2p^i - 1 \quad \text{and} \\ & \qquad \dim \xi_i = 2p^i - 2 \quad \text{for } p > 2. \end{cases}$$

Let $P_* \subset A_*$ be $P(\xi_1^2, \xi_2^2, \dots)$ for $p = 2$ and $P(\xi_1, \xi_2, \dots)$ for $p > 2$, and let $E_* = A_* \otimes_{P_*} \mathbf{Z}/(p)$, i.e. $E_* = E(\xi_1, \xi_2, \dots)$ for $p = 2$ and $E_* = E(\tau_0, \tau_1, \dots)$ for $p > 2$. Then we have

4.4.3. THEOREM. *With notation as above* (a)

$$\operatorname{Ext}_{E_*}(\mathbf{Z}/(p), \mathbf{Z}/(p)) = P(a_0, a_1, \dots)$$

with $a_i \in \operatorname{Ext}^{1,2p^i-1}$ *represented in the cobar complex* (A1.2.11) *by* $[\xi_i]$ *for* $p = 2$ *and* $[\tau_i]$ *for* $p > 2$,

(b) $P_* \to A_* \to E_*$ *is an extension of Hopf algebras* (A1.1.15) *and there is a Cartan–Eilenberg spectral sequence* (A1.3.15) *converging to* $\operatorname{Ext}_{A_*}(\mathbf{Z}/(p), \mathbf{Z}/(p))$ *with*

$$E_2^{s_1,s_2,t} = \operatorname{Ext}_{P_*}^{s_1}(\mathbf{Z}/(p), \operatorname{Ext}_{E_*}^{s_2,t}(\mathbf{Z}/(p), \mathbf{Z}(p)))$$

and

$$d_r\colon E_r^{s_1,s_2,t} \to E_r^{s_1+r,s_2-r+1,t},$$

(c) *the* P_**-coaction on* $\operatorname{Ext}_{E_*}(\mathbf{Z}/(p), \mathbf{Z}/(p))$ *is given by*

$$\psi(a_n) = \begin{cases} \sum_i \xi_{n-i}^{2^{i+1}} \otimes a_i & \text{for } p = 2 \\ \sum_i \xi_{n-i}^{p^i} \otimes a_i & \text{for } p > 2, \quad \text{and} \end{cases}$$

(*d*) *for* $p > 2$ *the Cartan–Eilenberg spectral sequence collapses from* E_2 *with no nontrivial extensions.*

PROOF. Everything is straightforward but (d). We can give A_* a second grading based on the number of τ_i's which are preserved by both the product and the coproduct (they do not preserve it at $p = 2$). This translates to a grading of Ext by the number of a_i's which must be respected by the differentials, so the spectral sequence collapses. □

For the algebraic Novikov spectral sequence, let $I = (p, v_1, v_2, \dots) \subset BP_*$. We filter $BP_*(BP)$ by powers of I and study the resulting spectral sequence (A1.3.9).

4.4.4. ALGEBRAIC NOVIKOV SS THEOREM (Novikov [**1**], Miller [**2**]). There is a spectral sequence converging to $\mathrm{Ext}_{BP_*(BP)}(BP_*, BP_*)$ with

$$E_1^{s,m,t} = \mathrm{Ext}_{P_*}^{s,t}(\mathbf{Z}/(p), I^m/I^{m+1})$$

and $d_r\colon E_r^{s,m,t} \to E_r^{s+1,r+m,t}$. The E_1^{***} of this spectral sequence coincides with the E_2^{***} of 4.4.3.

PROOF. A1.3.9 gives a spectral sequence with

$$E_1 = \mathrm{Ext}_{E_0BP_*(BP)}(E_0BP_*, E_0BP_*).$$

Now we have $BP_*(BP)/I = E_0BP_*(BP)\otimes_{E_0BP_*}\mathbf{Z}/(p) = P_*$. We apply the change-of-rings isomorphism A1.3.12 to the Hopf algebroid map $(E_0BP_*, E_0BP_*(BP)) \to (\mathbf{Z}/(p), P_*)$ and get

$$\begin{aligned}
&\mathrm{Ext}_{P_*}(\mathbf{Z}/(p), E_0BP_*)\\
&\qquad = \mathrm{Ext}_{E_0BP_*(BP)}(E_0BP_*, (E_0BP_*(BP)\otimes_{E_0BP_*}\mathbf{Z}/(p))\,\square_{P_*}\, E_0BP_*)\\
&\qquad = \mathrm{Ext}_{E_0BP_*(BP)}(E_0BP_*, P_*\,\square_{P_*}\, E_0BP_*)\\
&\qquad = \mathrm{Ext}_{E_0BP_*(BP)}(E_0BP_*, E_0BP_*).
\end{aligned}$$

The second statement follows from the fact that $E_0BP_* = \mathrm{Ext}_{E_*}(\mathbf{Z}/(p), \mathbf{Z}/(p))$. □

In order to use this spectral sequence we need to know its E_1-term. For $p > 2$, 4.4.3(d) implies that it is the cohomology of the Steenrod algebra, i.e., the classical Adams E_2-term suitably reindexed. This has been calculated in various ranges by May [**1**], and Liulevicius [**2**], but we will compute it here from scratch. Theorem 4.4.3(d) fails for $p = 2$ so we need another method, outlined in Miller [**2**] and used extensively by Aubry [**1**].

We start with $\mathrm{Ext}_{P_*}(\mathbf{Z}/(p), \mathbf{Z}/(p))$. For $p = 2$ we have $\mathrm{Ext}_{A_*}^{s,t}(\mathbf{Z}/(2), \mathbf{Z}/(2)) = \mathrm{Ext}_{P_*}^{s,2t}(\mathbf{Z}/(2), \mathbf{Z}(2))$, so the latter is known if we know the former through half the range of dimensions being considered. For $p > 2$ we will make the necessary calculation below.

Then we compute $\mathrm{Ext}_{P_*}(\mathbf{Z}/(p), E_0BP_*/I_n)$, by downward induction on n. To start the induction, observe that through any given finite range of dimensions $BP_*/I_n \simeq \mathbf{Z}/(p)$ for large enough n. For the inductive step we use the short exact sequence

$$0 \to \Sigma^{\dim v_n} E_0BP_*/I_n \to E_0BP_*/I_n \to E_0BP_*/I_{n+1} \to 0,$$

which leads to a Bockstein spectral sequence of the form

$$P(a_n)\otimes \mathrm{Ext}_{P_*}(\mathbf{Z}/(p), E_0BP_*/I_{n+1}) \Rightarrow \mathrm{Ext}_{P_*}(\mathbf{Z}/(p), E_0BP_*/I_n). \tag{4.4.5}$$

The method we will use in this section differs only slightly from the above. We will compute the groups $\mathrm{Ext}_{BP_*(BP)}(BP_*, BP_*/I_n)$ by downward induction on n; these will be abbreviated by $\mathrm{Ext}(BP_*/I_n)$. To start the induction we note that $\mathrm{Ext}^{s,t}(BP_*/I_n) = \mathrm{Ext}_{P_*}^{s,t}(\mathbf{Z}/(p), \mathbf{Z}/(p))$ for $t < 2(p^n - 1)$. For the inductive step we analyze the long exact sequence of Ext groups induced by the short exact sequence

$$0 \to \Sigma^{\dim v_n} BP_*/I_n \to BP_*/I_n \to BP_*/I_{n+1} \to 0, \tag{4.4.6}$$

either directly or via a Bockstein spectral sequence similar to 4.4.5. The long exact sequence and Bockstein spectral sequence are related as follows. The connecting homomorphism in the former has the form

$$\delta_n\colon\ \mathrm{Ext}^s(BP_*/I_{n+1}) \to \mathrm{Ext}^{s+1}(\Sigma^{2p^n-2}BP_*/I_n).$$

The target is a module over $\mathrm{Ext}^0(BP_*/I_n)$ which is $\mathbf{Z}/(p)[v_n]$ for $n > 0$ and $\mathbf{Z}_{(p)}$ for $n = 0$ by 4.3.2. Assume for simplicity that $n > 0$. For each $x \in \mathrm{Ext}(BP_*/I_{n+1})$ there is a maximal k such that $\delta_n(x) = v_n^k y$, i.e., such that $y \in \mathrm{Ext}(BP_*/I_n)$ is not divisible by v_n. (This y is not unique but is only determined modulo elements annihilated by v_n^k.) Let $\bar{y} \in \mathrm{Ext}(BP_*/I_{n+1})$ denote the image of y under the reduction map $BP_*/I_n \to BP_*/I_{n+1}$ Then in the Bockstein spectral sequence there is a differential $d_{1+k}(x) = a_n^{1+k}\bar{y}$.

Now we will start the process by computing $\mathrm{Ext}_{P_*}^{s,t}(\mathbf{Z}/(p), \mathbf{Z}/(p))$ for $p > 2$ and $t < (p^2 + p + 1)q$. In this range we have $P_* = P(\xi_1, \xi_2)$. We will apply the Cartan–Eilenberg spectral sequence (A1.3.15) to the Hopf algebra extension

$$P(\xi_1) \to P(\xi_1, \xi_2) \to P(\xi_2). \tag{4.4.7}$$

The E_2-term is $\mathrm{Ext}_{P(\xi_1)}(\mathbf{Z}/(p), \mathrm{Ext}_{P(\xi_2)}(\mathbf{Z}/(p), \mathbf{Z}/(p)))$. The extension is cocentral (A1.1.15) so we have

$$E_2 = \mathrm{Ext}_{P(\xi_1)}(\mathbf{Z}/(p), \mathbf{Z}/(p)) \otimes \mathrm{Ext}_{P(\xi_2)}(\mathbf{Z}/(p), \mathbf{Z}(p)).$$

By a routine calculation this is in our range of dimensions

$$E(h_{10}, h_{11}, h_{12}, h_{20}, h_{21}) \otimes P(b_{10}, b_{11}, b_{20})$$

with

$$h_{i,j} \in \mathrm{Ext}_{P(\xi_i)}^{1,2p^j(p^i-1)} \quad \text{and} \quad b_{i,j} \in \mathrm{Ext}_{P(\xi_i)}^{2,2p^{j+1}(p^i-1)}.$$

The differentials are (up to sign) $d_2(h_{2,j}) = h_{1,j}h_{1,j+1}$ and $d_3(b_{20}) = h_{12}b_{10} - h_{11}b_{11}$ [compare 4.3.22(a)]. The result is

4.4.8. THEOREM. *For $p > 2$ and $t < (p^2+p+1)q$, $\mathrm{Ext}_{P_*}^{s,t}(\mathbf{Z}/(p), \mathbf{Z}/(p))$ is a free module over $P(b_{10})$ on the following 10 generators: 1, h_{10}, h_{11}, $g_0 = \langle h_{11}, h_{10}, h_{10}\rangle$, $k_0 = \langle h_{11}, h_{11}, h_{10}\rangle$, $h_{10}k_0 = \pm h_{11}g_0$, h_{12}, $h_{10}h_{12}$, b_{11}, and $h_{10}b_{11}$. There is a multiplicative relation $h_{11}b_{11} = h_{12}b_{10}$ and (for $p = 3$) $h_{11}k_0 = \pm h_{10}b_{11}$.* □

The extra relation for $p = 3$ follows easily from A1.4.6. For $p > 3$ there is a corresponding Massey product relation $\langle k_0, h_{11}, \ldots, h_{11}\rangle = h_{10}b_{11}$ up to a nonzero scalar, where there are $p - 2$ factors h_{11}.

The alert reader may observe that the restriction $t < (p^2 + p + 1)q$ is too severe to give us $\mathrm{Ext}^{s,t}$ for $t - s < (p^2 + p)q$ because there are elements in this range with $s > q$, e.g., b_{10}^p. However, one sees easily that in a larger range all elements with $s > q$ are divisible by b_{10} and this division gets us back into the range $t < (p^2 + p + 1)q$. One could make this more precise, derive some vanishing lines, and prove the following result.

4.4.9. THEOREM. *Let $p > 2$.*
(a) *$\mathrm{Ext}_{P_*}^{s,t}(\mathbf{Z}/(p), \mathbf{Z}/(p)) = 0$ for $t - s < f(s)$ where*

$$f(s) = \begin{cases} (p^2 - p - 1)s & \text{for } s \text{ even} \\ 2p - 3 + (p^2 - p - 1)(s - 1) & \text{for } s \text{ odd.} \end{cases}$$

(b) *Let $R_* = P_*/(\xi_1, \xi_2)$. Then $\mathrm{Ext}_{R_*}^{s,t}(\mathbf{Z}/(p), \mathbf{Z}(p)) = 0$ for $t - s < g(s)$ where*

$$g(s) = \begin{cases} (p^4 - p - 1)s & \text{for } s \text{ even} \\ 2p^3 - 3 + (p^4 - p - 1)(s - 1) & \text{for } s \text{ odd.} \end{cases}$$

(c) *The map* $P(\xi_1, \xi_2) \to P_*$ *induces an epimorphism in* $\mathrm{Ext}^{s,t}$ *for* $(t-s) < h(s)$ *and an isomorphism for* $(t-s) < h(s-1) - 1$*, where*

$$\begin{aligned} h(s) &= 2p^3 - 3 + f(s-1) \\ &= \begin{cases} 2p^3 - 3 + (p^2-p-1)(s-1) & \text{for } s \text{ odd} \\ 2p^3 + 2p - 6 + (p^2-p-1)(s-2) & \text{for } s \text{ even.} \end{cases} \end{aligned}$$ □

This result is far more than we need, and we leave the details to the interested reader.

Now we start feeding in the generators v_n inductively. In our range 4.4.8 gives us $\mathrm{Ext}(BP_*/I_3)$. Each of the specified generators is easily seen to come from a cocycle in the cobar complex $C(BP_*/I_2)$ so we have

$$\mathrm{Ext}(BP_*/I_2) = \mathrm{Ext}(BP_*/I_3) \otimes P(v_2),$$

i.e., the Bockstein spectral sequence collapses in our range.

The passage to $\mathrm{Ext}(BP_*/I_1)$ is far more complicated. The following formulas in $C(BP_*/I_1)$ are relevant.

(4.4.10) $$\text{(a)} \quad d(v_2) = v_1 t_1^p - v_1^p t_1$$

and

$$\text{(b)} \quad d(t_2) = -t_1|t_1^p - v_1 b_{10}.$$

These follow immediately from 4.3.20 and 4.3.15. From 4.4.10(a) we get

(4.4.11) $$\text{(a)} \quad \delta_1(v_2^i) \equiv i v_2^{i-1} h_{11} \mod (v_1)$$

and

$$\text{(b)} \quad \delta_1(v_2^p) \equiv v_1^{p-1} h_{12} \mod (v_1^p).$$

Next we look at elements in $\mathrm{Ext}^1(BP_*/I_2)$. Clearly, h_{10}, h_{11}, and h_{12} are in $\ker \delta_i$ as are $v_2^i h_{11}$ for $i < p-1$ by the above calculation. This leaves $v_2^i h_{10}$ for $1 \le i \le p-1$ and $v_2^{p-1} h_{11}$. For the former 4.4.10 gives

$$\begin{aligned} d(v_2^i t_1 + i v_1 v_2^{i-1}(t_1^{1+p} - t_2)) &\equiv i v_1^2 v_2^{i-1} b_{10} \\ &\quad + \binom{i}{2} v_1^2 v_2^{i-1}(t_1^{2p}|t_1 - 2t_1^p|t_2 + 2t_1^p|t_1^{1+p}) \mod (v_1^3). \end{aligned}$$

The expression in the second term is a multiple of k_0, so we have

(4.4.12) $$\delta(v_2^i h_{10}) \equiv i v_1 v_2^{i-1} b_{10} \pm \binom{i}{2} v_1 v_2^{i-2} k_0 \mod (v_1^2).$$

To deal with $v_2^{p-1} h_{11}$ we use 4.4.10(a) to show

$$d\left(\sum_{0<i<p} \frac{1}{p}\binom{p}{i} v_2^{p-i} v_1^{i-1} t_1^{pi}\right) \equiv \sum_{0<i<p} \frac{1}{p}\binom{p}{i} v_1^{p-1} t_1^{pi}|t_1^{p^2-pi} \mod (v_1^p)$$

so

(4.4.13) $$\delta_1(v_2^{p-1} h_{11}) = \pm v_1^{p-2} b_{11}.$$

This is a special case of 4.3.22(b).

Now we move on to the elements in $\mathrm{Ext}^2(BP_*/I_2)$. They are $h_{10}h_{12}$, b_{11}, $v_2^i b_{10}$, $v_2^i g_0$, and $v_2^i k_0$ for suitable i. The first two are clearly in $\ker \delta_1$. Equation 4.4.12

eliminates the need to consider $v_2^i b_{10}$ for $i < p-1$, so that leaves $v_2^{p-1}b_{10}$, $v_2^i g_0$, and $v_2^i k_0$. Routine calculation with 4.4.10 gives

$$\text{(a)} \quad \delta_1(v_2^i g_0) \equiv \pm(v_2^i h_{10} b_{10} \pm i v_2^{i-1} h_{10} k_0) \mod (v_1^2)$$

and

$$\text{(b)} \quad \delta_1(v_2^i k_0) \equiv \pm v_2^i h_{11} b_{10} \mod (v_1^2).$$

We have to handle $v_2^{p-1}b_{10}$ more indirectly.

4.4.14. LEMMA. *$\delta_1(v_2^{p-1}b_{10} \pm \frac{1}{2}v_2^{p-2}k_0) = cv_1^{p-3}h_{10}b_{11}$ for some nonzero $c \in \mathbf{Z}/(p)$.*

PROOF. By 4.4.13, $v_1^{p-1}b_{11} = 0$ in $\mathrm{Ext}(BP_*/I_1)$, so $v_1^{p-1}h_{10}b_{11} = 0$ and $v_1^i h_{10} b_{11} = \delta_1(x)$ for some $i < p-1$ and some $x \in \mathrm{Ext}^2(BP_*/I_2)$. The only remaining x is the indicated one. □

From 4.4.14 we get $\delta_1(v_2^{p-1}h_{10}b_{10} \pm \frac{1}{2}v_2^{p-2}h_{10}k_0) \equiv 0 \mod (v_1^{p-2})$. All other elements in $\mathrm{Ext}^s(BP_*/I_2)$ for $s \geq 3$ are divisible by h_{10} or b_{10} and they can all be accounted for in such a way that the above element, which we denote by $\bar\phi$, must be in $\ker \delta_1$. Hence δ_1 is completely determined in our range.

Equivalently, we have computed all of the differentials in the Bockstein spectral sequence. However, there are some multiplicative extensions which still need to be worked out.

4.4.15. THEOREM. *For $p > 2$, $\mathrm{Ext}(BP_*/I_1) = P(v_1) \otimes E(h_{10}) \oplus M$, where M is a free module over $P(b_{10})$ on the following generators:*

$$\beta_i = \delta_1(v_2^i), \quad h_{10}\beta_i, \quad \bar\beta_i = v_1^{-1}\delta_1(v_2^i h_{10}) \quad (\textit{e.g.}, \ \bar\beta_1 = \pm b_{10}),$$
$$\textit{and} \quad h_{10}\bar\beta_i \quad \textit{for } 1 \leq i \leq p-1;$$
$$\beta_{p/i} = v_1^{1-i}\delta_i(v_2^p) \quad \textit{and} \quad h_{10}\beta_{p/i} \quad \textit{for } 1 \leq i \leq p;$$
$$\bar\beta_{p/i} = v_1^{2-i}\delta_1(v_2^{p-1}h_{11}) \quad \textit{for } 2 \leq i \leq p;$$
$$h_{10}\bar\beta_{p/i} \quad \textit{for } 3 \leq i \leq p; \quad \bar\phi \quad \textit{and} \quad \beta_1\bar\beta_{p/p}.$$

Here δ_1 is the connecting homomorphism for the short exact sequence

$$0 \to \Sigma^q BP_*/I_1 \xrightarrow{v_1} BP_*/I_1 \to BP_*/I_2 \to 0.$$

Moreover,

$$h_{10}\beta_i = v_1\bar\beta_i, \quad v_1\beta_i = 0, \quad v_1^p\beta_{p/p} = 0,$$
$$v_1^{p-1}\bar\beta_{p/p} = 0, \quad \textit{and} \quad v_1^{p-2}h_{10}\bar\beta_{p/p} = 0.$$

(*This description of the multiplicative structure is not complete.*)

PROOF. The additive structure of this Ext follows from the above calculations. The relations follow from the way the elements are defined. □

FIGURE 4.4.16 illustrates this result for $p = 5$. Horizontal lines indicate multiplication by v_1, and an arrow pointing to the right indicates that the element is free over $P(v_1)$. A diagonal line which increases s and t/q by one indicates multiplication by h_{10} and one which increases t/q by 4 indicates the Massey product operation $\langle -, h_{10}, h_{10}, h_{10}, h_{10}\rangle$. Thus two successive diagonal lines indicate multiplication by $b_{10} = \pm\langle h_{10}, h_{10}, h_{10}, h_{10}, h_{10}\rangle$. The broken line on the right indicates the limit of our calculation.

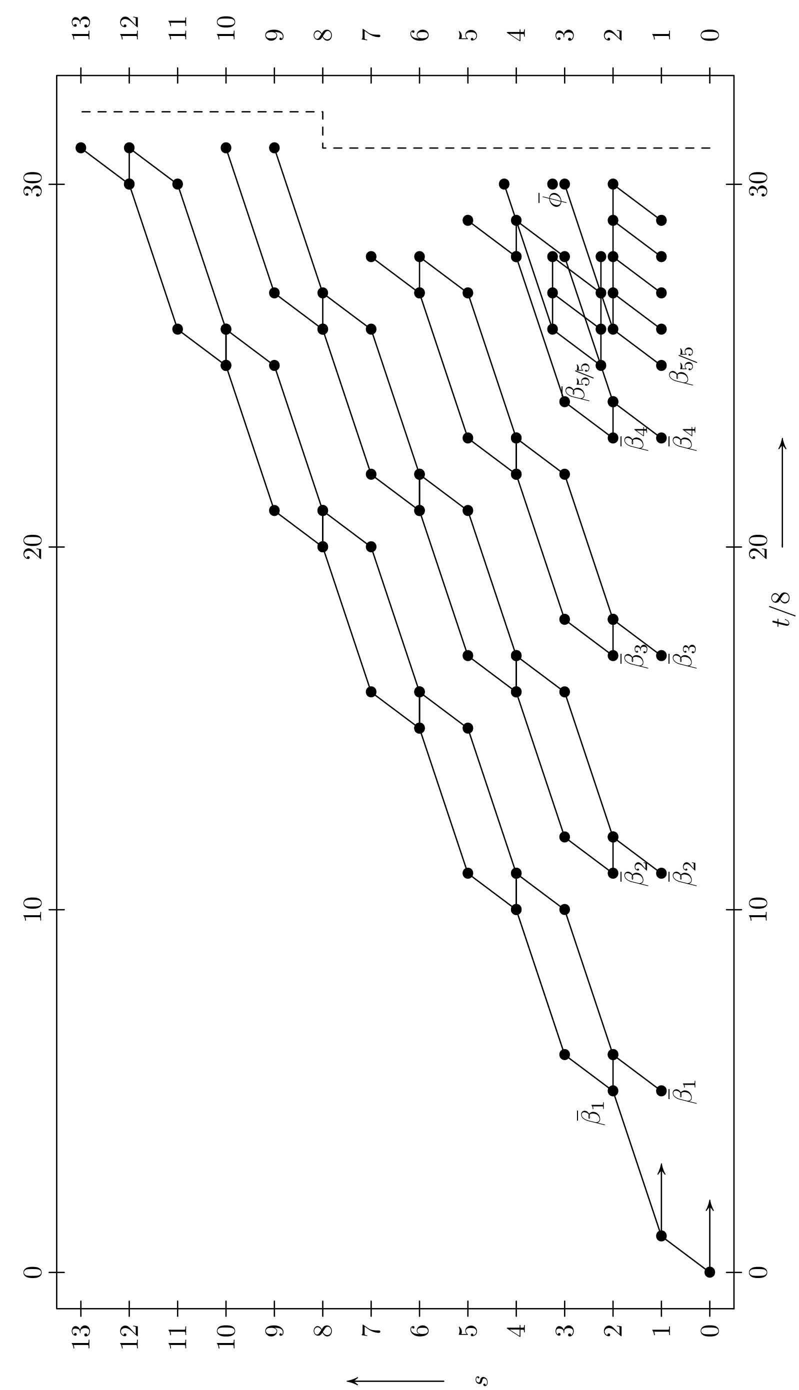


FIGURE 4.4.16. $\mathrm{Ext}^{s,t}_{BP_*(BP)}(BP_*, BP_*/I_1)$ for $p = 5$ and $t - s \leq 240$.

Now we have to consider the long exact sequence or Bockstein spectral sequence associated with

$$0 \to BP_* \xrightarrow{p} BP_* \to BP_*/I_1 \to 0.$$

First we compute $\delta_0(v_1^i)$. Since $d(v_1) = pt_1$ in $C(BP_*)$ we have $d(v_1^i) \equiv ipv_1^{i-1}t_1 \mod (ip^2)$, so

$$\delta_0(v_1^i) \equiv iv_1^{i-1}h_{10} \qquad \mod (ip^2). \tag{4.4.17}$$

Moving on to $\mathrm{Ext}^1(BP_*/I_1)$ we need to compute δ_0 on β_i and $\beta_{p/i}$. The former can be handled most easily as follows. $\delta_0(\bar{\beta}_i) = 0$ because there is no element in the appropriate grading in Ext^3. δ_0 is a derivation mod (p) so $\delta_0(v_1\bar{\beta}_i) = h_{10}\bar{\beta}_i$. Since $v_1\bar{\beta}_i = h_{10}\beta_i$ we have $h_{10}\bar{\beta}_i = \delta_0(h_{10}\beta_i) = h_{10}\delta_0(\beta_i)$ so

$$\delta_0(\beta_i) = \bar{\beta}_i. \tag{4.4.18}$$

Now $\beta_{p/p} = h_{12} - v_1^{p^2-p}h_{11}$ and $v_1^{p^2-p}h_{11}$ is cohomologous to $v_1^{p^2-1}h_{10}$, which by 4.4.16 is in $\ker \delta_0$. Hence

$$\delta_0(\beta_{p/p}) = \delta_0(h_{12}) = b_{11} = \pm\bar{\beta}_{p/p}. \tag{4.4.19}$$

It follows that

$$\delta_0(\beta_{p/p-i}) = \delta_0(v_1^i\beta_{p/p}) = iv_1^{i-1}h_{10}\beta_{p/p} \pm v_1^i\bar{\beta}_{p/p}.$$

This accounts for all elements in sight but $\delta_0(h_{10\beta_{p/1}})$ which vanishes mod (p). We will show that it is a unit multiple of $p\Phi$ below in 5.1.24.

Putting all this together gives

4.4.20. THEOREM. *For $p > 2$ and $t - s \le (p^2 + p)q$, $\mathrm{Ext}(BP_*)$ is as follows. $\mathrm{Ext}^0 = \mathbf{Z}_{(p)}$ concentrated in dimension zero. $\mathrm{Ext}^{1,qi} = \mathbf{Z}_{(p)}/(pi)$ generated by $\bar{\alpha}_i = i^{-1}\delta_0(v_1^i)$, where $\alpha_1 = h_{10}$. For $s \ge 2$ Ext^s generated by all $b_{10}^j x$, where x is one of the following: $\beta_i = \delta_0(\beta_i)$ (where $\beta_1 = \pm b_0$) and $\alpha_1\beta_i$ for $1 \le i \le p-1$; $\beta_{p/p-i} = \delta_0(\beta_{p/p-i})$ for $0 \le i \le p-1$; $\alpha_1\beta_{p/p-i}$ for $0 \le i \le p-3$; and $\phi = p^{-1}\delta_0(h_{10}\beta_{p/1})$ which has order p^2. ϕ is a unit multiple of $\langle \beta_{p/2}, \alpha_1, \alpha_1\rangle$ and $p\phi$ is a unit multiple $\alpha_1\beta_{p/1}$. Here $\beta_{i/j}$ denotes the image under δ_0 of the corresponding element in $\mathrm{Ext}(BP_*/I_1)$.* □

For $p = 5$ this is illustrated in FIG. 4.4.21, with notation similar to that of Fig. 4.4.16. It also shows differentials (long arrows originating at $\beta_{5/5}$ and $\beta_1\beta_{5/5}$), which we discuss now. By sparseness (4.4.2) $E_2 = E_{2p-1}$ and $d_{2p-1}\colon E_{2p-1}^{s,t} \to E_{2p-1}^{s+2p-1,t-2p+2}$. It is clear that in our range of dimensions $E_{2p} = E_\infty$ because any higher (than d_{2p-1}) differential would have a target whose filtration (the s-coordinate) would be too high. Naively, the first possible differential is $d_{2p-1}(\alpha_{p^2-1}) = c\beta_i^p$. However, d_{2p-1} respects multiplication by α_1 and $\alpha_1\alpha_{p^2-1}$ so $c\alpha_1\beta_1^p = 0$ and $c = 0$. Alternatively one can show (see 5.3.7) that each element in Ext^1 is a permanent cycle.

4.4.22. THEOREM (Toda [**2, 3**]). *$d_{2p-1}(\beta_{p/p}) = a\alpha_1\beta_1^p$ for some nonzero $a \in \mathbf{Z}/(p)$.* □

Toda shows that any $x \in \pi_*(S)$ of order p must satisfy $\alpha_1 x^p = 0$. For $x = \beta_1$ this shows $\alpha_1\beta_1^p = 0$ in homotopy. Since it is nonzero in E_2 it must be killed by a differential and our calculation shows that $\beta_{p/p}$ is the only possible source for it. We do not know how to compute the coefficient a, but its value seems to be of little consequence.

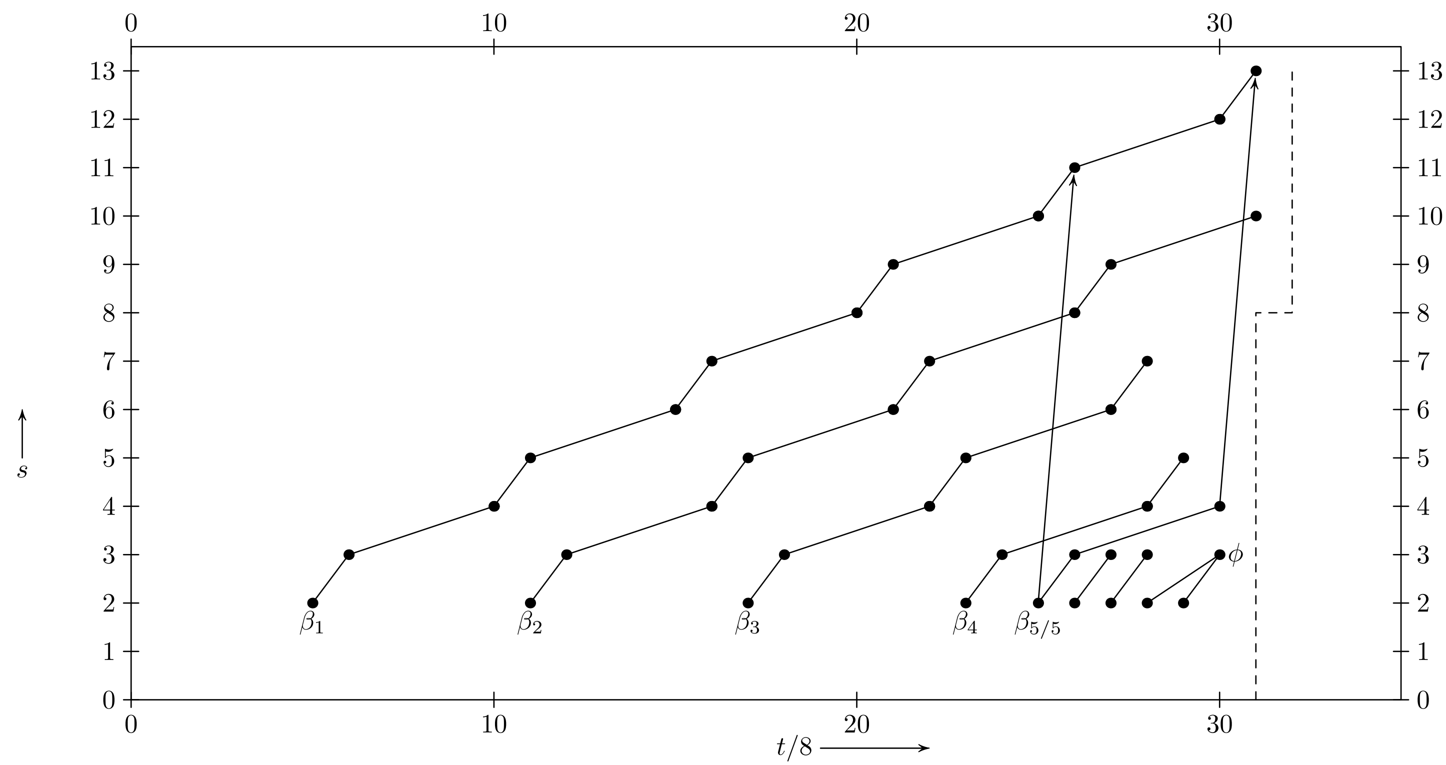


FIGURE 4.4.21. The Adams–Novikov spectral sequence for $p = 5$, $t - s \leq 240$, and $s \geq 2$.

Theorem 4.4.22 implies that $d_{2p-1}(\beta_1\beta_{p/p}) = \alpha_1\beta_1^{p+1}$. Inspection of 4.4.20 or 4.4.21 shows that there are no other nontrivial differentials.

Notice that the element $\alpha_1\beta_{p/p}$ survives to E_∞ even though $\beta_{p/p}$ does not. Hence the corresponding homotopy element, usually denoted by ε', is indecomposable. It follows easily from the definition of Massey products (A1.4.1) that $\langle\alpha_1, \alpha_1, \beta_1^p\rangle$ is defined in E_{2p}, has trivial indeterminacy, and contains a unit multiple of $\alpha_1\beta_{p/p}$. It follows from 7.5.4 that ε' is the corresponding Toda bracket. Using A1.4.6 we have

$$\langle\alpha_1, \ldots, \alpha_1, \varepsilon'\rangle = \langle\alpha_1, \ldots, \alpha_1\rangle\beta_1^p = \beta_1^{p+1}$$

with $p-2$ α_1's on the left and p α_1's on the right.

Looking ahead we can see this phenomenon generalize as follows. For $1 \le i \le p-1$ we have $d_{2p-1}(\beta_{p/p}^i) = ia\alpha_1\beta_{p/p}^{pi-1}$. For $i \le p-2$ this leads to $\langle\alpha_1, \ldots, \alpha_1\beta_1^{ip}\rangle$ [with $(i+1)$ α_1's] being a unit multiple of $\varepsilon^{(i)} = \alpha_1\beta_{p/p}^i$, and $\langle\alpha_1, \alpha_1, \ldots, \alpha_1\varepsilon^{(i)}\rangle$ [with $(p-i-1)$ α_1's] is a unit multiple of β_1^{1+ip}. In particular, $\alpha_1\varepsilon^{(p-2)}$ is a unit multiple of $\beta_1^{1+(p-2)p}$. Since $\alpha_1\beta_1^p = 0$ (4.4.22), $\beta_1^{p^2-p+1} = 0$ since it is a unit multiple of $\alpha_1\beta_1^p\varepsilon^{(p-2)}$. However, in the E_2-term all powers of β_1 are nonzero (Section 6.4), so $\beta_1^{p^2-p+1}$ must be killed by a differential, more precisely by $d_{(p-1)q+1}(\alpha_1\beta_{p/p}^{p-1})$.

Now we will make an analogous calculation for $p = 2$. The first three steps are shown FIG. 4.4.23. In (a) we have $\mathrm{Ext}_{P_*}(\mathbf{Z}/(2), \mathbf{Z}(2))$, which is $\mathrm{Ext}(BP_*/I_4)$ for $t-s \le 29$. Since differentials in the Bockstein spectral sequences and the Adams–Novikov spectral sequence all lower $t-s$ by 1, we lose a dimension with each spectral sequence. In (a) we give elements the same names they have in $\mathrm{Ext}_{A_*}(\mathbf{Z}/(2), \mathbf{Z}/(2))$. Hence we have $c_0 = \langle h_{11}, h_{12}^2, h_{11}\rangle$ and $Px = \langle x, h_{10}^4, h_{12}$. Diagonal lines indicate multiplication by h_{10}, h_{11}, and h_{12}. The arrow pointing up and to the right indicates that all powers of h_{10} are nontrivial.

The Bockstein spectral sequence for $\mathrm{Ext}(BP_*/I_3)$ collapses and the result is shown in Fig. 4.4.23(b). The next Bockstein spectral sequence has some differentials. Recall that δ_2 is the connecting homomorphism for the short exact sequence

$$0 \to \Sigma^6 BP_*/I_2 \xrightarrow{v_2} BP_*/I_2 \to BP_*/I_3 \to 0.$$

Since $\eta_R(v_3) \equiv v_3 + v_2t_1^4 + v_2^2t_1 \mod I_2$ by 4.3.1 we have

(4.4.24)
$$\begin{aligned}
&\text{(a)} \quad \delta_2(v_3h_{10}^i) = (h_{12} + v_2h_{10})h_{10}^i && \text{for } i \le 2,\\
&\text{(b)} \quad \delta_2(v_3h_{10}^i) = v_2h_{10}^{i+1} && \text{for } i \ge 3,\\
&\text{(c)} \quad \delta_2(v_3h_{12}^i) = h_{12}^{i+1} && \text{for } i = 1, 2,\\
&\text{(d)} \quad \delta_2(v_2^2) = v_2h_{13} + v_2^3h_{11}.
\end{aligned}$$

This accounts for all the nontrivial values of δ_2. In $\mathrm{Ext}(BP_*/I_2)$ we denote $\delta_2(v_3^i)$ by γ_i and $v_2^{-1}\delta_2(v_3^2)$ by $\gamma_{2/2}$. The elements $v_3h_{1,1}$, $v_3h_{1,0}h_{1,2} \in \mathrm{Ext}(BP_*/I_3)$ are in $\ker\delta_2$ and hence lift back to $\mathrm{Ext}(BP_*/I_2)$, where we denote them by ζ_2 and $x_{2,2}$, respectively. They are represented in $C(BP_*/I_2)$ by

(4.4.25)
$$\begin{aligned}
&\text{(a)} \quad \zeta_2 = v_3t_1^2 + v_2(t_2^2 + t_1^6) + v_2^2t_2 \quad \text{and}\\
&\text{(b)} \quad x_{22} = v_3t_1|t_1^4 + v_2(t_2|t_2^2 + t_1|t_1^2t_2^2 + t_2|t_1^6 + t_3|t_1^2 + t_1^5|t_1^4)\\
&\qquad\qquad + v_2^2(t_2|t_2 + t_1|t_1^2t_2 + t_1t_2|t_1^2).
\end{aligned}$$

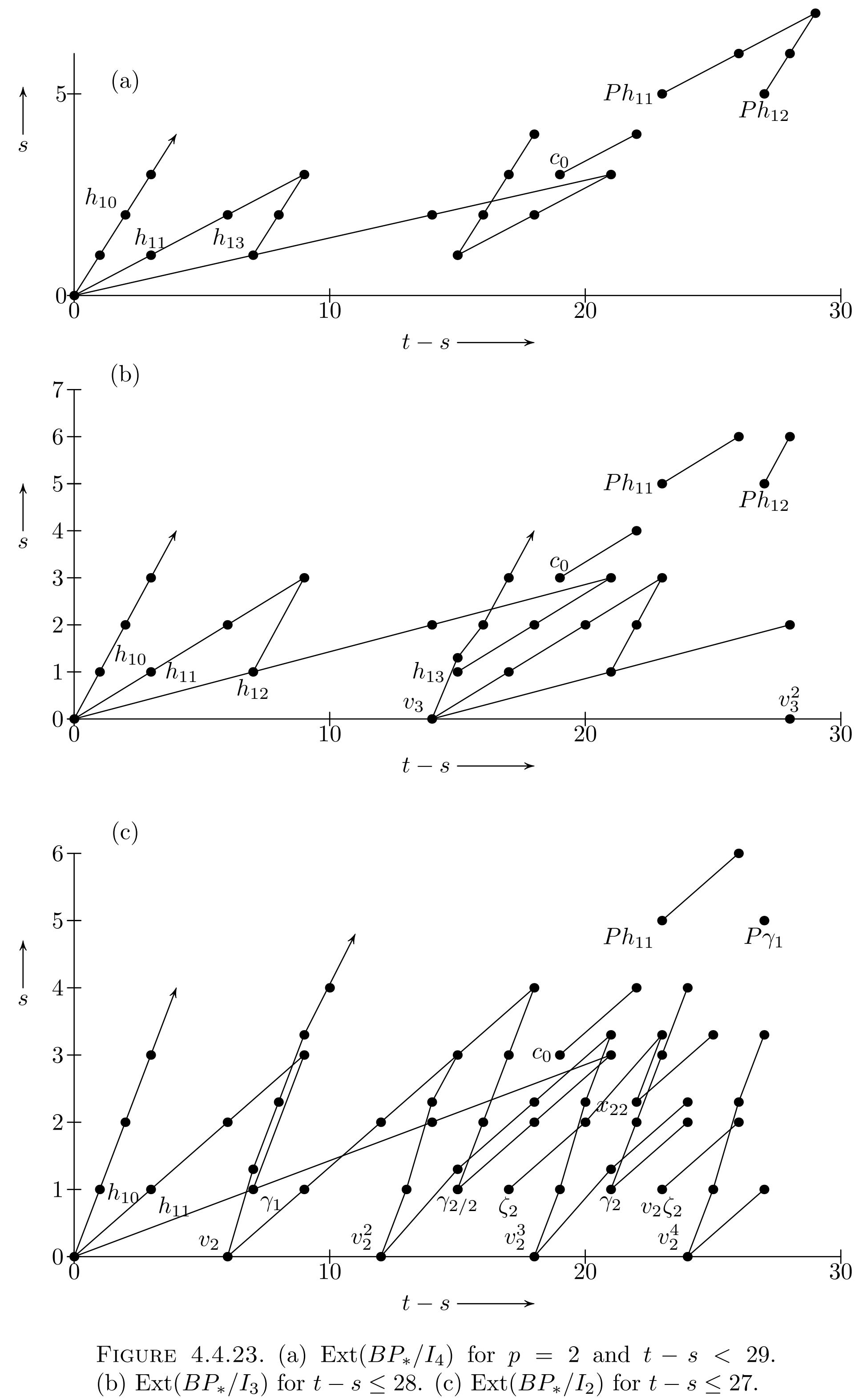


FIGURE 4.4.23. (a) $\mathrm{Ext}(BP_*/I_4)$ for $p = 2$ and $t - s < 29$. (b) $\mathrm{Ext}(BP_*/I_3)$ for $t - s \le 28$. (c) $\mathrm{Ext}(BP_*/I_2)$ for $t - s \le 27$.

Now we pass to $\mathrm{Ext}(BP_*/I_1)$. To compute δ_1 on $\mathrm{Ext}^0(BP_*/I_2)$ we have $\eta_R(v_2) \equiv v_2 + v_1t_1^2 + v_1^2t_1 \mod I_1$, so

$$\begin{aligned}
&\delta_1(v_2) \equiv h_{11} && \mod (v_1),\\
&\delta_1(v_2^2) \equiv v_1h_{12} \equiv v_1(\gamma_1 + v_2h_{10}) && \mod (v_1^2),\\
&\delta_1(v_2^3) \equiv v_2^2h_{11} && \mod (v_1),\\
&\delta_1(v_2^4) \equiv v_1^3h_{13} \equiv v_1^3(\gamma_{2/2} + v_2^2h_{11}) && \mod (v_1^4),\\
&\delta_1(v_2^5) \equiv v_2^4h_{11} && \mod (v_1),
\end{aligned} \tag{4.4.26}$$

This means that in $\mathrm{Ext}^1(BP_*/I_2)$ it suffices to compute δ_1 on v_2h_{10}, ζ_2, $v_2^3h_{10}$, γ_2, and $v_2\zeta_2$. We find $\delta_1(v_2h_{10})$ and the element pulls back to

$$x_7 = v_2t_1 + v_1(t_2 + t_1^3). \tag{4.4.27}$$

For ζ_2 we compute in $C(BP_*/I_1)$ and get

$$d(\zeta_2 + v_1t_1^2t_2^2) \equiv v_1(t_1^4|t_1^4 + v_2^2t_1|t_1) \quad \mod (v_1^2)$$

so

$$\delta_1(\zeta_2) \equiv \gamma_1^2 \quad \mod (v_1) \tag{4.4.28}$$

For $v_2^3h_{10}$ we compute

$$d(v_2^3t_1 + v_1v_2^2(t_2 + t_1^3) + v_1^2v_3t_1) \equiv v_1^2v_2^2t_1|t_1 \quad \mod (v_1^3)$$

so

$$\delta_1(v_2^3h_{10}) \equiv v_1v_2^2h_{10}^2 \quad \mod (v_1^2). \tag{4.4.29}$$

Similar calculations give

$$\delta_1(\gamma_2) \equiv h_{11}\gamma_{2/2} \quad \mod (v_1^2) \tag{4.4.30}$$

and

$$\delta_1(v_2\zeta_2) \equiv h_{11}\zeta_2 + v_2^3h_{10} \quad \mod (v_1^2)$$

In $\mathrm{Ext}^2\, BP_*/I_2)$ it suffices to compute δ_1 on x_{22}. We will show

$$\delta_1(x_{22}) = c_0 \tag{4.4.31}$$

using Massey products. Since x_{22} projects to $v_3h_{10}h_{12}$ we have $x_{22} \in \langle v_2, \gamma_1^2, h_{10}\rangle$, so $\delta_1(x_{22}) \in \langle \delta_1(v_2), \gamma_1^2, h_{10}\rangle$ by A1.4.11. This is $\langle h_{11}, \gamma_1^2, h_{10}\rangle$, which is easily seen to be c_0.

This completes our calculation of δ_1. The resulting value of $\mathrm{Ext}(BP_*/I_1)$ is shown in FIG. 4.4.32. The elements 1 and x_7 are free over $P(v_1, h_{10})$. As usual we denote $v_1^{1-j}\delta_1(v_2^i)$ by $\beta_{i/j}$. x_7 is defined by 4.4.27. η_1 and η_2 (not to be confused with the η_j of Mahowald [**6**]) denote $\delta_1(\zeta_2)$ and $\delta_1(v_2\zeta_2)$.

We must comment on some of the relations indicated in 4.4.32.

4.4.33. LEMMA. *In* $\mathrm{Ext}(BP_*/I_1)$ *for* $p = 2$ *the following relations hold.*
(a) $h_{10}\beta_3 = v_1\beta_{2/2}^2$
(b) $\beta_{2/2}^3 = \beta_1^2\beta_{4/4} + h_{10}^2\beta_{4/2}$
(c) $h_{10}^3x_7\beta_{4/4} = v_1P\beta_1$.

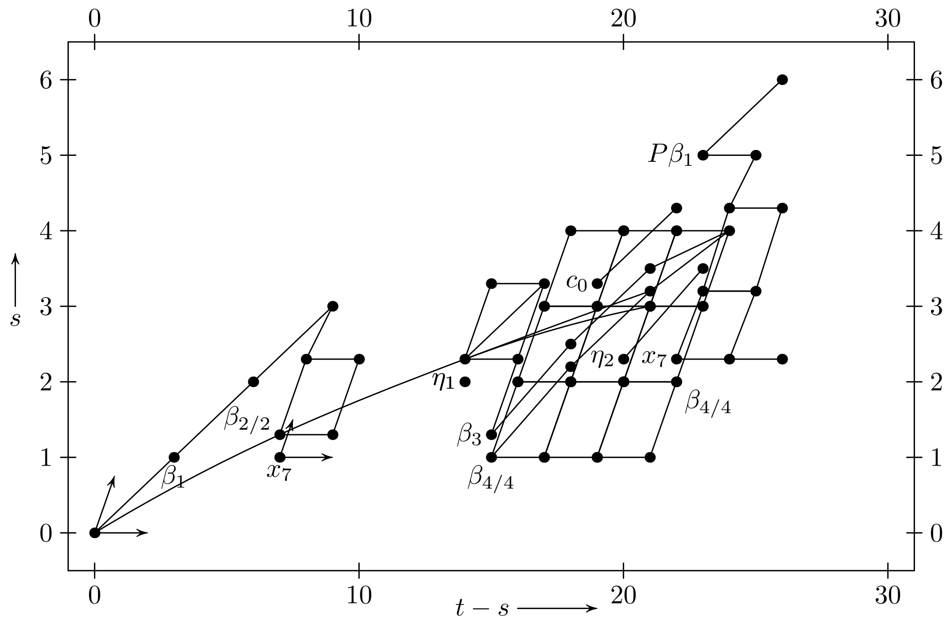


FIGURE 4.4.32. $\mathrm{Ext}(BP_*/I_1)$ for $p = 2$ and $t - s \le 26$

PROOF.

(a) $\beta_{2/2} = h_{12} \mod (v_1)$ so $v_1\beta_{2/2}^2 = \delta_1(v_2^2 h_{12})$ while $h_{10}\beta_3 = \delta_1(v_2^3 h_{10})$. Since $\eta_R(v_2v_3) \equiv v_2v_3 + v_2^2 t_1^4 + v_2^3 t_1 \mod I_2$, we have $v_2^2 h_{12} = v_2^3 h_{10}$ in $\mathrm{Ext}(BP_*/I_2)$.

(b) $\beta_{2/2} = h_{12} + v_1^2 h_{11} = h_{12} + v_1^3 h_{10}$ so

$$\begin{aligned} \beta_{2/2}^3 &\equiv h_{12}^3 \equiv h_{11}^2 h_{13} \equiv \beta_{4/4} h_{11}^2 \\ &\equiv \beta_{4/4}(\beta_1^2 + v_1^2 h_{10}^2) \equiv \beta_{4/4}\beta_1^2 + \beta_{4/2} h_{10}^2 \mod (v_1^3). \end{aligned}$$

(c) $$\begin{aligned} v_1 P\beta_1 &= v_1\langle \beta_1, h_{10}^4, \beta_{4/4}\rangle \\ &= \langle v_1, \beta_1, h_{10}^4\rangle \beta_{4/4} \quad \text{by A1.4.6} \\ &= \langle v_1, \beta_1, h_{10}\rangle h_{10}^3 \beta_{4/4} \quad \text{by A1.4.6}. \end{aligned}$$

The last Massey product is easily seen to contain x_7. □

Now we pass to $\mathrm{Ext}(BP_*)$ by computing δ_0, beginning with $\mathrm{Ext}^0(BP_*/I_1) = P(v_1)$. By direct calculation we have

(4.4.34) $$\begin{aligned} \delta_0(v_1^{2i+1}) &\equiv v_1^{2i} h_{10} \mod (2) \\ \delta_0(v_1^2) &= 2\beta_1 \end{aligned}$$

To handle larger even powers of v_1, consider the formal expression $u = v_1^2 - 4v_1^{-1}v_2$. Using the formula (in terms of Hazewinkel's generators A2.2.1)

$$\eta_R(v_2) = v_2 - 5v_1 t_1^2 - 3v_1^2 t_1 + 2t_2 - 4t_1^3,$$

we find that $d(u) = 8v_1^{-2}x_7$ in $C(v_1^{-1}BP_*/(2^4))$. It follows that

$$d(u^2 - 2^4 v_1^{-2} v_2^2) \equiv 2^4(x_7 + \beta_{2/2}) \mod (2^5)$$

and for $i > 2$

$$d(u^i) \equiv 8iv_1^{2i-4}x_7 \mod (16i)$$

so

$$\delta_0(v_1^4) \equiv 2^3(x_7 + \beta_{2/2}) \mod (2^4) \tag{4.4.35}$$

and

$$\delta_0(v_i^{2i}) \equiv 4iv_1^{2i-4}x_7 \mod (8i) \quad \text{for } i \geq 3.$$

Combining this with

$$\delta_0(v_1^{2i+1}h_0^j) = v_1^{2i}h_0^{j+1} \tag{4.4.36}$$

and

$$\delta_0(v_1^{2i+1}h_0^j x_7) = v_1^{2i}h_0^{j+1}x_7$$

accounts for all elements of the form $v_1^i h_0^j x_7^\varepsilon$ for $i, j \geq 0$ and $\varepsilon = 0, 1$ we have

4.4.37. THEOREM. *For $p = 2$ $\mathrm{Ext}^1(BP_*)$ is generated by $\bar{\alpha}_i$ for $i \geq 1$ where*

$$\bar{\alpha}_i = \begin{cases} \delta_0(v_1^i) & \text{for } i \text{ odd} \\ \frac{1}{2}\delta_0(v_1^2) & \text{for } i = 2 \\ (1/2i)\delta_0(v_1^i) & \text{for even } i \geq 4. \end{cases}$$

In particular $\bar{\alpha}_1 = h_{10}$. Moreover $\bar{\alpha}_1^j\bar{\alpha}_i \neq 0$ for all $j > 0$ and $i \neq 2$. □

Moving on to $\mathrm{Ext}^1(BP_*/I_1)$ we still need to compute δ_0 on h_{12}, v_1h_{12}, β_3, and $v_1^j h_{13}$ for $0 \leq j \leq 3$. An easy calculation gives

$$\begin{aligned} \delta_0(h_{12}) &\equiv h_{11}^2 && \mod (2), \\ \delta_0(v_1h_{12}) &\equiv h_{10}h_{12} && \mod (2), \\ \delta_0(h_{13}) &\equiv h_{12}^2 && \mod (2), \\ \delta_0(v_1h_{13}) &\equiv v_1h_{12}^2 + h_{10}h_{13} && \mod (2), \\ \delta_0(v_1^2h_{13}) &\equiv 2(h_{11} + v_1h_{10})h_{13} && \mod (4), \end{aligned} \tag{4.4.38}$$

and

$$\delta_0(v_1^3h_{13}) \equiv v_1^2h_{10}h_{13} \qquad \mod (2).$$

For β_3 we have

$$\delta_0(\beta_3) = \beta_{2/2}^2 + \eta_1. \tag{4.4.39}$$

The proof is deferred until the next chapter (5.1.25).

In $\mathrm{Ext}^2(BP_*/I_1)$ all the calculations are straightforward except η_2 and $x_7\beta_{4/j}$. The former gives

$$\delta_0(\eta_2) = c_0, \tag{4.4.40}$$

which we defer to 5.1.25. For the latter we have

$$\delta_0(x_7\beta_{4/4}) \equiv x_7h_{12}^2 \mod (2).$$

Computing in $C(BP_*/I_2)$ we get

$$d(t_1^2|t_3 + t_2|t_2^2 + t_1^3|t_2^2 + t_1^2t_2|t_1^4) = t_1|t_1^4|t_1^4 + v_2t_1^2|t_1^2|t_1^2$$

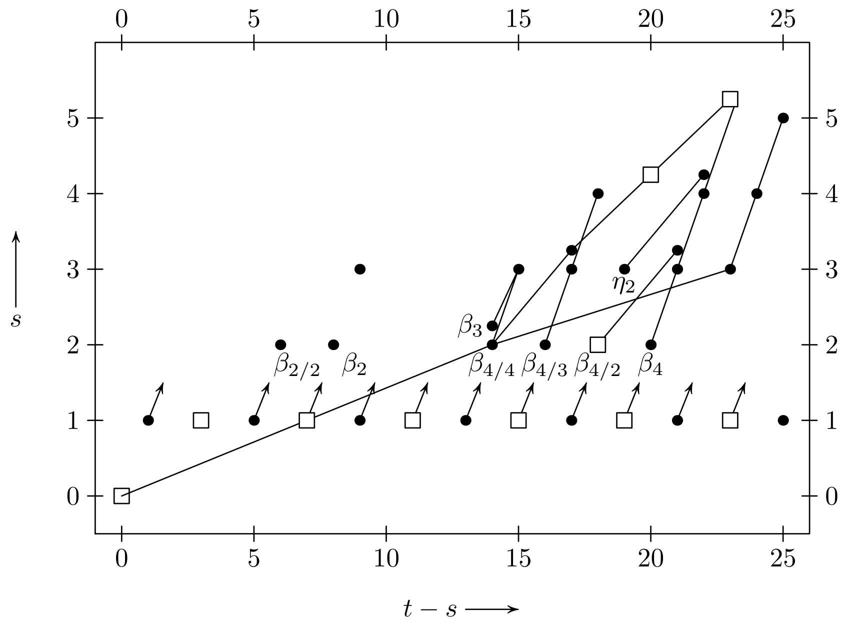


FIGURE 4.4.45. $\mathrm{Ext}(BP_*)$ for $p = 2$, $t - s \leq 25$.

so $x_7 h_{12}^2 \equiv \beta_3 \beta_1^2 \mod (v_1^2)$ and

$$\delta_0(x_7\beta_{4/4}) \equiv \beta_1^2\beta_3 + ch_{10}^2\beta_{4/2} \mod (2) \tag{4.4.41}$$

for $c = 0$ or 1. Note that

$$\delta_0(h_{10}\beta_4) = h_{10}^2\beta_{4/2}.$$

We also get from 4.4.37

$$\delta_0(x_7\beta_{4/3}) \equiv ch_{10}^2\beta_4 + h_{10}x_7\beta_{4/4} \mod (2). \tag{4.4.42}$$

$\delta_0(x_7\beta_{4/2})$ must be a multiple of $h_{10}x_7\beta_{4/3}$ but the latter is not in $\ker\delta_0$ so

$$\delta_0(x_7\beta_{4/2}) = 0. \tag{4.4.43}$$

Of the remaining calculations of δ_0 all are easy but $\beta_1^2\beta_{4/4}$ and $h_{10}^3\beta_4 = \beta_1^3\beta_{4/4}$. It is clear that $\delta_0(\beta_1^2\beta_{4/4})$ and $\delta_0(\beta_1^3\beta_{4/4})$ are multiples of elements which reduce to h_{10}^3 and $P\beta_1$, respectively. Since $\beta_1^2\beta_{2/2}^2 = 0$ and $\beta_1\beta h_{10}^2\beta_{4/3} = 0$ we have $\delta_0(\beta_1^2\beta_{4/4}) \equiv 0 \mod (2)$ and $\delta_0(\beta_1^2\beta_{4/4}) \equiv 0 \mod (4)$. Thus the simplest possible result is

$$\begin{aligned} \frac{1}{2}\delta_0(\beta_1^2\beta_{4/4}) &\equiv h_{10}^3\beta_{4/3} \mod (2), \\ \frac{1}{4}\delta_0(\beta_1^3\beta_{4/4}) &\equiv P\beta_1 \mod (2). \end{aligned} \tag{4.4.44}$$

We will see below that larger values of the corresponding Ext groups would lead to a contradiction.

The resulting value of $\mathrm{Ext}(BP_*)$ is shown in FIG. 4.4.45. Here squares denote elements of order greater than 2. The order of the elements in Ext^1 is given in 4.4.37. The generators of $\mathrm{Ext}^{2,20}$ and $\mathrm{Ext}^{4,24}$ have order 4 while that of $\mathrm{Ext}^{5,28}$ has order 8.

We compute differentials and group extensions in the Adams–Novikov spectral sequence for $p = 2$ by comparing it with the Adams spectral sequence. The E_2-term of the latter as computed by Tangora [**1**] is shown in FIG. 4.4.46. This procedure will determine all differentials and extensions in the Adams spectral sequence in this range as well.

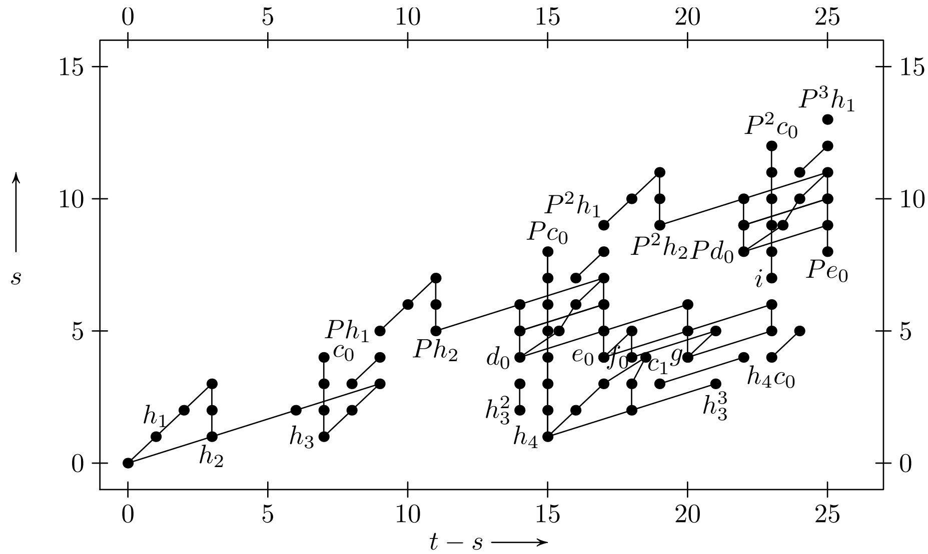


FIGURE 4.4.46. $\mathrm{Ext}_{A_*}(\mathbf{Z}/2, \mathbf{Z}/2)$ for $t - s \le 25$.

The Adams element h_1 corresponds to the Novikov $\bar\alpha_1$. Since $h_1^4 = 0$, α_1^4 must be killed by a differential, and it must be $d_3(\bar\alpha_3)$. It can be shown that the periodicity operator P in the Adams spectral sequence (see 3.4.6) corresponds to multiplication by v_1^4, so $P^i h_1$ corresponds to $\bar\alpha_{4i+1}$, so $d_3(\bar\alpha_{4i+3}) = \bar\alpha_1^3\bar\alpha_{4i+1}$. The relation $h_0^2 h_2 = h_1^3$ gives a group extension in the Adams–Novikov spectral sequence, $2\alpha_{4i+2} = \bar\alpha_1^2\bar\alpha_{4i+1}$ in homotopy. The element $P^i h_2$ for $i > 0$ corresponds to $2\bar\alpha_{4i+1}$. This element is not divisible by 2 in the Adams spectral sequence so we deduce $d_3(\bar\alpha_{4i+2}) = \alpha_1^3\bar\alpha_{4i}$ for $i > 0$. Summing up we have

4.4.47. THEOREM. *The elements in* $\mathrm{Ext}(BP_*)$ *for* $p = 2$ *listed in* 4.4.37 *behave in the Adams–Novikov spectral sequence as follows.* $d_3(\bar\alpha_{4i+3}) = \bar\alpha_1^3\bar\alpha_{4i+3}$ *for* $i \ge 0$ *and* $d_3(\bar\alpha_{4i+2}) = \bar\alpha_1^3\bar\alpha_{4i}$ *for* $i \ge 1$. *Moreover the homotopy element corresponding to* $\alpha_{4i+2} = r\bar\alpha_{4i+2}$ *does not have order* 2; *twice it is* $\bar\alpha_1^2\bar\alpha_{4i}$ *for* $i \ge 1$ *and* $\bar\alpha_1^3$ *for* $i = 0$. □

As it happens, there are no other Adams–Novikov spectral sequence differentials in this range, although there are some nontrivial extensions.

These elements in the Adams–Novikov spectral sequence E_∞-term correspond to Adams elements near the vanishing line. The towers in dimensions congruent to $7 \mod (8)$ correspond to the groups generated by $\bar\alpha_{4i}$. Thus the order of $\bar\alpha_{4i}$ determines how many elements in the tower survive to the Adams E_∞-term. For example, the tower in dimension 15 generated by h_4 has 8 elements. $\bar\alpha_8$ has order 2^5 so only the top elements can survive. From this we deduce $d_3(h_0^i h_4) = h_0^i d_0$ for

$i = 1, 2$ and either $d_3(h_4) = d_0$ or $d_2(h_4) = h_0h_3^2$. To determine which of these two occurs we consult the Adams–Novikov spectral sequence and see that β_3 and $\beta_{4/4}$ must be permanent so $\pi_{14}^s = \mathbf{Z}/(2) \oplus \mathbf{Z}/(2)$. If $d_3(h_4) = d_0$ the Adams spectral sequence would give $\pi_{14}^s = \mathbf{Z}/(4)$, so we must have $d_2(h_4) = h_0h_3^2$.

One can also show that P^ic_0 corresponds to $\alpha_1\bar{\alpha}_{4i+4}$ for $i > 1$ and this leads to a nontrivial multiplicative extension in the Adams spectral sequence. For example, the homotopy element corresponding to Pc_0 is α_1 times the one corresponding to $h_0^3h_4$.

The correspondence between Adams–Novikov spectral sequence and Adams spectral sequence permanent cycles is shown in the following table.

TABLE 4.4.48. Correspondence between Adams–Novikov spectral sequence and Adams spectral sequence permanent cycles for $p = 2$, $14 \le t - s \le 24$

Adams–Novikov element	*Adams element*	*Adams–Novikov element*	*Adams element*
$\beta_{4/4}$	h_3^2	β_4	g
β_3	d_0	$\frac{1}{2}\bar{\alpha}_2^2\beta_{4/4}$	h_0g
$\beta_{4/3}$	h_1h_4	$\frac{1}{4}\alpha_1^3\beta_4$	h_2g
$\frac{1}{2}\beta_{4/2}$	h_2h_4	$\beta_{4/4}\bar{\alpha}_4$	h_4c_0
η_2	c_1		

4.4.49. COROLLARY. *The Adams–Novikov spectral sequence has nontrivial group extensions in dimensions* 18 *and* 20 *and the homotopy product* $\beta_4\bar{\alpha}_2$ *is detected in filtration* 4. □

4.4.50. COROLLARY. *For* $14 \le t - s \le 24$ *the following differentials occur in the Adams spectral sequence for* $p = 2$.

$$d_2(h_4) = h_0h_3^2, \quad d_3(h_0h_4) = h_0d_0, \quad d_2(e_0) = h_1^2d_0,$$
$$d_2(f_0) = h_0^2e_0, \quad d_2(i) = h_0Pd_0, \quad \textit{and} \quad d_2(Pe_0) = h_1^2Pd.$$

There are nontrivial multiplicative extensions as follows:

$$h_1 \cdot h_0^3h_4 = Pc_0, \quad h_1 \cdot h_1g = Pd_0, \quad \textit{and} \quad h_0 \cdot h_2^2e_0 = h_1Pd_0 = h_2 \cdot h_2^2d_0. \quad \square$$

CHAPTER 5

The Chromatic Spectral Sequence

The spectral sequence of the title is a mechanism for organizing the Adams–Novikov E_2-term and ultimately $\pi_*(S^0)$ itself. The basic idea is this. If an element x in the E_2-term, which we abbreviate by $\operatorname{Ext}(BP_*)$ (see 5.1.1), is annihilated by a power of p, say p^i, then it is the image of some $x' \in \operatorname{Ext}(BP_*/p^i)$ under a suitable connecting homomorphism. In this latter group one has multiplication by a suitable power of v_1 (depending on i), say v_1^m. x' may or may not be annihilated by some power of v_1^m, say v_1^{mj}. If not, we say x is v_1-*periodic*; otherwise x' is the image of some $x'' \in \operatorname{Ext}(BP_*/(p^i, v_1^{mj}))$ and we say it is v_1-*torsion*. In this new Ext group one has multiplication by v_2^n for some n. If x is v_1-torsion, it is either v_2-periodic or v_2-torsion depending on whether x'' is killed by some power of v_2^n. Iterating this procedure one obtains a complete filtration of the original Ext group in which the nth subgroup in the v_n-torsion and the nth subquotient is v_n-periodic. This is *the chromatic filtration* and it is associated with the *chromatic spectral sequence* of 5.1.8. The chromatic spectral sequence is like a spectrum in the astronomical sense in that it resolves stable homotopy into periodic components of various types.

Recently we have shown that this algebraic construction has a geometric origin, i.e., that there is a corresponding filtration of $\pi_*(S^0)$. The chromatic spectral sequence is based on certain inductively defined short exact sequences of comodules 5.1.5. In Ravenel [**9**] we show that each of these can be realized by a cofibration

$$N_n \to M_n \to N_{n+1}$$

with $N_0 = S^0$ so we get an inverse system

$$S^0 \leftarrow \Sigma^{-1} N_1 \leftarrow \Sigma^{-2} N_2 \leftarrow \cdots .$$

The filtration of $\pi_*(S^0)$ by the images of $\pi_*(\Sigma^{-n} N^n)$ is the one we want. Applying the Novikov Ext functor to this diagram yields the chromatic spectral sequence, and applying homotopy yields a geometric form of it. For more discussion of this and related problems see Ravenel [**8**].

The chromatic spectral sequence is useful computationally as well as conceptually. In 5.1.10 we introduce the chromatic cobar complex $CC(BP_*)$. Even though it is larger than the already ponderous cobar complex $C(BP_*)$, it is easier to work with because many cohomology classes (e.g., the Greek letter elements) have far simpler cocycle representatives in CC than in C.

In Section 1 the basic properties of the chromatic spectral sequence are given, most notably the change-of-rings theorem 5.1.14, which equates certain Ext groups with the cohomology of certain Hopf algebras $\Sigma(n)$, the nth Morava stabilizer algebra. This isomorphism enables one to compute these groups and was the original motivation for the chromatic spectral sequence. These computations will be the subject of the next chapter. Section 1 also contains various computations (5.1.20–5.1.22 and 5.1.24) which illustrate the use of the chromatic cobar complex.

In Section 2 we compute various Ext^1 groups (5.2.6, 5.2.11, 5.2.14, and 5.2.17) and recover as a corollary the Hopf invariant one theorem (5.2.8), which says almost all elements in the Adams spectral sequence $E_2^{1,*}$ are not permanent cycles. Our method of proof is to show they are not in the image of the Adams–Novikov $E_2^{1,*}$ after computing the latter.

In Section 3 we compute the v_1-periodic part of the Adams–Novikov spectral sequence and its relation to the J-homomorphism and the μ-family of Adams [**1**]. The main result is 5.3.7, and the resulting pattern in the Adams–Novikov spectral sequence for $p = 2$ is illustrated in 5.3.8.

In Section 4 we describe Ext^2 for all primes (5.4.5), referring to the original papers for the proofs, which we cannot improve upon. Corollaries are the nontriviality of γ_t, (5.4.4) and a list of elements in the Adams spectral sequence $E_2^{2,*}$ which cannot be permanent cycles (5.4.7). This latter result is an analog of the Hopf invariant one theorem. The Adams spectral sequence elements not so excluded include the Arf invariant and η_j families. These are discussed in 5.4.8–5.4.10.

In Section 5 we compile all known results about which elements in Ext^2 are permanent cycles, i.e., about the β-family and its generalizations. We survey the relevant work of Smith and Oka for $p \geq 5$, Oka and Toda for $p = 3$, and Davis and Mahowald for $p = 2$.

In Section 6 we give some fragmentary results on Ext^s for $s \geq 3$. We describe some products of α's and β's and their divisibility properties. We close the chapter by describing a possible obstruction to the existence of the δ-family.

Since the appearance of the first edition, many computations related to the chromatic spectral sequence have been made by Shimomura. A list of some of them can be found in Shimomura [**2**]. A description of the first three columns of the chromatic spectral sequence (meaning the rational, v_1- and v_2-periodic parts) for the sphere can be found in Shimomura and Wang [**3**] for $p = 2$, in Shimomura and Wang [**4**] for $p = 3$, and in Shimomura and Yabe [**5**] for $p \geq 5$. Analogous computations for the mod p Moore spectrum can be found in Shimomura [**6**] for $p = 2$, in Shimomura [**7**] for $p = 3$ and in Shimomura [**8**] for $p \geq 5$.

1. The Algebraic Construction

In this section we set up the chromatic spectral sequence converging to the Adams–Novikov E_2-term, and use it to make some simple calculations involving Greek letter elements (1.3.17 and 1.3.19). The chromatic spectral sequence was originally formulated by Miller, Ravenel, and Wilson [**1**]. First we make the following abbreviation in notation, which will be in force throughout this chapter: given a $BP_*(BP)$ comodule M (A1.1.2), we define

(5.1.1) $$\mathrm{Ext}(M) = \mathrm{Ext}_{BP_*(BP)}(BP_*, M).$$

To motivate our construction recall the short exact sequence of comodules given by 4.3.2(c)

(5.1.2) $$0 \to \Sigma^{2(p^n-1)} BP_*/I_n \xrightarrow{v_n} BP_*/I_n \to BP_*/I_{n+1} \to 0$$

and let

$$\delta_n\colon\ \mathrm{Ext}^s(BP_*/I_{n+1}) \to \mathrm{Ext}^{s+1}(BP_*/I_n)$$

denote the corresponding connecting homomorphism.

5.1.3. DEFINITION. *For $t, n > 0$ let*

$$\alpha_t^{(n)} = \delta_0\delta_1 \cdots \delta_{n-1}(v_n^t) \in \mathrm{Ext}^n(BP_*). \qquad \square$$

Here $\alpha^{(n)}$ stands for the nth letter of the Greek alphabet. The status of these elements in π_*^S is described in 1.3.11, 1.3.15, and 1.3.18. The invariant prime ideals in I_n in 5.1.2 can be replaced by invariant regular ideals, e.g., those provided by 4.3.3. In particular we have

5.1.4. DEFINITION. $\alpha_{sp^i/i+1} \in \mathrm{Ext}^{1,qsp^i}(BP_*)$ *(where $q = 2p - 2$) is the image of $v_1^{sp^i}$ under the connecting homomorphism for the short exact sequence*

$$0 \to BP_* \xrightarrow{p^{i+1}} BP_* \to BP_*/(p^{i+1}) \to 0. \qquad \square$$

We will see below that for $p > 2$ these elements generate $\mathrm{Ext}^1(BP_*)$ (5.2.6) and that they are nontrivial permanent cycles in $\mathrm{im}\, J$. We want to capture all of these elements from a single short exact sequence; those of 5.1.4 are related by the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & BP_* \arrow[r, "p^i"] \arrow[d, equal] & BP_* \arrow[r] \arrow[d, "p"] & BP_*/(p^i) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & BP_* \arrow[r, "p^{i+1}"] & BP_* \arrow[r] & BP_*/(p^{i+1}) \arrow[r] & 0
\end{tikzcd}$$

Taking the direct limit we get

$$0 \to BP_* \to \mathbf{Q} \otimes BP_* \to \mathbf{Q}/\mathbf{Z}_{(p)} \otimes BP_* \to 0;$$

we denote these three modules by N^0, M^0, and N^1, respectively. Similarly, the direct limit of the sequences

$$0 \to BP_*/(p^{i+1}) \xrightarrow{v_1^{p^{i+j}}} \Sigma^{-qp^{i+j}} BP_*/(p^{i+1}) \to \Sigma^{-1p^{i+j}} BP_*/(p^{i+1}, v_1^{p^{i+j}}) \to 0$$

gives us

$$0 \to BP_*/(p^\infty) \to v_1^{-1}BP_*/(p^\infty) \to BP_*/(p^\infty, v_1^\infty) \to 0$$

and we denote these three modules by N^1, M^1, and N^2, respectively. More generally we construct short exact sequences

$$0 \to N^n \to M^n \to N^{n+1} \to 0 \tag{5.1.5}$$

inductively by $M^n = v_n^{-1}BP_* \otimes_{BP_*} N^n$. Hence N^n and M^n are generated as $\mathbf{Z}_{(p)}$-modules by fractions $\frac{x}{y}$ where $x \in BP_*$ for N^n and $v_n^{-1}BP_*$ for M^n and y is a monomial in the ideal $(pv_1 \cdots v_{n-1})$ of the subring $\mathbf{Z}_{(p)}[v_1, \ldots, v_{n-1}]$ of BP_*. The BP_*-module structure is such that $vx/y = 0$ for $v \in BP_*$ if this fraction when reduced to lowest terms does not have its denominator in the above ideal. For example, the element $\frac{1}{p^i v_1^j} \in N^2$ is annihilated by the ideal (p^i, v_1^j).

5.1.6. LEMMA. 5.1.5 *is a short exact sequence of $BP_*(BP)$-comodules.*

PROOF. Assume inductively that N^n is a comodule and let $N' \subset N^n$ be a finitely generated subcomodule. Then N' is annihilated by some invariant regular ideal with n generators given by 4.3.3. It follows from 4.3.3 that multiplication by some power of v_n, say v_n^k, is a comodule map, so

$$v_n^{-1}N' = \varinjlim_{v_n^k} \Sigma^{-\dim v_n^{ki}} N'$$

is a comodule. Alternatively, N' is annihilated by some power of I_n, so multiplication by a suitable power of v_n is a comodule map by Proposition 3.6 of Landweber [7] and $v_n^{-1}N'$ is again a comodule. Taking the direct limit over all such N' gives us a unique comodule structure on M^n and hence on the quotient N^{n+1}. $\square$

5.1.7. DEFINITION. *The* chromatic resolution *is the long exact sequence of comodules*

$$0 \to BP_* \to M^0 \xrightarrow{d_e} M^1 \xrightarrow{d_e} \cdots$$

obtained by splicing the short exact sequences of 5.1.5. $\square$

The associated resolution spectral sequence (A1.3.2) gives us

5.1.8. PROPOSITION. *There is a chromatic spectral sequence converging to* $\mathrm{Ext}(BP_*)$ *with* $E_1^{n,s} = \mathrm{Ext}^s(M^n)$ *and* $d_r\colon E_r^{n,s} \to E_r^{n+r,s+1-r}$ *where* d_1 *is the map induced by* d_e *in* 5.1.7. $\square$

5.1.9. REMARK. There is a chromatic spectral sequence converging to $\mathrm{Ext}(F)$ where F is any comodule which is flat as a BP_*-module, obtained by tensoring the resolution of 5.1.7 with F.

5.1.10. DEFINITION. *The* chromatic cobar complex $CC(BP_*)$ *is given by*

$$CC^u(BP_*) = \bigoplus_{s+n=u} C^s(M^n),$$

where $C(\ )$ *is the cobar complex of A1.2.11, with* $d(x) = d_e^*(x) + (-1)^n d_i(x)$ *for* $x \in C^s(M^n)$ *where* d_e^* *is the map induced by* d_e *in 5.1.7 (the external component of* d*) and* d_i *(the internal component) is the differential in the cobar complex* $C(M_n)$*.* $\square$

It follows from 5.1.8 and A1.3.4 that $H(CC(BP_*)) = H(C(BP_*)) = \mathrm{Ext}(BP_*)$. The embedding $BP_* \to M^0$ induces an embedding of the cobar complex $C(BP_*)$ into the chromatic cobar complex $CC(BP_*)$. Although $CC(BP_*)$ is larger than $C(BP_*)$, we will see below that it is more convenient for certain calculations such as identifying the Greek letter elements of 5.1.3.

This entire construction can be generalized to BP_*/I_m as follows.

5.1.11. DEFINITION. *Let* $N_m^0 = BP_*/I_m$ *and define* BP_**-modules* N_m^n *and* M_m^n *inductively by short exact sequences*

$$0 \to N_m^n \to M_m^n \to N_m^{n+1} \to 0$$

where $M_m^n = v_{m+n}^{-1} BP_* \otimes_{BP_*} N_m^n$. $\square$

Lemma 5.1.6 can be generalized to show that these are comodules. Splicing them gives a long exact sequence

$$0 \to BP_*/I_m \to M_m^0 \xrightarrow{d_e} M_m^1 \xrightarrow{d_e} \cdots$$

and a chromatic spectral sequence as in 5.1.8. Moreover BP_*/I_m can be replaced by any comodule L having an increasing filtration $\{F_i L\}$ such that each subquotient

F_i/F_{i-1} is a suspension of BP_*/I_m, e.g., $L = BP_*/I_m^k$. We leave the details to the interested reader.

Our main motivation here, besides the Greek letter construction, is the computability of $\operatorname{Ext}(M_n^0)$; it is essentially the cohomology of the automorphism group of a formal group law of height n (1.4.3 and A2.2.18). This theory will be the subject of Chapter 6. We will state the first major result now. We have $M_n^0 = v_n^{-1}BP_*/I_n$, which is a comodule algebra (A1.1.2), so $\operatorname{Ext}(M_n^0)$ is a ring (A1.2.14). In particular it is a module over $\operatorname{Ext}^0(M_n^0)$. The following is an easy consequence of the Morava–Landweber theorem, 4.3.2.

5.1.12. Proposition. *For $n > 0$, $\operatorname{Ext}^0(M_n^0) = \mathbf{Z}/(p)[v_n, v_n^{-1}]$. We denote this ring by $K(n)_*$. [The case $n = 0$ is covered by 5.2.1, so it is consistent to denote $\mathbf{Q}$ by $K(0)_*$.]* $\square$

5.1.13. Definition. *Make $K(n)_*$ a BP_*-module by defining multiplication by v_i to be trivial for $i \neq n$. Then let $\Sigma(n) = K(n)_* \otimes_{BP_*} BP_*(BP) \otimes_{BP_*} K(n)_*$.* $\square$

$\Sigma(n)$, the nth Morava stabilizer algebra, is a Hopf algebroid which will be closely examined in the next chapter. It has previously been called $K(n)_*K(n)$, e.g., in Miller, Ravenel, and Wilson [**1**], Miller and Ravenel [**5**], and Ravenel [**5, 6**]. $K(n)_*$ is also the coefficient ring of the nth Morava K-theory; see Section 4.2 for references. We have changed our notation to avoid confusion with $K(n)_*(K(n))$, which is $\Sigma(n)$ tensored with a certain exterior algebra.

The starting point of Chapter 6 is

5.1.14. Change-of-Rings Theorem (Miller and Ravenel [**5**]).

$$\operatorname{Ext}(M_n^0) = \operatorname{Ext}_{\Sigma(n)}(K(n)_*, K(n)_*). \qquad \square$$

We will also show (6.2.10)

5.1.15. Morava Vanishing Theorem. *If $(p-1) \nmid n$ then $\operatorname{Ext}^s(M_n^0) = 0$ for $s > n^2$.* $\square$

Moreover this Ext satisfies a kind of Poincaré duality, e.g.,

$$\operatorname{Ext}^s(M_n^0) = \operatorname{Ext}^{n^2-s}(M_n^0),$$

and it is essentially the cohomology of a certain n stage nilpotent Lie algebra of rank n^2. If we replace $\Sigma(n)$ with a quotient by a sufficiently large finitely generated subalgebra, then this Lie algebra becomes abelian and the Ext [even if $(p-1)$ divides n] becomes an exterior algebra over $K(n)_*$ on n^2 generators of degree one.

To connect these groups with the chromatic spectral sequence we have

5.1.16. Lemma. *There are short exact sequences of comodules*

$$0 \to M_{m+1}^{n-1} \xrightarrow{j} \Sigma^{\dim v_m} M_m^n \xrightarrow{v_m} M_m^n \to 0$$

and Bockstein spectral sequences converging to $\operatorname{Ext}(M_m^n)$ with

$$E_1^{s,*} = \operatorname{Ext}^s(M_{m+1}^{n-1}) \otimes P(a_m)$$

where multiplication by a_m in the Bockstein spectral sequence corresponds to division by v_m in $\operatorname{Ext}(M_m^n)$. d_r is not a derivation but if $d_r(a_m^r x) = y \neq 0$ then $d_r(a_m^{r+i}x) = v_m^i y$.

Proof. The spectral sequence is that associated with the increasing filtration of M_m^n defined by $F_iM_m^n = \ker v_m^i$ (see A1.3.9). Then $E^0M_m^n = M_{m+1}^{n-1} \otimes P(a_m)$. $\square$

Using 5.1.16 n times we can in principle get from $\mathrm{Ext}(M_n^0)$ to $\mathrm{Ext}(M_0^n) = \mathrm{Ext}(M^n)$ and hence compute the chromatic E_1-term (5.1.8). In practice these computations can be difficult.

5.1.17. REMARK. We will not actually use the Bockstein spectral sequence of 5.1.16 but will work directly with the long exact sequence

$$\to \mathrm{Ext}^s(M_{m+1}^{n-1}) \xrightarrow{j} \mathrm{Ext}^s(M_m^n) \xrightarrow{v_m} \mathrm{Ext}^s(\Sigma^{-2p^m+2} M_m^n) \xrightarrow{\delta} \mathrm{Ext}^{s+1}(M_{m+1}^{n-1}) \to \cdots$$

by induction on s. Given an element $x \in \mathrm{Ext}(M_{m+1}^{n-1})$ which we know not to be in $\operatorname{im}\delta$, we try to divide $j(x)$ by v_m as many times as possible. When we find an $x' \in \mathrm{Ext}(M_m^n)$ with $v_m^r x' = j(x)$ and $\delta(x') = y \neq 0$ then we will know that $j(x)$ cannot be divided any further by v_m. Hence δ serves as reduction mod I_{m+1}. This state of affairs corresponds to $d_r(a_m^r x) = y$ in the Bockstein spectral sequence of 5.1.16. We will give a sample calculation with δ below (5.1.20).

We will now make some simple calculations with the chromatic spectral sequence starting with the Greek letter elements of 5.1.3. The short exact sequence of 5.1.2 maps to that of 5.1.5, i.e., we have a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & BP_*/I_n \arrow[r, "v_n"] \arrow[d, "i"] & \Sigma^{-\dim v_n} BP_*/I_n \arrow[r] \arrow[d] & \Sigma^{-\dim v_n} BP_*/I_{n+1} \arrow[r] \arrow[d, "i"] & 0 \\
0 \arrow[r] & N^n \arrow[r] & M^n \arrow[r] & N^{n+1} \arrow[r] & 0
\end{tikzcd}$$

with

$$i(v_{n+1}^t) = \frac{v_{n+1}^t}{pv_1 \cdots v_n}.$$

Hence $\alpha_t^{(n)}$ can be defined as the image of $i(v_n^t)$ under the composite of the connecting homomorphisms of 5.1.5, which we denote by $\alpha\colon \mathrm{Ext}^0(N^n) \to \mathrm{Ext}^n(BP_*)$. On the other hand, the chromatic spectral sequence has a bottom edge homomorphism

$$\begin{tikzcd}
\mathrm{Ext}^0(M^n) \arrow[r, equal] & E_1^{n,0} & & \\
\mathrm{Ext}^0(N^n) \arrow[u] \arrow[r] & \ker d_1 \arrow[u] \arrow[r] & E_\infty^{n,0} \arrow[r] & \mathrm{Ext}^n(BP_*)
\end{tikzcd}$$

which we denote by

$$\kappa\colon \mathrm{Ext}^0(N^n) \to \mathrm{Ext}^n(BP_*).$$

κ and α differ by sign, i.e.,

5.1.18. PROPOSITION. $\kappa = (-1)^{[(n+1)/2]}\alpha$, *where* $[x]$ *is the largest integer not exceeding* x.

PROOF. The image y_0 of $i(v_n^t)$ in M^n is an element in the chromatic complex (5.1.10) cohomologous to some class in the cobar complex $C(BP_*)$. Inductively we can find $x_s \in C^s(M^{n-s-1})$, and $y_s \in C^s(M^{n-s})$ such that $d_e(x_s) = y_s$ and $d_i(x_s) = y_{s+1}$. Moreover $y_n \in C^n(M^0)$ is the image of some $x_n \in C^n(BP_*)$. It follows from the definition of the connecting homomorphism that x_n is a cocycle representing $\alpha(i(v_n^t)) = \alpha_t^{(n)}$. On the other hand, y_s is cohomologous to $(-1)^{n-s}y_{s+1}$ in $CC(BP_*)$ by 5.1.10 and $\prod_{s=0}^{n-1}(-1)^{n-s} = (-1)^{[n+1/2]}$ so x_n represents $(-1)^{[n+1/2]}\kappa(i(v_n^t))$. $\square$

5.1.19. DEFINITION. *If $x \in \mathrm{Ext}^0(M^n)$ is in the image of $\mathrm{Ext}^0(N^n)$ (and hence gives a permanent cycle in the chromatic spectral sequence) and has the form*

$$\frac{v_n^t}{p^{i_0} v_1^{i_1} \cdots v_{n-1}^{i_{n-1}}} \bmod I_n$$

(i.e., x is the indicated fraction plus terms with larger annihilator ideals) then we denote $\alpha(x)$ by $\alpha^{(n)}_{t/i_{n-1},\ldots,i_0}$; if for some $m < n$, $i_k = 1$ for $k \le m$ then we abbreviate $\alpha(x)$ by $\alpha^{(n)}_{t/i_{n-1},\ldots,i_{m+1}}$. □

5.1.20. EXAMPLES AND REMARKS. We will compute the image of β_t in $\mathrm{Ext}^2(BP_*/I_2)$ for $p > 2$ in two ways.

(a) We regard β_t as an element in $\mathrm{Ext}^0(M^2)$ and compute its image under connecting homomorphisms δ_0 to $\mathrm{Ext}^1(M_1^1)$ and then δ_1 to $\mathrm{Ext}^2(M_2^0)$, which is $E_1^{0,2}$ in the chromatic spectral sequence for $\mathrm{Ext}(BP_*/I_2)$. To compute δ_0, we pick an element in $x \in M^2$ such that $px = \beta_t$, and compute its coboundary in the cobar complex $C(M^2)$. The result is necessarily a cocycle of order p, so it can be pulled back to $\mathrm{Ext}^1(M_1^1)$. To compute δ_1 on this element we take a representative in $C^1(M_1^1)$, divide it by v_1, and compute its coboundary.

Specifically β_t is $\frac{v_2^t}{pv_1} \in M^2$, so we need to compute the coboundary of $x = \frac{v_2^t}{p^2 v_1}$. It is convenient to write x as $\frac{v_1^{p-1} v_2^t}{p^2 v_1^p}$, then the denominator is the product of elements generating an invariant regular ideal, which means that we need to compute η_R on the numerator only. We have

$$\eta_R(v_1^{p-1}) \equiv v_1^{p-1} - p v_1^{p-2} t_1 \mod (p^2)$$

and

$$\eta_R(v_2^t) \equiv v_2^t + t v_2^{t-1}(v_1 t_1^p + p t_2) \mod (p^2, pv_1, v_1^2).$$

These give

$$d\left(\frac{v_t^{p-1} v_2^t}{p^2 v_1^p}\right) = \frac{-v_2^t t_1}{p v_1^2} + \frac{t v_2^{t-1}}{p v_1}(t_2 - t_1^{1+p}).$$

This is an element of order p in $C^1(M^2)$, so it is in the image of $C^1(M_1^1)$. In this group the p in the denominator is superfluous, since everything has order p, so we omit it. To compute δ_1 we divide by v_1 and compute the coboundary; i.e., we need to find

$$d\left(\frac{-v_2^t t_1}{v_1^3} + \frac{t v_2^{t-1}(t_2 - t_1^{1+p})}{v_1^2}\right).$$

Recall (4.3.15)

$$\Delta(t_2) = t_2 \otimes 1 + t_1 \otimes t_1^p + 1 \otimes t_2 + v_1 b_{10}$$

where

$$b_{10} = -\sum_{0<i<p} \frac{1}{p}\binom{p}{i} t_1^i \otimes t_1^{p-i}$$

as in 4.3.14. From this we get

$$\begin{aligned} d\left(\frac{-v_2^t t_1}{v_1^3}+\frac{tv_2^{t-1}}{v_1^2}(t_2-t_1^{1+p})\right) =&\frac{-tv_2^{t-1}t_1^p|t_1}{v_1^2}-\binom{t}{2}\frac{v_2^{t-2}}{v_1}t_1^{2p}|t_1\\ &+t(t-1)\frac{v_2^{t-2}}{t_1^p}|(t_2-t_1^{1+p})\\ &+t\frac{v_2^{t-1}}{v_1^2}(-v_1b_{10}+t_1^p|t_1)\\ =&\binom{t}{2}\frac{v_2^{t-2}}{v_1}(2t_1^p|t_2-2t_1^p|t_1^{1+p}-t_1^{2p}|t_1)\\ &-t\frac{v_2^{t-1}}{v_1}b_{10}. \end{aligned}$$

We will see below that $\mathrm{Ext}^2(M_2^0)$ has generators k_0 represented by $2t_1^p|t_2-2t_1^p|t_1^{1+p}-t_1^{2p}|t_1$ and b_{10}. Hence the mod I_2 reduction of $-\beta_t$ is

$$\binom{t}{2}v_2^{t-2}k_0+tv_2^{t-1}b_{1,0}.$$

(b) In the chromatic complex $CC(BP_*)$ (5.1.10), $\beta_t\in M^2$ is cohomologous to elements in $C^1(M^1)$ and $C^2(M^0)$. These three elements pull back to N^2, $C^1(N^1)$, and $C^2(N^0)$, respectively. In theory we could compute the element in $C^2(N^0)=C^2(BP_*)$ and reduce mod I_2, but this would be very laborious. Most of the terms of the element in $C^0(BP_*)$ are trivial mod I_2, so we want to avoid computing them in the first place. The passage from $C^0(N^2)$ to $C^2(BP_*)$ is based on the four-term exact sequence

$$0\to BP_*\to M^0\to M^1\to N^2\to 0.$$

Since $\frac{v_2^t}{pv_1}\in N^2$ is in the image of $\Sigma^{-q}BP_*/I_2$, we can replace this sequence with

$$0\to BP_*\xrightarrow{p}BP_*\xrightarrow{v_1}\Sigma^{-q}BP_*/I_1\to\Sigma^{-q}BP_*/I_2\to 0.$$

We are going to map the first BP_* to BP_*/I_2; we can extend this to a map of sequences to

$$0\to BP_*/I_2\xrightarrow{p}BP_*/(p^2,pv_1,v_1^2)\xrightarrow{v_1}\Sigma^{-q}BP_*/(p,v_1^3)\to\Sigma^{-q}BP_*/I_2\to 0,$$

which is the identity on the last comodule. [The reader may be tempted to replace the middle map by

$$BP_*/(p^2,v_1)\xrightarrow{v_1}\Sigma^{-q}BP_*/(p,v_1^2)$$

but $BP_*/(p^2,v_1)$ is not a comodule.] This sequence tells us which terms we can ignore when computing in the chromatic complex, as we will see below.

Specifically we find (ignoring signs) that $\frac{v_2^t}{pv_1}\in M^2$ is cohomologous to

$$\frac{tv_2^{t-1}t_1^p}{p}+\binom{t}{2}\frac{v_1v_2^{t-2}}{p}t_1^{2p}+\text{higher terms}.$$

Note that the first two terms are divisible by v_1 and v_1^2 respectively in the image of $C^1(\Sigma^{-q}BP_*/(p))$ in $C^1(M^1)$. The higher terms are divisible by v_1^3 and can therefore be ignored.

In the next step we will need to work mod I_2^2 in the image of $C^2(BP_*)$ in $C^2(M^0)$ via multiplication by p. From the first term above we get

$$t(t-1)v_2^{t-2}t_2|t_1^p + tv_2^{t-1}b_{10},$$

while the second term gives

$$\binom{t}{2}v_2^{t-2}t_1|t_1^{2p}$$

and their sum represents the same element obtained in (a).

Our next result is

5.1.21. Proposition. *For $n \geq 3$,*

$$\alpha_1^{(n)} = (-1)^n \alpha_1 \alpha_{p-1}^{(n-1)}. \qquad \square$$

For $n = 3$ this gives $\gamma_1 = -\alpha_1\beta_{p-1}$. In the controversy over the nontriviality of γ_1 (cf. the paragraph following 1.3.18) the relevant stem was known to be generated by $\alpha_1\beta_{p-1}$, so what follows is an easy way (given all of our machinery) to show $\gamma_1 \neq 0$.

Proof of 5.1.2. α_1 is easily seen to be represented by t_1 in $C(BP_*)$, while $\alpha_1^{(n)}$ and $\alpha_{p-1}^{(n-1)}$ are represented by

$$(-1)^{[n+1/2]}\frac{v_n}{pv_1\cdots v_{n-1}} \in M^n \quad \text{and} \quad (-1)^{[n/2]}\frac{v_{n-1}^{p-1}}{pv_1\cdots v_{n-2}} \in M^{n-1},$$

respectively. Hence $(-1)^n\alpha_1\alpha_{p-1}^{(n-1)} = -\alpha_{p-1}^{(n-1)}\alpha_1$ is represented by

$$(-1)^{[n/2]}\frac{v_{n-1}^{p-1}t_1}{pv_1\cdots v_{n-2}} \in C^1(M^{n-1}) \subset CC^n(BP_*)$$

and it suffices to show that this element is cohomologous to $\frac{(-1)^{[n+1/2]}v_n}{(pv_1\cdots v_{n-1})}$ in $CC(BP_*)$.

Now consider

$$x = \frac{v_{n-1}^{-1}v_n}{pv_1\cdots v_{n-2}} - \frac{v_{n-1}^p}{pv_1\cdots v_{n-3}v_{n-2}^{1+p}} \in M^{n-1}.$$

Clearly

$$d_e(x) = \frac{v_n}{pv_1\cdots v_{n-1}}.$$

To compute $d_i(x)$ we need to know $\eta_R(v_{n-1}^{-1}v_n)$ mod I_{n-1} and $\eta_R(v_{n-1}^p)$ mod $(p, v_1, \ldots, v_{n-3}, v_{n-2}^{1+p})$ since $d_i(x) = \eta_R(x) - x$. We know

$$\eta_R(v_n) \equiv v_n + v_{n-1}t_1^{p^{n-1}} - v_{n-1}^p t_1 \mod I_{n-1}$$

by 4.3.21, so

$$\eta_R(v_{n-1}^p) \equiv v_{n-1}^p + v_{n-2}^p t_1^{p^{n-1}} - v_{n-2}^{p^2}t_1^p \mod I_{n-2}.$$

Hence

$$\eta_R(v_{n-1}^{-1}v_n) - v_{n-1}^{-1}v_n \equiv t_1^{p^{n-1}} - v_{n-1}^{p-1}t_1 \mod I_{n-1}$$

and

$$\eta_R(v_{n-1}^p) - v_{n-1}^p \equiv v_{n-2}^p t_1^{p^{n-1}} \mod (p, v_1, \ldots, v_{n-3}, v_{n-2}^{1+p}).$$

It follows that

$$d_i(x) = \frac{-v_{n-1}^{p-1}t_1}{pv_1\cdots v_{n-2}}$$

so

$$d(x) = \frac{v_n}{pv_1\cdots v_{n-1}} + (-1)^n\frac{v_{n-1}^{p-1}t_1}{pv_1\cdots v_{n-2}}$$

and a simple sign calculation gives the result. □

For $p = 2$ 5.1.21 says $\alpha_1^{(n)} = \alpha_1^{n-2}\alpha_1^{(2)}$ for $n \geq 2$. We will show that each of these elements vanishes and that they are killed by higher differentials (d_{n-1}) in the chromatic spectral sequence. We do not know if there are nontrivial d_r's for all $r \geq 2$ for odd primes.

5.1.22. THEOREM. *In the chromatic spectral sequence for $p = 2$ there are elements $x_n \in E_{n-1}^{1,n-2}$ for $n \geq 2$ such that*

$$d_{n-1}(x_n) = \frac{v_n}{2v_1\cdots v_{n-1}} \in E_{n-1}^{n,0}.$$

PROOF. Fortunately we need not worry about signs this time. Equation 4.3.1 gives $\eta_R(v_1) = v_1 - 2t_1$ and $\eta_R(v_2) \equiv v_2 + v_1t_1^2 + v_1^2t_1 \mod (2)$. We find then that

$$x_2 = \frac{v_1^2 + 4v_1^{-1}v_2}{8}$$

has the desired property. For $n > 2$ we represented x_n by

$$\frac{[(t_2 - t_1^3 + v_1^{-1}v_2t_1)|t_1|\cdots|t_1|]}{2} \in C^{n-2}(M^1)$$

with $n-3$ t_1's. To compute $d_{n-1}(x_n)$ let

$$\tilde{x}_n = x_n + \sum_{i=1}^{n-2}\frac{(v_{i+1}^2 - v_i^2v_{i+1}^{-1}v_{i+2})t_1|\cdots|t_1}{2v_1\cdots v_{i-1}v_i^3} \in CC(BP_*),$$

where the ith term has $(n-2-i)$ t_1's. Then one computes

$$d(\tilde{x}_n) = \frac{v_n}{2v_1\cdots v_{n-1}},$$

so

$$d_{n-1}(x_n) = \frac{v_n}{2v_1\cdots v_{n-1}}$$

unless this element is killed by an earlier differential, in which case x_n, would represent a nontrivial element in $\mathrm{Ext}^{n-1,2n}(BP_*)$, which is trivial by 5.1.23 below. □

5.1.23. EDGE THEOREM.
(a) For all primes p $\mathrm{Ext}^{s,t}(BP_*) = 0$ for $t < 2s$,
(b) for $p = 2$ $\mathrm{Ext}^{s,2s}(BP_*) = \mathbf{Z}/(2)$ for $s \geq 1$, and
(c) for $p = 2$ $\mathrm{Ext}^{s,2s+2}(BP_*) = 0$ for $s \geq 2$.

PROOF. We use the cobar complex $C(BP_*)$ of A1.2.11. Part (a) follows from the fact that $C^{s,t}$ for $t < 2s$. $C^{s,2s}$ is spanned by $t_1|\cdots|t_1$ while $C^{s,2s+2}$ is spanned by $v_1t_1|\cdots|t_1$ and $e_j = t_1|\cdots t_1|t_1^2|t_1\cdots t_1$ with t_1^2 in the jth position, $1 \leq j \leq s$. Since $d(t_1^3) = -3t_1|t_1^2 - 3t_1^2|t_1$, the e_j's differ by a coboundary up to sign. Part (b) follows from

$$d(e_1) = 2t_1|\cdots|t_1 = -d(v_1t_1|\cdots|t_1)$$

and (c) follows from

$$d(t_2|t_1|\cdots|t_1) = -v_1t_1|\cdots|t_1 - e_1.$$

$\square$

We conclude this section by tying up some loose ends in Section 4.4. For $p > 2$ we need

5.1.24. LEMMA. *For odd primes, $\alpha_1\beta_p$ is divisible by p but not by p^2. (This gives the first element of order p^2 in $\operatorname{Ext}^s(BP_*)$ for $s \geq 2$.)*

PROOF. Up to sign $\alpha_1\beta_p$ is represented by $\frac{v_2^pt_1}{pv_1}$. Now $\frac{v_2^pt_1}{p^2v_1}$ is not a cocycle, but if we can get a cocycle by adding a term of order p then we will have the desired divisibility. It is more convenient to write this element as $\frac{v_1^{p-1}v_2^pt_1}{p^2v_1^p}$; then the factors of the denominator form an invariant sequence [i.e., $\eta_R(v_1^p) \equiv v_1^p \mod (p^2)$], so to compute the coboundary it suffices to compute $\eta_R(v_1^{p-1}v_2) \mod (p^2, v_1^p)$. We find

$$d\left(\frac{v_1^{p-1}v_2^p}{p^2v_1^p}\, t_1\right) = \frac{-v_2^pt_1|t_1}{pv_1^2} = \frac{1}{2}\, d\left(\frac{v_2^pt_1^2}{pv_1^2}\right)$$

so the desired cocycle is

$$\frac{v_1^{p-1}v_2^pt_1}{p^2v_1^p} - \frac{1}{2}\,\frac{v_2^pt_1^2}{pv_1^2}.$$

This divisibility will be generalized in (5.6.2).

To show that $\alpha_1\beta_p$ is not divisible by p^2 we compute the mod (p) reduction of our cocycle. More precisely we compute its image under the connecting homomorphism associated with

$$0 \to M_1^1 \to M_0^2 \xrightarrow{p} M_0^2 \to 0$$

(see 5.1.16). To do this we divide by p and compute the coboundary. Our divided (by p) cocycle is

$$\frac{v_1^{p^2-1}v_2^pt_1}{p^3v_1^{p^2}} - \frac{1}{2}\,\frac{v_1^{p-2}v_2^pt_1^2}{p^2v_1^p}$$

and its coboundary is

$$\frac{v_2^p(t_1^2|t_1 + t_1|t_1^2)}{pv_1^3} + \frac{v_2^{p-1}t_2|t_1}{pv_1} - \frac{1}{2}\frac{v_2^{p-1}t_1^p|t_1^2}{pv_1} - \frac{v_2^{p-1}t_1^{1+p}|t_1}{pv_1}$$

We can eliminate the first term by adding $\frac{1}{3}\frac{v_2^pt_1^3}{pv_i^3}$ (even if $p = 3$). For $p > 3$ the resulting element in $C^2(M_1^1)$ is

$$\frac{v_2^{p-1}(t_2|t_1 - t_1^{1+p}|t_1 - t_1^p|t_2)}{v_1}.$$

Reducing this mod I_2 in a similar fashion gives a unit multiple of $\bar{\phi}$ in 4.1.14. For $p = 3$ we add $\frac{-v_2t_1^6}{3v_1^2}$ to the divided cocycle and get

$$\frac{v_2^{p-1}(t_2|t_1 - \cdots)}{v_1} + \frac{v_2}{v_1}(t_1^3|t_1^6 + t_1^6|t_1^3),$$

which still gives a nonzero element in $\operatorname{Ext}^2(M_1^1)$. $\square$

For $p = 2$ we need to prove 4.4.38 and 4.4.40, i.e., 5.1.25. LEMMA. *In the notation of* 4.4.32 *for* $p = 2$ (a) $\delta_0(\beta_3) \equiv \beta^2_{2/2} + \eta_1 \mod (2)$,

(b) $\delta_0(\eta_2) \equiv c_0 \mod (2)$.

PROOF. For (a) we have

$$d\left(\frac{v_1v_2^3}{4v_1^2} + \frac{v_2v_3}{2v_1^2}\right) = \frac{v_2^2t_1^4}{2v_1^2} + \frac{v_3t_1^2 + v_2t_2^2 + v_2^2t_2 + v_2t_1^6}{2v_1},$$

which gives the result.

For (b) we use Massey products. We have $\eta_2\langle\eta_1, v_1, \beta_1\rangle$ so by A1.4.11 we have $\delta_0(\eta_2) \equiv \langle\eta_1, h_{10}, \beta_1\rangle \mod (2)$. Hence we have to equate this product with c_0, which by 4.4.31 is represented by $\frac{x_{22}}{v_1}$, where x_{22} is defined by 4.4.25. To expedite this calculation we will use a generalization of Massey products not given in A1.4 but fully described by May [**3**]. We regard η_1 as an element in $\mathrm{Ext}^1(M_1^1)$, and h_{10}, and β_1 as elements in $\mathrm{Ext}^1(BP_*/I_1)$ and use the pairing $M_1^1 \otimes BP_*/I_1 \to M_1^1$ to define the product. Hence the cocycles representing η_1, h_{10} and β_1 are

$$\frac{v_3t_1^2 + v_2(t_2^2 + t_1^6) + v_2^2t_2}{v_1}, \quad t_1, \quad \text{and} \quad t_1^2 + v_1t_1,$$

respectively. The cochains whose coboundaries are the two successive products are

$$\frac{v_3(t_2 + t_1^3) + v_2(t_3 + t_1t_2^2 + t_1^4t_2 + t_1^7) + v_2^2(t_1^4 + t_1t_2)}{v_1} \quad \text{and} \quad t_2.$$

If we alter the resulting cochain representative of the Massey product by the coboundary of

$$\frac{v_3t_1^2t_2 + v_2(t_2^3 + t_2t_1^6 + t_1^9) + v_2^2(t_1^6 + t_2^2)}{v_1} + \frac{v_2^4(t_2 + t_1^3)}{v_1^4} + \frac{v_2^5t_1}{v_1^5}$$

we get the desired result. □

2. $\mathrm{Ext}^1(BP_*/I_n)$ and Hopf Invariant One

In this section we compute $\mathrm{Ext}^1(BP_*/I_n)$ for all n. For $n > 0$ our main results are 5.2.14 and 5.2.17. For $n = 0$ this group is $E_2^{1,*}$ in the Adams–Novikov spectral sequence and is given in 5.2.6. In 5.2.8 we will compute its image in the classical Adams spectral sequence, thereby obtaining proofs of the essential content of the Hopf invariant one theorems 1.2.12 and 1.2.14. More precisely, we will prove that the specified h_i's are not permanent cycles, but we will not compute $d_2(h_i)$. The computation of $\mathrm{Ext}^1(BP_*/I_n)$ is originally due to Novikov [**1**] for $n = 0$ and to Miller and Wilson [**3**] for $n > 0$ (except for $n = 1$ and $p \geq 2$).

To compute $\mathrm{Ext}^1(BP_*)$ with the chromatic spectral sequence we need to know $\mathrm{Ext}^1(M^0)$ and $\mathrm{Ext}^0(M^1)$. For the former we have

5.2.1. THEOREM. (a) $\mathrm{Ext}^{s,t}(M^0) = \begin{cases} \mathbf{Q} & \text{if } s = t = 0 \\ 0 & \text{otherwise} \end{cases}$

(b) $\mathrm{Ext}^{0,t}(BP_*) = \begin{cases} \mathbf{Z}_{(p)} & \text{if } t = 0 \\ 0 & \text{otherwise} \end{cases}$.

PROOF. (a) Since $M^0 = \mathbf{Q} \otimes BP_*$, we have $\mathrm{Ext}(M^0) = \mathrm{Ext}_\Gamma(A, A)$ where $A = M^0$ and $\Gamma = \mathbf{Q} \otimes BP_*(BP)$. Since t_n is a rational multiple of $\eta_R(v_n) - v_n$ modulo decomposables, Γ is generated by the image of η_R and η_L and is therefore a unicursal Hopf algebroid (A1.1.11). Let $\bar{v}_n = \eta_R(v_n)$, so $\Gamma = A[\bar{v}_1, \bar{v}_2, \dots]$. The coproduct in Γ is given by $\Delta(v_n) = v_n \otimes 1$ and $\Delta(\bar{v}_n) = 1 \otimes \bar{v}_n$. The map $\eta_R\colon A \to \Gamma = A \otimes_A \Gamma$ makes A a right Γ-comodule. Let R be the complex $\Gamma \otimes E(y_1, y_2, \dots)$ where $E(y_1, y_2, \dots)$ is an exterior algebra on generators y_i of degree 1 and dimension $2(p^i - 1)$. Let the coboundary d be a derivation with $d(y_n) = d(\bar{v}_n) = 0$ and $d(v_n) = y_n$. Then R is easily seen to be acyclic with $H^0(R) = A$. Hence R is a suitable resolution for computing $\operatorname{Ext}_\Gamma(A, A)$ (A1.2.4). We have $\operatorname{Hom}_\Gamma(A, R) = A \otimes E(y_1, \dots)$ and this complex is easily seen to be acyclic and gives the indicated Ext groups for M^0.

For (b) $\operatorname{Ext}^0 BP_* = \ker d_e \subset \operatorname{Ext}^0(M_0^0)$ and $d_e(x) \neq 0$ if x is a unit multiple of a negative power of p. $\square$

To get at $\operatorname{Ext}(M^1)$ we start with

5.2.2. Theorem.

(a) *For $p > 2$, $\operatorname{Ext}(M_1^0) = K(1)_* \otimes E(h_0)$ where $h_0 \in \operatorname{Ext}^{1,q}$ is represented by t_1 in $C^1(M_1^0)$ (see 5.1.12) and $q = 2p - 2$ as usual.*

(b) *For $p = 2$, $\operatorname{Ext}(M_1^0) = K(1)_* \otimes P(h_0) \otimes E(\rho_1)$, where h_0 is as above and $\rho_1 \in \operatorname{Ext}^{1,0}$ is represented by $v_1^{-3}(t_2 - t_1^3) + v_1^{-4} v_2 t_1$.* $\square$

This will be proved below as 6.3.21.

Now we use the method of 5.1.17 to find $\operatorname{Ext}^0(M^1)$; in the next section we will compute all of $\operatorname{Ext}(M^1)$ in this way. From 4.3.3 we have $\eta_R(v_1^{p^i}) \equiv v_1^{p^i} \mod (p^{i+1})$, so $\frac{v_1^{sp^i}}{p^{i+1}} \in \operatorname{Ext}^0(M^1)$. For p odd we have

$$\eta_R(v_1^{sp^i}) \equiv v_1^{sp^i} + sp^{i+1} v_1^{sp^i - 1} t_1 \qquad \mod (p^{i+2}) \tag{5.2.3}$$

so in 5.1.17 we have

$$\delta\left(\frac{v_1^{sp^i}}{p^{i+1}}\right) = s v_1^{sp^i - 1} h_0 \in \operatorname{Ext}^1(M_1^0)$$

for $p \nmid s$, and we can read off the structure of $\operatorname{Ext}^0(M_1^0)$ below.

For $p = 2$, 5.2.3 fails for $i > 0$, e.g.,

$$\eta_R(v_1^2) = v_1^2 + 4 v_1 t_1 + 4 t_1^2 \mod (8).$$

The element $t_1^2 + v_1 t_1 \in C^1(M_1^0)$ is the coboundary of $v_1^{-1} v_2$, so

$$\alpha_{2/3} = \frac{(v_1^2 + 4 v_1^{-1} v_2)}{8} \in \operatorname{Ext}^0(M^1);$$

i.e., we can divide by at least one more power of p than in the odd primary case. In order to show that further division by 2 is not possible we need to show that $\alpha_{2/3}$ has a nontrivial image under δ (5.1.17). This in turn requires a formula for $\eta_R(v_2)$ mod (4). From 4.3.1 we get

$$\eta_R(v_2) = v_2 + 13 v_1 t_1^2 - 3 v_1^2 t_1 - 14 t_2 - 4 t_1^3. \tag{5.2.4}$$

[This formula, as well as $\eta_R(v_1) = v_1 - 2t_1$, are in terms of the v_i defined by Araki's formula A2.2.2. Using Hazewinkel's generators defined by A2.2.1 gives $\eta_R(v_1) = v_1 + 2t_1$ and $\eta_R(v_2) = v_2 - 5 v_1 t_1^2 - 3 v_1^2 t_1 + 2 t_2 - 4 t_1^3$.]

Let $x_{1,1} = v_1^2 + 4 v_1^{-1} v_2$. Then 5.2.4 gives

$$\eta_R(x_{1,1}) \equiv x_{1,1} + 8(v_1^{-1} t_2 + v_1^{-1} t_1^3 + v_1^{-2} v_2 t_1) \quad \mod (16) \tag{5.2.5}$$

so $\delta(\alpha_{2/3}) = v_1^2 \rho_1 \neq 0 \in \operatorname{Ext}^1(M_1^0)$.

5.2.6. THEOREM.
(a) *For p odd*

$$\operatorname{Ext}^{0,t}(M^1) = \begin{cases} 0 & \text{if } q \nmid t \text{ where } q = 2p-2 \\ \mathbf{Q}/\mathbf{Z}_{(p)} & \text{if } t = 0 \\ \mathbf{Z}/(p^{i+1}) & \text{if } t = sp^i q \text{ and } p \nmid s \end{cases}$$

These groups are generated by

$$\frac{v_1^{sp^i}}{p^{i+1}} \in M^1.$$

(b) *For p odd*

$$\operatorname{Ext}^{1,t}(BP_*) = \begin{cases} \operatorname{Ext}^{0,t}(M^1) & \text{if } t > 0 \\ 0 & \text{if } t = 0 \end{cases}$$

(c) *For $p = 2$*

$$\operatorname{Ext}^{0,t}(M^1) = \begin{cases} 0 & \text{if } t \text{ is odd} \\ \mathbf{Q}/\mathbf{Z}_{(2)} & \text{if } t = 0 \\ \mathbf{Z}/(2) & \text{if } t \equiv 2 \mod 4 \\ \mathbf{Z}/(2^{i+3}) & \text{if } t = 2^{i+2}s \text{ for odd } s \end{cases}$$

These groups are generated by $\frac{v_1^s}{2}$ *and* $\frac{x_{1,1}^{2^i s}}{2^{i+3}} \in M^1$ *where* $x_{1,1}$ *is as in* 5.2.5.
(d) *For $p = 2$*

$$\operatorname{Ext}^{1,t}(BP_*) = \begin{cases} 0 & \text{if } t \leq 0 \\ \operatorname{Ext}^{0,t}(M^1) & \text{if } t > 0 \text{ and } t \neq 4 \\ \mathbf{Z}/(4) & \text{if } t = 4 \end{cases}$$

and $\operatorname{Ext}^{1,4}(BP_*)$ *is generated by* $\alpha_{2,2} = \pm\frac{v_1^2}{4}$.

We will see in the next section (5.3.7) that in the Adams–Novikov spectral sequence for $p > 2$, each element of $\operatorname{Ext}^1(BP_*)$ is a permanent cycle detecting an element in the image of the J-homomorphism (1.1.13). For $p = 2$ the generators of $\operatorname{Ext}^{1,2t}$ are permanent cycles for $t \equiv 0$ and $1 \mod (4)$ while for $t \equiv 2$ and 3 the generators support nontrivial d_3's (except when $t = 2$) and the elements of order 4 in $\operatorname{Ext}^{1,8t+4}$ are permanent cycles. The generators of $E_4^{1,4t} = E_\infty^{1,4t}$ detect elements in $\operatorname{im} J$ for all $t > 0$.

PROOF OF 5.2.6. Part (a) was sketched above. We get $\mathbf{Q}/\mathbf{Z}_{(p)}$ in dimension zero because $1/p^i$ is a cocycle for all $i > 0$. For (b) the chromatic spectral sequence gives a short exact sequence

$$0 \to E_\infty^{1,0} \to \operatorname{Ext}^1(BP_*) \to E_\infty^{0,1} \to 0$$

and $E_\infty^{0,1}$ by 5.2.1. $E_\infty^{1,0} = E_2^{1,0} = \ker d_e / \operatorname{im} d_e$. An element in $E_1^{1,0} = \operatorname{Ext}^0(M^1)$ has a nontrivial image under d_e iff it has terms involving negative powers of v_1, so $\ker d_e \subset E_1^{1,0}$ is the subgroup of elements in nonnegative dimensions. The zero-dimensional summand $\mathbf{Q}/\mathbf{Z}_{(p)}$ is the image of d_e, so $E_2^{1,0} = \operatorname{Ext}^1(BP_*)$ is as stated.

For (c) the computation of $\mathrm{Ext}^0(M_1^0)$ is more complicated for $p = 2$ since 5.2.3 no longer holds. From 5.2.5 we get

$$\text{(5.2.7)} \qquad \eta_R(x_{1,1}^{2^i s}) \equiv x_{1,1}^{2^i s} + 2^{i+3} x_{1,1}^{2^i s - 1}(v_1^{-1} t_2 + v_1^{-1} t_1^3 + v_1^{-2} v_2 t_1) \qquad \mod (2^{i+4})$$

for odd s, from which we deduce that $\frac{x_{1,1}^{2^i s}}{2^{i+3}}$ is a cocycle whose image under δ (see 5.1.17) is $v_1^{2^{i+1} s}\rho_1$. Equation 5.2.3 does hold for $p = 2$ when $i = 0$, so $\mathrm{Ext}^{0,2s}(M_1^0)$ is generated by $\frac{v_1^s}{2}$ for odd s. This completes the proof of (c).

For (d) we proceed as in (b) and the situation in nonpositive dimensions is the same. We need to compute $d_e\Big(\frac{x_{1,1}^{2^i s}}{2^{i+3}}\Big)$. Since $x_{1,1} = v_1^2 + 4v_1^{-1} v_2$, we have

$$\frac{x_{1,1}^{2^i s}}{2^{i+3}} = \frac{v_1^{2^{i+1} s} + 2^{i+2} v_1^{2^i s - 3} v_2}{2^{i+3}}.$$

For $2^i s = 1$ (but for no $2^i s > 1$) this expression has a negative power of v_1 and we get

$$d_e\Big(\frac{x_{1,1}}{8}\Big) = \frac{v_2}{2v_1} \in M^2.$$

This gives a chromatic d_1 (compare 5.1.21) and accounts for the discrepancy between $\mathrm{Ext}^{0,4}(M^1)$ and $\mathrm{Ext}^{1,4}(BP_*)$. $\square$

Now we turn to the Hopf invariant one problem. Theorems 1.2.12 and 1.2.14 say which elements of filtration 1 in the classical Adams spectral sequence are permanent cycles. We can derive these results from our computation of $\mathrm{Ext}^1(BP_*)$ as follows. The map $BP \to H/(p)$ induces a map Φ from the Adams–Novikov spectral sequence to the Adams spectral sequence. Since both spectral sequences converge to the same thing there is essentially a one-to-one correspondence between their E_∞-terms. A nontrivial permanent cycle in the Adams spectral sequence of filtration s corresponds to one in the Adams–Novikov spectral sequence of filtration $\leq s$.

To see this consider BP_* and mod (p) Adams resolutions (2.2.1 and 2.1.3)

$$\begin{tikzcd}
S^0 \arrow[r, equal] & X_0 \arrow[d] & X_1 \arrow[l] \arrow[d] & \cdots \arrow[l] \\
S^0 \arrow[r, equal] & Y_0 & Y_1 \arrow[l] & \cdots \arrow[l]
\end{tikzcd}$$

where the vertical maps are the ones inducing Φ. An element $x \in \pi_*(S^0)$ has Adams filtration s if it is in $\operatorname{im} \pi_*(Y_s)$ but not in $\operatorname{im} \pi_*(Y_{s+1})$. Hence it is not in $\operatorname{im} \pi_*(X_{s+1})$ and its Novikov filtration is at most s.

We are concerned with permanent cycles with Adams filtration 1 and hence of Novikov filtration 0 or 1. Since $\mathrm{Ext}^0(BP_*)$ is trivial in positive dimensions [5.2.1(b)] it suffices to prove

5.2.8. Theorem. *The image of*

$$\Phi\colon\ \mathrm{Ext}^1(BP_*) \to \mathrm{Ext}^1_{A_*}(\mathbf{Z}/(p), \mathbf{Z}/(p))$$

is generated by h_1, h_2, *and* h_3, *for* $p = 2$ *and by* $h_0 \in \mathrm{Ext}^{1,q}$ *for* $p > 2$. (*These elements are permanent cycles*; *cf.* 1.2.11 *and* 1.2.13.)

PROOF. Recall that $A_* = \mathbf{Z}/(p)[t_1, t_2, \dots] \otimes E(e_0, e_1, \dots)$ with

$$\Delta(t_n) = \sum_{0 \le i \le n} t_i \otimes t_{n-i}^{p^i} \qquad \text{and} \qquad \Delta(e_n) = 1 \otimes e_n + \sum_{1 \le i \le n} e_i \otimes t_{n-i}^{p^i}$$

where $t_0 = 1$. Here t_n and e_n are the conjugates of Milnor's ξ_n and τ_n (3.1.1). The map $BP_*(BP) \to A_*$ sends $t_n \in BP_*(BP)$ to $t_n \in A_*$.

Now recall the I-adic filtration of 4.4.4. We can extend it to the comodules M^n and N^n by saying that a monomial fraction $\frac{v^I}{v^J}$ is in F^k iff the sum of the exponents in the numerator exceeds that for the denominator by at least k. (This k may be negative and there is no k such that $F^k M^n = M^n$ or $F^k N^n = N^n$. However, there is such a k for any finitely generated subcomodule of M^n or N^n.) For each $k \in \mathbf{Z}$ the sequence

$$0 \to F^k N^n \to F^k M^n \to F^k N^{n+1} \to 0$$

is exact. It follows that $\alpha\colon \operatorname{Ext}^s(N^n) \to \operatorname{Ext}^{s+n}(BP_*)$ (5.1.18) preserves the I-adic filtration and that if $x \in F^1 \operatorname{Ext}^0(N^1)$ then $\Phi\alpha(x) = 0$.

Easy inspection of 5.2.6 shows that the only elements in $\operatorname{Ext}^0(M^1)$ not in F^1 are α_1 and, for $p = 2$, $\alpha_{2/2}$, and $\alpha_{4/4}$, and the result follows. □

Now we turn to the computation of $\operatorname{Ext}^1(BP_*/I_n)$ for $n > 0$; it is a module over $\operatorname{Ext}^0(BP_*/I_n)$ which is $\mathbf{Z}/(p)[v_n]$ by 4.3.2. We denote this ring by $k(n)_*$. It is a principal ideal domain and $\operatorname{Ext}^1(BP_*/I_n)$ has finite type so the latter is a direct sum of cyclic modules, i.e., of free modules and modules of the form $k(n)_*/(v_n^i)$ for various $i > 0$. We call these the v_n-torsion free and v_n-torsion summands, respectively. The rank of the former is obtained by inverting v_n, i.e., by computing $\operatorname{Ext}^1(M_n^0)$. The submodule of the v_n-torsion which is annihilated by v_n is precisely the image of $\operatorname{Ext}^0(BP_*/I_{n+1}) = k(n+1)_*$ under the connecting homomorphism for the short exact sequence

$$0 \to \Sigma^{\dim v_n} BP_*/I_n \xrightarrow{v_n} BP_*/I_n \to BP_*/I_{n+1} \to 0. \tag{5.2.9}$$

We could take these elements in $\operatorname{Ext}^1(BP_*/I_n)$ and see how far they can be divided by v_n by analyzing the long exact sequence for 5.2.9, assuming we know enough about $\operatorname{Ext}^1(BP_*/I_{n+1})$ to recognize nontrivial images of elements of $\operatorname{Ext}^1(BP_*/I_n)$ when we see them. This approach was taken by Miller and Wilson [**3**].

The chromatic spectral sequence approach is superficially different but one ends up having to make the same calculation either way. From the chromatic spectral sequence for $\operatorname{Ext}(BP_*/I_n)$ (5.1.11) we get a short exact sequence

$$0 \to E_\infty^{1,0} \to \operatorname{Ext}^1(BP_*/I_n) \to E_\infty^{0,1} \to 0, \tag{5.2.10}$$

where $E_\infty^{1,0} = E_2^{1,0}$ is a subquotient of $\operatorname{Ext}^0(M_{n+1}^1)$ and is the v_n-torsion summand, while $E_\infty^{0,1} = E_3^{0,1} \operatorname{Ext}^1(M_n^0)$ is the v_n-torsion free quotient. To get at $\operatorname{Ext}^0(M_{n+1}^1)$ we study the long exact sequence for the short exact sequence

$$0 \to M_{n+1}^0 \xrightarrow{j} \Sigma^{\dim v_n} M_n^1 \xrightarrow{v_n} M_n^1 \to 0$$

as in 5.1.17; this requires knowledge of $\operatorname{Ext}^0(M_{n+1}^0)$ and $\operatorname{Ext}^1(M_{n+1}^0)$. To determine the subgroup $E_\infty^{0,1}$ of $\operatorname{Ext}^1(M_n^0)$ we need the explicit representatives of generators of the latter constructed by Moreira [**1, 3**].

The following result (to be proved later as 6.3.12) then is relevant to both $E_\infty^{0,1}$ and $E_\infty^{1,0}$ in 5.2.10.

5.2.11. THEOREM. *$\mathrm{Ext}^1(M_n^0)$ for $n > 0$ is the $K(n)_*$-vector space generated by $h_i \in \mathrm{Ext}^{1,p^iq}$ for $0 \le i \le n-1$ represented by $t_1^{p^i}$, $\zeta_n \in \mathrm{Ext}^{1,0}$ (for $n \ge 2$)represented for $n = 2$ by $v_2^{-1}t_2 + v_2^{-p}(t_2^p - t_1^{p^2+p}) - v_2^{-1-p}v_3t_1^p$, and (if $p = 2$ and $n \ge 1$) $\rho_n \in \mathrm{Ext}^{1,0}$. ($\zeta_n$ and ρ_n will be defined in 6.3.11).* $\square$

5.2.12. REMARK. For $i \ge n$, h_i does not appear in this list because the equation

$$\eta_R(v_{n+1}) \equiv v_{n+1} + v_n t_1^{p^n} - v_n^p t_1 \mod I_n$$

leads to a cohomology between h_{n+i} and $v_n^{(p-1)p^i}h_i$.

Now we will describe $\mathrm{Ext}^0(M_n^1)$ and $E_\infty^{1,0}$. The groups are v_n-torsion modules. The submodule of the former annihilated by v_n is generated by $\left\{\frac{v_{n+1}^t}{v_n} : t \in \mathbf{Z}\right\}$. Only those elements with $t > 0$ will appear in $E_\infty^{1,0}$; if $t = 0$ the element is in $\operatorname{im} d_1$, and $\ker d_1$ is generated by those elements with $t \ge 0$. We need to see how many times we can divide by v_n and (still have a cocycle). An easy calculation shows that if $t = sp^i$ with $p \nmid s$, then $\frac{v_{n+1}^t}{v_n^{p^i}}$ is a cocycle whose image in $\mathrm{Ext}^1(M_{n+1}^0)$ is $sv_{n+1}^{(s-1)p^i}h_{n+i}$, but by 5.2.12 these are not linearly independent, so this is not the best possible divisibility result. For example, for $n = 1$ we find that

$$\frac{v_2^{p^2}}{v_1^{1+p^2}} - \frac{v_2^{p^2-p+1}}{v_1^2} - \frac{v_2^{-p}v_3^p}{v_1}$$

is a cocycle.

The general result is this.

5.2.13. THEOREM. *As a $k(n)_*$-module, $\mathrm{Ext}^0(M_n^1)$ is the direct sum of*
(i) *the cyclic submodules generated by $\frac{x_{n+1,i}^s}{v_n^{a_{n+1,i}}}$ for $i \ge 0$, $p \nmid s$; and*
(ii) *$K(n)_*/k(n)_*$, generated by $\frac{1}{v_n^j}$ for $j \ge 1$.*
The $x_{n,i}$ are defined as follows.

$$\begin{aligned}
x_{1,0} &= v_1, \\
x_{1,1} &= v_1^p \quad \text{if } p > 2 \quad \text{and} \quad v_1^2 + 4v_1^{-1}v_2 \quad \text{if } p = 2, \\
x_{1,i} &= x_{1,i-1}^p && \text{for } i \ge 2, \\
x_{2,0} &= v_2, \\
x_{2,1} &= v_2^p - v_1^p v_2^{-1} v_3, \\
x_{2,2} &= x_{2,1}^p - v_1^{p^2-1} v_2^{p^2-p+1} - v_1^{p^2+p-1} v_2^{p^2-2p} v_3, \\
x_{2,i} &= x_{2,i-1}^2 && \text{for } i \ge 3 \quad \text{if } p = 2
\end{aligned}$$

and

$$x_{2,i-1}^p - 2v_1^{b_{2,i}} v_2^{(p-1)p^{i-1}+1} \qquad \text{for } i \ge 3 \quad \text{if } p > 2,$$

where

$$
\begin{aligned}
b_{2,i} &= (p+1)(p^{i-1}-1), \\
x_{n,0} &= v_n && \text{for} \quad n>2, \\
x_{n,1} &= v_n^p - v_{n-1}^p v_n^{-1} v_{n+1}, \\
x_{n,i} &= x_{n,i-1}^p && \text{for} \quad i>1 \quad \text{and} \quad i \not\equiv 1 \mod (n-1), \\
x_{n,i} &= x_{n,i-1}^p - v_{n-1}^{b_{n,i}} v_n^{p^i-p^{i-1}+1} && \text{for} \quad i>1, \quad \text{and} \quad i \equiv 1 \mod (n-1)
\end{aligned}
$$

where

$$b_{n,i} = \frac{(p^{i-1}-1)(p^n-1)}{p^{n-1}-1} \quad \text{for} \quad i \equiv 1 \mod (n-1).$$

The $a_{n,i}$ *are defined by*

$$
\begin{aligned}
a_{1,0} &= 1 \\
a_{1,i} &= i+2 && \text{for } p=2 \quad \text{and} \quad i \geq 1, \\
a_{i,1} &= i+1 && \text{for } p>2 \quad \text{and} \quad i \geq 1, \\
a_{2,0} &= 1, \\
a_{2,i} &= p^i + p^{i-1} - 1 && \text{for } p>2 \quad \text{and} \quad i \geq 1 \quad \text{or } p=2 \quad \text{and} \quad i=1, \\
a_{2,i} &= 3 \cdot 2^{i-1} && \text{for } p=2 \text{ and } i>1, \\
a_{n,0} &= 1 && \text{for } n>2, \\
a_{n,1} &= p, \\
a_{n,i} &= pa_{n,i-1} && \text{for } i>1 \quad \text{and} \quad i \not\equiv 1 \mod (n-1),
\end{aligned}
$$

and

$$a_{n,i} = pa_{n,i} + p - 1 \qquad \text{for } i>1 \quad \text{and} \quad i \equiv 1 \mod (n-1).$$

□

This is Theorem 5.10 of Miller, Ravenel, and Wilson [**1**], to which we refer the reader for the proof.

Now we need to compute the subquotient $E_2^{1,0}$ of $\mathrm{Ext}^0(M_n^1)$. It is clear that the summand of (ii) above is in the image of d_1 and that ker d_1 is generated by elements of the form $\frac{x_{n+1,i}^s}{v_1^j}$ for $s \geq 0$. Certain of these elements for $s>0$ are not in $\ker d_1$; e.g., we saw in 5.2.6 that $d_1\left(\frac{x_{1,1}}{8}\right) \neq 0$. More generally we find $d_1\left(\frac{x_{n+1,i_s}}{v_n^j}\right) \neq 0$ iff $s=1$ and $p^i < j \leq a_{n+1,i}$ (see Miller and Wilson [**3**]), so we have

5.2.14. Corollary. *The* v_n*-torsion summand of* $\mathrm{Ext}^1(BP_*/I_n)$ *is generated by the elements listed in* 5.2.13(i) *for* $s>0$ *with* (*when* $s=1$) $\frac{x_{n+1,i}}{v_n^{a_{n+1,i}}}$ *replaced by* $\frac{x_{n+1,i}}{v_n^{p^i}}$. □

Now we consider the $k(n)_*$-free summand $E_\infty^{0,1} \subset \mathrm{Ext}^1(M_n^0)$. We assume $n>1$ ($n=1$ is the subject of 5.2.2); 5.2.11 tells us that $E_\infty^{0,1}$ has rank $n+1$ for $p>2$ and $n+2$ for $p=2$. We need to determine the image of $\mathrm{Ext}^1(BP_*/I_n)$ in $\mathrm{Ext}^1(M_n^0)$. To show that an element in the former is not divisible by v_n we must show that it has a nontrivial image in $\mathrm{Ext}^1(BP_*/I_{n+1})$. The elements $h_i \in \mathrm{Ext}^1(M_n^0)$ clearly are in the image of $\mathrm{Ext}^1(BP_*/I_n)$ and have nontrivial images in $\mathrm{Ext}^1(BP_*/I_{n+1})$. The elements ζ_n and ρ_n are more complicated. The formula given in 5.2.11 for ζ_2 shows that $v_2^{1+p}\zeta_2$ pulls back to $\mathrm{Ext}^1(BP_*/I_2)$ and projects to $v_3h_1 \in \mathrm{Ext}^1(BP_*/I_3)$. This element figures in the proof of 5.2.13 and in the computation of $\mathrm{Ext}^2(BP_*)$ to be described in Section 4.

The formula of Moreira [**1**] for a representative of ζ_n is

$$T_n = \sum_{1\le i\le j\le k\le n} u_{2n-k}^{p^{k-i}} t_j^{p^{n-i}} c(t_{k-j})^{p^{n-i+j}} \tag{5.2.15}$$

where the $u_{n+i} \in M_n^0$ are defined by

$$u_n = v_n^{-1} \quad \text{and} \quad \sum_{0\le i\le k} u_{n+i} v_{n+k-i}^{p^i} = 0 \quad \text{for } k > 0. \tag{5.2.16}$$

One sees from 5.2.16 that $u_{n+i-1}v_n^{(p^i-1)/(p-1)} \in BP_*/I_n$ so $\widehat{T}_n = v_n^{(p^n-1)/(p-1)}T_n \in BP_*(BP)/I_n$. In 5.2.15 the largest power of v_n^{-1} occurs in the term with $i = j = k = 1$; in T_n this term is $v_n^{(p^n-1)/(p-1)}u_{2n-1}t_1^{p^{n-1}}$ and its image in $\mathrm{Ext}^1(BP_*/I_{n+1})$ is $(-1)^{n+1}v_{n+1}^{(p^{n-1}-1)/(p-1)}h_{n-1}$.

The formula of Moreira [**3**] for a representative U_n of ρ_n is very complicated and we will not reproduce it. From it one sees that $v_n^{2^{2n}-1+2^{n-1}}U_n \in BP_*(BP)/I_n$ reduces to $v_{n+1}^{2^{2n-1}-1}t_1^{2^{n-1}} \in BP_*(BP)/I_{n+1}$.

Combining these results gives

5.2.17. THEOREM. *The $k(n)_*$-free quotient $E_\infty^{0,1}$ of $\mathrm{Ext}^1(BP_*/I_n)$ for $n \ge 1$ is generated by $h_i \in \mathrm{Ext}^{1,p^iq}$ for $0 \le i \le n-1$, $\hat\zeta = v_n^{(p^n-1)/(p-1)}\zeta_n$, and (for $p = 2$) $\hat\rho_n = v_n^{2^{2n}+2^{n-1}-1}\rho_n$. The imagas of $\hat\zeta_n$ and $\hat\rho_n$ in $\mathrm{Ext}^1(BP_*/I_{n+1})$ are $(-1)^{n+1}v_{n+1}^{(p^{n-1}-1)/(p-1)}h_{n-1}$ and $v_{n+1}^{2^{2n-1}-1}h_{n-1}$, respectively.* □

3. $\mathrm{Ext}(M^1)$ and the J-Homomorphism

In this section we complete the calculation of $\mathrm{Ext}(M^1)$ begun with 5.2.6 and describe the behavior of the resulting elements in the chromatic spectral sequence and then in the Adams–Novikov spectral sequence. Then we will show that the elements in $\mathrm{Ext}^1(BP_*)$ (and, for $p = 2$, Ext^2 and Ext^3) detect the image of the homomorphism $J\colon \pi_*(SO) \to \pi_*^S$ (1.1.12). This proof will include a discussion of Bernoulli numbers. Then we will compare these elements in the Adams–Novikov spectral sequence with corresponding elements in the Adams spectral sequence.

We use the method of 5.1.17 to compute $\mathrm{Ext}(M^1)$; i.e., we study the long exact sequence of Ext groups for

$$0 \to M_1^0 \xrightarrow{j} M^1 \xrightarrow{p} M^1 \to 0. \tag{5.3.1}$$

$\mathrm{Ext}(M_1^0)$ is described in 5.2.6 and the computation of $\mathrm{Ext}^0(M^1)$ is given in 5.2.6 Let δ be the connecting homomorphism for 5.3.1. Then from the proof of 5.2.6 we have

5.3.2. COROLLARY. *The image of δ in $\mathrm{Ext}^1(M_1^0)$ is generated by* (a) *$v_1^th_0$ for all $t \ne 1$ when p is odd and*

(b) *$v_1^th_0$ for all even t and $v_1^t\rho_1$ for all $t \ne 0$ when $p = 2$.* □

For odd primes this result alone determines all of $\mathrm{Ext}(M^1)$. $\mathrm{Ext}^s(M_1^0) = 0$ for $s > 1$ and there is only one basis element of $\mathrm{Ext}^1(M_1^0)$ not in $\operatorname{im}\delta$, namely

$v_1^{-1}h_0$. Its image under j is represented by $\frac{v_1^{-1}t_1}{p}$. Since $\mathrm{Ext}^2(M_1^0)=0$, there is no obstruction to dividing $j(v_1^{-1}h_0)$ by any power of p, so we have

$$\mathrm{Ext}^{1,t}(M^1)=\begin{cases}\mathbf{Q}/\mathbf{Z}_{(p)} & \text{for } t=0\\ 0 & \text{for } t\neq 0\end{cases}\tag{5.3.3}$$

for any odd prime p. We can construct a representative of an element of order p^k in $\mathrm{Ext}^{1,0}(M^1)$ as follows. From 4.3.1 we have $\eta_R(v_1)=v_1=put_1$ where $u=1-p^{p-1}$. Then a simple calculation shows that

$$y_k=-\sum_{i\geq 1}(-1)^i\frac{v_1^{-i}u^it_1^i}{ip^{k+1-i}}\tag{5.3.4}$$

is the desired cocycle. (This sum is finite although the ith term for some $i>k$ could be nonzero if $p\mid i$.) The group $\mathrm{Ext}^{1,0}(M^1)+E_1^{1,1,0}$ cannot survive in the chromatic spectral sequence because it would give a nontrivial $\mathrm{Ext}^{2,0}(BP_*)$ contradicting the edge theorem, 5.1.23. It can be shown (lemma 8.10 of Miller, Ravenel, and Wilson [**1**]) that this group in fact supports a d_1 with trivial kernel. Hence we have

5.3.5. THEOREM.
(a) *for $p>2$ the group $\mathrm{Ext}^{s,t}(M^1)$ is*

$\mathbf{Q}/\mathbf{Z}_{(p)}$ *generated by* $\frac{1}{p^k}$ *for* $(s,t)=(0,0)$.

$\mathbf{Z}/(p^{i+1})$ *generated by* $\frac{v_1^{rp^i}}{p^{i+1}}$ *for* $p\nmid r$ *and* $(s,t)=(0,rp^iq)$,

$\mathbf{Q}/\mathbf{Z}_{(p)}$ *generated by* y_k (5.3.4) *for* $(s,t)=(1,0)$ *and*

0 *otherwise.*

(b) *In the chromatic spectral sequence, where* $\mathrm{Ext}^{s,t}(M^1)=E_1^{1,s,t}E_1^{1,0,0}\subset \operatorname{im} d_1$ *and* $\ker d_1\bigoplus_{t\geq 0}E_1^{1,0,t}$, *so* $E_\infty^{1,*}=\mathrm{Ext}^1(BP_*)$ *and* $\ker d_1=\bigoplus_{t\geq 0}E_1^{1,0,t}$, *so* $E_\infty^{1,*}=\mathrm{Ext}^1(BP_*)$ *is generated by the groups* $\mathrm{Ext}^{0,t}(M^1)$ *for* $t>0$. □

We will see below that each generator of $\mathrm{Ext}^1(BP_*)$ for $p>2$ is a permanent cycle in the Adams–Novikov spectral sequence detecting an element in the image of J (1.1.12).

The situation for $p=2$ is more complicated because $\mathrm{Ext}(M_1^0)$ has a polynomial factor not present for odd primes. We use 5.3.2 and 5.2.2 to compute $\mathrm{Ext}^s(M^1)$ for $s>1$. The elements of order 2 in $\mathrm{Ext}^{1,0}(M^1)$ are the images under j (5.3.1) of $v_1^th_0$ for t odd and $v_1^t\rho_1$ for t odd and $t=0$.

We claim $j(\rho_1)$ is divisible by any power of 2, so $\mathrm{Ext}^{1,0}(M^1)$ contains a summand isomorphic to $\mathbf{Q}/\mathbf{Z}_{(2)}$ as in the odd primary case. To see this use 5.2.4 to compute

$$d\Big(\frac{v_1^{-3}v_2+2v_1^{-6}v_2^2}{4}\Big)=\frac{v_1^{-1}t_1+v_1^{-2}t_1^2+2v_1^{-4}t_1^4}{4}+\frac{v_1^{-4}(v_2t_1+v_1t_1^3+v_1t_2)}{2},$$

showing that y_2 (5.3.4) represents $j(\rho_1)$; the same calculation shows that

$$y_1=\frac{v_1^{-1}t_1+v_1^{-2}t_1^2}{2}$$

is a coboundary. Hence the y_k for $k\geq 2$ give us the cocycles we need.

Next we have to deal with $j(v_1^t h_0)$ and $j(v_1^t \rho_1)$ for odd t. These are not divisible by 2 since an easy calculation gives $\delta_j(v_1^t x) = v_1^{t-1} h_0 x$ for t odd and $x = h_0^{i+1}$ or $h_0^i \rho_1$ for any $i \geq 0$. Indeed this takes care of all the remaining elements in the short exact sequence for 5.3.1 and we get

5.3.6. THEOREM.
(a) *For $p = 2$, $\operatorname{Ext}^{s,t}(M^1)$ is*

$$\begin{cases} \mathbf{Q}/\mathbf{Z}_{(2)} \textit{ generated by } \frac{1}{2^j} & \\ \qquad\qquad \textit{for } (s,t) = (0,0), & \\ \mathbf{Z}/(2) \textit{ generated by } \frac{v_1^r}{2} & \\ \qquad\qquad \textit{for } (s,t) = (0,2r) \textit{ and } r \textit{ odd}, & \\ \mathbf{Z}/(2^{i+3}) \textit{ generated by } \frac{x_{1,1}^{r2^i}}{2^{i+3}} & \\ \qquad\qquad \textit{for } (s,t) = (0, r2^{i+2}) \textit{ and } r \textit{ odd}, & \\ \mathbf{Q}/\mathbf{Z}_{(2)} \oplus \mathbf{Z}/(2) \textit{ generated by } y_k \ (k \geq 2) \textit{ and } \frac{v_1^{-1} t_1}{2} & \\ \qquad\qquad \textit{for } (s,t) = (1,0), & \\ \mathbf{Z}/(2) \textit{ generated by } j(v_1^r h_0^s) & \\ \qquad\qquad \textit{for } s > 0,\ t = 2(r+s),\ r \textit{ odd, and } (s,t) \neq (1,0) & \\ \mathbf{Z}/(2) \textit{ generated by } j(v_1^r \rho_1 h_0^{s-1}) & \\ \qquad\qquad \textit{for } s > 0,\ t + 2(r+s-1), \textit{ and } r \textit{ odd}, & \\ \qquad\qquad\qquad \textit{and} & \\ 0 \qquad\qquad \textit{otherwise.} & \end{cases}$$

(b) *In the chromatic spectral sequence for $p = 2$, $E_\infty^{1,2,t}$ is*

$$\begin{cases} \operatorname{Ext}^{s,t}(M^1) & \textit{for } t = 2s + 2r \textit{ and } r \geq 1,\ r \neq 2, \\ \mathbf{Z}/(4) \textit{ generated by } \frac{v_1^2}{4} & \textit{for } (s,t) = (0,4), \textit{ and } \mu_{2t-1} \in \pi_{8t+1}^S, \\ & \qquad \textit{and} \\ 0 & \textit{otherwise} \end{cases}$$

(*See* 5.1.22 *for a description of differentials originating in* $E_r^{1,s,2s+4}$.) *In other words the subquotient of* $\operatorname{Ext}(BP_*)$ *corresponding to* $E_\infty^{1,*}$ *is generated by* $\operatorname{Ext}^1(BP_*)$ (5.2.6) *and products of its generators* (*excluding* $\alpha_{2/2} \in \operatorname{Ext}^{1,4}$) *with all positive powers of* $\alpha_1 \in \operatorname{Ext}^{1,2}$.

PROOF. Part (a) was proved above. For (b) the elements said to survive, i.e., those in $E_1^{1,0}$ and $j(v_1^r \rho_1 h_0^{s-1})$ for $s > 0$ with odd $r \geq 5$ and $j(v_1^r h_0^s)$ for $s > 0$ with odd $r \geq 1$, are readily seen to be permanent cycles. The other elements in $E_1^{1,s}$ for $s > 0$ have to support nontrivial differentials by the edge theorem, 5.1.23. $\square$

Now we describe the behavior of the elements of 5.3.5(b) and 5.3.6(b) in the Adams–Novikov spectral sequence. The result is

5.3.7. THEOREM.
(a) *For $p > 2$, each element in $\operatorname{Ext}^1(BP_*)$ is a permanent cycle in the Adams–Novikov spectral sequence represented by an element of* $\operatorname{im} J$ (1.1.13) *having the same order.*

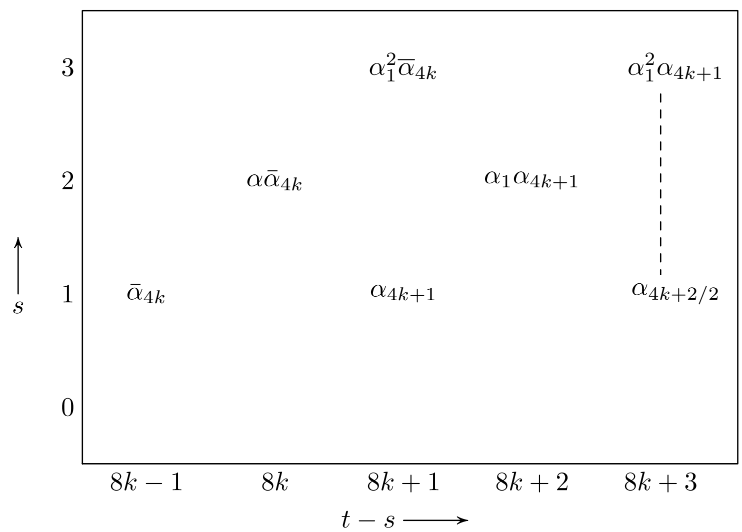
(b) *For $p = 2$ the behavior of $\operatorname{Ext}^{1,2t}(BP_*)$ in the Adams–Novikov spectral sequence depends on the residue of $t \bmod (4)$ as follows. If $t \equiv 1 \bmod 4$ the generator α_t is a permanent cycle represented by the element $\mu_{2t-1} \in \pi_{8t-1}^S$ of order 2 constructed by Adams* [1]. *In particular α_1 is represented by η* (1.1.13). *$\alpha_1 \alpha_t$ is* *represented by* $\mu_{2t} = \eta\mu_{2t-1}$ *and* $\alpha_1^2\alpha_t$ *is represented by an element of order* 2 *in* $\operatorname{im} J \subset \pi_{2t+1}^S$ (*the order of this group is an odd multiple of* 8). $\alpha^{s+3}\alpha_t = d_3(\alpha_1^s\alpha_{t+2})$ *for all* $s \geq 0$.

If $t \equiv 0 \mod (4)$ *then the generator* $\bar{\alpha}_t$ *of* $\operatorname{Ext}^{1,2t}(BP_*)$ *is a permanent cycle represented by an element of* $\operatorname{im} J$ *having the same order, as are* $\alpha_1\bar{\alpha}_t$, *and* $\alpha_1^2\bar{\alpha}_t$. $\alpha_1^{s+3}\bar{\alpha}_t = d_3(\alpha_1^s\alpha_{t+2/3})$ *for* $s \geq 0$. *In particular* y_4 *is represented by* $\sigma \in \pi_7^S$ (1.1.13).

If $t \equiv 2 \mod (4)$, $\alpha_{t/2}$ (*twice the generator except when* $t = 2$) *is a permanent cycle represented by an element in* $\operatorname{im} J$ *of order* 8. ($\alpha_{t/2}$ *has order* 4 *and* 4 *times the generator of* $\operatorname{im} J$ *represents* $\alpha_1^2\alpha_{t-2}$ *as remarked above*). *In particular* $\alpha_{2/2}$ *is represented by* $\nu \in \pi_3^S$ (1.1.13). □

This result says that the following pattern occurs for $p = 2$ in the Adams–Novikov spectral sequence E_∞-term as a direct summand for all $k > 0$:

(5.3.8)



Where all elements have order 2 except $\alpha_{4k+2/2}$, which has order 4, and $\bar{\alpha}_{4k}$, whose order is the largest power of 2 dividing $16k$; the broken vertical line indicates a nontrivial group extension. The image of J represents all elements shown except α_{4k+1} and $\alpha_1\alpha_{4k+1}$.

Our proof of 5.3.7 will be incomplete in that we will not prove that im J actually has the indicated order. This is done up to a factor of 2 by [**1**] Adams [**1**], where it is shown that the ambiguity can be removed by proving the Adams conjecture, which was settled by Quillen [**1**] and Sullivan [**1**].

We will actually use the complex J-homomorphism $J\colon \pi_*(U) \to \pi_*^S$, where U is the unitary group. Its image is known to coincide up to a factor of 2 with that of the real J-homomorphism. We will comment more precisely on the difference between them in due course.

An element $x \in \pi_{2t-1}(U)$ corresponds to a stable complex vector bundle ξ over S^{2t}. Its Thom spectrum $T(\xi)$ is a 2-cell CW-spectrum $S^0 \cup e^{2t}$ with attaching map $J(x)$ and there is a canonical map $T(\xi) \to MU$. We compose it with the standard map $MU \to BP$ and get a commutative diagram

(5.3.9)
$$\begin{tikzcd}
S^0 \arrow[r] \arrow[d] & T(\xi) \arrow[r] \arrow[d] & S^{2t} \arrow[d] \\
S^0_{(p)} \arrow[r] & BP \arrow[r] & \overline{BP} \arrow[d] \\
& & BP \wedge \overline{BP}
\end{tikzcd}$$

where the two rows are cofibre sequences. The map $S^{2t} \to \overline{BP}$ is not unique but we do get a unique element $e(x) \in \pi_{2t}(BP \wedge \overline{BP})/\operatorname{im}\pi_{2t}(BP)$. Now $E_2^{1,2t}$ of the Adams–Novikov spectral sequence is by definition a certain subgroup of this quotient containing $e(x)$, so we regard the latter as an element in $\mathrm{Ext}^{1,2t}(BP_*)$. Alternatively, the top row in 5.3.9 gives a short exact sequence of comodules which is the extension corresponding to $e(x)$. We need to show that if x generates $\pi_{2t-1}(U)$ then $e(x)$ generates $\mathrm{Ext}^1(BP_*)$ up to a factor of 2.

For a generator x_t of $\pi_{2t-1}(U)$ we obtain a lower bound on the order of $e(x)$ as follows. If $je(x_t) = 0$ for some integer j then for the bundle given by $x = jx_t \in \pi_{2t-1}(U)$ the map $S^{2t} \to \overline{BP}$ in 5.3.9 lifts to BP, so we get an element in $\pi_{2t}(BP)$. Now consider the following diagram

(5.3.10)
$$\begin{tikzcd}
\pi_*(BU) \arrow[d] & \pi_*(MU) \arrow[r, "\theta"] \arrow[d] & \mathbf{Z} \arrow[d] \\
H_*(BU) \arrow[r, phantom, "\cong"] & H_*(MU) \arrow[r] & \mathbf{Q}
\end{tikzcd}$$

where the two left-hand vertical maps are the Hurewicz homomorphisms and θ is some ring homomorphism; it extends as indicated since $\pi_*(MU)\otimes\mathbf{Q} \cong H_*(MU)\otimes\mathbf{Q}$ by 3.1.5. Let ϕ be the composite map (not a ring homomorphism) from $\pi_*(BU)$ to $\mathbf{Q}$. If $\phi(x_t)$ has denominator j_t, then j_t divides the order of $e(x_t)$.

According to Bott [**2**] the image of x_t in $H_{2t}(BU)$ is $(t-1)!\, s_t$ where s_t is a primitive generator of $H_{2t}(BU)$. By Newton's formula

$$s(z) = \frac{z}{b(z)}\frac{db(z)}{dz},$$

where $s(z) = \sum_{t\geq 0} s_t z^t$ and $b(z) = \sum_{t\geq 0} b_t z^t$, the b_t being the multiplicative generators of $H_*(BU) \cong H_*(MU)$ (3.1.4).

Now by Quillen's theorem, 4.1.6, θ defines a formal group law over $\mathbf{Z}$ (see Appendix 2), and by 4.1.11

$$\theta(b(z)) = \frac{\exp(z)}{z}$$

so

$$\theta(s(z)) = \frac{z}{\exp(z)}\frac{d\exp(z)}{dz} - 1,$$

where $\exp(z)$ is the exponential series for the formal group law defined by θ, i.e., the functional inverse of the logarithm (A2.1.5).

The θ we want is the one defining the multiplicative formal group law (A2.1.4) $x+y+xy$. An easy calculation shows $\exp(z)=e^z-1$ so

$$\theta(s(z))=\frac{ze^z}{e^z-1}-1.$$

This power series is essentially the one used to define Bernoulli numbers (see appendix B of Milnor and Stasheff [**5**]), i.e., we have

$$\theta(s(z))=\frac{z}{2}+\sum_{k\geq 1}(-1)^{k+1}\frac{B_k z^{2k}}{(2k)!}$$

where B_k is the kth Bernoulli number. Combining this with the above formula of Bott we get

5.3.11. Theorem. *The image of a generator x_t of $\pi_{2t-1}(U)=\pi_{2t}(BU)$ under the map $\phi\colon \pi_*BU\to \mathbf{Q}$ of 5.3.10 is $\frac{1}{2}$ if $t=1$, 0 for odd $t>1$, and $\pm B_k/2k$ for $t=2k$. Hence the order of x_t in $\mathrm{Ext}^1(BP_*)$ is divisible by 2 for $t=1$, 1 for $t>1$, and the denominator j_{2k} of $B_k/2k$ for $t=2k$.* □

This denominator j_{2k} is computable by a theorem of von Staudt proved in 1845; references are given in Milnor and Stasheff [**5**]. The result is that $p\mid j_{2k}$ iff $(p-1)\mid 2k$ and that if p^i is the highest power of such a prime which divides $2k$ then p^{i+1} is the highest power of p dividing j_{2k}. Comparison with 5.2.6 shows that $\mathrm{Ext}^{1,4k}(BP_*)$ also has order p^{i+1} except when $p=2$ and $k>1$, in which case it has order 2^{i+2}. This gives

5.3.12. Corollary. *The subgroup of $\mathrm{Ext}^{1,2t}(BP_*)$ generated by $e(x_t)$ (5.3.9), i.e., by the image of the complex J-homomorphism, has index 1 for $t=1$ and 2. and 1 or 2 for $t\geq 3$. Moreover each element in this subgroup is a permanent cycle in the Adams–Novikov spectral sequence.* □

This completes our discussion of $\mathrm{im}\,J$ for odd primes. We will see that the above index is actually 2 for all $t\geq 3$, although the method of proof depends on the congruence class of $t \mod (4)$. We use the fact that the complex J-homomorphism factors through the real one. Hence for $t\equiv 3 \mod (4)$, $e(x_t)=0$ because $\pi_{2t-1}(SO)=0$.

For $t\equiv 0$ the map $\pi_{2t-1}(U)\to\pi_{2t-1}(SO)$ has degree 2 in Bott [**1**] (and for $t\equiv 2$ it has degree 1) so $e(x_t)$ is divisible by 2 and the generator y of $\mathrm{Ext}^1(BP_*)$ is as claimed in 5.3.7. This also shows that ηy_t, and $\eta^2 y_t$, detect elements in $\mathrm{im}\,J$. Furthermore η^3 kills the generator of $\pi_{2t-1}(SO)$ by 3.1.26, so $\alpha_1^3 y_t$ must die in the Adams–Novikov spectral sequence. It is nonzero at E_2, so it must be killed by a higher differential and the only possibility is $d_3(\alpha_{t+2/3})=\alpha_1^3 y_t$ [here we still have $t\equiv 0 \mod (4)$].

For $t\equiv 1$ the generator of $\pi_{2t-1}(SO)=\mathbf{Z}/(2)$ is detected by $\eta^2 y_{t-1}$ as observed above, so $e(x_t)=0$. For $t\equiv 2$ we just saw that the generator $\alpha_{t/3}$ of $\mathrm{Ext}^{1,2t}$ supports a nontrivial d_3 for $t>2$, so we must have $e(x_t)=\alpha_{t/2}$.

To complete the proof of 5.3.7 we still need to show three things: for $t\equiv 1 \mod (4)$, α_t is a permanent cycle, for $t\equiv 3$, $d_3(\alpha_t)=\alpha_t^3\alpha_{t-2}$, and for $t\equiv 2\mathrm{m}$ α_t is represented by an element of order 4 whose double is detected by $\alpha_1^2\alpha_{t-1}$. To do this we must study the Adams–Novikov spectral sequence for the mod (2) Moore spectrum $M(2)$. Since $BP_*(M(2))=BP_*/(2)$ is a comodule algebra, the Adams–Novikov E_2-term for $M(2)$, $\mathrm{Ext}(BP_*/(2))$, is a ring (A1.2.14). However, since $M(2)$

is not a ring spectrum, the Adams–Novikov spectral sequence differentials need not respect this ring structure. The result we need is

5.3.13. THEOREM. (a) *$\mathrm{Ext}(BP_*/(2))$ contains $\mathbf{Z}/(2)[v_1, h_0] \otimes \{1, u\}$ as a direct summand where $v_1 \in \mathrm{Ext}^{0,2}$, $h_0 \in \mathrm{Ext}^{1,2}$, and $u \in \mathrm{Ext}^{1,8}$ are represented by v_1, t_1, and $t_1^4 + v_1t_1^3, v_1^2t_1^2 + v_1t_2 + v_2t_1$ respectively. This summand maps isomorphically to $E_\infty^{0,*}$ in the chromatic spectral sequence for $\mathrm{Ext}(BP_*/(2))$ (5.1.11).*

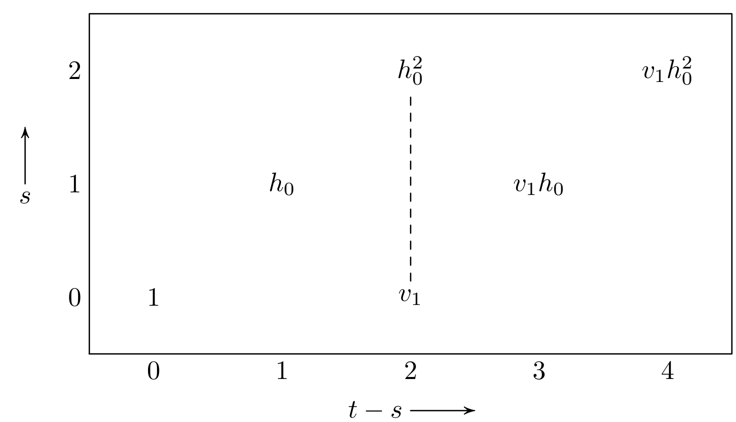
(b) *In the Adams–Novikov spectral sequence for $M(2)$, $v_1^t h_0^s u^e$ is a permanent cycle for $s \geq 0$, $e = 0, 1$, and $t \equiv 0$ or $1 \mod (4)$. If $t \equiv 2$ or 3 then $d_3(v_1^t h_0^s u^e) = v_1^{t-2} h_0^{s+3} u^e$. For $t \equiv 3$, $v_1^t u^e$ is represented by an element of order 4 in $\pi_{2t+7e}(M(2))$ whose double is detected by $h_0^2 v_1^{t-1} u^e$.*

(c) *Under the reduction map $BP_* \to BP_*/(2)$ induced by $S^0 \to M(2)$, if t is odd then the generator α_t of $\mathrm{Ext}^{1,2t}(BP_*)$ maps to $v_1^{t-1}h_0$. If t is even and at least 4 then the generator y_t of $\mathrm{Ext}^{1,2t}(BP_*)$ maps to $v_1^{t-4}u$.*

(d) *Under the connecting homomorphism $\delta\colon \mathrm{Ext}^s(BP_*/(2)) \to \mathrm{Ext}^{s+1}(BP_*)$ induced by $M(2) \to S^1$ (2.3.4), v_1^t maps to $\alpha_t \in \mathrm{Ext}^{1,2t}(BP_*)$ for all $t > 0$; uv_1 maps to $\alpha_1 y_{t+3}$ if t is odd and to 0 if t is even.* □

In other words, the Adams–Novikov E_∞-term for $M(2)$ has the following pattern as a summand in low dimensions:

(5.3.14)



where the broken vertical line represents a nontrivial group extension. [Compare this with 3.1.28(a) and 5.3.8.] The summand of (a) also contains the products of these elements with $v_1^{4t}u^e$ for $t \geq 0$ and $e = 0, 1$. The only other generators of $\mathrm{Ext}^{s,t}(BP_*/(2))$ for $t - s \leq 13$ are $\beta_1 \in \mathrm{Ext}^{1,4}$, $\beta_1^2 \in \mathrm{Ext}^{2,8}$, $h_0^s\beta_{2/2} \in \mathrm{Ext}^{1+s,8+2s}$ for $s = 0, 1, 2$ (where $h_0^2\beta_{2/2} = \beta_1^3$), and $h_0^s\beta_2 \in \mathrm{Ext}^{1+s,10+2s}$ for $s = 0, 1$.

Before proving this we show how it implies the remaining assertions of 5.3.7 listed above. For $t \equiv 1 \mod (4)$, $\alpha_t = \delta(v_1^t)$ by (d) and is therefore a permanent cycle by (b). For $t \equiv 3$, $\alpha_1 = \delta(v_1^t)$ and δ commutes with differentials by 2.3.4, so

$$\begin{aligned} d_3(\alpha_t) &= \delta d_3(v_1^t) = \delta(h_0^3 v_1^{t-2}) \\ &= \alpha_1^3 \alpha_{t-2}. \end{aligned}$$

For the nontrivial group extension note that for $t \equiv 1$ $\alpha_1^2\alpha_t$ maps to an element killed by a differential so it is represented in $\pi_*(S^0)$ by an element divisible by 2. Alternatively, α_{t+1} is not the image under δ of a permanent cycle so it is not represented by an element of order 2.

Proof of 5.3.13. Recall that in the chromatic spectral sequence converging to $\mathrm{Ext}(BP_*/(2))$, $\mathrm{Ext}_1^{0,*} = \mathrm{Ext}(M_1^0)$, which is described in 5.2.2. Once we have determined the subgroup $E_\infty^{1,*} \subset E_1^{1,*}$ then (c) and (d) are routine calculations, which we will leave to the reader. Our strategy for proving (b) is to make low-dimensional computations by brute force (more precisely by comparison with the Adams spectral sequence) and then transport this information to higher dimensions by means of a map $\alpha\colon \Sigma^8 M(2) \to M(2)$ which induces multiplication by v_1^4 in BP-homology. [For an odd prime p there is a map $\alpha\colon \Sigma^q M(p) \to M(p)$ inducing multiplication by v_1. v_1^4 is the smallest power of v_1 for which such a map exists at $p = 2$.]

To prove (a), recall (5.2.2) that $\mathrm{Ext}(v_1^{-1}BP_*/(2)) = K(1)_*[h_0, \rho_1]/(\rho_1^2)$ with $h_0 \in \mathrm{Ext}^{1,2}$ and $\rho_1 \in \mathrm{Ext}^{1,0}$. We will determine the image of $\mathrm{Ext}(BP_*/(2))$ in this group. The element u maps to $v_1^4\rho_1$. [Our representative of u differs from that of $v_1^4\rho_1$ given in 5.2.2 by an element in the kernel of this map. We choose this u because it is the mod (2) reduction of $y_4 \in \mathrm{Ext}^{1,8}(BP_*)$.] It is clear that the image contains the summand described in (a). If the image contains $v_1^{-1}h_0^s$ or $v_1^{4-t}h_0^2\rho_1$ for any $t > 0$, then it also contains that element times any positive power of h_0. One can show then that such a family of elements in $\mathrm{Ext}(BP_*/(2))$ would contradict the edge theorem, 5.1.23.

To prove (b) we need some simple facts about $\pi_*(S^0)$ in dimensions ≤ 8 which can be read off the Adams spectral sequence (3.2.11). First we have $\eta^3 = 4\nu$ in $\pi_3(S^0)$. This means $h_0^3 x$ must be killed by a differential in the Adams–Novikov spectral sequence for $M(2)$ for any permanent cycle x. Hence we get $d_3(v_1^2) = h_0^3$ and $d_3(v_1^3) = v_1 h_0^3$. Next, if we did not have $\pi_2(M(2)) = \mathbf{Z}/(4)$ then $v_1 \in \pi_1(M(2))$ would extend to a map $\Sigma^2(M(2)) \to M(2)$ and by iterating it we could show that all powers of v_1 are permanent cycles, contradicting the above.

Now suppose we can show that v_1^4 and u are permanent cycles representing elements of order 2 in $\pi_*(M(2))$, i.e., maps $S^n \to M(2)$ which extend to self-maps $\Sigma^n M(2) \to M(2)$. Then we can iterate the resulting $\alpha\colon \Sigma^8 M(2) \to M(2)$ and compare with the map extending u to generalize the low-results above to all of (b).

A simple calculation with the Adams spectral sequence shows that $\pi_7(M(2))$ and $\pi_8(M(2))$ both have exponent 2 and contain elements representing u and v_1^4, respectively, so we have both the desired self-maps. $\square$

4. Ext^2 and the Thom Reduction

In this section we will describe $\mathrm{Ext}^2(BP_*)$ and what is known about its behavior in the Adams–Novikov spectral sequence. We will not give all the details of the calculation; they can be found in Miller, Ravenel, and Wilson [**1**] for odd primes and in Shimomura [**1**] for $p = 2$. The main problem is to compute $\mathrm{Ext}^0(M^2)$ and the map d_e^* from it to $\mathrm{Ext}^0(M^3)$. From this will follow (5.4.4) that the $\gamma_t \in \mathrm{Ext}^3(BP_*)$ are nontrivial for all $t > 0$ if p is odd. (We are using the notation of 5.1.19.) They are known to be permanent cycles for $p \geq 7$ (1.3.18).

We will also study the map Φ from Ext^2 to $E_2^{2,*}$ of the Adams spectral sequence as in 5.2.8 to show that most of the elements in the latter group, since they are not $\mathrm{im}\,\Phi$, cannot be permanent cycles (5.4.7). The result is that $\mathrm{im}\,\Phi$ is generated by

$$\{\Phi(\beta_{p^n/p^n-1}), \Phi(\beta_{p^n/p^n})\colon n \geq 1\}$$

and a certain finite number of other generators. It is known that for $p = 2$ the $\Phi(\beta_{p^n/p^n-1})$ are permenet cycles. They are the $\eta_{n+2} \in \Pi^S_{2^{n+2}}$ constructed by Mahowald [**6**] using Brown–Gitler spectra. For odd primes it follows that some element closely resembling β_{p^n/p^n-i} for $1 \le i \le p^n - 1$ is a nontrivial permanent cycle (5.4.9) and there is a similar more complicated result for $p = 2$ (5.4.10).

For $p = 2$, $\Phi(\beta_{2^n/2^n}) = b^2_{n+1}$ is known to be a permanent cycle iff there is a framed $(2^{n+2}-2)$-manifold with Kervaire invariant one (Browder [**1**]), and such are known to exist for $0 \le n \le 4$ (Barratt *et al.* [**2**]). The resulting element in $\pi_{2^{j+1}-2}$ is known as θ_j and its existence is perhaps the greatest outstanding problem in homotopy theory. It is known to have certain ramifications in the EHP sequence (1.5.29).

For odd primes the situation with $\Phi(\beta_{p^n/p^n})$ is quite different. We showed in Ravenel [**7**] that this element is *not* a permanent cycle for $p \ge 5$ and $n \ge 1$, and that β_{p^n/p^n} itself is *not* a permanent cycle in the Adams–Novikov spectral sequence for $p \ge 3$ and $n \ge 1$; see 6.4.1.

To compute Ext^2 with the chromatic spectral sequence we need to know $E^{0,2}_\infty$, $E^{1,1}_\infty$, and E^0_∞. The first vanishes by 5.2.1; the second is given by 5.3.5 for $p > 2$ and 5.3.6 for $p = 2$. For odd primes $\mathrm{Ext}^1(M^1) = E^{1,1}_1$ vanishes in positive dimensions; for $p = 2$ it gives elements in $\mathrm{Ext}^2(BP_*)$ which are products of α_1 with generators in $\mathrm{Ext}^2(BP_*)$. The main problem then is to compute $E^{0,2}_1 = \mathrm{Ext}^0(M^2)$. We use the short exact sequence

$$0 \to M^1_1 \to M^2 \xrightarrow{p} M^2 \to 0$$

and our knowledge of $\mathrm{Ext}^0(M^1_1)$ (5.2.13). The method of 5.1.17 requires us to recognize nontrivial elements in $\mathrm{Ext}^1(M^1_1)$. This group is not completely known but we have enough information about it to compute $\mathrm{Ext}^0(M^2)$. We know $\mathrm{Ext}^1(M^0_2)$ by 5.2.11, and in proving 5.2.13 one determines the image of $\mathrm{Ext}^0(M^1_1)$ in it. Hence we know all the elements in $\mathrm{Ext}^1(M^1_1)$ which are annihilated by v_1, so any other element whose product with some v^i_1 is one of these must be nontrivial.

To describe $\mathrm{Ext}^0(M^2)$ we need some notation from 5.2.13. We treat the odd primary case first. There we have

$$\begin{aligned}
x_{2,0} &= v_2,\\
x_{2,1} &= v_2^p - v_1^p v_2^{-1} v_3,\\
x_{2,2} &= x_{2,1}^p - v_1^{p^2-1} v_2^{p^2-p+1} - v_1^{p^2+p-1} v_2^{p^2-2p} v_3, \quad \text{and}\\
x_{2,i} &= x_{2,i-1}^p - 2v_1^{b_{2,i}} v_2^{(p-1)p^{i-1}+1} \quad \text{for } i \ge 3,
\end{aligned}$$

where $b_{2,i} = (p + 1)(p^{i-1} - 1)$. Also $a_{2,0} = 1$ and $a_{2,i} = p^i + p^{i-1} - 1$ for $i \ge 1$. Then

5.4.1. THEOREM (Miller, Ravenel, and Wilson [**1**]). *For odd primes p, $\mathrm{Ext}^0(M^2)$ is the direct sum of cyclic p-groups generated by*

(i) $\frac{x^s_{2,i}}{p^{k+1}v^j_1}$ *with $p \nmid s$, $j \ge 1$, $k \ge 0$ such that $p^k \mid j$ and $j \le a_{2,i-k}$ and either $p^{k+1} \nmid j$ or $a_{i-k-1} < j$; and*

(ii) $\frac{1}{p^{k+1}v^j_1}$ *for $k \ge 0$, $p^k \mid j$, and $j \ge 1$.* $\square$

Note that s may be negative.

For $p=2$ we define $x_{2,i}$ as above for $0 \le i \le 2$, $x_{2,i} = x_{2,i-1}^2$ for $i \ge 3$, $a_{2,0} = 1$, $a_{2,1} = 2$, and $a_{2,i} = 3 \cdot 2^{i-1}$ for $i \ge 2$. We also need $x_{1,0} = v_1$, $x_{1,1} = v_1^2 + 4v_1^{-1}v_2$, and $x_{1,i} = x_{1,i-1}^2$ for $i \ge 2$. In the following theorem we will describe elements in $\operatorname{Ext}^0(M^2)$ as fractions with denominators involving $x_{1,i}$, i.e., with denominators which are not monomials. These expressions are to be read as shorthand for sums of fractions with monomial denominators. For example, in $\frac{1}{8x_{1,1}}$ we multiply numerator and denominator by $x_{1,1}$ to get $\frac{x_{1,1}}{8x_{1,1}^2}$. Now $x_{1,1}^2 \equiv v_1^4 \mod (8)$ so we have

$$\frac{1}{8x_{1,1}} = \frac{v_1^2 + 4v_1^{-1}v_2}{8v_1^4} = \frac{1}{8v_1^2} + \frac{v_2}{2v_1^5}.$$

5.4.2. Theorem (Shimomura [**1**]). *For $p = 2$, $\operatorname{Ext}^0(M^2)$ is the direct sum of cyclic 2-groups generated by*

(i) $\frac{v_2^s}{2v_1}$, $\frac{x_{2,1}^s}{2v_1^j}$, $\frac{x_{2,2}^s}{2v_1^k}$, *and* $\frac{x_{2,2}^s}{8x_{11}}$ *for s odd, $j = 1$ or 2 and $k = 1$, 3, 4, 5, or 6 ($k = 2$ is excluded because $\beta_{4s/2}$ is divisible by 2);*

(ii) $\frac{x_{2,i}^s}{2v_1^j}$ *for s odd, $i \ge 3$, $j \le a_{2,i}$, and either j is odd or $a_{2,i-1} < j$;*

(iii) $\frac{x_{2,j}^s}{2^{k+1}v_1^{j2^k}}$ *for s odd, $j, k \ge 1$, $i \ge 3$, and $a_{2,i-k-1} < j2^k \le a_{2,i-k}$;*

(iv) $\frac{x_{2,i}^s}{2^{k+2}x_{1,k}^j}$ *for s odd $i \ge 3$, $k \ge 1$, j odd and ≥ 1, and $2^k j \le a_{2,i-k-1}$; and*

(v) $\frac{1}{2v_1^j}$, $\frac{1}{2^{k+2}x_{1,k}^j}$ *for j odd and ≥ 1 and $k \ge 1$.* □

This result and the subsequent calculation of $\operatorname{Ext}^2(BP_*)$ for $p = 2$ were obtained independently by S. A. Mitchell.

These two results give us $E_1^{2,0}$ in the chromatic spectral sequence. The image of d_1 is the summand of 5.4.1(ii) and 5.4.2(v) and, for $p = 2$, the summand generated by β_1; this is the same d_1 that we needed to find $\operatorname{Ext}^1(BP_*)$ (5.2.6). We know that $\operatorname{im} d_2 = 0$ since its source, $E_2^{0,1}$, is trivial by 5.2.1. The problem then is to compute $d_1\colon E_1^{2,0} \to E_1^{3,0}$. Clearly it is nontrivial on all the generators with negative exponent s. The following result is proved for $p > 2$ as lemma 7.2 in Miller, Ravenel, and Wilson [**1**] and for $p = 2$ in section 4 of Shimomura [**1**].

5.4.3. Lemma. *In the chromatic spectral sequence, $d_1\colon E_1^{2,0} \to E_1^{3,0}$ is trivial on all of the generators listed in 5.4.1 and 5.4.2 except the following*:

(i) *all generators with $s < 0$;*

(ii) $\frac{x_{2,i}}{pv_1^j}$ *with $p^i < j \le a_{2,i}$, and $i \ge 2$, the image of this generator being* $\frac{-v_3^{p^{i-1}}}{pv_1^{j-p^i}v_2^{p^{i-1}}}$; *and*

(iii) *(for $p = 2$ only)* $\frac{x_{2,2}}{8x_{1,1}}$, *whose image is* $\frac{v_3^2}{2v_1v_2}$. □

From this one easily read off both the structure of $\operatorname{Ext}^2(BP_*)$ and the kernel of $\alpha\colon \operatorname{Ext}^0(N^3) \to \operatorname{Ext}^3(BP_*)$, i.e., which Greek letter elements of the γ-family are trivial. We treat the latter case first. The kernel of α consists of $\operatorname{im} d_1 \oplus \operatorname{im} d_2 \oplus \operatorname{im} d_3$. For $p = 2$ we know that $\gamma_1 \in \operatorname{im} d_2$ by 5.1.22. d_2 for $p > 2$ and d_3 for all primes are trivial because $E_2^{1,1}$ (in positive dimensions) and $E_3^{0,2}$ are trivial by 5.3.5 and 5.2.1, respectively.

5.4.4. Corollary. *The kernel of $\alpha\colon \operatorname{Ext}^0(N^3) \to \operatorname{Ext}^3(BP_*)$ (5.1.18) is generated by $\gamma_{p^i/p^i,j}$ for $i \ge 1$ with $1 \ge j \ge p^i - 1$ for $p > 2$ and $1 \le j \le p^i$ for $p = 2$;*

and (for $p = 2$ only) γ_1 and γ_2. In particular $0 \neq \gamma_t \in \mathrm{Ext}^3(BP_*)$ for all $t > 0$ if $p > 2$ and for all $t > 2$ if $p = 2$.

5.4.5. COROLLARY.

(a) *For p odd, $\mathrm{Ext}^2(BP_*)$ is the direct sum of cyclic p-groups generated by $\beta_{sp^i/j,1+\phi(i,j)}$ for $s \geq 1$, $p \nmid s$, $j \geq 1$, $i \geq 0$, and $\phi(i,j) \geq 0$ where $\phi(i,j)$ is the largest integer k such that $p^k \mid j$ and*

$$j \leq \begin{cases} a_{2,i-k} & \text{if } s > 1 \text{ or } k > 0 \\ p^i & \text{if } s = 1 \text{ and } k = 0 \end{cases}$$

This generator has order $p^{1+\phi(i,j)}$ and internal dimension $2(p^2-1)sp^i - 2(p-1)j$. It is the image under α (5.1.18) of the element $\frac{x_{2,i}^s}{p^{1+\phi(i,j)}v_1^j}$ of 5.4.1.

(b) *For $p = 2$, $\mathrm{Ext}^2(BP_*)$ is the direct sum of cyclic 2-groups generated by $\alpha_1\bar{\alpha}_t$, where $\bar{\alpha}_t$ generates $\mathrm{Ext}^{1,2t}(BP_*)$ for $t \geq 1$ and $t \neq 2$ (see 5.2.6), and by $\beta_{s2^i/j,1+\phi(i,j)}$ for $s \geq 1$, s odd, $j \geq 1$, $i \geq 0$, and $\phi(i,j) \geq 0$ where*

$$\phi(i,j) = \begin{cases} 0 & \text{if } 2 \mid j \text{ and } a_{2,i-1} < j \leq a_{2,i}, \\ 0 & \text{if } j \text{ is odd and } j \leq a_{2,i}, \\ 2 & \text{if } j = 2 \text{ and } i = 2, \\ k \geq 2 & \text{if } j \equiv 2^{k-1} \mod (2^k),\ j \leq a_{2,i-k}, \text{ and } i \geq 3, \\ k \geq 1 & \text{if } 2^k \mid j,\ a_{2,i-k-1} < j \leq a_{2,i-k}, \text{ and } i \geq 3, \\ -1 & \text{otherwise} \end{cases}$$

unless $s = 1$, in which case $a_{2,i}$ is replaced by 2^i in cases above where $\phi(i,j) = 0$, $\phi(2,2) = 1$, and β_1 is omitted. The order, internal dimension, and definition of this generator are as in (a). □

For example when $p = 2$, $i = 3$ and s is odd with $s > 1$, we have generators

$$\begin{cases} \beta_{8s/j,2} = \dfrac{x_{2,2}^{2s}}{4v_1^j} & \text{for } j = 2,4,6 \\ \beta_{8s/j} = \dfrac{x_{2,2}^{2s}}{2v_1^j} & \text{for } 1 \leq j \leq 12 \text{ and } j \neq 2,4,6, \end{cases}$$

but $\beta_{8/j}$ is not defined for $9 \leq j \leq 12$. Similarly when $p > 2$, $i = 4$ and s is prime to p with $s > 1$, we have generators

$$\begin{cases} \beta_{p^4s/p^2,3} = \dfrac{x_{2,4}^s}{p^3v_1^{p^2}} & \\ \beta_{p^4s/j,2} = \dfrac{x_{2,4}^s}{p^2v_1^j} & \text{for } p|j,\ j \neq p^2 \text{ and } j \leq p^3+p^2-1 \\ \beta_{p^4s/j} = \dfrac{x_{2,4}^s}{pv_1^j} & \text{for other } j \leq p^4+p^3-1, \end{cases}$$

but $\beta_{p^4/j}$ is not defined for $p^4 < j \leq p^4 + p^3 - 1$.

Next we study the Thom reduction map Φ from $\mathrm{Ext}^2(BP_*)$ to $E_2^{2,*}$ in the classical Adams spectral sequence. This map on Ext^1 was discussed in 5.2.8. The group $E_2^{2,*}$ was given in 3.4.1 and 3.4.2. The result is

5.4.6. THEOREM. *The generators of* $\mathrm{Ext}^2(BP_*)$ *listed in* 5.4.5 *map to zero under the Thom reduction map* $\Phi\colon \mathrm{Ext}(BP_*) \to \mathrm{Ext}_{A_*}(\mathbf{Z}/(p), \mathbf{Z}/(p))$ *with the following exceptions.*

(a) (S. A. Mitchell). *For* $p = 2$

$$\Phi(\alpha_1^2) = h_1^2, \qquad \Phi(\alpha_1\alpha_{4/4}) = h_1h_3,$$
$$\Phi(\beta_{2^j/2^j}) = h_{j+1}^2 \qquad \text{for } j \geq 1,$$
$$\Phi(\beta_{2^j/2^j-1}) = h_1h_{j+2} \qquad \text{for } j \geq 2,$$
$$\Phi(\beta_{4/2,2}) = h_2h_4 \quad \text{and} \quad \Phi(\beta_{8/6,2}) = h_2h_5.$$

(b) (Miller, Ravenel, and Wilson [**1**]). *For* $p > 2$ $\Phi(\beta_{p^j/p^j}) = -b_j$ *for* $j \geq 0$*;* $\Phi(\beta_{p^j/p^j-1}) = h_0h_{j+1}$ *for* $j \geq 1$*, and* $\Phi(\beta_2) = \pm k_0$*.*

PROOF. We use the method of 5.2.8. For (a) we have to consider elements of $\mathrm{Ext}^1(N^1)$ as well as $\mathrm{Ext}^0(N^2)$. Recall (5.3.6) that the former is spanned by $\frac{v_1^s\rho_1}{2}$ for odd $s \geq 5$ and $\frac{v_1^st_1}{2}$ for odd $s \geq 1$. We are looking for elements with I-adic filtration ≥ 0, and the filtrations of t_1 and ρ_1 are 0 and -4, respectively. Hence we need to consider only $\frac{v_1^5\rho_1}{2}$ and $\frac{v_1t_1}{2}$, which give the first two cases of (a).

The remaining cases come from $\mathrm{Ext}^0(N^2)$. The filtration of $x_{2,i}$ is p^i so $\beta_{i/j,k}$ has filtration $i-j-k$, and this number is positive in all cases except those indicated above. We will compute $\Phi(\beta_{2/2})$ and $\Phi(\beta_{4/2,2})$, leaving the other cases of (a) and (b) to the reader. [The computation of $\Phi(\beta_1)$ and $\Phi(\beta_2)$ for $p > 2$ were essentially done in 5.1.20.] Using the method of 5.1.20(a), we find that $\beta_{2/2}$ reduces to $\frac{v_2t_1^2}{v_1} \mod (2)$, which in turn reduces to $t_1^2|t_1^2 \mod I_2$, which maps under Φ to h_2^2. Similarly, $\beta_{4/2,2}$ reduces to $\frac{v_2^3t_1^2}{v_1} + \frac{v_2^4(t_1^2+v_1t_1)}{v_1^4} \mod (2)$ and to $v_2^2t_1^2|t_1^2 + t_1^8|t_1^2 \mod I_2$, which maps under Φ to h_2h_4. $\square$

This result limits the number of elements in $\mathrm{Ext}_{A_*}^2(\mathbf{Z}/(p), \mathbf{Z}/(p))$ which can be permanent cycles. As remarked above (5.2.8), any such element must correspond to one having Novikov filtration ≤ 2. Theorem 5.4.6 tells us which elements in $\mathrm{Ext}(BP_*)^2$ map nontrivially to the Adams spectral sequence. Now we need to see which elements in $\mathrm{Ext}^1(BP_*)$ correspond to elements of Adams filtration 2. This amounts to looking for elements in $\mathrm{Ext}^0(N^1)$ with I-adic filtration 1. From 5.2.8 we see that $\alpha_{2/2}$ and $\alpha_{4/4}$ for $p = 2$ have I-adic filtration 0, so α_2 and $\alpha_{4/3}$ have filtration 1 and correspond to h_0h_2 and h_0h_3, respectively. More generally, α_t for all primes has filtration $t-1$ and therefore corresponds to an element with Adams filtration $\geq t$. Hence we get

5.4.7. COROLLARY. *Of the generators of* $\mathrm{Ext}_{A_*}^2(\mathbf{Z}/(p), \mathbf{Z}(p))$ *listed in* 3.4.1 *and* 3.4.2*, the only ones which can be permanent cycles in the Adams spectral sequence are*

(a) *for* $p = 2$*,* h_0^2*,* h_0h_2*,* h_0h_3*,* h_j^2 *for* $j \geq 1$*,* h_1h_j *for* $j \geq 3$*,* h_2h_4*, and* h_2h_5*; and*

(b) *for* $p > 2$*,* a_0^2*,* b_j *for* $j \geq 0$*,* a_1*,* a_0h_1 *for* $p = 3$*,* h_0h_j *for* $j \geq 2$*, and* k_0*.* $\square$

Part (a) was essentially proved by Mahowald and Tangora [**8**], although their list included h_3h_6. In Barratt, Mahowald, and Tangora [**1**] it was shown that h_2h_5 is not a permanent cycle. It can be shown that $d_3(\beta_{8/6,2}) \neq 0$, while $\beta_{4/2,2}$ is a permanent cycle. The elements h_0^2, h_0h_2, h_0h_3 for $p=2$ and a_0^2, a_1, a_0h_1 ($p=3$) for odd primes are easily seen to be permanent cycles detecting elements in $\operatorname{im} J$.

This leaves two infinite families to be considered: the b_j (or h_{j+1}^2 for $p=2$) for $j \geq 0$ and the h_0h_j (or h_1h_{j+1} for $p=2$) for $j \geq 1$. These are dealt with in 3.4.4 and 4.4.22. In Section 6.4 we will generalize the latter to

5.4.8. THEOREM. (a) *In the Adams–Novikov spectral sequence for $p \geq 3$,*

$$d_{2p-1}(\beta_{p^j/p^j}) \equiv \alpha_1\beta^p_{p^{j-1}/p^{j-1}} \neq 0$$

modulo a certain indeterminacy for $j \geq 1$.

(b) *In the Adams spectral sequence for $p \geq 5$, b_j is not a permanent cycle for $j \geq 1$.* □

The restriction on p in 5.4.8(b) is essential; we will see (6.4.11) that b_2 is a permanent cycle for $p=3$.

The proof of 3.4.4(b) does not reveal which element in $\mathrm{Ext}^2(BP_*)$ detects the constructed homotopy element. 5.4.5 implies that $\mathrm{Ext}^{2,(1+p^j)q}$ is a $\mathbf{Z}/(p)$ vector space of rank $[j/2]$; i.e., it is spanned by elements of the form $\delta_0(x)$ for $x \in \mathrm{Ext}^1(BP_*/(p))$. (This group is described in 5.2.14 and 5.2.17.) The x that we want must satisfy $v_1^{p^{j-1}-2}x = \delta_1(v_2^{p^j})$. ($\delta_0$ and δ_1 are defined in 5.1.2.) The fact that the homotopy class has order p, along with 2.3.4, means that x itself [as well as $\delta_0(x)$] is a permanent cycle, i.e., that the map $f\colon S^m \to S^0$ for $m = q(1+p^j)-3$ given by 3.3.4(d) fits into the diagram

$$\begin{tikzcd} S^m \arrow[r, "f"] \arrow[d] & S^0 \\ \Sigma^m M(p) \arrow[r, "\tilde{f}"] & \Sigma^{-1}M(p) \arrow[u] \end{tikzcd}$$

where $M(p)$ denotes the mod (p) Moore spectrum and the vertical maps are inclusion of the bottom cell and projection onto the top cell. Now $\tilde{f}$ can be composed with any iterate of the map $\alpha\colon \Sigma^q M(p) \to M(p)$ inducing multiplication by v_1 in BP-homology, and the result is a map $S^{m+qt} \to S^0$ detected by $\delta_0(v_1^t x)$. This gives

5.4.9. THEOREM. (R. Cohen [**3**]) *Let $\zeta_{j-1} \in \pi^S_{m-1}$ be the element given by* 3.4.4(d), *where $m = (1+p^j)q-2$. It is detected by an element $y_{j-1} \in \mathrm{Ext}^{3,2+m}(BP_*)$ congruent to $\alpha_1\beta_{p^{j-1}/p^{j-1}}$ modulo elements of higher I-adic filtration* (*i.e., modulo* $\ker \Phi$). *Moreover for $j \geq 3$ and $0 < i < p^{j-1}-p^{j-2}-p^{j-3}$ $\zeta_{j-1,i} \in \langle \zeta_{j-1}, p, \alpha_1\rangle \subset \pi^S_{m-1+qi}$, obtained as above, is nontrivial and detected by an element in $\mathrm{Ext}^{3,2+m+qi}(BP_*)$ congruent to $\alpha_1\beta_{p^{j-1}/p^{j-1}-i}$.* □

The range of i in 5.4.9(b) is smaller than in (a) because $\alpha_1\beta_{p^j/p^{j-1}+p^{j-2}} = 0$ for $j \geq 2$. To see this compute the coboundary of $\frac{v_1v_2^{p^j}}{p^2v_1^{(p+1)p^{j-2}}}$.

The analogous results for $p=2$ are more complicated. $\eta_j \in \pi^S_{2^j}$ is *not* known to have order 2, so we cannot extend it to a map $\Sigma^{2^j}M(2) \to S^0$ and compose with elements in $\pi_*(M(2))$ as we did in the odd primary case above. In fact, there is reason to believe the order of η_j is 4 rather than 2. To illustrate the results one might expect, suppose $\beta_{2^j/2^j}$ is a permanent cycle represented by an element of order 2. (This would imply that the Kervaire invariant element θ_{j+1} exists; see

1.5.29.) Then we get a map $f\colon \Sigma^{2^{j+2}-2}M(2) \to S^0$ which we can compose with the elements of $\pi_*(M(2))$ given by 5.3.13. In particular, fv_1^{4k} would represent $\beta_{2^j/2^j-4k}$, which is nontrivial for $k < 2^{j-2}$, fv_1 would represent $\beta_{2^j/2^j-1}$ (i.e., would be closely related to η_{j+2}), and $2fv_1$ would represent $\alpha_1^2\beta_{2^j/2^j}$, leading us to expect η_{j+2} to have order 4. Since the elements of 5.3.13 have filtration ≤ 3, the composites with f would have filtration ≤ 5. Hence their nontriviality in $\mathrm{Ext}(BP_*)$ is not obvious.

Now 5.3.13 describes 12 families of elements in $\mathrm{Ext}(BP_*/(2))$ (each family has the form $\{v_1^{4k}x\colon k \geq 0\}$) which are nontrivial permanent cycles: the six shown in 5.3.14 and their products with u. Since we do not know θ_{j+1} exists we cannot show that these are permanent cycles directly. However, five of them ($v_1\alpha_1$, $v_1\alpha_1^2$, uv_1 $uv_1\alpha_1$, and $uv_1\alpha_1^2$) can be obtained by composing v_1 with mod (2) reductions of permanent cycles in $\mathrm{Ext}(BP_*)$, and hence correspond to compositions of η_{j+1} with elements in π_*^S. Four of these five families have been shown to be nontrivial by Mahowald [**10**] without use of the Adams–Novikov spectral sequence.

5.4.10. THEOREM (Mahowald [**10**]). *Let $\mu_{8k+1} \in \pi_{8k-1}^S$ be the generator constructed by Adams [**1**] and detected by $\alpha_{4k+1} \in \mathrm{Ext}^{1,8k+2}(BP_*)$, and let $\rho_k \in \pi_{8k-1}^S$ be a generator of $\mathrm{im}\,J$ detected by a generator y_{4k} of $\mathrm{Ext}^{1,8k}(BP_*)$. Then for $0 < k < 2^{j-4}$ the compositions $\mu_{8k+1}\eta_j$, $\eta\mu_{8k+1}\eta_j$, $\rho_k\eta_j$, and $\eta\rho_k\eta_j$ are essential. They are detected in the Adams spectral sequence respectively by $P^kh_1^2h_j$, $P^kh_1^3h_j$, $P^{k-1}c_0h_j$, and $P^{k-1}c_0h_1h_j$.* □

This result provides a strong counterexample to the "doomsday conjecture", which says that for each $s \geq 0$, only finitely many elements of $E_2^{s,*}$ are permanent cycles (e.g., 1.5.29 is false). This is true for $s = 0$ and 1 by the Hopf invariant one theorem, 1.2.12, but 5.4.10 shows it is false for each $s \geq 2$.

5. Periodic Families in Ext^2

This section is a survey of results of other authors concerning which elements in $\mathrm{Ext}^2(BP_*)$ are nontrivial permanent cycles. These theorems constitute nearly all of what is known about systematic phenomena in the stable homotopy groups of spheres.

First we will consider elements various types of β's. The main result is 5.5.5. Proofs in this area tend to break down at the primes 2 and 3. These difficulties can sometimes be sidestepped by replacing the sphere with a suitable torsion-free finite complex. This is the subject of 5.5.6 ($p = 3$) and 5.5.7 ($p = 2$).

In 5.5.8 we will treat decomposable elements in Ext^2.

The proof of Smith [**1**] that β_t is a permanent cycle for $p \geq 5$ is a model for all results of this type, the idea being to show that the algebraic construction of β_t can be realized geometrically. There are two steps here. First, show that the first two short exact sequences of 5.1.2 can be realized by cofiber sequences, so there is a spectrum $M(p, v_1)$ with $BP_*(M(p, v_1)) = BP_*/I_2$, denoted elsewhere by $V(1)$. [Generally if $I = (q_0, q_1, \ldots, q_{n-1}) \in BP_*$ is an invariant regular ideal and there is a finite spectrum X with $BP_*(X) = BP_*/I$ then we will denote X by $M(q_0, \ldots, q_{n-1})$.] This step is quite easy for any odd prime and we leave the details to the reader. It cannot be done for $p = 2$. Easy calculations (e.g., 5.3.13) show that the map $S^2 \to M(2)$ realizing v_1 does not have order 2 and hence does not extend to the required map $\Sigma^2M(2) \to M(2)$. Alternatively, one could show that $H^*(M(2, v_1); \mathbf{Z}/(2))$, if it existed, would contradict the Adem relation $Sq^2Sq^2 = Sq^3Sq^1$.

The second step, which fails for $p = 3$, is to show that for all $t > 0$, $v_1^t \in \mathrm{Ext}^0(BP_*/I_2)$ is a permanent cycle in the Adams–Novikov spectral sequence for $M(p, v_1)$. Then 2.3.4 tells us that $\beta_t = \delta_0\delta_1(v_2^t)$ detects the composite

$$S^{2t(p^2-1)} \xrightarrow{v_2^t} M(p, v_1) \to \Sigma^{q+1}M(p) \to S^{q+2},$$

where $q = 2p-2$ as usual. One way to do this is to show that the third short exact sequence of 5.1.2 can be realized, i.e., that there is a map $\beta\colon \Sigma^{2(p^2-1)}M(p, v_1) \to M(p, v_1)$ realizing multiplication by v_2. This self-map can be iterated t times and composed with inclusion of the bottom cell to realize v_2^t. To construct β one must first show that v_2 is a permanent cycle in the Adams–Novikov spectral sequence for $M(p, v_1)$. One could then show that the resulting map $S^{2(p^2-1)} \to M(p, v_1)$ extends cell by cell to all of $\Sigma^{2(p^2-1)}M(p, v_1)$ by obstruction theory. However, this would be unnecessary if one knew that $M(p, v_1)$ were a ring spectrum, which it is for $p \geq 5$ but not for $p = 3$. Then one could smash v_2 with the identity on $M(p, v_1)$ and compose with the multiplication, giving

$$\Sigma^{2(p^2-1)}M(p, v_1) \to M(p, v_1) \wedge M(p, v_1) \to M(p, v_1),$$

which is the desired map β.

Showing that $M(p, v_1)$ is a ring spectrum, i.e., constructing the multiplication map, also involves obstruction theory, but in lower dimensions than above.

We will now describe this calculation in detail and say what goes wrong for $p = 3$. We need to know $\mathrm{Ext}^{s,t}(BP_*/I_2)$ for $t - s \leq 2(p^2 - 1)$. This deviates from $\mathrm{Ext}(BP_*/I) = \mathrm{Ext}_{P_*}(\mathbf{Z}/(p), \mathbf{Z}/(p))$ only by the class $v_2 \in \mathrm{Ext}^{0,2(p^2-1)}$. It follows from 4.4.8 that there are five generators in lower dimensions, namely $1 \in \mathrm{Ext}^{0,0}$, $h_0 \in \mathrm{Ext}^{1,q}$, $b_0 \in \mathrm{Ext}^{2,pq}$, $h_0b_0 \in \mathrm{Ext}^{3(p+1)q}$, $h_1 \in \mathrm{Ext}^{1,pq}$, and $\mathrm{Ext}^{s,t} = 0$ for $t - s = 2(p^2 - 1) - 1$ so v_2 is a permanent cycle for any odd prime.

To show $M(p, v_1)$ is a ring spectrum we need to extend the inclusion $S^0 \to M(p, v_1)$ to a suitable map from $X = M(p, v_1) \wedge M(p, v_1)$. We now assume $p = 5$ for simplicity. Then X has cells in dimensions 0, 1, 2, 9, 10, 11, 18, 19, and 20, so obstructions occur in $\mathrm{Ext}^{s,t}$ for $t - s$ one less than any of these numbers. The only one of these groups which is nontrivial is $\mathrm{Ext}^{0,0} = \mathbf{Z}/(p)$. In this case the obstruction is p times the generator (since the 1-cells in X are attached by maps of degree p), which is clearly zero. Hence for $p \geq 5$ $M(p, v_1)$ is a ring spectrum and we have the desired self-map β needed to construct the β_t's.

However, for $p = 3$ obstructions occur in dimensions 10 and 11, where the Ext groups are nonzero. There is no direct method known for calculating an obstruction of this type when it lies in a nontrivial group. In Toda [**1**] it is shown that the nontriviality of one of these obstructions follows from the nonassociativity of the multiplication of $M(3)$.

We will sketch another proof now. If $M(3, v_1)$ is a ring spectrum then each β_t is a permanent cycle, but we will show that β_4 is not. In $\mathrm{Ext}^{6,84}(BP_*)$ one has $\beta_1^2\beta_4$ and $\beta_1\beta_{3/3}^2$. These elements are actually linearly independent, but we do not need this fact now. It follows from 4.4.22 that $d_5(\beta_1\beta_{3/3}^2) = \pm\alpha_1\beta_1^4\beta_{3/3} \neq 0$. The nontriviality of this element can be shown by computing the cohomology of P_* in this range.

Now $\beta_2^3 \in \mathrm{Ext}^{6,84}(BP_*)$ is a permanent cycle since β_2 is. If we can show

$$\beta_2^3 = \pm\beta_1\beta_{3/3}^2 \pm \beta_1^2\beta_4 \tag{5.5.1}$$

then $\beta_1^2\beta_4$ and hence β_4 will have to support a nontrivial d_5. We can prove 5.5.1 by reducing to $\mathrm{Ext}(BP_*/I_2)$. By 5.1.20 the images of β_1, β_2, and β_4 in this group are $\pm b_{10}$, $\pm v_2 b_{10} \pm k_0$, and $\pm v_2^3 b_{10}$, respectively, and the image of $\beta_{3/3}$ is easily seen to be $\pm b_{11}$. Hence the images of $\beta_1^2\beta_4$, $\beta_1\beta_{3/3}^2$, and β_2^3 are $\pm v_2^3 b_{10}^3$, $\pm b_{10}b_{11}^2$ and $\pm v_2^3 b_{10}^3 \pm k_0^3$ respectively. Thus 5.5.1 will follow if we can show $k_0^3 = \pm b_{10}b_{11}^2$. (At any larger prime p we would have $k_0^p = 0$.) k_0 is the Massey product $\pm\langle h_0, h_1, h_1\rangle$. Using A1.4.6 we have up to sign

$$\begin{aligned} k_0^2 &= \langle h_0, h_1, h_1\rangle\langle h_0, h_1, h_1\rangle \\ &= \langle h_0\langle h_0, h_1, h_1\rangle, h_1, h_1\rangle \\ &= \langle\langle h_0, h_0, h_1\rangle h_1, h_1, h_1\rangle \\ &= \langle h_0, h_0, h_1\rangle\langle h_1, h_1, h_1\rangle \\ &= g_0 b_{11} \end{aligned}$$

and

$$\begin{aligned} k_0^3 &= \langle h_0, h_1, h_1\rangle\langle h_0, h_0, h_1\rangle b_{11} \\ &= \langle h_0\langle h_0, h_0, h_1\rangle, h_1, h_1\rangle b_{11} \\ &= \langle\langle h_0, h_0, h_0\rangle h_1, h_1, h_1\rangle b_{11} \\ &= \langle h_0, h_0, h_0\rangle\langle h_1, h_1, h_1\rangle b_{11} \\ &= b_{10}b_{11}^2 \quad \text{as claimed.} \end{aligned}$$

5.5.2. THEOREM (Smith [**1**]). *Let $p \geq 5$*

(a) $\beta_t \in \mathrm{Ext}^{2,q((p+1)t-1)}$ *is a nontrivial permanent cycle in the Adams–Novikov spectral sequence for all $t > 0$.*

(b) *There is a map $\beta\colon \Sigma^{2(p^2-1)}M(p, v_1) \to M(p, v_1)$ inducing multiplication by v_2 in BP-homology. β_t detects the composite*

$$S^{2t(p^2-1)} \to \Sigma^{2t(p^2-1)}M(p, v_1) \xrightarrow{\beta^t} M(p, v_1) \to S^{2p}.$$

(c) *$M(p, v_1)$ is a ring spectrum.* □

5.5.3. THEOREM (Behrens and Pemmaraju [**1**]). (a) *For $p = 3$ the complex $V(1)$ admits a self-map realizing multiplication by v_2^9 in BP-homology.*

(b) *The element $\beta_t \in \mathrm{Ext}^{2,q((p+1)t-1)}$ is a nontrivial permanent cycle in the Adams–Novikov spectral sequence for t congruent to 0, 1,2, 5, or 6 modlulo 9.*

To realize more general elements in $\mathrm{Ext}^2(BP_*)$ one must replace I_2 in the above construction by an invariant regular ideal. For example a self-map β of $M(p^2, v_1^p)$ inducing multiplication by $v_2^{p^2}$ (such a map does not exist) would show that $\beta_{tp^2/p,2}$ is a permanent cycle for each $t > 0$. Moreover we could compose β^t on the left with maps other than the inclusion of the bottom cell to get more permanent cycles. $\mathrm{Ext}^0(BP_*/(p^2, v_1^p))$ contains pv_1^i for $0 \leq i < p$, and each of these is a permanent cycle and using it we could show that $\beta_{tp^2/p-i}$ is a permanent cycle.

It is easy to construct $M(p^{i+1}, v_1^{sp^i})$ for $s > 0$ and p odd. Showing that it is a ring spectrum and constructing the appropriate self-map is much harder. The following result is a useful step.

5.5.4. THEOREM. (a) (Mahowald [**11**]). *$M(4, v_1^{4t})$ is a ring spectrum for $t > 0$.*

(b) (Oka [**7**]). *$M(2^{i+2}, v_1^{2^i t} + 2^{i+1} t v_1^{2^i t-3} v_2)$ is a ring spectrum for $i \geq 2$ and $t \geq 2$.*

(c) (Oka [**7**]). *For $p > 2$, $M(p^{i+1} v_1^{p^i t})$ is a ring spectrum for $i \geq 0$ and $t \geq 2$ [Recall $M(p, v_1)$ is a ring spectrum for $p \geq 5$ by 5.5.2(c).]* □

Note that $M(p^i, v_1^j)$ is not unique; the theorem means that there is a finite ring spectrum with the indicated BP-homology.

Hence we have a large number of four-cell ring spectra available, but it is still hard to show that the relevant power of v_2 is a permanent cycle in Ext^0.

5.5.5. THEOREM.

(a) (Davis and Mahowald [**1**], Theorem 1.3). *For $p = 2$, there is a map $\Sigma^{48} M(2, v_1^4) \to M(2, v_1^4)$ inducing multiplication by v_2^8, so $\beta_{8t/4}$ and $\beta_{8t/3}$ are permanent cycles for all $t > 0$.*

(b) *For $p \geq 5$ the following spectra exist: $M(p, v_1^{p-1}, v_2^p)$* (Oka [**4, 1**], Smith [**2**], Zahler [**2**]); *$M(p, v_2^p, v_2^{tp})$ for $t \geq 2$* (Oka [**5**]); *$M(p, v_1^{2p-2}, v_2^{p^2})$* (Oka [**6**]); *$M(p, v_2^{2p}, v_2^{tp^2})$ for $t \geq 2$* (Oka [**6**]); *$M(p^2, v_1^p, v_2^{tp^2})$ for $t \geq 2$* (Oka [**6**]); *and consequently the following elements in $\mathrm{Ext}^2(BP_*)$ are nontrivial permanent cycles: $\beta_{tp/i}$ for $t > 0$, $1 \leq i \leq p-1$; $\beta_{tp/p}$ for $t \geq 2$; $\beta_{tp^2/i}$ for $t > 0$, $1 \leq i \leq 2p-2$; $\beta_{tp^2/2p}$ and $\beta_{tp^2/2p-1}$ for $t \geq 2$; and $\beta_{tp^2/p,2}$ for $t \geq 2$.*

(c) (Oka [**10**]). *For $p \geq 5$ the spectra $M(p, v_1^{2n-2p}, v_2^{p^n t})$ for $t \geq 2$ and $n \geq 3$, and $M(p, v_1^{2^{n-3}p}, v_2^{p^n})$ for $n \geq 3$ exist. Consequently the following elements are nontrivial permanent cycles: $\beta_{p^n t/s}$ for $t \geq 2$, $n \geq 3$, and $1 \leq s \leq 2^{n-2}p$; and $\beta_{p^n t/s}$ for $t \geq 1$, $n \geq 3$, and $1 \leq s \leq 2^{n-3}p$. In particular the p-rank of π_k^S can be arbitrarily large.* □

Note that in (a) $M(2, v_1^4)$ is not a ring spectrum since $M(2)$ is not, so the proof involves more than just showing that $v_2^8 \in \mathrm{Ext}^0(BP_*/(2, v_1^4))$ is a permanent cycle.

When a spectrum $M(p^i, v_1^j, v_2^k)$ for an invariant ideal $(p^i, v_1^j, v_2^k) \subset BP_*$ does not exist one can look for the following sort of substitute for it. Take a finite spectrum X with torsion-free homology and look for a finite spectrum $XM(p^i, v_1^j, v_2^k)$ whose BP homology is $BP_*(X) \otimes_{BP_*} BP_*/(p^i, v_1^j, v_2^k)$. Then the methods above show that the image $\bar{\beta}_{k/j,i}$ of $\beta_{k/j,i}$ induced by the inclusion $S^0 \to X$ [assuming X is (-1)-connected with a single 0-cell] is a permanent cycle. The resulting homotopy class must "appear" on some cell of X, giving us an element in π_*^S which is related to $\beta_{k/j,i}$. The first example of such a result is

5.5.6. THEOREM (Oka and Toda [**8**]). *Let $p = 3$ and $X = S^0 \cup_{\beta_1} e^{11}$, the mapping cone of β_1.*

(a) *The spectrum $XM(3, v_1, v_2)$ exists so $\bar{\beta}_t \in \mathrm{Ext}^2(BP_*(X))$ is a permanent cycle for each $t > 0$.*

(b) *The spectrum $XM(3, v_1^2, v_2^3)$ exists so $\bar{\beta}_{3t/2} \in \mathrm{Ext}^2(BP_*(X))$ is a permanent cycle for each $t > 0$.*

Let $p = 5$ and $X = S^0 \cup_{\beta_1} e^{39}$.

(c) *The spectrum* $XM(5, v_1, v_2, v_3)$ *exists so* $\bar{\gamma}_t \in \mathrm{Ext}^3(BP_*(X))$ *is a permanent cycle for all* $t > 0$. □

Hence $\bar{\beta}_t$ detects an element in $\pi_{16t-6}(X)$ which we also denote by $\bar{\beta}_t$. The cofibration defining X gives an long exact sequence

$$\cdots \to \pi_n(S^0) \xrightarrow{i} \pi_n(X) \xrightarrow{j} \pi_{n-11}(S^0) \xrightarrow{\beta_1} \pi_{n-1}(S^0) \to \cdots$$

where the last map is multiplication by $\beta_1 \in \pi_{10}(S^0)$. If $\bar{\beta}_t \notin \operatorname{im} i$ then $j(\beta_t) \neq 0$, so for each $t > 0$ we get an element in either π^S_{16t-6} or π^S_{16t-17}. For example, in the Adams–Novikov spectral sequence for the sphere one has $d_5(\beta_4) = \alpha_1\beta_1^2\beta_{3/3}$ so $\beta_4 \notin \operatorname{im} i$ and $j(\beta_4) \in \pi^S_{47}$ is detected by $\alpha_1\beta_1\beta_{3/3}$, i.e., $j(\beta_4) = \beta_1\varepsilon'$ (see 5.1.1). We can regard $j(\beta_t)$ as a substitute for β_t when the latter is not a permanent cycle.

In the above example we had $BP_*(X) = BP_* \oplus \Sigma^i BP_*$ as a comodule, so $\mathrm{Ext}(BP_*)$ is a summand of $\mathrm{Ext}(BP_*(X))$. In the examples below this is not the case, so it is not obvious that $\beta_{k/j,i} \neq 0$.

5.5.7. THEOREM (Davis and Mahowald [**1**] and Mahowald [**12**]). *Let* $p = 2$, $X = S^0 \cup_\eta e^2$, $W = S^0 \cup_\nu e^4$, *and* $Y = X \wedge W$. *Part* (a) *below is essentially theorem* 1.4 *of Davis and Mahowald* [**1**], *while the numbers in succeeding statements refer to theorems in Mahowald* [**12**]. *Their* Y *and* A_1 *are* $XM(2)$ *and* $XM(2, v_1)$ *in our notation.*

(a) $XM(2, v_1, v_2^8)$ *exists and* $\bar{\beta}_{8t} \in \mathrm{Ext}^2(BP_*(X))$ *is a nontrivial permanent cycle.*

(b) (1.4) *In the Adams–Novikov spectral sequence for* S^0, β_{8t} *is not a permanent cycle and* $\bar{\beta}_{8t} \in \pi_{4qt-4}(X)$ *projects under the pinching map* $X \to S^2$ *to an element detected by* $\alpha_1^2\beta_{8t/3}$ *if this element is nontrivial.*

(c) (1.5) $\bar{v}_2^{8t+1} \in \mathrm{Ext}^0(BP_*(X)/I_2)$ *and* $\bar{\beta}_{8t+1} \in (BP_*(X))$ *are nontrivial permanent cycles.* $\beta_{8t+1} \in \mathrm{Ext}^2(BP_*)$ *is not a permanent cycle and* $\beta_{8t+1} \in \pi_{48t+2}(X)$ *projects to an element detected by* $\alpha_1\alpha_{4/4}\beta_{8t/3} \in \mathrm{Ext}^4(BP_*)$ *if this element is nontrivial.*

PROOF. (a) Davis and Mahowald [**1**] showed that $XM(2, v_1)$ admits a self-map realizing v_2^8. This gives the spectrum and the permanent cycles. To show $\bar{\beta}_{8t} \neq 0$ it suffices to observe that $\beta_{8t} \in \mathrm{Ext}^2(BP_*)$ is not divisible by α_1.

(b) Mahowald [**12**] shows that $\bar{\beta}_{8t} \in \pi_{48t-4}(X)$ projects nontrivially to π^S_{48t-6}. By duality there is a map $f\colon \Sigma^{48t-4}(X) \to S^0$ which is nontrivial on the bottom cell. From 5.3.13 one can construct a map $\Sigma^{48t-4}X \to \Sigma^{48t-10}M(2)$ which is $v_1\eta^2$ on the bottom cell and such that the top cell is detected by $v_1^3 \in \mathrm{Ext}^0(BP_*/(2))$. Now compose this with the extension of $\beta_{8t/4}\Sigma^{48t-10}M(2) \to S^0$ given by 5.5.4(a). The resulting map $g\colon \Sigma^{48t-4}X \to S^0$ is $\alpha_1^2\beta_{8t/3}$ on the bottom cell and the top cell is detected by β_{8t}. Hence this map agrees with f modulo higher Novikov filtration. If $\alpha_1^2\beta_{8t/3} \neq 0 \in \mathrm{Ext}^4(BP_*)$ it follows that the bottom cell on f is detected by that element. [It is likely that $\alpha_1^3\beta_{8t/3} = 0$ (this is true for $t = 1$), so the differential on β_{8t} is not a d_3.]

(c) As in (b) Mahowald [**12**] shows the projection of β_{8t+1} in π^S_{48t} is nontrivial. To show that $\alpha_1\alpha_{4/4}\beta_{8t/3}$ detects our element if it is nontrivial we need to make a low-dimensional computation in the Adams–Novikov spectral sequence for $M(2, v_1^4)$ where we find that $v_1^3v_2 \in \mathrm{Ext}^{0,12}(BP_*/(2, v_1^4))$ supports a differential hitting $v_1\alpha_{4/4}\alpha_1^2 \in \mathrm{Ext}^{3,14}$. It follows that ${}_1\sigma\eta \in \pi_{11}(M(2, v_1^4))$ extends to a map

$\Sigma^{10}X \to M(2, v_1^4)$ with the top cell detected by $v_2v_1^3$. Suspending $48t-10$ times and composing with the extension of $\beta_{8t/4}$ to $\Sigma^{48t-10}M(2, v_1^4)$ gives the result. $\square$

Now we consider products of elements in Ext^1.

5.5.8. THEOREM. *Let $\bar{\alpha}_t$ be a generator of* $\mathrm{Ext}^{1,qt}(BP_*)$ (*see* 5.2.6).
(a) (Miller, Ravenel, and Wilson [**1**]). *For $p>2$, $\bar{\alpha}_s\bar{\alpha}_t = 0$ for all $s, t > 0$.*
(b) *For $p = 2$*

(i) *If s or t is odd and neither is 2 then $\bar{\alpha}_1\bar{\alpha}_t = \alpha_1\bar{\alpha}_{s+t-1} \neq 0$.*
(ii) $\bar{\alpha}_2^2 = \beta_{2/2}$.
(iii) $\bar{\alpha}_4^2 = \beta_{4/4} + \beta_3$.

(*Presumably, all other products of this sort vanish.*)

In an earlier version (iii) read $\bar{\alpha}_4^2 = \beta_{4/4}$ instead of the above. This error was brought to my attention by Hans-Werner Henn in 2016.

PROOF. Part (a) is given in Miller, Ravenel, and Wilson [**1**] as theorem 8.18. The method used is similar to the proof of (b) below.

For (b)(i) assume first that s and t are both odd. Then $\bar{\alpha}_s = \frac{v_1^s}{2}$ and the mod (2) reduction of $\bar{\alpha}_t$ is $v_1^{t-1}t_1$. Hence $\bar{\alpha}_s\bar{\alpha}_t = \frac{v_1^{s+t-1}}{2}t_1 = \bar{\alpha}_{s+t-1}\bar{\alpha}_1$.

For s odd and $t = 2$ we have

$$\alpha_s\bar{\alpha}_2 = \frac{v_1^s}{2}(t_1^2 + vt_1) = d\left(\frac{v_1^{s-t}v_2}{2}\right) \quad \text{so} \quad \alpha_s\bar{\alpha}_2 = 0.$$

For t even and $t > 2$, recall that

$$\bar{\alpha}_t = \frac{x^{t/2}}{4t} \quad \text{where} \quad x = v_1^2 - 4v_1^{-1}v_2$$

and

$$d(x) = 8\rho$$

where

$$\rho \equiv v_1^{-2}v_2t_1 - v_1^{-1}(t_2 + t_1^3) + 2(v_1t_1 + v_1^{-2}t_1^4 + v_1^{-2}t_1t_2 + v_1^{-3}v_2t_1^3) \mod (4).$$

Hence for even $t > 2$ the mod (2) reduction of $\bar{\alpha}_t$ is $v_1^{t-2}\rho$ and for odd s

$$\bar{\alpha}_s\bar{\alpha}_t = \frac{v_1^{s+t-2}}{2}\rho = \frac{v_1x^{(s+t-1)/2}}{2}\rho.$$

Since

$$d\left(\frac{v_1x^{(s+t+1)/2}}{2^3(s+t+1)}\right) = \frac{v_1}{2}x^{(s+t-1)/2}\rho + \frac{x^{(s+t+1)/2}t_1}{2^2(s+t+1)}$$

so $\bar{\alpha}_s\bar{\alpha}_t = \bar{\alpha}_1\bar{\alpha}_{s+t-1}$ as claimed.

For (ii) we have $\bar{\alpha}_2^2 = \frac{v_1^2(t_1^2+v_1t_1)}{4}$. The coboundary of $\frac{v_1^4}{2^5} + \frac{v_1^{-2}v_2^2}{2}$ shows this is cohomologous to $\beta_{2/2}$.

For (iii) we have $\alpha_{4/4}^2 \in \mathrm{Ext}^{2,16}$ which is $(\mathbf{Z}/(2))^3$ generated by $\alpha_1\alpha_7$, β_3, and $\beta_{4/4}$. $\alpha_1\alpha_7$ is not a permanent cycle (5.3.7) so $\alpha_{4/4}^2$ must be a linear combination of $\beta_{4/4}$ and β_3. Their reductions mod I_2, $t_1^4|t_1^4$ and $v_2t_1^4|t_1$, are linearly independent so it suffices to compute $\alpha_{4/4}^2 \mod I_2$. The mod I_2 reduction of $\alpha_{4/4}$ is $t_1^4 + v_2t_1$, so the result follows. $\square$

6. Elements in Ext^3 and Beyond

We begin by considering products of elements in Ext^2 with those in Ext^1 and Ext^2. If x and y are two such elements known to be permanent cycles, then the nontriviality of xy in Ext implies that the corresponding product in homotopy is nontrivial, but if $xy = 0$ then the homotopy product could still be nontrivial and represent an element in a higher Ext group. The same is true of relations among and divisibility of products of permanent cycles; they suggest but do not imply (without further argument) similar results in homotopy.

Ideally one should have a description of the subalgebra of $\mathrm{Ext}(BP_*)$ generated by Ext^1 and Ext^2 for all primes p. Our results are limited to odd primes and fall into three types (see also 5.5.8). First we describe the subgroup of Ext^3 generated by products of elements in Ext^1 with elements of order p in Ext^2 (5.6.1). Second we note that certain of these products are divisible by nontrivial powers of p (5.6.2). These two results are due to Miller, Ravenel, and Wilson [**1**], to which we refer for most of the proofs.

Our third result is due to Oka and Shimomura [**9**] and concerns products of certain elements in Ext^2 (5.4.4–5.4.7). They show further that in certain cases when a product of permanent cycles is trivial in Ext^4, then the corresponding product in homotopy is also trivial.

This brings us to γ's and δ's. Toda [**1**] showed that γ_t is a permanent cycle for $p \geq 7$ (1.3.18), but left open the case $p = 5$. In Section 7.5 we will make calculations to show that γ_3 does not exist. We sketch the argument here. As remarked in Section 4.4, 4.4.22 implies that $d_{33}(\alpha_1\beta_{5/5}^4) = \beta_1^{21}$ (up to a nonzero scalar). Calculations show that $\alpha_1\beta_{5/5}^4$ is a linear combination of $\beta_1^3\gamma_3$ and $\beta_1\langle\alpha_1\beta_3, \beta_4, \gamma_2\rangle$. Hence if the latter can be shown to be a permanent cycle then we must have $d_{33}(\gamma_3) = \beta_1^{18}$. Each of the factors in the above Massey product is a permanent cycle, so it suffices to show that the products $\alpha_1\beta_3\beta_4 \in \pi_{323}(S^0)$ and $\beta_4\gamma_2 \in \pi_{619}(S^0)$ both vanish. Our calculation shows that both of these stems have trivial 5-torsion.

To construct δ_t one could proceed as in the proof of 5.5.2. For $p \geq 7$ there is a finite complex $V(3)$ with $BP_*(V(3)) = BP_*/I_4$. According to Toda [**1**] it is a ring spectrum for $p \geq 11$. Hence there is a self-map realizing multiplication by v_4 iff there is a corresponding element in $\pi_*(V(3))$. We will show (5.6.13) that the group $\mathrm{Ext}^{2p-1,2(p^4+p-2)}(BP_*/I_4)$ is nonzero for all $p \geq 3$, so it is possible that $d_{2p-1}(v_4) \neq 0$.

The following result was proved in Miller, Ravenel, and Wilson [**1**] as theorem 8.6.

5.6.1. THEOREM. *Let $m \geq 0$, $p \nmid s$, $s \geq 1$, $1 \leq j \leq a_{2,m}$ (where $a_{2,m}$ is as in* 5.4.1*) for $s > 1$ and $1 \leq j \leq p^m$ for $s = 1$. Then $\alpha_1\beta_{sp^m/j} \neq 0$ in $\mathrm{Ext}^3(BP_*)$ iff one of the following conditions holds*

(i) *$j = 1$ and either $s \not\equiv -1 \mod (p)$ or $s \equiv -1 \mod (p^{m+2})$.*

(ii) *$j = 1$ and $s = p - 1$.*

(iii) *$j > 1 + a_{2,m-\nu(j-1)-1}$.*

In case (ii)*, we have $\alpha_1\beta_{p-1} = -\gamma_1$ and for $m \geq 1$, $2\alpha_1\beta_{(p-1)p^m} = -\gamma_{p^m/p^m,p^m}$. The onfy linear relations among these classes are*

$$\alpha_1\beta_{sp^2/p+2} = s\alpha_1\beta_{sp^2-1},$$

and

$$\alpha_1\beta_{sp^{2m+2}/2+a_{2,m+1}} = 2s\alpha_1\beta_{sp^{2m+2}-p^m} \quad \textit{for} \quad m \geq 1. \qquad \square$$

This result implies that some of these products vanish and therefore certain Massey products (A1.4.1) are defined. For example, $\alpha_1\beta_{(tp-1)p^m} = 0$ if $t > 1$ and $p^{m+2} \nmid t$ so we have Massey products such as $\langle \beta_{2p-1}, \alpha_1, \alpha_1\rangle$ represented up to nonzero scalar multiplication by

$$\frac{v_2^{2p}t_1}{pv_1^{1+p}} + \frac{v_2^{2p-1}t_1^2 - 2v_2^{p-1}v_3t_1}{pv_1}.$$

This product has order p^2 but many others do not. For example, $\alpha_1\beta_{p/2} = 0$ and $\langle \beta_{p/2}, \alpha_1, \alpha_1\rangle$ is represented by

$$\frac{2v_1^{p-1}v_2^pt_1}{p^2v_1^p} - \frac{v_2^pt_1^2}{pv_1^2}$$

which has order p^2 and $p\langle \beta_{p/2}, \alpha_1, \alpha_1\rangle = \alpha_1\beta_p$ up to nonzero scalar multiplication. Similarly, one can show

$$\alpha_1\beta_{p^2} = p\langle \beta_{p^2/2}, \alpha_1, \alpha_1\rangle = p^2\langle \beta_{p^2/3}, \alpha_1, \alpha_1, \alpha_1\rangle.$$

The following results were 2.8(c) and 8.17 in Miller, Ravenel, and Wilson [**1**].

5.6.2. THEOREM. *With notation as in 5.6.1, if $\alpha_1\beta_{sp^m/j} \neq 0$ in $\mathrm{Ext}^3(BP_*)$, then it is divisible by at least p^i whenever $0 < i \leq m$ and $j \geq a_{2,m-i}$.* $\square$

5.6.3. THEOREM. *With notation as above and with t prime to p, $\alpha_{sp^k/k+1}\beta_{tp^m/j} = s\alpha_1\beta_{tp^m/j-sp^k+1}$ in $\mathrm{Ext}^3(BP_*)$.* $\square$

Now we consider products of elements in Ext^2, which are studied in Oka and Shimomura [**9**].

5.6.4. THEOREM. *For $p \geq 3$ we have $ij\beta_s\beta_t = st\beta_i\beta_j$ in Ext^4 for $i + j = s + t$.*

PROOF. To compute $\beta_s\beta_t$ we need the mod I_2 reduction of β_t, which was computed in 5.1.20. Hence we find $\beta_s\beta_t$ is represented by

$$\frac{-tv_2^{s+t-1}b_{10} + \binom{t}{2}v_2^{s+t-2}k_0}{pv_1}.$$

Now let

$$u_m = \frac{v_2^mv_1^{p-1}t_1^p}{p^2v_1^p} - \frac{v_2^mt_2}{pv_1^2} + \frac{kv_2^{k-1}}{pv_1}(t_1^pt_2 - t_1^{2p+1}).$$

A routine computation gives

$$d\Big(\frac{t}{2}\,u_{s+t-1}\Big) = \frac{sv_2^{s+t-1}b_{10}}{pv_1} - \frac{s}{2}(s+t-1)v_2^{s+t-2}k_0$$

and hence $\beta_s\beta_t$ is represented by $-\frac{st}{2}\frac{v_2^{s+t-2}k_0}{pv_1}$ and the result follows. $\square$

The analogous result in homotopy for $p \geq 5$ was first proved by Toda [**7**]. The next three results are 6.1, A, and B of Oka and Shimomura [**9**].

5.6.5. THEOREM. *For $p \geq 3$ the following relations hold in* Ext^4 *for $s, t > 0$.*
(i) $\beta_s\beta_{tp^k/r} = 0$ *for $k \geq 1$, $t \geq 2$ and $r < a_{2,k}$.*
(ii) $\beta_s\beta_{tp^2/p,2} = \beta_{s+t(p^2-p)}\beta_{tp/p}$.
(iii) *For $t, k \geq 2$,*

$$\begin{aligned} \beta_2\beta_{tp^k/a_{2,k}} &= \beta_{s+(tp-1)(p^{k-1}-p)}\beta_{tp^2/a_{2,2}} \\ &= (t/2)\beta_{s+(tp-1)p^{k-1}-(2p-1)p}\beta_{2p^2/a_{2,2}}. \quad \square \end{aligned}$$

5.6.6. THEOREM. *For $p \geq 5$, $0 < r \leq p$, with $r \leq p-1$ if $t = 1$, the element $\beta_s\beta_{tp/r}$ is trivial in $\pi_*(S^0)$ if one of the following conditions holds.*
(i) $r \leq p-2$.
(ii) $r = p-1$ *and* $s \not\equiv -1 \mod (p)$.
(iii) $r = p-1$ *or* p *and* $t \equiv 0 \mod (p)$. $\square$

5.6.7. THEOREM. *For $p \geq 5$, $s \not\equiv 0$ or 1, $t \not\equiv 0 \mod (p)$, and $t \geq 2$, the elements $\beta_s\beta_{tp/p}$ and $\beta_s\beta_{tp^2/p,2}$ are nontrivial.* $\square$

Now we will display the obstruction to the existence of $V(4)$, i.e., a nontrivial element in $\mathrm{Ext}^{2p-1,2(p^4+p-2)}(BP_*/I_4)$. This group is isomorphic to the corresponding Ext group for $P_* = P[t_1, t_2, \dots]$, the dual to the algebra of Steenrod reduced powers. To compute this Ext we use a method described in Section 3.5. Let $P(1)_* = P/(t_1^{p^2}, t_2^p, t_3, \dots)$, the dual to the algebra generated by P^1 and P^p. We will give P_* a decreasing filtration so that $P(1)_*$ is a subalgebra of E_0P_*. We let $t_1, t_2 \in F_0$, and $t_i^{p^2}, t_{i+1}^p, t_{i+2} \in F^{(p^i-1)/(p-1)}$ for $i \geq 1$. Then as an algebra we have

$$E_0P_* = P(1)_* \otimes T(t_{i+2,0}, t_{i+1,1}) \otimes P(t_{i,2}), \tag{5.6.8}$$

where $i \geq 1$, $t_{i,j}$ corresponds to $t_i^{p^j}$, and T denotes the truncated polynomial algebra of height p. Let R denote the tensor product of the second two factors in 5.6.8. Then

$$P(1)_* \to E_0P_* \to R \tag{5.6.9}$$

is an extension of Hopf algebras (A1.1.5) for which there is a Cartan–Eilenberg spectral sequence (A1.3.14) converging to

$$\mathrm{Ext}_{E_0P_*}(\mathbf{Z}/(p), \mathbf{Z}/(p))$$

with

$$E_2 = \mathrm{Ext}_{P(1)_*}(\mathbf{Z}/(p), \mathrm{Ext}_R(\mathbf{Z}/(p), \mathbf{Z}/(p))). \tag{5.6.10}$$

The filtration of P_* gives a spectral sequence (A1.3.9) converging to

$$\mathrm{Ext}_{P^*}(Z/(p), Z/(p))$$

with

$$E_2 = \mathrm{Ext}_{E_0P_*}(\mathbf{Z}/(p), \mathbf{Z}/(p)). \tag{5.6.11}$$

In the range of dimensions we need to consider, i.e., for $t - s \leq 2(p^4-1)$ Ext_R is easy to compute. We leave it to the reader to show that it is the cohomology of the differential $P(1)_*$-comodule algebra

$$E(h_{12}, h_{21}, h_{30}, h_{13}, h_{22}, h_{31}, h_{40}) \otimes P(b_{12}, b_{21}, b_{30})$$

with $d(h_{22}) = h_{12}h_{13}$, $d(h_{31}) = h_{21}h_{13}$, and $d(h_{40}) = h_{30}h_{13}$. In our range this cohomology is

$$E(h_{12}, h_{21}, h_{30}, h_{13})/h_{13}(h_{12}, h_{21}, h_{30}) \otimes P(b_{12}, b_{21}, b_{30}), \tag{5.6.12}$$

where the nontrivial action of $P(1)$ is given by

$$P^1 h_{30} = h_{21}, \quad P^p h_{21} = h_{12}, \quad \text{and} \quad P^p b_{30} = b_{21}.$$

We will not give all of the details of the calculations since our aim is merely to find a generator of $\mathrm{Ext}_*^{2p-1,2(p^4+p-2)}$. The element in question is

$$b_{20}^{p-3} h_{11} h_{20} h_{12} h_{21} h_{30}. \tag{5.6.13}$$

We leave it to the interested reader to decipher this notation and verify that it is a nontrivial cocycle.

CHAPTER 6

Morava Stabilizer Algebras

In this chapter we develop the theory which is the mainspring of the chromatic spectral sequence. Let $K(n)_* = \mathbf{Z}/(p)[v_n, v_n^{-1}]$ have the BP_*-module structure obtained by sending all v_i, with $i \neq n$ to 0. Then define $\Sigma(n)$ to be the Hopf algebra $K(n)_* \otimes_{BP_*} BP_*(BP) \otimes_{BP_*} K(n)_*$. We will describe this explicitly as a $K(n)_*$-algebra below. Its relevance to the Adams–Novikov spectral sequence is the isomorphism (6.1.1)

$$\operatorname{Ext}_{BP_*(BP)}(BP_*, v_n^{-1}BP_*/I_n) \cong \operatorname{Ext}_{\Sigma(n)}(K(n)_*, K(n)_*),$$

which is input needed for the chromatic spectral sequence machinery described in Section 5.1. In combination with 6.2.4, this is the result promised in 1.4.9. Since $\Sigma(n)$ is much smaller than $BP_*(BP)$, this result is a great computational aid. We will prove it along with some generalizations in Section 1, following Miller and Ravenel [**5**] and Morava [**2**].

In Section 2 we study $\Sigma(n)$, the nth Morava stabilizer algebra. We will show (6.2.5) that it is closely related to the $\mathbf{Z}/(p)$-group algebra of a pro-p-group S_n (6.2.3 and 6.2.4). S_n is the strict automorphism group [i.e., the group of automorphisms $f(x)$ having leading term x] of the height n formal group law F_n (see A2.2.18 for a description of the corresponding endomorphism ring). We use general theorems from the cohomology of profinite groups to show S_n is either p-periodic (if $(p-1) \mid n$) or has cohomological dimension n^2 (6.2.10).

In Section 3 we study this cohomology in more detail. The filtration of 4.3.24 leads to a May spectral sequence studied in 6.3.3 and 6.3.4. Then we compute H^1 (6.3.12) and H^2 (6.3.14) for all n and p. The section concludes with computations of the full cohomology for $n = 1$ (6.3.21), $n = 2$ and $p > 3$ (6.3.22), $n = 2$ and $p = 3$ (6.3.24), $n = 2$ and $p = 2$ (6.3.27), and $n = 3$, $p > 3$ (6.3.32).

The last two sections concern applications of this theory. In Section 4 we consider certain elements β_{p^i/p^i} in $\operatorname{Ext}^2(BP_*)$ for $p > 2$ analogous to the Kervaire invariant elements $\beta_{2^i/2^i}$ for $p = 2$. We show (6.4.1) that these elements are *not* permanent cycles in the Adams–Novikov spectral sequence. A crucial step in the proof uses the fact that S_{p-1} has a subgroup of order p to detect a lot of elements in Ext. Theorem 6.4.1 provides a test that must be passed by any program to prove the Kervaire invariant conjecture: it must not generalize to odd primes!

In Section 5 we construct ring spectra $T(m)$ satisfying $BP_*(T(m)) = BP_*[t_1, \dots, t_m]$ as comodules. The algebraic properties of these spectra will be exploited in the next chapter. We will show (6.5.5, 6.5.6, 6.5.11, and 6.5.12) that its Adams–Novikov E_2-term has nice properties.

1. The Change-of-Rings Isomorphism

Our first objective is to prove

6.1.1. THEOREM (Miller and Ravenel [**5**]). *Let M be a $BP_*(BP)$-comodule annihilated by $I_n = (p, v_1, \ldots, v_{n-1})$, and let $\overline{M} = M \otimes_{BP_*} K(n)_*$. Then there is a natural isomorphism*

$$\operatorname{Ext}_{BP_*(BP)}(BP_*, v_n^{-1}M) = \operatorname{Ext}_{\Sigma(n)}(K(n)_*, \overline{M}).$$

Here $v_n^{-1}M$ denotes $v_n^{-1}BP_* \otimes_{BP_*} M$, which is a comodule (even though $v_n^{-1}BP_*$ is not) by 5.1.6. $\square$

This result can be generalized in two ways. Let

$$E(n)_* = v_n^{-1}BP_*/(v_{n+i} \colon i > 0)$$

and

$$E(n)_*(E(n)) = E(n)_* \otimes_{BP_*} BP_*(BP) \otimes_{BP_*} E(n)_*.$$

It can be shown, using the exact functor theorem of Landweber [**3**], that $E(n)_* \otimes_{BP_*} BP_*(X)$ is a homology theory on X represented by a spectrum $E(n)$ with $\pi_*(E(n)) = E(n)_*$, and with $E(n)_*(E(n))$ being the object defined above. We can generalize 6.1.1 by replacing $\Sigma(n)$ with $E(n)_*(E(n))$ and relaxing the hypothesis on M to the condition that it be I_n-nil, i.e., that each element (but not necessarily the entire comodule) be annihilated by some power of I_n. For example, N^n of Section 5.1 is I_n-nil. Then we have

6.1.2. THEOREM (Miller and Ravenel [**5**]). *Let M be I_n-nil and let $\overline{M} = M \otimes_{BP_*} E(n)_*$. Then there is a natural isomorphism*

$$\operatorname{Ext}_{BP_*(BP)}(BP_*, v_n^{-1}M) = \operatorname{Ext}_{E(n)_*(E(n))}(E(n)_*, \overline{M}).$$

There is another variation due to Morava [**2**]. Regard BP_* as a $\mathbf{Z}/2(p^n-1)$-graded object and consider the homomorphism $\theta \colon BP_* \to \mathbf{Z}/(p)$ given by $\theta(v_n) = 1$ and $\theta(v_i) = 0$ for $i \neq n$. Let $I \subset BP_*$ be $\ker\theta$ and let V_θ and VT_θ denote the I-adic completions of BP_* and $BP_*(BP)$. Let $E_\theta = V_\theta(v_{n+i} \colon i > 0)$ and $EH_\theta = E_\theta \otimes_{V_\theta} VT_\theta \otimes_{V_\theta} E_\theta$.

6.1.3. THEOREM (Morava [**2**]). *With notation as above there is a natural isomorphism*

$$\operatorname{Ext}_{VT_\theta}(V_\theta, M) \cong \operatorname{Ext}_{EH_\theta}(E_\theta, \overline{M})$$

where M is a VT_θ-comodule and $\overline{M} = M \otimes_{V_\theta} E_\theta$. $\square$

Of these three results only 6.1.1 is relevant to our purposes so we will not prove the others in detail. However, Morava's proof is more illuminating than that of Miller and Ravenel [**5**] so we will sketch it first.

Morava's argument rests on careful analysis of the functors represented by the Hopf algebroids VT_θ and EH_θ. First we need some general nonsense.

Recall that a groupoid is a small category in which every morphism is invertible. Recall that a Hopf algebroid (A, Γ) over K is a cogroupoid object in the category of commutative K-algebras; i.e., it represents a covariant groupoid-valued functor. Let $\alpha, \beta \colon \mathbf{G} \to \mathbf{H}$ be maps (functors) from the groupoid $\mathbf{G}$ to the groupoid $\mathbf{H}$. Since $\mathbf{G}$ is a category it has a set of objects, $Ob(\mathbf{G})$, and a set of morphisms, $\operatorname{Mor}(\mathbf{G})$, and similarly for $\mathbf{H}$.

6.1.4. DEFINITION. *The functors* $\alpha, \beta\colon G \to H$ *are* equivalent *if there is a map* $\theta\colon Ob(\mathbf{G}) \to \mathrm{Mor}(\mathbf{H})$ *such that for any morphism* $g\colon g_1 \to g_2$ *in* $\mathbf{G}$ *the diagram*

$$\begin{tikzcd} \alpha(g_1) \arrow[r, "\alpha(g)"] \arrow[d, "\theta(g_1)"'] & \alpha(g_2) \arrow[d, "\theta(g_2)"'] \\ \beta(g_1) \arrow[r, "\beta(g)"] & \beta(g_2) \end{tikzcd}$$

commutes. Two maps of Hopf algebroids $a, b\colon (A, \Gamma) \to (B, \Sigma)$ *are* naturally equivalent *if the corresponding natural transformations of groupoid-valued functors are naturally equivalent in the above sense. Two Hopf algebroids* (A, Γ) *and* (B, Σ) *are* equivalent *if there are maps* $f\colon (A, \Gamma) \to (B, \Sigma)$ *and* $h\colon (B, \Sigma) \to (A, \Gamma)$ *such that* hf *and* fh *are naturally equivalent to the appropriate identity maps.* □

Now we will show that a Hopf algebroid equivalence induces an isomorphism of certain Ext groups. Given a map $f\colon (A, \Gamma) \to (B, \Sigma)$ and a left Γ-comodule M, define a Σ-comodule $f^*(M)$ to be $B \otimes_A M$ with coactions

$$B \otimes_A M \to B \otimes_A \Gamma \otimes_A M \to B \otimes_B \Sigma \otimes_A M = \Sigma \otimes_B B \otimes_A M.$$

6.1.5. LEMMA. *Let* $f\colon (A, \Gamma) \to (B, \Sigma)$ *a Hopf algebroid equivalence. Then there is a natural isomorphism* $\mathrm{Ext}_\Gamma(A, M) \cong \mathrm{Ext}_\Sigma(B, f^*(M))$ *for any* Γ*-comodule* M.

PROOF. It suffices to show that equivalent maps induce the same homomorphisms of Ext groups. An equivalence between the maps $a, b\colon (A, \Gamma) \to (B, \Sigma)$ is a homomorphism $\phi\colon \Gamma \to B$ with suitable properties, including $\phi\eta_R = a$ and $\phi\eta_L = b$. Since η_R and η_L are related by the conjugation in Γ, it follows that the two A-module structures on B are isomorphic and that $a^*(M)$ is naturally isomorphic to $b^*(M)$. We denote them interchangeably by M'. The maps a and b induce maps of cobar complexes (A1.2.11) $C_\Gamma(M) \to C_\Sigma(M')$. A tedious routine verification shows that ϕ induces the required chain homotopy. □

Now we consider the functors represented by VT_θ and EH_θ. Recall that an *Artin local ring* is a commutative ring with a single maximal ideal satisfying the descending chain condition, i.e., the maximal ideal is nilpotent. If A is such a ring with finite residue field k then it is $W(k)$-module, where $W(k)$ is the Witt ring of A2.2.16. Let $\mathbf{Art}_\theta$ denote the category of $\mathbf{Z}/(2(p^n-1))$-graded Artin local rings whose residue field is an $\mathbf{F}_p$-algebra. Now let $m_\theta = \ker\theta \subset BP_*$. Then BP_*/m_θ^n with the cyclic grading is is object in $\mathbf{Art}_\theta$, so $V_\theta = \varprojlim BP_*/m_\theta^n$ is an inverse limit of such objects as is VT_θ. For any $A \in \mathbf{Art}_\theta$, we can consider $\mathrm{Hom}^c(VT_\theta, A)$, the set of continuous ring homomorphisms from VT_θ to A. It is a groupoid-valued functor on $\mathbf{Art}_\theta$ pro-represented by VT_θ. (We have to say "pro-represented" rather than "represented" because VT_θ is not in $\mathbf{Art}_\theta$.)

6.1.6. PROPOSITION. VT_θ *pro-represents the functor* $\mathbf{lifts}_\theta$ *from* $\mathbf{Art}_\theta$ *to the category of groupoids, defined as follows. Let* $A \in \mathbf{Art}_\theta$ *have residue field* k. *The objects in* $\mathbf{lifts}_\theta(A)$ *are* p*-typical liftings to* A *of the formal group law over* k *induced by the composite* $BP_* \xrightarrow{\theta} \mathbf{F}_p \to k$, *and morphisms in* $\mathbf{lifts}_\theta(A)$ *are strict isomorphisms between such liftings.* □

6.1.7. DEFINITION. *Let* $m_A \subset A$ *be the maximal ideal for* $A \in \mathbf{Art}_\theta$. *Given a homomorphism* $f\colon F \to G$ *of formal group laws over* A, *let* $\bar{f}\colon \overline{F} \to \overline{G}$ *denote their reductions* mod m_A. f *is a* $*-$isomorphism *if* $\bar{f}(x) = x$.

6.1.8. LEMMA. *Let F and G be objects in* $\mathbf{lifts}_\theta(A)$. *Then the map* $\mathrm{Hom}(F,G) \to \mathrm{Hom}(\overline{F},\overline{G})$ *is injective.*

PROOF. Suppose $\bar{f} = 0$, i.e., $f(x) = 0 \mod m_A$. We will show that $f(x) \equiv 0 \mod m_A^r$ implies $f(x) \equiv 0 \mod m_A^{r+1}$ for any $r > 0$, so $f(x) = 0$ since m_A is nilpotent. We have

$$G(f(x), f(y)) \equiv f(x) + f(y) \mod m_A^{2r}$$

since

$$G(x,y) \equiv x + y \mod (x,y)^2.$$

Consequently,

$$[p]_G(f(x)) \equiv pf(x) \mod m_A^{2r} \equiv 0 \mod m_A^{r+1}$$

since $p \in m_A$. On the other hand

$$[p]_G(f(x)) = f([p]_F(x))$$

and we know $[p]_F(x) \equiv x^{p^n} \mod m_A$ by A2.2.4. Hence $f([p]_F(x)) \equiv 0 \mod m_A^{r+1}$ gives the desired congruence $f(x) \equiv 0 \mod m_A^{r+1}$. □

Now suppose $f_1, f_2\colon F \to G$ are $*$-isomorphisms (6.1.7) as is $g\colon G \to F$. Then $gf_1 = gf_2$ by 6.1.8 so $f_1 = f_2$; i.e. $*$-isomorphisms are unique. Hence we can make

6.1.9. DEFINITION. $\mathbf{lifts}^*_\theta(A)$ *is the groupoid of $*$-isomorphism classes of objects in* $\mathbf{lifts}_\theta A$.

6.1.10. LEMMA. *The functors* $\mathbf{lifts}_\theta$ *and* $\mathbf{lifts}^*_\theta$ *are naturally equivalent.*

PROOF. There is an obvious natural transformation $\alpha\colon \mathbf{lifts}_\theta \to \mathbf{lifts}^*_\theta$, and we need to define $\beta\colon \mathbf{lifts}^*_\theta \to \mathbf{lifts}_\theta$, of each $*$-isomorphism class. Having done this, $\alpha\beta$ will be the identity on $\mathbf{lifts}^*_\theta$ and we will have to prove $\beta\alpha$ is equivalent (6.1.4) to the identity on $\mathbf{lifts}_\theta$.

The construction of β is essentially due to Lubin and Tate [**3**]. Suppose $G_1 \in \mathbf{lifts}_\theta(A)$ is induced by $\theta_1\colon BP_* \to A$. Using A2.1.26 and A2.2.6 one can show that there is a unique $G_2 \in \mathbf{lifts}_\theta(A)$ $*$-isomorphic to G_1 and induced by θ_2 satisfying $\theta(v_{n+i}) = 0$ for all $i > 0$. We leave the details to the interested reader. As remarked above, the $*$-isomorphism from G_1 to G_2 is unique. The verification that $\beta\alpha$ is equivalent to the identity is straightforward. □

To prove 6.1.3, it follows from 6.1.5 and 6.1.10 that it suffices to show EH_θ pro-represents $\mathbf{lifts}^*_\theta$. In the proof of 6.1.10 it was claimed that any suitable formal group law over A is canonically $*$-isomorphic to one induced by $\theta\colon BP_* \to A$ which is such that it factors through E_θ. In the same way it is clear that the morphism set of $\mathbf{lifts}^*_\theta(A)$ is represented by EH_θ, so 6.1.3 follows.

Now we turn to the proof of 6.1.1. We have a map $BP_*(BP) \to \Sigma(n)$ and we need to show that it satisfies the hypotheses of the general change-of-rings isomorphism theorem A1.3.12, i.e., of A1.1.19. These conditions are

(6.1.11) (i) the map $\Gamma' = BP_*(BP) \otimes_{BP_*} K(n)_* \to \Sigma(n)$ is onto and

(ii) $\Gamma' \,\square_{\Sigma(n)}\, K(n)_*$ is a $K(n)_*$-summand of Γ'.

Part (i) follows immediately from the definition $\Sigma(n) = K(n)_* \otimes_{BP_*} \Gamma'$. Part (ii) is more difficult. We prefer to replace it with its conjugate,

(ii) $K(n)_* \square_{\Sigma(n)} K(n)_*(BP)$ is a $K(n)_*$ summand of $K(n)_*BP$ which is defined to be $K(n)_* \otimes_{BP_*} BP_*(BP)$. Let $B(n)_*$ denote $v_n^{-1}BP_*/I_n$. Then the right BP_*-module structure on $K(n)_*(BP)$ induces a right $B(n)_*$- module structure.

6.1.12. Lemma. *There is a map*

$$K(n)_*BP \to \Sigma(n) \otimes_{K(n)_*} B(n)_*$$

which is an isomorphism of $\Sigma(n)$*-comodules and of* $B(n)_*$*-modules, and which carries* 1 *to* 1*.*

Proof. Our proof is a counting argument, and in order to meet requirements of connectivity and finiteness, we pass to suitable "valuation rings". Thus let

$$\begin{aligned} k(0)_* &= \mathbf{Z}_{(p)} \subset K(0)_*, \\ k(n)_* &= \mathbf{F}_p[v_n] \subset K(n)_*, \quad n > 0, \\ k(n)_*BP &= k(n)_* \otimes_{BP_*} BP_*(BP) \subset K(n)_*BP, \\ b(n)_* &= k(n)_*[u_1, u_2, \dots] \subset B(n)_*, \end{aligned}$$

where $u_k = v_n^{-1}v_{n+k}$.

It follows from A2.2.6 that in $k(n)_*BP$,

$$\eta_R(v_{n+k}) \equiv v_n t_k^{p^n} - v_n^{p^k} t_k \mod (\eta_R(v_{n+1}), \dots, \eta_R(v_{n+k-1})). \tag{6.1.13}$$

Hence $\eta_R\colon BP_* \to k(n)_*BP$ factors through an algebra map $b(n)_* \to k(n)_*BP$. It is clear from 6.1.13 that as a right $b(n)_*$-module, $k(n)_*BP$ is free on generators $t^\alpha = t_1^{\alpha_1} t_2^{\alpha_2} \dots$ where $0 \le \alpha_i < p^n$ and all but finitely many α_i are 0; in particular, it is of finite type over $b(n)_*$.

Now define

$$\sigma(n) = k(n)_*BP \otimes_{b(n)_*} k(n)_* \subset \Sigma(n);$$

by the above remarks $\sigma(n) = k(n)_*[t_1, t_2, \dots]/(t_k^{p^n} - v_n^{p^k-1} t_k\colon k \ge 1)$ as an algebra. $(k(n)_*, \sigma(n))$ is clearly a sub-Hopf algebroid of $(K(n)_*, \Sigma(n))$, so $\sigma(n)$ is a Hopf algebra over the principal ideal domain $k(n)_*$.

The natural map $BP_*(BP) \to \sigma(n)$ makes $BP_*(BP)$ a left $\sigma(n)$-comodule, and this induces a left $\sigma(n)$-comodule structure on $k(n)_*BP$. We will show that the latter is an *extended* left $\sigma(n)$-comodule.

Define a $b(n)_*$-linear map $f\colon k(n)_*BP \to b(n)_*$ by

$$f(t^\alpha) = \begin{cases} 1 & \text{if } \alpha = (0, 0, \dots) \\ 0 & \text{otherwise.} \end{cases}$$

Then f satisfies the equations

$$\begin{aligned} f\bar{\eta}_R &= id\colon b(n)_* \to b(n)_*, \\ f \otimes_{b(n)_*} k(n)_* &= \varepsilon\colon \sigma(n) \to k(n)_*. \end{aligned}$$

Now let $\tilde{f}$ be the $\sigma(n)$-comodule map lifting f:

$$\begin{tikzcd} k(n)_*BP \arrow[r, "\psi"] \arrow[dr, "\tilde{f}"'] & \sigma(n) \otimes_{k(n)^*} k(n)_*BP \arrow[d, "\sigma(n)\otimes f"] \\ & \sigma(n) \otimes_{k(n)^*} b(n)_* \end{tikzcd} \tag{6.1.14}$$

Since $\psi\bar{\eta}_R(x) = 1 \otimes \bar{\eta}_R(x)$, ψ is $b(n)_*$-linear, so $\tilde{f}$ is too. We claim $\tilde{f}$ is an isomorphism. Since both sides are free of finite type over $b(n)_*$ it suffices to prove that $\tilde{f} \otimes_{b(n)_*} k(n)$ is an isomorphism. But 6.1.14 is then reduced to

$$\begin{tikzcd} \sigma(n) \arrow[r, "\Delta"] \arrow[dr, "\tilde{f}\otimes_{b(n)_*} k(n)_*"'] & \sigma(n) \otimes_{k(n)_*} \sigma(n) \arrow[d, "1\otimes\varepsilon"] \\ & \sigma(n) \otimes_{k(n)_*} k(n)_* \end{tikzcd}$$

so the claim follows from unitarity of Δ.

Now the map $K(n)_* \otimes_{k(n)_*} \tilde{f}$ satisfies the requirements of the lemma. $\square$

6.1.15. COROLLARY. *$\bar{\eta}_R\colon B(n)_* \to K(n)_* \,\square_{\Sigma(n)}\, K(n)_*BP$ is an isomorphism of $B(n)_*$-modules.*

PROOF. The natural isomorphism

$$B(n)_* \to K(n)_* \,\square_{\Sigma(n)}\, (\Sigma(n) \otimes_{K(n)_*} B(n)_*)$$

is $B(n)_*$-linear and carries 1 to 1. Hence

$$\begin{tikzcd} & K(n)_* \,\square_{\Sigma(n)}\, (\Sigma(n) \otimes_{K(n)_*} B(n)_*) \\ B(n)_* \arrow[ur, "\cong"] \arrow[dr, "\bar{\eta}_R"'] & \\ & K(n)_* \,\square_{\Sigma(n)}\, K(n)_*BP \arrow[uu, "\cong"'] \end{tikzcd}$$

commutes, and $\bar{\eta}_R$ is an isomorphism. $\square$

Hence 6.1.11(ii) follows from the fact that $K(n)_*$ is a summand of $\Sigma(n)$, and 6.1.1 is proved. From the proof of 6.1.12 we get an explicit description of $\Sigma(n)$, namely

6.1.16. COROLLARY. *As an algebra*

$$\Sigma(n) = K(n)_*[t_1, t_2, \dots]/(v_n t_i^{p^n} - v_n^{p^i} t_i \colon i > 0).$$

Its coproduct is inherited from $BP_*(BP)$, i.e., a suitable reduction of 4.3.13 holds.

2. The Structure of $\Sigma(n)$

To study $\Sigma(n)$ it is convenient to pass to the corresponding object graded over $\mathbf{Z}/2(p^n - 1)$. Make $\mathbf{F}_p$ a $K(n)_*$-module by sending v_n to 1, and let $S(n) = \Sigma(n) \otimes_{K(n)_*} \mathbf{F}_p$. For a $\Sigma(n)$-comodule M let $\overline{M} = M \otimes_{K(n)_*} \mathbf{F}_p$, which is easily seen to be an $S(n)$-comodule. The categories of $\Sigma(n)$- and $S(n)$-comodules are equivalent and we have

6.2.1. PROPOSITION. *For a $\Sigma(n)$-comodule M,*

$$\operatorname{Ext}_{\Sigma(n)}(K(n)_*, M) \otimes_{K(n)_*} \mathbf{F}_p \cong \operatorname{Ext}_{S(n)}(\mathbf{F}_p, \overline{M}). \qquad \square$$

We will see below (6.2.5) that if we regard $S(n)$ and $\overline{M}$ as graded merely over $\mathbf{Z}/(2)$, there is a way to recover the grading over $\mathbf{Z}/2(p^n-1)$. If M is concentrated in even dimensions (which it is in most applications) then we can regard $\overline{M}$ and $S(n)$ as ungraded objects. Our first major result is that $S(n)\otimes\mathbf{F}_{p^n}$ (ungraded) is the continuous linear dual of the $\mathbf{F}_{p^n}$-group algebra of a certain profinite group S_n to be defined presently.

6.2.2. DEFINITION. *The topological linear dual $S(n)^*$ of $S(n)$ is as follows. [In Ravenel [**5**] $S(n)^*$ and $S(n)$ are denoted by $S(n)$ and $S(n)_*$, respectively.] Let $S(n)_{(i)}$ be the sub-Hopf algebra of $S(n)$ generated by $\{t,\dots,t_i\}$. It is a vector space of rank p^{ni} and $S(n)=\varinjlim S(n)_{(i)}$. Then $S(n)^*=\varprojlim \operatorname{Hom}(S(n)_{(i)},\mathbf{F}_p)$, equipped with the inverse limit topology. The product and coproduct in $S(n)$ give maps of $S(n)^*$ to and from the completed tensor product*

$$S(n)^*\mathbin{\widehat{\otimes}}S(n)^*=\varprojlim \operatorname{Hom}(S(n)_{(i)}\otimes S(n)_j,\mathbf{F}_p).$$

To define the group S_n recall the $\mathbf{Z}_p$-algebra E_n of A2.2.17, the endomorphism ring of a height n formal group law. It is a free $\mathbf{Z}_p$-algebra of rank n^2 generated by ω and S, where ω is a primitive (p^n-1)th root of units, $S\omega=\omega^pS$, and $S^n=p$. $S_n\subset E_n^\times$, is the group of units congruent to $1 \mod (S)$, the maximal ideal in E_n. S_n is a profinite group, so its group algebra $\mathbf{F}_{p^n}[S_n]$ has a topology and is a profinite Hopf algebra. S_n is also a p-adic Lie group; such groups are studied by Lazard [**4**].

6.2.3. THEOREM. *$S(n)^*\otimes\mathbf{F}_q\cong\mathbf{F}_q[S_n]$ as profinite Hopf algebras, where $q=p^n$, S_n is as above, and we disregard the grading on $S(n)^*$.*

PROOF. First we will show $S(n)^*\otimes\mathbf{F}_q$, is a group algebra. According to Sweedler [**1**], Proposition 3.2.1, a cocommutative Hopf algebra is a group algebra iff it has a basis of group-like elements, i.e., of elements x satisfying $\Delta x=x\otimes x$. This is equivalent to the existence of a dual basis of idempotent elements $\{y\}$ satisfying $y_i^2=y_i$, and $y_iy_j=0$ for $i\neq j$. Since $S(n)\otimes\mathbf{F}_q$, is a tensor product of algebras of the form $R=\mathbf{F}_q[t]/(t^q-t)$, it suffices to find such a basis for R. Let $a\in\mathbf{F}_q^\times$ be a generator and let

$$r_i=\begin{cases}-\sum\limits_{0<j<q}(a^it)^j & \text{for } 0<i<q,\\ 1-t^{q-1} & \text{for } i=0.\end{cases}$$

Then $\{r_i\}$ is such a basis, so $S(n)^*\otimes\mathbf{F}_q$, is a group algebra.

Note that tensoring with $\mathbf{F}_q$ cannot be avoided, as the basis of R is not defined over $\mathbf{F}_p$.

For the moment let G_n denote the group satisfying $\mathbf{F}_p[G_n]\cong S(n)^*\otimes\mathbf{F}_q$. To get at it we define a completed left $S(n)$-comodule structure on $\mathbf{F}_q[[x]]$, thereby defining a left G_n-action. Then we will show that it coincides with the action of S_n as formal group law automorphisms given by A2.2.18.

We now define the comodule structure map

$$\psi\colon \mathbf{F}_q[[x]]\to S(n)\mathbin{\widehat{\otimes}}\mathbf{F}_q[[x]]$$

to be an algebra homomorphism given by

$$\psi(x)=\sum_{i\geq 0}{}^F t_i\otimes x^{p^i},$$

where $t_0 = 1$ as usual. To verify that this makes sense we must show that the following diagram commutes.

$$\begin{tikzcd}
\mathbf{F}_q[[x]] \arrow[r, "\psi"] \arrow[d, "\psi"'] & S(n) \mathbin{\widehat{\otimes}} \mathbf{F}_q[[x]] \arrow[d, "\Delta\otimes 1"] \\
S(n) \mathbin{\widehat{\otimes}} \mathbf{F}_q[[x]] \arrow[r, "1\otimes\psi"] & S(n) \mathbin{\widehat{\otimes}} S(n) \mathbin{\widehat{\otimes}} \mathbf{F}_q[[x]]
\end{tikzcd}$$

for which we have

$$\begin{aligned}
(\Delta\otimes 1)\psi(x) &= (\Delta\otimes 1)\sum_{i\geq 0}^{F} t_i \otimes x^{p^i} \\
&= \sum_{i\geq 0}^{F}\left(\sum_{j-k=i}^{F} t_j \otimes t_k^{p^j}\right)\otimes x^{p^i} \\
&= \sum_{j,k\geq 0}^{F} t_j \otimes t_k^{p^j} \otimes x^{p^{j+k}}
\end{aligned}$$

This can be seen by inserting x as a dummy variable in 4.3.12. We also have

$$\begin{aligned}
(1\otimes\psi)\psi(x) &= (1\otimes\psi)\left(\sum_{j\geq 0}^{F} t_i \otimes x^{p^i}\right) \\
&= \sum_{j\geq 0}^{F} t_i \otimes \left(\sum_{j\geq 0}^{F} t_j \otimes x^{p^i}\right)^{p^i} \\
&= \sum_{i,j\geq 0}^{F} t_i \otimes t_j^{p^i} \otimes x^{p^{i+j}}.
\end{aligned}$$

The last equality follows from the fact that $F(x^p, y^p) = F(x,y)^p$. The linearity of ψ follows from A2.2.21(b), so ψ defines an $S(n)\otimes\mathbf{F}_q$-comodule structure on $\mathbf{F}_q[[x]]$.

We can regard the t_i, as continuous $\mathbf{F}_q$-valued functions on G_n and define an action of G_n on the algebra $\mathbf{F}_q[[x]]$ by

$$g(x) = \sum_{i\geq 0}^{F} t_i(g)x^{p^i}$$

for $g \in G_n$. Hence $G(x) = x$ iff $g = 1$, so our representation is faithful.

We can embed G_n in the set of all power series of the form $\sum_{i\geq 0}^{F} a_i x^{p^i}$ which is E_n by A2.2.21 so the result follows. $\square$

6.2.4. Corollary. *If M is an ungraded $S(n)$-comodule, then 6.2.3 gives a continuous S_n-action on $M\otimes\mathbf{F}_q$, and*

$$\operatorname{Ext}^*_{S(n)}(\mathbf{F}_p, M)\otimes\mathbf{F}_q = H^*_c(G_n, M\otimes\mathbf{F}_q)$$

where H^*_c denotes continuous group cohomology. $\square$

To recover the grading on $S(n)\otimes M$, we have an action of the cyclic group of order $q-1$ generated by $\bar{\omega}^i\omega^i$ via conjugation in E_n.

6.2.5. Proposition. *The eigenspace of $S(n)\otimes\mathbf{F}_q$ with eigenvalue $\bar{\omega}^i$ is the component $S(n)_{2i}\otimes\mathbf{F}_q$ of degree $2i$.*

PROOF. The eigenspace decomposition is multiplicative in the sense that if x and y are in the eigenspaces with eigenvalues $\bar{\omega}^i$ and $\bar{\omega}^j$, respectively, the xy is in the eigenspace with eigenvalue $\bar{\omega}^{i+j}$. Hence it suffices to show that t_k is in the eigenspace with eigenvalue $\bar{\omega}^{p^k-1}$.

To see this we compute the conjugation of $t_kS^k \in E_n$ by ω and we have $\omega^{-1}(t_kS^k)\omega = \omega^{-1}t_k\omega^{p^k}S^k = \omega^{p^k-1}t_kS^k$. □

Corollary 6.2.4 enables us to apply certain results from group cohomology theory to our situation. First we give a matrix representation of E_n over $W(\mathbf{F}_q)$.

6.2.6. PROPOSITION. *Let $e = \sum_{0\leq i<n} e_iS^i$ with $e_i \in W(\mathbf{F}_q)$ be an element of E_n. Define an $n\times n$ matrix $(e_{i,j})$ over $W(\mathbf{F}_q)$ by*

$$e_{i+1,j+1} = \begin{cases} e_{j-i}^{\sigma^i} & \text{for } i \leq j \\ pe_{j+n-i}^{\sigma^i} & \text{for } i > j. \end{cases}$$

Then (a) *this defines a faithful representation of E_n;* (b) *the determinant $|e_{i,j}|$ lies in $\mathbf{Z}_p$.*

PROOF. Part (a) is straightforward. For (b) it suffices to check that ω and S give determinants in $\mathbf{Z}_p$. □

We can now define homomorphisms $c\colon \mathbf{Z}_p \to S_n$ and $d\colon S_n \to \mathbf{Z}_p$ for $p>2$, and $c\colon \mathbf{Z}_2^\times \to S_n$ and $d\colon \mathbf{Z}_2^\times$ for $p=2$ by identifying S_n with the appropriate matrix group. ($\mathbf{Z}_p$ is to be regarded here as a subgroup of $\mathbf{Z}_p^\times$.) Let d be the determinant for all primes. For $p>2$ let $c(x) = \exp(px)I$, where I is the $n\times n$ identity matrix and $x\in\mathbf{Z}_p$; for $p=2$ let $c(x)=xI$ for $x\in\mathbf{Z}_2^\times$.

6.2.7. THEOREM. *Let $S_n^1 = \ker d$.*
(a) *If $p>2$ and $p\nmid n$ then $S_n \cong \mathbf{Z}_p \oplus S_n^1$.*
(b) *If $p=2$ and n is odd then $S_n \cong S_n^1 \oplus \mathbf{Z}_2^\times$.*

PROOF. In both cases one sees that $\operatorname{im} c$ lies in the center of S_n (in fact $\operatorname{im} c$ is the center of S_n) and is therefore a normal subgroup. The composition dc is multiplication by n which is an isomorphism for $p\nmid n$, so we have the desired splitting. □

We now describe an analogous splitting for $S(n)$. Let $A^* = \mathbf{F}_p[\mathbf{Z}_p]$ for $p>2$ and $A^* = \mathbf{F}_2[\mathbf{Z}_2^\times]$ for $p=2$. Let A_* be the continuous linear dual of A.

6.2.8. PROPOSITION. *As an algebra $A = \mathbf{F}_p[u_1, u_2, \ldots]/(u_i - u_i^p)$. The coproduct Δ is given by*

$$\sum_{i\geq 0}^G \Delta(u_i) = \sum_{i,j\geq 0}^G u_i\otimes u_j$$

where $u_0 = 1$ and G is the formal group law with

$$\log_G(X) = \sum \frac{x^{p^i}}{p^i}.$$

PROOF. Since $A \cong \mathbf{F}_p[S_1]$, this follows immediately from 6.2.3. □

We can define Hopf algebra homomorphisms $c_*\colon S(n)\otimes\mathbf{F}_q \to A\otimes\mathbf{F}_q$ and $d_*\colon A\otimes\mathbf{F}_q \to S(n)\otimes\mathbf{F}_q$ dual to the group homomorphisms c and d defined above.

6.2.9. THEOREM. *There exist maps $c_*\colon S(n)\to A$ and $d_*\colon A\to S(n)$ corresponding to those defined above, and for $p\nmid n$, $S(n)\cong A\otimes B$, where $B\otimes \mathbf{F}_q$, is the continuous linear dual of $\mathbf{F}_q[S_n^1]$, where S_n^1, is defined in 6.2.7.*

PROOF. We can define c_* explicitly by

$$c_*t_i=\begin{cases} u_{i/n} & \text{if } n\mid i\\ 0 & \text{otherwise.}\end{cases}$$

It is straightforward to check that this is a homomorphism corresponding to the c_* defined above. In lieu of defining d_* explicitly we observe that the determinant of $\sum_{i\geq 0} t_iS^i$, where $t_i\in W(\mathbf{F}_q)$ and $t_i=t_i^q$, is a power series in p whose coefficients are polynomials in the t_i over $\mathbf{Z}_p$. It follows that d_* can be defined over $\mathbf{F}_p$. The splitting then follows as in 6.2.7. □

Our next result concerns the size of $\mathrm{Ext}_{S(n)}(\mathbf{F}_p,\mathbf{F}_p)$, which we abbreviate by $H^*(S(n))$.

6.2.10. THEOREM.

(a) *$H^*(S(n))$ is finitely generated as an algebra.*

(b) *If $(p-1)\nmid n$, then $H^i(S(n))=0$ for $i>n^2$ and $H^i(S(n))=H^{n^2-i}(S(n))$ for $0\leq i\leq n^2$, i.e., $H^*(S(n))$ satisfies Poincaré duality.*

(c) *If $(p-1)\mid n$, then $H^*(S(n))$ is p-periodic, i.e., there is some $x\in H^i(S(n))$ such that $H^*(S(n))$ above some finite dimension is a finitely generated free module over $\mathbf{F}_p[x]$.* □

We will prove 6.2.10(a) below as a consequence of the open subgroup theorem (6.3.6), which states that every sufficiently small open subgroup of S_n has the same cohomology as $\mathbf{Z}_p^{n^2}$. Then (c) and the statement in (b) of finite cohomological dimension are equivalent to saying that the Krull dimension of $H^*(S(n))$ is 1 or 0, respectively. Recall that the Krull dimension of a Noetherian ring R is the largest d such that there is an ascending chain $p_0\subset p_1\subset\cdots\subset p_d$ of nonunit prime ideals in R. Roughly speaking, d is the number of generators of the largest polynomial algebra contained in R. Thus $d=0$ iff every element in R is nilpotent, which in view of (a) implies (b). If $d=1$ and R is a graded $\mathbf{F}_p$-algebra, then every element in R has a power in $\mathbf{F}_p[x]$ for a fixed $x\in R$. R is a module over $\mathbf{F}_p[x]$, which is a principal ideal domain. Since $H^*(S(n))$ is graded and finitely generated, it is a direct sum of cyclic modules over $\mathbf{F}_p[x]$. More specifically it is a direct sum of a torsion module (where each element is annilhilated by some power of x) and a free module. Since it is finitely generated, the torsion must be confined to low dimensions, and $H^*(S(n))$ is therefore a free $\mathbf{F}_p[x]$-module in high dimensions, so (a) implies (c).

The following result helps determine the Krull dimension.

6.2.11. THEOREM (Quillen [**3**]). *For a profinite group G the Krull dimension of $H^*(G;\mathbf{F}_p)$ is the maximal rank of an elementary abelian p-subgroup of G, i.e., subgroup isomorphic to $(\mathbf{Z}/(p))^d$.* □

To determine the maximal elementary abelian subgroup of S_n, we use the fact that $D_n=E_n\otimes\mathbf{Q}$ is a division algebra over $\mathbf{Q}_p$ (A2.2.17), so if $G\subset S_n$ is abelian, then the $\mathbf{Q}_p$-vector space in D_n spanned by the elements of G is a subfield $K\subset D_n$. Hence the elements of G are all roots of unity, G is cyclic, and the Krull dimension is 0 or 1.

6.2.12. THEOREM. *A degree m extension K of $\mathbf{Q}_p$ embeds in D_n iff $m \mid n$.*

PROOF. See Serre [**1**, p. 202] or Cassels and Fröhlich [**1**, p. 138]. □

By 6.2.11 $H^*(S(n))$ has Krull dimension 1 iff S_n contains pth roots of unity. Since the field K obtained by adjoining such roots to $\mathbf{Q}_p$ has degree $p-1$, 6.2.12 gives 6.2.10(c) and the finite cohomological dimension statement in (b). For the rest of (b) we rely on theorem V.2.5.8 of Lazard [**4**], which says that if S_n (being an analytic pro-p-group of dimension n^2) has finite cohomological dimension, then that dimension is n^2 and Poincaré duality is satisfied.

The following result identifies some Hopf algebra quotients of $S(n)\otimes\mathbf{F}_{p^n}$. These are related to the graded Hopf algebras $\Sigma_A(n)$ discussed in Ravenel [**10**]. More precisely, $S(d,f)_a$ is a nongraded form of $\Sigma_A(d/f)$, where A is the ring of integers in an extension K (depending on a) of $\mathbf{Q}_p$ of degree fn/d and residue degree f.

6.2.13. THEOREM. *Let $a \in \overline{\mathbf{F}}_p$ be a (p^n-1)th root of unity, let d divide n, and let f divide d. Then there is a Hopf algebra*

$$S(d,f)_a = \mathbf{F}_{p^n}[t_f, t_{2f}, \dots]/(t_{if}^{p^d} - a_i t_{if} : i > 0)$$

where $a_i = a^{p^{id}-1}$, and a surjective homomorphism

$$\theta : S(n) \otimes \mathbf{F}_{p^n} \to S(d,f)_a$$

given by

$$t_i \mapsto \begin{cases} t_i & \text{if } f|i \\ 0 & \text{otherwise.} \end{cases}$$

The coproduct on $S(d,f)_a$ is determined by the one on $S(n)$. This Hopf algebra is cocommutative when $f = d$.

PROOF. We first show that the algebra structure on $S(d,f)_a$ is compatible with that on $S(n)$. The relation $t_{if}^{p^d} = a_i t_{if}$ implies

$$\begin{aligned} t_{if}^{p^{2d}} &= (a_i t_{if})^{p^d} = a_i^{(p^{2d}-1)/(p^d-1)} t_{if} = a^{(p^{id}-1)(p^{2d}-1)/(p^d-1)} t_{if} \\ t_{if}^{p^{3d}} &= a^{(p^{id}-1)(p^{3d}-1)/(p^d-1)} t_{if} \\ &\vdots \\ t_{if}^{p^n} &= a^{(p^{id}-1)(p^n-1)/(p^d-1)} t_{if} = t_{if}, \end{aligned}$$

so θ exists as an algebra map.

For the coproduct in $S(n)$ we have

$$\sum_{i\ge 0}^F \Delta(t_i) x^{p^i} = \sum_{i,j\ge 0}^F t_i \otimes t_j^{p^i} x^{p^{i+j}}$$

(where x is a dummy variable) which induces

$$\sum_{i\ge 0}^F \Delta(t_{if}) x^{p^{if}} = \sum_{i,j\ge 0}^F t_{if} \otimes t_{jf}^{p^{if}} x^{p^{(i+j)f}}$$

in $S(d,f)_a$. We need to show that this is compatible with the multiplicative relations. We can write $if = kd + \ell f$ with $0 \le \ell f < d$, so we can rewrite the above as

$$\begin{aligned}
\sum_{i\geq 0}^{F} \Delta(t_{if})x^{p^{if}} &= \sum_{i,j\geq 0}^{F} t_{if}\otimes t_{jf}^{p^{kd+\ell f}} x^{p^{(i+j)f}} \\
&= \sum_{i,j\geq 0}^{F} a_j^{p^{\ell f}(p^{kd}-1)/(p^d-1)} t_{if}\otimes t_{jf}^{p^{\ell f}} x^{p^{(i+j)f}} \\
&= \sum_{i,j\geq 0}^{F} a^{p^{\ell f}(p^{jd}-1)(p^{kd}-1)/(p^d-1)} t_{if}\otimes t_{jf}^{p^{\ell f}} x^{p^{(i+j)f}},
\end{aligned}$$

which gives a well defined coproduct in $S(d,f)_a$.

If $f=d$ then the right hand side simplifies to

$$\sum_{i,j\geq 0}^{F} a^{(p^{jd}-1)(p^{if}-1)/(p^d-1)} t_{if}\otimes t_{jf} x^{p^{(i+j)f}},$$

which is cocommutative as claimed. □

3. The Cohomology of $\Sigma(n)$

In this section we will use a spectral sequence (A1.3.9) based on the filtration of $\Sigma(n)$ induced by the one on $BP_*(BP)/I_n$ given in 4.3.24. We have

6.3.1. THEOREM. *Define integers $d_{n,i}$ by*

$$d_{n,i} = \begin{cases} 0 & \text{if } i\leq 0 \\ \max(i, pd_{n,i-n}) & \text{for } i>0. \end{cases}$$

Then there is a unique increasing filtration of the Hopf algebra $S(n)$ with $\deg t_i^{p^j} = d_{n,i}$ for $0\leq j<n$. □

The following is a partial description of the coproduct in the associated graded object $E^0S(n)$. For large i we need only partial information about the coproduct on $t_{i,j}$ in order to prove Theorem 6.3.3. I am grateful to Agnes Beaudry for finding an error in an earlier version of the following.

6.3.2. THEOREM. *Let $E^0S(n)$ denote the associated bigraded Hopf algebra. Its algebra structure is*

$$E^0S(n) = T(t_{i,j}\colon i>0,\quad j\in \mathbf{Z}/(n)),$$

where $T(\cdot)$ denotes the truncated polynomial algebra of height p on the indicated elements and $t_{i,j}$ corresponds to $t_i^{p^j}$. The coproduct is induced by the one given in 4.3.34. Explicitly, let $m=pn/(p-1)$. Then

$$\Delta(t_{i,j}) = \begin{cases} \sum\limits_{0\leq k\leq i} t_{k,j}\otimes t_{i-k,k+j} & \text{if } i<m, \\ \sum\limits_{0\leq k\leq i} t_{k,j}\otimes t_{i-k,k+j} + \bar{b}_{i-n,j+n-1} & \text{if } i=m, \\ t_{i,j}\otimes 1 + 1\otimes t_{i,j} + \bar{b}_{i-n,j+n-1} & \\ \qquad \operatorname{mod}(t_{k,\ell}\colon k\leq i-n-1) & \text{if } i>m, \end{cases}$$

where $t_{0,j}=1$ and $\bar{b}_{i,j}$ corresponds to the $b_{i,j}$ of 4.3.14. □

As in the case of the Steenrod algebra, the dual object $E_0S(n)^*$ is primitively generated and is the universal enveloping algebra of a restricted Lie algebra $L(n)$. $L(n)$ has basis $\{x_{i,j} \colon i > 0,\ j \in \mathbf{Z}/(n)\}$, where $x_{i,j}$ is dual to $t_{i,j}$.

6.3.3. THEOREM. *$E_0S(n)^*$ is the restricted enveloping algebra on primitives $x_{i,j}$ with bracket*

$$[x_{i,j}, x_{k,l}] = \begin{cases} \delta^l_{i+j} x_{i+k,j} - \delta^j_{k+1} x_{i+k,l} & \text{for } i+k \le m, \\ 0 & \text{otherwise,} \end{cases}$$

where m is the largest integer not exceeding $pn/(p-1)$, and $\delta^s_t = 1$ iff $s \equiv t \mod (n)$ and $\delta^s_t = 0$ otherwise. The restriction ξ is given by

$$\xi(x_{i,j}) = \begin{cases} x_{i+n,j+1} & \text{if } i > n/(p-1) \\ & \qquad \text{or } i = n/(p-1) \text{ and } p > 2 \\ x_{2n,j} + x_{2n,j+1} & \text{if } i = n \text{ and } p = 2 \\ 0 & \text{if } i < n/p-1. \quad \square \end{cases}$$

The formula for the restriction was given incorrectly in the first edition, and this error led to an incorrect description in 6.3.24 of the multiplicative structure of $H^*(S(2))$ for $p = 3$. The correct description is due to Henn [**1**] and will be given below. The corrected restriction formula was given to me privately by Ethan Devinatz.

PROOF OF 6.3.3. The formula for the bracket can be derived from 6.3.2 as follows. The primitive $x_{i,j}$ is dual to $t_{i,j}$. The bracket has the form

$$[x_{i,j}, x_{k,l}] = \sum_{m,n} c^{a,b}_{i,j,k,\ell} x_{a,b},$$

where the coefficient $c^{a,b}_{i,j,k,\ell}$ is nonzero only if the coproduct expansion on $t_{a,b}$ contains a term of the form $t_{i,j} \otimes t_{k,\ell}$ or $t_{k,\ell} \otimes t_{i,j}$. This can happen only when the the two expressions have the same bidegree. This means that

$$\begin{aligned} d_{n,a} &= d_{n,i} + d_{n,k} \\ \text{and} \qquad 2p^b(p^a-1) &\equiv 2p^j(p^i-1) + 2p^\ell(p^k-1) \qquad \mod 2(p^n-1) \end{aligned}$$

This happens only when $a = i+k \le m$ and $b = j$ or ℓ. Inspection of the coproduct formula leads to indicated Lie bracket.

The restriction requires more care. For finding the restriction on $x_{i,j}$ it suffices to work in the subalgebra of $E_0S(n)^*$ generated by $x_{k,\ell}$ for $k \ge i$.

It is also dual to passing to the quotient of $E^0S(n)$ obtained by killing $t_{k,\ell}$ for $k < i$. Hence description of $\Delta(t_{i,j})$ for $i > m$ given in 6.3.2 is sufficient for our purposes.

When $i > m$ we have

$$\begin{aligned} \Delta(t_{i,j}) &= t_{i,j} \otimes 1 + 1 \otimes t_{i,j} + \overline{b}_{i-n,j-1} \\ &= t_{i,j} \otimes 1 + 1 \otimes t_{i,j} - \sum_{0<\ell<p} p^{-1} \binom{p}{\ell} t^{\ell}_{i-n,j-1} \otimes t^{p-\ell}_{i-n,j-1}, \end{aligned}$$

so for $i > n/(p-1)$,

$$\Delta(t_{i+n,j+1}) \equiv t_{i+n,j+1} \otimes 1 + 1 \otimes t_{i+n,j+1} - \sum_{0<\ell<p} p^{-1}\binom{p}{\ell} t_{i,j}^{\ell} \otimes t_{i,j}^{p-\ell}$$
$$\bmod (t_{k,\ell} \colon k \leq i-1).$$

For brevity let $B = E^0 S(n)/(t_{k,\ell} \colon k \leq i-1)$ and let $\overline{B} = B/\mathbf{F}_p$ denote the unit coideal, the dual of the augmentation ideal in B^*.

It follows that under the reduced iterated coproduct

$$B \xrightarrow{\Delta^{p-1}} B^{\otimes p} \longrightarrow \overline{B}^{\otimes p}$$

we have

$$t_{i+n,j+1} \mapsto t_{i,j} \otimes t_{i,j} \otimes \cdots \otimes t_{i,j},$$

which leads to the desired value of $\xi(x_{i,j})$ for $i > n/(p-1)$. The argument for $i = n/p-1$ and p odd is similar.

For the case $p = 2$ and $i = n$, 6.3.2 gives

$$\begin{aligned}
\Delta(t_{2n,j}) &= \sum_{0\leq k\leq 2n} t_{k,j} \otimes t_{2n-k,k+j} + \overline{b}_{n,j-1} \\
&= t_{n,j-1} \otimes t_{n,j-1} + \sum_{0\leq k\leq 2n} t_{k,j} \otimes t_{2n-k,k+j} \\
&= t_{n,j-1} \otimes t_{n,j-1} + t_{n,j} \otimes t_{n,j} \\
&\qquad + \sum_{0\leq k<n} (t_{k,j} \otimes t_{2n-k,j+k} + t_{2n-k,j} \otimes t_{k,j-k}),
\end{aligned}$$

and the formula for $\xi(x_{n,j})$ follows.

For $i < n/(p-1)$ there are no terms in $\Delta(t_{i+n,k})$ for any k that would lead to a nontrivial restriction on $x_{i,j}$. □

Recall that Theorem 6.2.3 identifies $S(n)^* \otimes \mathbf{F}_q$ with the group ring $\mathbf{F}_q[S_n]$ and that S_n is the group of units in the $\mathbf{Z}_p$-algebra E_n congruent to 1 modulo the maximal ideal (S). Killing the first few t_is in $S(n)$ as we did in the proof above corresponds to replacing the group S_n by the subgroup of units congruent to 1 modulo a power of (S).

Let $L(n)$ be the Lie algebra without restriction with basis $x_{i,j}$ and bracket as above. We now recall the main results of May [**2**].

6.3.4. THEOREM. *There are spectral sequences*
(a) $E_2 = H^*(L(n)) \otimes P(b_{i,j}) \Rightarrow H^*(E_0 S(n)^*)$,
(b) $E_2 = H^*(E_0 S(n)^*) \Rightarrow H^*(S(n))$,
where $b_{i,j} \in H^{2pd_i}(E_0 S(n)^*)$ *with internal degree* $2p^{j+1}(p^i-1)$ *and* $P(\cdot)$ *is the polynomial algebra on the indicated generators.* □

Now let $L(n,k)$ be the quotient of $L(n)$ obtained by setting $x_{i,j} = 0$ for $i > k$. Then our first result is

6.3.5. THEOREM. *The* E_2*-term of the first May spectral sequence* [6.3.4(a)] *may be replaced by* $H^*(L(n,m)) \otimes P(b_{i,j} \colon i \leq m-n)$*, where* $m = [pn/(p-1)]$ *as before.*

PROOF. By 6.3.3 $L(n)$ is the product of $L(n,m)$ and an abelian Lie algebra, so

$$H^*(L(n)) \cong H^*(L(n,m)) \otimes E(h_{i,j} \colon i > m),$$

where $E(\cdot)$ denotes the exterior algebra on the indicated generators and $h_{i,j} \in H^1L(n)$ is the element corresponding to $x_{i,j}$. It also follows from 6.3.4 that the appropriate differential will send $h_{i,j}$ to $-b_{i-n,j-1}$ for $i > m$. It follows that the entire spectral sequence decomposes as a tensor product of two spectral sequences, one with the E_2-term indicated in the statement of the theorem, and the other having $E_2 = E(h_{i,j}) \otimes P(b_{i-n,j})$ with $i > m$ and $E_\infty = \mathbf{F}_p$. □

If $n < p-1$ then 6.3.5 gives a spectral sequence whose E_2-term is $H^*(L(n,n))$, showing that $H^*(S(n))$ has cohomological dimension n^2 as claimed in 6.2.10(b).

In Ravenel [**6**] we claimed erroneously that the spectral sequence of 6.3.4(b) collapses for $n < p-1$. The argument given there is incorrect. For example, we have reason to believe that for $p = 11$, $n = 9$ the element

$$(h_{1,0}h_{2,0}\cdots h_{7,0})(h_{2,8}h_{3,7}\cdots h_{7,3})$$

supports a differential that hits a nonzero multiple of

$$h_{1,0}h_{2,0}(h_{1,8}h_{2,7}\cdots h_{6,3})(h_{2,1}h_{3,1}\cdots h_{6,1}).$$

We know of no counterexample for smaller n or p.

Now we will prove 6.2.10(a), i.e., that $H^*(S(n))$ is finitely generated as an algebra. For motivation, the following is a special case of a result in Lazard [**4**].

6.3.6. Open Subgroup Theorem. *Every sufficiently small open subgroup of S_n is cohomologically abelian in the sense that it has the same cohomology as $\mathbf{Z}_p^{n^2}$, i.e., an exterior algebra on n^2 generators.* □

We will give a Hopf algebra theoretic proof of this for a cofinal set of open subgroups, namely the subgroups of elements in E_n congruent to 1 modulo (S^i) for various $i > 0$. The corresponding quotient group (which is finite) is dual the subalgebra of $S(n)$ generated by $\{t_k \colon k < i\}$. Hence the ith subgroup is dual to $S(n)/(t_k \colon k < i)$, which we denote by $S(n,i)$.

The filtration of 6.3.1 induces one on $S(n,i)$ and analogs of the succeeding four theorems hold for it.

6.3.7. Theorem. *If $i \geq n$ and $p > 2$, or $i > n$ and $p = 2$, then*

$$H^*(S(n,i)) = E(h_{k,j} \colon i \leq k < i+n,\ j \in \mathbf{Z}/(n)).$$

Proof. The condition on i is equivalent to $i > n-1$ and $i > m/2$, where as before $m = pn/(p-1)$. In the analog of 6.3.3 we have $i, k > m/2$ so $i + k > m$ so the Lie algebra is abelian. We also see that the restriction ξ is injective, so the spectral sequence of 6.3.5 has the E_2-term claimed to be $H^*(S(n,i))$. This spectral sequence collapses because $h_{k,j}$ corresponds to $t_k^{p^j} \in S(n,i)$, which is primitive for each k and j. □

Proof of 6.2.10(a). Let $A(i)$ be the Hopf algebra corresponding to the quotient of S_n by the ith congruence subgroup, so we have a Hopf algebra extension (A1.1.15)

$$A(i) \to S(n) \to S(n,i).$$

The corresponding Cartan–Eilenberg spectral sequence (A1.3.14) has

$$E_2 = \operatorname{Ext}_{A(i)}(\mathbf{F}_p, H^*(S(n,i)))$$

and converges to $H^*(S(n))$ with $d_r\colon E_r \to E_r^{s+r,t-r+1}$. Each E_r-term is finitely generated since $A(i)$ and $H^*(S(n,i))$ are finite-dimensional for $i > m/2$. Moreover, $E_{n^2} = E_\infty$, so E_∞ and $H^*(S(n))$ are finitely generated. □

Now we continue with the computation of $H^*(S(n))$. Theorem 6.3.5 indicates the necessity of computing $H^*(L(n,k))$ for $k \le m$, and this may be done with the Koszul complex, i.e.,

6.3.8. THEOREM. *$H^*(L(n,k))$ for $k \le m$ is the cohomology of the exterior complex $E(h_{i,j})$ on one-dimensional generators $h_{i,j}$ with $i \le k$ and $j \in \mathbf{Z}/(n)$, with coboundary*

$$d(h_{i,j}) = \sum_{0<s<i} h_{s,j}h_{i-s,s+j}.$$

The element $h_{i,j}$ corresponds to the element $x_{i,j}$ and therefore has filtration degree i and internal degree $2p^j(p^i-1)$.

PROOF. This follows from standard facts about the cohomology of Lie algebras (Cartan and Eilenberg [**1**, XII, Section 7]). □

Since $L(n,k)$ is nilpotent its cohomology can be computed with a sequence of change-of-rings spectral sequences analogous to A1.3.14.

6.3.9. THEOREM. *There are spectral sequences with*

$$E_2 = E(h_{k,j}) \otimes H^*(L(n,k-1)) \Rightarrow H^*(L(n,k))$$

and $E_3 = E_\infty$.

PROOF. The spectral sequence is that of Hochschild–Serre (see Cartan and Eilenberg [**1**, pp. 349–351] for the extension of Lie algebras

$$A(n,k) \to L(n,k) \to L(n,k-1)$$

where $A(n,k)$ is the abelian Lie algebra on $x_{k,j}$. Hence $H^*(A(n,k)) = E(h_{k,j})$. The E_2-term, $H^*(L,(n,k-1), H^*(A(n,k))$ is isomorphic to the indicated tensor product since the extension is central.

For the second statement, recall that the spectral sequence can be constructed by filtering the complex of 6.3.8 in the obvious way. Inspection of this filtered complex shows that $E_3 = E_\infty$. □

In addition to the spectral sequence of 6.3.4(a), there is an alternative method of computing $H^(*E_0S(n)^*)$. Define $\tilde{L}(n,k)$ for $k \le m$ to be the quotient of $PE_0S(n)^*$ by the restricted sub-Lie algebra generated by the elements $x_{i,j}$ for $k < i \le m$, and define $F(n,k)$ to be the kernel of the extension

$$0 \to F(n,k) \to \tilde{L}(n,k) \to \tilde{L}(n,k-1) \to 0.$$

Let $H^*(\tilde{L}(n,k))$ denote the cohomology of the restricted enveloping algebra of $\tilde{L}(n,k)$. Then we have

6.3.10. THEOREM. *There are change-of-rings spectral sequences converging to $H^*(\tilde{L}(n,k))$ with*

$$E_2 = H^*(F(n,k)) \otimes H^*(\tilde{L}(n,k-1))$$

where

$$H^*(F(n,k)) = \begin{cases} E(h_{k,j}) & \text{for } k > m-n \\ E(h_{k,j}) \otimes P(b_{k,j}) & \text{for } k \le m-n \end{cases}$$

and $H^*(\tilde{L}(n,m)) = H^*(E_0 S(n)^*)$.

PROOF. Again the spectral sequence is that given in Theorem XVI.6.1 of Cartan and Eilenberg [1]. As before, the extension is cocentral, so the E_2-term is the indicated tensor product. The structure of $H^*(F(n,k))$ follows from 6.3.3 and the last statement is a consequence of 6.3.5. $\square$

We begin the computation of $H^1(S(n))$ with:

6.3.11. LEMMA. $H^1(E_0 S(n)^*) = H^1(E^0 S(n))$ *is generated by*

$$\zeta_n = \sum_j h_{n,j} \quad \text{and} \quad \rho_n = \sum_j h_{2n,j} \quad \text{for } p = 2;$$

and for $n > 1$, $h_{1,j}$ *for each* $j \in \mathbf{Z}/(n)$.

PROOF. By 6.3.4(a) and 6.3.5 $H^1(E_0 S(n)) = H^1 L(n,m))$. The indicated elements are nontrivial cycles by 6.3.8. It follows from 6.3.3 that $L(n,m)$ can have no other generators since $[x_{1,j}, x_{i-1,j+1}] = x_{i,j} - \delta^j_{i+j} x_{i,j+1}$. $\square$

In order to pass to $H^1(S(n))$ we need to produce primitive elements in $S(n)_*$ corresponding to ζ_n and ρ_n (the primitive $t_1^{p^j}$ corresponds to $h_{1,j}$). We will do this with the help of the determinant of a certain matrix. Recall from (6.2.3) that $S(n) \otimes \mathbf{F}_{p^n}$ was isomorphic to the dual group ring of S_m which has a certain faithful representation over $W(\mathbf{F}_{p^n})$ (6.2.6). The determinant of this representation gave a homomorphism of $S(n)$ into $\mathbf{Z}_p^\times$, the multiplicative group of units in the p-adic integers. We will see that in H^1 this map gives us ζ_n and ρ_n.

More precisely, let $M = (m_{i,j})$ be the n by n matrix over $\mathbf{Z}_p[t_1, t_2, \dots]/(t_i - t_i^{p^n})$ given by

$$m_{i,j} = \begin{cases} \sum\limits_{k \geq 0} p^k t_{kn+j-i}^{p^i} & \text{for } i \leq j \\ \sum\limits_{k \geq 0} p^{k+1} t_{kn+j-i}^{p^i} & \text{for } i > j \end{cases}$$

where $t_0 = 1$.

Now define $T_n \in S(n)_*$ to be the mod (p) reduction $p^{-1}(\det M - 1)$ and for $p = 2$ define $U_n \in S(n)_*$ to be the mod (2) reduction of $\frac{1}{8}(\det M^2 - 1)$. Then we have

6.3.12. THEOREM. *The elements* $T_n \in S(n)$ *and, for* $p = 2$, $U_n \in S(n)$ *are primitive and represent the elements* ζ_n *and* $\rho_n + \zeta_n \in H^1(S(n))$, *respectively. Hence* $H^1(S(n))$ *is generated by these elements and for* $n > 1$ *by the* $h_{1,j}$ *for* $j \in \mathbf{Z}/(n)$.

PROOF. The statement that T_n and U_n are primitive follows from 6.2.6. That they represent ζ_n and $\rho_n + \zeta_n$ follows from the fact that

$$T_n \equiv \sum_j t_n^{p^j} \mod (t_1, t_2, \dots, t_{n-1})$$

and

$$U_n \equiv \sum_j t_{2n}^{2^j} + t_n^{2^j} \mod (t_1, t_2, \dots, t_{n-1}). \qquad \square$$

EXAMPLES.

$$T_1 = t_1, \quad U_1 = t_1 + t_2, \quad T_2 = t_2 + t_2^p - t_1^{1+p},$$
$$U_2 = t_4 + t_4^2 + t_1 t_3^2 + t_1^2 t_3 + t_1^3 t_2 + t_1^3 t_2^2,$$

and

$$T_3 = t_3 + t_3^p + t_3^{p^2} + t_1^{1+p+p^2} - t_1 t_2^p - t_1^p t_2^{p^2} - t_1^{p^2} t_2.$$

Moreira [**1, 3**] has found primitive elements in $BP_*(BP)/I_n$ which reduce to our T_n. The following result is a corollary of 6.2.7.

6.3.13. PROPOSITION. *If $p \nmid n$, then $H^*(S(n))$ decomposes as a tensor product of an appropriate subalgebra with $E(\zeta_n)$ for $p > 2$ and $P(\zeta_n) \otimes E(\rho_n)$ for $p = 2$.* □

We now turn to the computation of $H^2(S(n))$ for $n > 2$. We will compute all of $H^*S((n))$ for $n = 2$ below.

6.3.14. THEOREM. *Let $n > 2$*

(a) *For $p = 2$, $H^2(S(n))$ is generated as a vector space by the elements ζ_n^2, $\rho_n\zeta_n$, ζ_n, $\zeta_n h_{1,j}$, $\rho_n h_{1,j}$, and $h_{1,i}h_{1,j}$ for $i \neq j \pm 1$, where $h_{1,i}h_{1,j} = h_{1,j}h_{1,i}$ and $h_{1,i}^2 \neq 0$.*

(b) *For $p > 2$, $H^2(S(n))$ is generated by the elements*

$$\zeta_n h_{1,i},\ b_{1,i},\ g_i = \langle h_{1,i}, h_{1,i+1}, h_{1,i}\rangle, \quad k_i = \langle h_{1,i+1}, h_{1,i+1}, h_{1,i}\rangle$$

and $h_{1,i}h_{1,j}$ for $i \neq j \pm 1$, where $h_{1,i}h_{1,j} + h_{1,j}h_{1,i} = 0$. □

Both statements require a sequence of lemmas. We treat the case $p = 2$ first.

6.3.15. LEMMA. *Let $p = 2$ and $n > 2$.*

(a) *$H^1(L(n, 2))$ is generated by $h_{1,i}$ for $i \in \mathbf{Z}/(n)$.*

(b) *$H^2(L(n, 2))$ is generated by the elements $h_{1,i}h_{1,j}$ for $i \neq j \pm 1$, g_i, k_i, and $e_{3,i} = \langle h_{1,i}, h_{1,i+1}, h_{1,i+2}\rangle$. The latter elements are represented by $h_{1,i}h_{2,i}$, $h_{1,i+1}h_{2,i}$, and $h_{1,i}h_{2,i+1} + h_{2,i}h_{1,i+2}$, respectively.*

(c) *$e_{3,i}h_{1,i+1} = h_{1,i}e_{3,i+1} + e_{3,i}h_{1,i}h_{1,i+3} = 0$, and these are the only relations among the elements $h_{1,i}e_{3,j}$.*

PROOF. We use the spectral sequence of 6.3.9 with $E_2 = E(h_{1,i}, h_{2,i})$ and $d_2(h_{2,i}) = h_{1,i}h_{1,i+1}$. All three statements can be verified by inspection. □

6.3.16. LEMMA. *Let $p = 2$, $n > 2$, and $2 < k \leq 2n$.*

(a) *$H^1(L(n, k))$ is generated by the elements $h_{1,i}$ along with ζ_n for $k \geq n$ and ρ_n for $k = 2n$.*

(b) *$H^2(L(n, k))$ is generated by products of elements in $H^1(L(n, k))$ subject to $h_{1,i}h_{1,i+1} = 0$, along with*

$$g_i = \langle h_{1,i}, h_{1,i}, h_{1,i+1}\rangle, \quad k_i = \langle h_{1,i}, h_{1,i+1}, h_{1,i+1}\rangle,$$
$$\alpha_i = \langle h_{1,i}, h_{1,i+1}, h_{1,i+2}, h_{1,i+1}\rangle, \quad \text{and}$$
$$e_{k+1,i} = \langle h_{1,i}, h_{1,i+1}, \ldots, h_{1,i+k}\rangle.$$

The last two families of elements can be represented by $h_{3,i}h_{1,i+1} + h_{2,i}h_{2,i+1}$ and $\Sigma_s h_{s,i}h_{k+1-s,i+1}$ respectively.

(c) *$h_{1,i}e_{k+1,i+1} + e_{k+1,i}h_{1,i+1+k} = 0$ and no other relations hold among products of the $e_{k+1,i}$ with elements of H^1.*

PROOF. Again we use 6.3.9 and argue by induction on k, using 6.3.15 to start the induction. We have $E_2 = E(h_{k,i}) \otimes H^*(L(n, k-1))$ with $d_2(h_{k,i}) = e_{k,i}$. The existence of the α_i follows from the relation $e_{3,i}h_{1,i+1} = 0$ in $H^3(L(n, 2))$ and that of $e_{k+1,i}$ from $h_{1,i}e_{k,i+1}h_{1,i+k} = 0$ in $H^3(L(n, k-1))$. The relation (c) for $k < 2$ is formal; it follows from a Massey product identity A1.4.6 or can be verified by direct calculation in the complex of 6.3.8. No combination of these products can be in the image of d_2 for degree reasons. □

6.3.17. *Let $p = 2$ and $n > 2$. Then $H^2(E_0S(n)^*)$ is generated by the elements $\rho_n\zeta_n$, $\rho_n h_{1,i}$, $\zeta_n h_{1,i}$, $h_{1,i}h_{1,j}$ for $i \neq j \pm 1$, α_i, and $h_{i,j}^2 = b_{i,j}$ for $1 \leq i \leq n, j \in$* $\mathbf{Z}/(n)$.

PROOF. We use the modified first May spectral sequence of 6.3.5. We have $m = 2n$ and $H^2(L(n, m))$ is given by 6.3.16. By easy direct computation one sees that $d_2(g_i) = b_{1,i}h_{1,i+1}$ and $d_2(k_i) = h_{1,i}b_{1,i+1}$. We will show that $d_2(e_{2n+1,i}) = h_{1,i}b_{n,i} + h_{1,i+n}b_{n,i-1}$.

$$\Delta(t_{2n+1}) = \sum t_j \otimes t_{2n+1-j}^{p^j} + b_{n+1,n-1}$$

modulo terms of lower filtration by 4.3.15. Then by 4.3.22

$$d(b_{n+1,n-1}) = t_1 \otimes b_{n,n} + b_{n,n-1} \otimes t_1$$

modulo terms of lower filtration and the nontriviality of $d_2(e_{2n+1,i})$ follows. □

PROOF OF 6.3.14(a). We now consider the second May spectral sequence (6.3.4(b)). By 4.3.22 we have $d_2(b_{i,j}) = h_{1,j+1}b_{i-1,j+1} + h_{1,i+j}b_{i-1,j} \neq 0$ for $i > 1$. The remaining elements of $H^2E_0S(n)$ survive either for degree reasons or by 6.3.12. □

For $p > 2$ we need an analogous sequence of lemmas. We leave the proofs to the reader.

6.3.18. LEMMA. *Let $n > 2$ and $p > 2$.*

(a) *$H^1(L(n, 2))$ is generated by $h_{1,i}$.*

(b) *$H^2(L(n, 2))$ is generated by the elements $h_{1,i}h_{1,j}$ (with $h_{1,i}h_{1,i+1} = 0$). $g_i = h_{1,i}h_{2,i}$, $k_i = h_{1,i+1}h_{2,i}$ and $e_{3,i} = h_{1,i}h_{2,i+1}h_{2,i}h_{1,i+2}$.*

(c) *The only relations among the elements $h_{1,i}e_{3,j}$ are $h_{1,i}e_{3,i+1} - e_{3,i}h_{1,i+3} = 0$.* □

6.3.19. LEMMA. *Let $n > 2$, $p > 2$, and $2 < k \leq m$. Then*

(a) *$H^1(L(n, k))$ is generated by $h_{1,i}$ and, for $k \geq n$, ζ_n.*

(b) *$H^2(L(n, k))$ is generated by $h_{1,i}h_{1,j}$ (with $h_{1,i}h_{1,i+1} = 0$), g_i, h_i,*

$$e_{k+1,i} = \sum_{0<j<k+1} h_{j,i}h_{k+1-j,i+j},$$

and, for $k \geq n$, $\zeta_n h_{1,i}$.

(c) *The only relations among products of elements in H^1 with the $e_{k+1,i}$ are $h_{1,i}e_{k+1,i+1} - e_{k+1,i}h_{1,k+1} = 0$.* □

6.3.20. LEMMA. *Let $n > 2$ and $p > 2$. Then $H^2(E_0S(n)^*)$ is generated by the elements $b_{i,j}$ for $i \leq m - n$ and by the elements of $H^2(L(n, m))$.*

PROOF OF 6.3.14(b). Again we look at the spectral sequence of 6.3.4(b). By arguments similar to those for $p = 2$ one can show that

$$d_p(b_{i,j}) = h_{1,i+j}b_{i-1,j} - h_{1,j+1}b_{i-1,j+1} \quad \text{for } i > 1$$

and

$$d_s(e_{m+1,i}) = h_{1,m+1+i-n}b_{m-n,i-1} - h_{1,i}b_{m-n,j} \quad \text{where } s = 1 + pn - (p-1)m,$$

and the remaining elements of $H^2(E_0S(n)^*)$ survive as before. □

Now we will compute $H^*(S(n))$ at all primes for $n \le 2$ and at $p > 3$ for $n = 3$.

6.3.21. THEOREM.
(a) $H^*(S(1)) = P(h_{1,0}) \otimes E(\rho_1)$ *for* $p = 2$;
(b) $H^*(S(1)) = E(h_{1,0})$ *for* $p > 2$
[*note that* $S(1)$ *is commutative and that* $\zeta_1 = h_{1,0}$].

PROOF. This follows immediately from 6.3.3, 6.3.5, and routine calculation. □

6.3.22. THEOREM. *For* $p > 3$, $H^*(S(2))$ *is the tensor product of* $E(\zeta_2)$ *with the subalgebra with basis* $\{1, h_{1,0}, h_{1,1}, g_0, g_1, g_0h_{1,1}\}$ *where*

$$g_i = \langle h_{1,i}, h_{1,i+1}, h_{1,i}\rangle,$$
$$h_{1,0}g_1 = g_0h_{1,1}, \quad h_{1,0}g_0 = h_{1,1}g_1 = 0,$$

and

$$h_{1,0}h_{1,1} = h_{1,0}^2 = h_{1,1}^2 = 0.$$

In particular, the Poincaré series is $(1+t)^2(1+t+t^2)$.

PROOF. The computation of $H^*(L(2,2))$ by 6.3.8 or 6.3.9 is elementary, and there are no algebra extension problems for the spectral sequences of 6.3.9 or 6.3.4(b). □

We will now compute $H^*(S(2))$ for $p = 3$. Our description of it in the first edition was incorrect, as was pointed out by Henn [**1**]. The computation given here is influenced by Henn but self-contained. Henn showed that there are two conjugacy classes of subgroups of order 3 in the group S_2. In each case the centralizer is the group of units congruent to one modulo the maximal ideal in the ring of integers of an embedded copy of the field $K = \mathbf{Q}_3[\zeta]$, where ζ is a primitive cube root of unity. Let C_1 and C_2 denote these two centralizers. Henn showed that the resulting map

$$H^*(S_2) \to H^*(C_1) \oplus H^*(C_2)$$

is a monomorphism.

We will describe this map in Hopf algebraic terms. Choose a fourth root of unity $i \in \mathbf{F}_9$, let $a = \pm i$, and consider the two quotients

$$\overline{S(2)}_+ = S(1,1)_i \qquad \text{and} \qquad \overline{S(2)}_- = S(1,1)_{-i},$$

where $S(1,1)_a$ is the quotient of $S(2) \otimes \mathbf{F}_9$ described in 6.2.13. Henn's map is presumably equivalent to

$$H^*(S(2)) \otimes \mathbf{F}_9 \to H^*(\overline{S(2)}_+) \oplus H^*(\overline{S(2)}_-). \tag{6.3.23}$$

In any case we will show that this map is a monomorphism.

We have the following reduced coproducts in $\overline{S(2)}_{\pm}$.

$$\begin{array}{rcl} \bar{t}_1 & \mapsto & 0 \\ \bar{t}_2 & \mapsto & a\bar{t}_1 \otimes \bar{t}_1 \\ \bar{t}_3 & \mapsto & \bar{t}_1 \otimes \bar{t}_2 + \bar{t}_2 \otimes \bar{t}_1 - a^3(\bar{t}_1^2 \otimes \bar{t}_1 + \bar{t}_1 \otimes \bar{t}_1^2) \end{array}$$

It follows that $\bar{t}_2 + a\bar{t}_1^2$ and $\bar{t}_3 - \bar{t}_1\bar{t}_2$ are primitive. The filtration of 6.3.1 induces one on $\overline{S(2)}_{\pm}$, and the methods of this section lead to

$$H^*(\overline{S(2)}_{\pm}) = E(\bar{h}_{1,0},\, \bar{h}_{2,0},\, \bar{h}_{3,0}) \otimes P(\bar{b}_{1,0})$$

with the evident notation.

6.3.24. THEOREM. *For $p = 3$, $H^*(S(2))$ is a free module over*

$$E(\zeta_2) \otimes P(b_{1,0})$$

on the generators

$$\{1,\, h_{1,0},\, h_{1,1},\, b_{1,1},\, \xi,\, a_0,\, a_1,\, b_{1,1}\xi\}\,,$$

where the elements $\xi \in H^2$ and $a_0, a_1 \in H^3$ will be defined below. The algebra structure is indicated in the following multiplication table.

1	$h_{1,0}$	$h_{1,1}$	$b_{1,1}$	ξ	a_0	a_1
$h_{1,0}$	0	0	$-b_{1,0}h_{1,1}$	0	$-b_{1,1}\xi$	$-b_{1,0}\xi$
$h_{1,1}$		0	$b_{1,0}h_{1,0}$	0	$-b_{1,0}\xi$	$b_{1,1}\xi$
$b_{1,1}$			$-b_{1,0}^2$	$b_{1,1}\xi$	$-b_{1,0}a_1$	$b_{1,0}a_0$
ξ				0	0	0
a_0					0	0
a_1						0

In particular, the Poincaré series is

$$(1+t)^2(1+t^2)/(1-t).$$

Moreover the map of (6.3.23) is a monomorphism.

PROOF. Our basic tools are the spectral sequences of 6.3.10 and some Massey product identities from A1.4. We have $H^*(\tilde{L}(2,1)) \cong E(h_{1,0}, h_{1,1}) \otimes P(b_{1,0}, b_{1,1})$, and a spectral sequence converging to $H^*(\tilde{L}(2,2))$ with $E_2 = E(\zeta_2, \eta) \otimes H^*(\tilde{L}(2,1))$, where

$$\begin{array}{rclcrcl} \zeta_2 & = & h_{2,0} + h_{2,1}, & \qquad & \eta & = & h_{2,1} - h_{2,0}, \\ d_2(\zeta_2) & = & 0, & & d_2(\eta) & = & h_{1,0}h_{1,1}, \end{array}$$

and $E_3 = E_\infty$. Hence E_∞ is a free module over $E(\zeta_2) \otimes P(b_{1,0}, b_{1,1})$ on generators

$$\{1,\, h_{1,0},\, h_{1,1},\, g_0,\, g_1,\, h_{1,0}g_1 = h_{1,1}g_0,\}\,,$$

where $g_i = \langle h_{1,i}, h_{1,i+1}, h_{1,i}\rangle$. This determines the additive structure of $H^*(\tilde{L}(2,2))$, but there are some nontrivial extensions in the multiplicative structure. We know by 6.3.13 that we can factor out $E(\zeta_2)$, and we can write $b_{1,i}$ as the Massey product $-\langle h_{1,i}, h_{1,i}, h_{1,i}\rangle$. Then by A1.4.6 we have $h_{1,i}g_i = -b_{1,i}h_{1,i+1}$, $g_i^2 = -b_{1,i}g_{i+1}$, $g_ig_{i+1} = b_{1,i}b_{1,i+1}$. These facts along with the usual $h_{1,i}^2 = h_{1,0}h_{1,1} = 0$ determine $H^*(\tilde{L}(2,2))$ as an algebra.

This algebra structure allows us to embed $H^*(\tilde{L}(2,2))$ in the ring

$$R = E(\zeta_2,\, h_{1,0},\, h_{1,1}) \otimes P(s_0,\, s_1)/(h_{1,0}h_{1,1},\, h_{1,0}s_1 - h_{1,1}s_0)$$

by sending ζ_2 and $h_{1,i}$ to themselves and

$$\begin{aligned} b_{1,i} &\mapsto -s_i^3 \\ g_0 &\mapsto s_0^2 s_1 \\ g_1 &\mapsto s_0 s_1^2. \end{aligned}$$

Here the cohomological degree of s_i is $2/3$, and $H^*(\tilde{L}(2,2))$ maps isomorphically to the subring of R consisting of elements of integral cohomological degree.

Next we have the spectral sequence of 6.3.10 converging to

$$H^*(\tilde{L}(2,3)) \cong H^*(E^0 S(2))$$

with $E_2 = E(h_{3,0}, h_{3,1}) \otimes H^*(\tilde{L}(2,2))$, and $d_2(h_{3,i}) = g_i - b_{1,i+1}$. We will see shortly that $E_3 = E_\infty$ for formal reasons. Tensoring this over $H^*(\tilde{L}(2,2))$ with R gives a spectral sequence with

$$\begin{aligned} E_2 &= E(h_{3,0}, h_{3,1}) \otimes R \\ \text{and} \qquad d_2(h_{3,0}) &= s_1(s_0^2 + s_1^2) \\ d_2(h_{3,1}) &= s_0(s_0^2 + s_1^2). \end{aligned}$$

This can be simplified by tensoring with $\mathbf{F}_9$ (which contains $i = \sqrt{-1}$) and defining

$$\begin{aligned} x_0 &= h_{1,0} + ih_{1,1} & x_1 &= h_{1,0} - ih_{1,1} \\ y_0 &= s_0 + is_1 & y_1 &= s_0 - is_1 \\ z_0 &= ih_{3,0} + h_{3,1} & z_1 &= -ih_{3,0} + h_{3,1} \end{aligned}$$

The Galois group of $\mathbf{F}_9$ over $\mathbf{F}_3$ acts here by conjugating scalars and permuting the two subscripts. Then we have

$$R \otimes \mathbf{F}_9 = E(\zeta_2,\, x_0,\, x_1) \otimes P(y_0,\, y_1)/(x_0x_1,\, x_0y_1 - x_1y_0),$$

where the cohomological degrees of x_i and y_i are 1 and $2/3$ respectively. In the spectral sequence we have

$$d_2(z_0) = y_0^2 y_1 \qquad \text{and} \qquad d_2(z_1) = y_0 y_1^2. \tag{6.3.25}$$

The image of $H^*(\tilde{L}(2,2)) \otimes \mathbf{F}_9$ in $R \otimes \mathbf{F}_9$ is a free module over the ring

$$B = E(\zeta_2) \otimes P(y_0^3, y_1^3)$$

on the following set of six generators.

$$C = \left\{1,\, x_0,\, x_1,\, y_0^2 y_1,\, y_0 y_1^2,\, x_0 y_0 y_1^2 = x_1 y_0^2 y_1\right\}$$

Hence the image of $E(h_{3,0},\, h_{3,1}) \otimes H^*(\tilde{L}(2,2)) \otimes \mathbf{F}_9$ is a free B-module on the set

$$\{1,\, z_0,\, z_1,\, z_0 z_1\} \otimes C,$$

but it is convenient to replace this basis by the set of elements listed in the following table.

1	z_0	z_1	$z_0 z_1$
x_0	$x_0 z_0$	$\beta = x_0 z_1 - x_1 z_0$	$-x_0 z_0 z_1$
x_1	$\delta = -x_1 z_0 - x_0 z_1$	$x_1 z_1$	$x_1 z_0 z_1$
$y_0^2 y_1$	$\alpha_1 = y_0^2 y_1 z_0 - y_0^3 z_1$	$\varepsilon = y_0^2 y_1 z_1 - y_0 y_1^2 z_0$	$y_0^2 y_1 z_0 z_1$
$y_0 y_1^2$	$\gamma = -y_0 y_1^2 z_0 - y_0^2 y_1 z_1$	$\alpha_0 = y_0 y_1^2 z_1 - y_1^3 z_0$	$-y_0 y_1^2 z_0 z_1$
$x_0 y_0 y_1^2$	$-x_0 \varepsilon$	$x_1 \varepsilon$	$x_0 y_0 y_1^2 z_0 z_1$

This basis is Galois invariant up to sign, i.e., the Galois image of each basis element is another basis element. The elements 1, $x_0y_0y_1^2$, δ, and γ are self-conjugate, while β, ε, z_0z_1 and $x_0y_0y_1^2z_0z_1$ are antiself-conjugate. The remaining elements form eight conjugate pairs.

In the spectral sequence the following twelve differentials (listed as six Poincaré dual pairs) are easily derived from (6.3.25) and account for each of these 24 basis elements.

$$\begin{array}{rclrcl} d_2(z_0) & = & y_0^2y_1 & d_2(x_1z_0z_1) & = & x_1\varepsilon \\ d_2(z_1) & = & y_0y_1^2 & d_2(-x_0z_0z_1) & = & -x_0\varepsilon \\ d_2(z_0z_1) & = & \varepsilon & d_2(\delta) & = & x_0y_0y_1^2 \\ d_2(x_0z_0) & = & y_0^3(x_1) & d_2(y_0^2y_1z_0z_1) & = & y_0^3(\alpha_0) \\ d_2(x_1z_1) & = & y_1^3(x_0) & d_2(-y_0y_1^2z_0z_1) & = & y_1^3(\alpha_1) \\ d_2(\gamma) & = & y_0^3y_1^3(1) & d_2(x_0y_0y_1^2z_0z_1) & = & y_0^3y_1^3(\beta) \end{array}$$

The spectral sequence collapses from E_3 since there are no elements in $E_3^{*,t}$ for $t > 1$. The image of $H^*(\tilde{L}(2,3)) \otimes \mathbf{F}_9$ in the E_∞-term is the B-module generated by

$$\{1,\, x_0,\, x_1,\, \alpha_0,\, \alpha_1,\, \beta\}$$

subject to the module relations

$$\begin{array}{rclcrcl} y_0^3y_1^3(1) & = & 0, & & y_0^3y_1^3(\beta) & = & 0, \\ y_0^3(x_1) & = & 0, & & y_0^3(\alpha_0) & = & 0, \\ y_1^3(x_0) & = & 0, & \text{and} & y_1^3(\alpha_1) & = & 0. \end{array}$$

The only nontrivial products among these six elements are

$$x_0\alpha_1 = -y_0^3\beta \qquad \text{and} \qquad x_1\alpha_0 = y_1^3\beta.$$

Equivalently the image is the free module over $E(\zeta_2) \otimes P(y_0^3 + y_1^3)$ on the eight generators

$$\left\{1,\, x_0,\, x_1,\, y_1^3,\, \beta,\, \alpha_0,\, \alpha_1,\, y_1^3\beta\right\} \tag{6.3.26}$$

with suitable algebra relations.

It follows that $H^*(E^0S(2))$ itself is a free module over $E(\zeta_2) \otimes P(b_{1,0})$ on the eight generators

$$\{1,\, h_{1,0},\, h_{1,1},\, b_{1,1},\, \xi,\, a_0,\, a_1,\, b_{1,1}\xi\}\,.$$

where

$$\xi = i\beta, \qquad a_0 = \alpha_0 + \alpha_1, \qquad \text{and} \qquad a_1 = i(\alpha_0 - \alpha_1).$$

It also follows that $E^0H^*(S(2))$ has the relations stated in the theorem. The absence of nontrivial multiplicative extensions in $H^*(S(2))$ will follow from the the fact that the map of (6.3.23) is monomorphic and there are no extensions in its target.

Now we will determine the images of the elements of (6.3.26) under the map of (6.3.23). Recall that

$$H^*(\overline{S(2)}_\pm) = E(\overline{h}_{1,0},\, \overline{h}_{2,0},\, \overline{h}_{3,0}) \otimes P(\overline{b}_{1,0})$$

As before it is convenient to adjoin a cube root $\overline{s}_0$ of $-\overline{b}_{1,0}$ and let

$$\overline{R}_\pm = E(\overline{h}_{1,0},\, \overline{h}_{2,0}) \otimes P(\overline{s}_0).$$

The map

$$H^*(S(2)) \otimes \mathbf{F}_9 \to E(\overline{h}_{3,0}) \otimes \overline{R}_+ \oplus E(\overline{h}_{3,0}) \otimes \overline{R}_-$$

behaves as follows.

$$\begin{array}{rclrcl}
x_0 & \mapsto & (0,\ -\overline{h}_{1,0}) & x_1 & \mapsto & (-\overline{h}_{1,0},\ 0) \\
y_0 & \mapsto & (0,\ -\overline{s}_0) & y_1 & \mapsto & (-\overline{s}_0,\ 0) \\
z_0 & \mapsto & (-i\overline{h}_{3,0},\ 0) & z_1 & \mapsto & (0,\ i\overline{h}_{3,0}) \\
\beta & \mapsto & (-i\overline{h}_{1,0}\overline{h}_{3,0},\ -i\overline{h}_{1,0}\overline{h}_{3,0}) & & & \\
\alpha_0 & \mapsto & (i\overline{s}_0^3\overline{h}_{3,0},\ 0) & \alpha_1 & \mapsto & (0,\ -i\overline{s}_0^3\overline{h}_{3,0})
\end{array}$$

It follows that Henn's map is a monomorphism. □

We now turn to the case $n = p = 2$. We will only compute $E^0H^*(S(2))$, so there will be some ambiguity in the multiplicative structure of $H^*(S(2))$. In order to state our result we need to define some classes. Recall (6.3.12) that $H^1(S(2))$ is the $\mathbf{F}_2$-vector space generated by $h_{1,0}$, $h_{1,1}$, ζ_2 and ρ_2. Let

$$\alpha_0 \in \langle \zeta_2, h_{1,0}, h_{1,1} \rangle, \quad \beta \in \langle h_{1,0}, \zeta_2, \zeta_2^2, h_{1,1} \rangle, \quad g = \langle h, h^2, h, h^2 \rangle,$$

where $h = h_{1,0} + h_{1,1}$, $\tilde{x} = \langle x, h, h^2 \rangle$ for $x = \zeta_2$, α_0, ζ_2^2, and $\alpha_0\zeta_2$ (more precise definitions of α_0 and β will be given in the proof).

6.3.27. THEOREM. *$E^0H^*(S(2))$ for $p = 2$ is a free module over $P(g) \otimes E(\rho_2)$ on 20 generators*: 1, $h_{1,0}$, $h_{1,1}$, $h_{1,0}^2$, $h_{1,1}^2$, $h_{1,0}^3$, β, $\beta h_{1,0}$, $\beta h_{1,1}$, $\beta h_{1,0}^2$, $\beta h_{1,1}^2$, $\beta h_{1,0}^3$, ζ_2, α_0, ζ_2^2, $\alpha_0\zeta_2$, $\tilde{\zeta}_2$, $\tilde{\alpha}_0$, $\tilde{\zeta}_2^2$, $\alpha_0\tilde{\zeta}_2$, *where $\alpha_0 \in H^2(S(2))$ and has filtration degree 4, $\beta \in H^3(S(2))$ and has filtration degree 8, $g \in H^4(S(2))$ and has filtration degree 8, and the cohomological and filtration degrees of $\tilde{x}$ exceed those of x by 2 and 4, respectively. Moreover $h_{1,0}^3 = h_{1,1}^3$, $\alpha_0^2 = \tilde{\zeta}_2^2$, and all other products are zero. The Poincaré series is $(1+t)^2(1-t^5)/(1-t)^2(1+t^2)$.*

PROOF. We will use the same notation for corresponding classes in the various cohomology groups we will be considering along the way.

Again our basic tool is 6.3.10. It follows from 6.3.5 that $H^*(E_0S(2)^*)$ is the cohomology of the complex

$$P(h_{1,0}, h_{1,1}, \zeta_2, h_{2,0}) \otimes E(h_{3,0}, h_{3,1}, \rho_2, h_{4,0})$$

with

$$\begin{aligned}
&d(h_{1,i}) = d(\zeta_2) = d(\rho_2) = 0, \\
&d(h_{3,i}) = h_{1,i}\zeta_2, \quad d(h_{2,0}) = h_{1,0}h_{1,1},
\end{aligned}$$

and

$$d(h_{4,0}) = h_{1,0}h_{3,1} + h_{1,1}h_{3,0} + \zeta_2^2.$$

This fact will enable us to solve the algebra extension problems in the spectral sequences of 6.3.10.

For $H^*(\tilde{L}(2,2))$ we have a spectral sequence with $E_2 = P(h_{1,0}, h_{1,1}, \zeta_2, h_{2,0})$ with $d_2(\zeta_2) = 0$ and $d_2(h_{2,0}) = h_{1,0}h_{1,1}$. It follows easily that

$$H^*(\tilde{L}(2,2)) = P(h_{1,0}, h_{1,1}, \zeta_2, b_{2,0})/(h_{1,0}h_{1,1})$$

where $b_{2,0} = h_{2,0}^2 = \langle h_{1,0}, h_{1,1}, h_{1,0}, h_{1,1} \rangle$.

For $H^*(\tilde{L}(2,3))$ we have a spectral sequence with

$$E_2 = E(h_{3,0}, h_{3,1}) \otimes H^*(\tilde{L}(2,2))$$

and $d_2(h_{3,i}) = h_{1,i}\zeta_2$. Let

$$\alpha_i = h_{1,i+1}h_{3,i} + \zeta_2 h_{2,i} \in \langle \zeta_2, h_{1,i}, h_{1,i+1}\rangle.$$

Then $H^*(\tilde{L}(2,3))$ as a module over $H^*(\tilde{L}(2,2))$ is generated by 1, α_0, and α_1 with

$$\zeta_2 h_{1,i} = \zeta_2(\alpha_0 + \alpha_1 + \zeta_2^2) = h_{1,i}\alpha_i = \zeta_2 h_{1,i+1}\alpha_i = 0$$

and

$$\alpha_0^2 = \zeta_2^2 b_{2,0}, \quad \alpha_1^2 = \zeta_2^2(\zeta_2^2 + b_{2,0}), \quad \alpha_0\alpha_1 = \zeta_2^2(\alpha_0 + b_{2,0}).$$

The Poincaré series for $H^*(\tilde{L}(2,3))$ is $(1+t+t^2)/(1-t^2)$.

For $H^*(\tilde{L}(2,4))$ we have a spectral sequence with

$$E_2 = E(h_{4,0}, \rho_2) \otimes H^*(\tilde{L}(2,3)),$$

$d_2(\rho_2) = 0$, and $d_2(h_{4,0}) = \alpha_0 + \alpha_1$. Define $\beta \in H^3(\tilde{L}(2,4))$ by

$$\beta = h_{4,0}(\alpha_0 + \alpha_1 + \zeta_2^2) + \zeta_2 h_{3,0}h_{3,1} \in \langle h_{1,0}, \zeta_2, \zeta_2^2, h_{1,1}\rangle.$$

Then $H^*(\tilde{L}(2,4))$ is a free module over $E(\rho_2) \otimes P(b_{2,0})$ on generators 1, $h_{1,i}^t$, ζ_2, ζ_2^2, α_0, $\alpha_0\zeta_2$, β, and $\beta h_{1,i}^t$, where $t > 0$. As a module over $H^*(\tilde{L}(2,3)) \otimes E(p_2)$ it is generated by 1 and β, with $(\alpha_0 + \alpha_1)1 = \zeta_2^3(1) = \alpha_0\zeta_2^2(1) = 0$. To solve the algebra extension problem we observe that $\beta\zeta_2 = 0$ for degree reasons; $\beta\alpha_i = \beta\langle \zeta_2, h_{1,i}, h_{1,i+1}\rangle = \langle \beta, \zeta_2, h_{1,i}\rangle h_{1,i+1} = 0$ since $\langle \beta, \zeta_2, h_{1,i}\rangle = 0$ for degree reasons; and $E(\rho_2)$ splits off multiplicatively by the remarks at the beginning of the proof.

This completes the computation of $H^*(E_0S(2)^*)$. Its Poincaré series is $(1+t)^2/(1-t)^2$. We now use the second May spectral sequence [6.3.4(b)] to pass to $E^0H^*(S(2))$. $H^*(E_0S(2)^*)$ is generated as an algebra by the elements $h_{1,0}$, $h_{1,1}$, ζ_2, ρ_2, α_0, $b_{2,0}$, and β. The first four of these are permanent cycles by 6.3.12.

By direct computation in the cobar resolution we have

$$d(t_3 + t_1 t_2^2) = \zeta_2 \otimes t_1, \tag{6.3.28}$$

so the Massey product for α_0 is defined in $H^*(S(2))$ and the α_0 is a permanent cycle. We also have

$$d(t_2 \otimes t_2 + t_1 \otimes t_1^2 t_2 + t_1 t_2 \otimes t_1^2) = t_1 \otimes t_1 \otimes t_1 + t_1^2 \otimes t_1^2 \otimes t_1^2,$$

so $d_2(b_{2,0}) = h_{1,0}^2 + h_{1,1}^3$. Inspection of the E_3 term shows that $b_{2,0}^2 = \langle h, h^2, h, h^2\rangle$, (where $h = h_{1,0} + h_{1,1}$) is a permanent cycle for degree reasons.

We now show that $\beta = \langle h_{1,0}, \zeta_2, \zeta_2^2, h_{1,1}\rangle$ is a permanent cycle by showing that its Massey product expression is defined in $E^0H^*(S(2))$. The products $h_{1,0}\zeta_2$ and $\zeta_2^2 h_{1,1}$ are zero by 6.3.28 and we have

$$\begin{aligned} d(\tilde{t}_3 \otimes \tilde{t}_3^2 + T_2\tilde{t}_3 \otimes t_1^2 + T_2 \otimes t_4 + T_2 \otimes t_2^3 + T_2 \otimes t_1^3(1 + t_2 + t_2^2)) \\ = T_2 \otimes T_2 \otimes T_2, \end{aligned} \tag{6.3.29}$$

where $\tilde{t}_3 = t_3 + t_1 t_2^2$ and $T_2 = t_2 + t_2^2 + t_1^3$, so $\zeta_2^3 = 0$ in $H^*(S(2))$. Inspection of $H^3(E_0S(2)^*)$ shows there are no elements of internal degree 2 or 4 and filtration degree > 7, so the triple products $\langle h_{1,0}, \zeta_2, \zeta_2^2\rangle$ and $\langle \zeta_2, \zeta_2^2, h_{1,1}\rangle$ must vanish and β is a permanent cycle.

Now the E_3 term is a free module over $E(\rho_2) \otimes P(b_{2,0}^2)$ on 20 generators: 1, $h_{1,0}$, $h_{1,1}$, $h_{1,0}^2$, $h_{1,1}^2$, $h_{1,0}^3 = h_{1,1}^3$, β, $\beta h_{1,0}$, $\beta h_{1,1}$, $\beta h_{1,1}^2$, βj_{10}^2, $\beta h_{1,0}^3$, ζ_2, α_0, ζ_2^2, $\alpha_0\zeta_2$, $\zeta_2 b_{1,0}$, $\alpha_0 b_{2,0}$, $\zeta_2^2 b_{2,0}$, $\zeta_2\alpha_0 b_{2,0}$. The last four in the list now have Massey product expressions $\langle \zeta_2, h, h^2\rangle$, $\langle \alpha_0, h, h^2\rangle$, $\langle \zeta_2^2, h, h^2\rangle$, and $\langle \alpha_0, \zeta_2, h, h^2\rangle$, respectively. These elements have to be permanent cycles for degree reasons, so $E_3 = E_\infty$, and we have determined $E^0H^*(S(2))$. □

We now describe an alternative method of computing $H^*(S(2) \otimes \mathbf{F}_4)$, which is quicker than the previous one, but yields less information about the multiplicative structure. By 6.3.4, this group is isomorphic to $H^*(S_2; \mathbf{F}_4)$, the continuous cohomology of certain 2-adic Lie group with trivial coefficients in $\mathbf{F}_4$, S_2 is the group of units in the degree 4 extension E_2 of $\mathbf{Z}_2$ obtained by adjoining ω and S with $\omega^2 + \omega + 1 = 0$, $S^2 = 2$ and $S\omega = \omega^2 S$.

Let Q denote the quaternion group, i.e., the multiplicative group (with 8 elements) of quaternionic integers of modulus 1.

6.3.30. Proposition. *There is a split short exact sequence of groups*

$$1 \to G \xrightarrow{i} S_2 \xrightarrow{j} Q \to 1. \tag{6.3.31}$$

The corresponding extension of dual group algebras over is

$$Q_* \xrightarrow{j_*} S(2) \xrightarrow{i_*} G_*$$

where $Q_* \cong \mathbf{F}_4[x, y]/(x^4 - x,\, y^2 - y)$ *and* $G_* \cong S(2)/(t_1, t_2 + \omega t_2^2)$ *as algebras where* $j_*(x) = t_1$, $j_*(y) = \bar{\omega} t_2 + \bar{\omega} x^2 t_2^2$, *and* $\bar{\omega}$ *is the residue class of* ω.

Proof. The splitting follows the theory of division algebras over local fields (Cassels and Fröhlich [**1**, pp. 137–138]]) which implies that $E_2 \otimes \mathbf{Q}_2$ is isomorphic to the 2-adic quaternions. We leave the remaining details to the reader. □

6.3.32.
(a) $H^*(Q; \mathbf{F}_2) = P(h_{1,0}, h_{1,1}, g)/(h_{1,0}h_{1,1}, h_{1,0}^3 + h_{1,1}^3)$.
(b) $H^*(G; \mathbf{F}_2) = E(\zeta_2, \rho_2, h_{3,0}, h_{3,1})$.

Proof. Part (a) is an easy calculation with the change-to-rings spectral sequence (A1.3.14) for $\mathbf{F}_2[x]/(x^4 + x) \to Q_* \to \mathbf{F}_2[y]/(y^2 + y)$. For (b) the filtration of $S(2)$ induces one on G_*. It is easy to see that E^0G_* is cocommutative and the result follows with no difficulty. □

6.3.33. Proposition. *In the Cartan–Eilenberg spectral sequence for* 6.3.31, $E_3 = E_\infty$ *and we get the same additive structure for* $H^*(S(2))$ *as in* 6.3.27.

Proof. We can take $H^*(G) \otimes H^*(Q)$ as our E_1-term. Each term is a free module over $E(\rho_2) \otimes P(g)$. We leave the evaluation of the differentials to the reader. □

Finally, we consider the case $n = 3$ and $p \geq 5$. We will not make any attempt to describe the multiplicative structure. An explict basis of $E^0H^*(S(3))$ will be given in the proof, from which the multiplication can be read off by the interested reader. It seems unlikely that there are any nontrivial multiplicative extensions.

6.3.34. Theorem. *For* $p \geq 5$, $H^*(S(3))$ *has the following Poincaré series*: $(1+t)^3(1+t+6t^2+3t^3+6t^4+t^5+t^6)$.

Proof. We use the spectral sequences of 6.3.9 to compute $H^*(L(3,2))$ and $H^*(L(3,3))$. For the former the E_2-term is $H(h_{1,i}) \otimes E(h_{2,i})$ with $i \in \mathbf{Z}/(3)$, $d_2(h_{1,i}) = 0$ and $d_2(h_{2,i}) = h_{1,i}h_{1,i+1}$. The Poincaré series for $H^*(L(3,2))$ is $(1+t)^2(1+t+5t^2+t^3+t^4)$ and it is generated as a vector space by the following elements and their Poincaré duals: 1, $h_{1,i}$, $g_i = h_{1,i}h_{2,i}$, $k_i = h_{2,i}h_{1,i+1}$, $e_{3,i} =$

$h_{1,i}h_{2i+1} + h_{2,i}h_{1,i+2}$ (where $\sum_i e_{3,i} = 0$), $g_i h_{1,i+1} = h_{1,i}k_i = h_{1,i}h_{2,i}h_{1,i+1}$, and $h_{1,i}e_{3,i} = g_i h_{1,i+2} = h_{1,i}h_{2,i}h_{1,i+2}$.

For $H^*(L(3,3))$ we have $E_2 = E(h_{3,i} \otimes H^*(L(3,2)))$ with $d_2(h_{3,i}) = e_{3,i}$, so $d_2(\sum h_{3,i}) = 0$. $H^*(L(3,3))$ has the indicated Poincaré series and is a free module over $E(\zeta_3)$, where $\zeta_3 = \sum h_{3,i}$, on the following 38 elements and the duals of their products with ζ_3:

$$1,\ h_{1,i},\ g_i,\ k_i,\ b_{1,i+2} = h_{1,i}h_{3,i} + h_{2,i}h_{2,i+2} + h_{3,i}h_{1,i},$$
$$g_i h_{1,i+1} = h_{1,i},\ k_i,\ h_{1,i}h_{2,i}h_{2,i+2},\ h_{1,i}h_{2,i}h_{2,i+1} + h_{1,i}h_{1,i+1}h_{3,i},$$
$$h_{1,i}h_{2,i}h_{3,i},\ h_{1,i}h_{2,i+2}h_{3,i+1},\ \sum_i (h_{1,i}h_{2,i+1} - h_{1,i+1}h_{2,i+2})h_{3,i},\ h_{1,i}k_i h_{3,j}$$

(where $h_{1,i}k_i \sum_j h_{3,j}$ is divisible by ζ_3), and $h_{1,i+2}h_{1,i}h_{2,i}(h_{3,i} + h_{3,i+1}) \pm h_{1,i}h_{2,0}h_{2,1}h_{2,2}$. $\square$

4. The Odd Primary Kervaire Invariant Elements

The object of this section is to apply the machinery above to show that the Adams–Novikov element $\beta_{p^i/p^i} \in \mathrm{Ext}^2$ (see 5.1.19) is not a permanent cycle for $p > 2$ and $i > 0$. This holds for the corresponding Adams element b_i (4.3.2) for $p > 3$ and $i > 0$; by 5.4.6 we know β_{p^i/p^i} maps to b_i. The latter corresponds to the secondary cohomology operation associated with the Adem relation $P^{(p-1)p^i}P^{p^i} = \cdots$. The analogous relation for $p = 2$ is $Sq^{2^i}Sq^{2^i} = \cdots$, which leads to the element h_i^2, which is related to the Kervaire invariant by Browder's theorem, hence the title of the section. To *stress this analogy we will denote β_{p^i/p^i} by θ_i.*

We know by direct calculation (e.g., 4.4.20) that θ_0 is a permanent cycle corresponding to the first element in coker J. By Toda's theorem (4.4.22) we know θ_1 is not a permanent cycle; instead we have $d_{2p-1}(\theta_1) = \alpha_1\theta_0^p$ (up to nonzero scalar multiplication) and this is the first nontrivial differential in the Adams–Novikov spectral sequence. Our main result is

6.4.1. Odd Primary Kervaire Invariant Theorem. *In the Adams–Novikov spectral sequence for $p > 2$ $d_{2p-1}(\theta_{i+1}) \equiv \alpha_1\theta_i^p \mod \ker \theta_0^{\alpha_i}$ (up to nonzero scalar multiplication) where $a_i = p(p^i - 1)/(p-1)$ and $\alpha_1\theta_i^p$ is nonzero modulo this indeterminacy.* $\square$

Our corresponding result about the Adams spectral sequence fails for $p = 3$, where b_2 is a permanent cycle even though b_1 is not.

6.4.2. Theorem. *In the Adams spectral sequence for $p \geq 5$ b_i is not a permanent cycle for $i \geq 1$.* $\square$

From 6.4.1 we can derive the nonexistence of certain finite complexes which would be useful for constructing homotopy elements with Novikov filtration 2.

6.4.3. Theorem. *There is no connective spectrum X such that*

$$BP_*(X) = BP_*/(p, v_1^{p^i}, v_2^{p^i})$$

for $i > 0$ and $p > 2$.

Proof. Using methods developed by Smith [1], one can show that such an X must be an 8-cell complex and that there must be cofibrations

(i) $\Sigma^{2p^i(p^2-1)}Y \xrightarrow{f} Y' \to X$,
(ii) $\Sigma^{2p^i(p-1)}V(0) \xrightarrow{g} V(0) \to Y$,
(iii) $\Sigma^{2p^i(p-1)}V(0) \xrightarrow{g'} V(0) \to Y'$,

where $V(0)$ is the mod (p) Moore spectrum, g and g' induce multiplication by $v_1^{p^i}$ in $BP_*(V(0)) = BP_*/(p)$, and f induces multiplication by $v_2^{p^i}$ in

$$BP_*(Y) = BP_*(Y') = BP_*/(p, v_1^{p^i}).$$

$V(0)$ and the maps g, g' certainly exist; e.g., Smith showed that there is a map

$$\alpha\colon \Sigma^{2(p-1)}V(0) \to V(0)$$

which includes multiplication by v_1, hence α^{p^i} induces multiplication by $v_1^{p^i}$, but it may not be the only map that does so.

Hence we have to show that the existence of f leads to a contradiction. Consider the composite

$$S^{2p^i(p^2-1)} \xrightarrow{j} \Sigma^{2p^i(p^2-1)}Y \xrightarrow{f} Y' \xrightarrow{k} S^{2+2p^i(p-1)},$$

where j is the inclusion of the bottom cell and k is the collapse onto the top cell. We will show that the resulting element in $\pi^s_{2p^{i+1}(p-1)-2}$ would be detected in the Novikov spectral sequence by θ_i, thus contradicting 6.4.1. The cofibrations (ii) and (iii) induce the following short exact sequence of BP^* modules

$$0 \to \Sigma^{2p^i(p-1)}BP_*/(p) \xrightarrow{v_1^{p^i}} BP_*/(p) \to BP_*/(p, v_1^{p^i}) \to 0,$$

and the cofibration

$$S^0 \xrightarrow{p} S^0 \to V(0)$$

induces

$$0 \to BP_* \xrightarrow{p} BP_* \to BP_*/(p) \to 0.$$

Hence we get connecting homomorphisms

$$\delta_1\colon \operatorname{Ext}^0(BP_*/(p, v_1^{p^i})) \to \operatorname{Ext}^1(BP_*/(p))$$

and

$$\delta_0\colon \operatorname{Ext}^1(BP_*/(p)) \to \operatorname{Ext}^2(BP_*).$$

The element $fj \in \pi_{2p^i(p^2-1)}(Y')$ is detected by $v_2^{p^i} \in \operatorname{Ext}^0(BP_*/(p, v_1^{p^i}))$. We know (5.1.19) that

$$\theta_i = \delta_0\delta_1(v_2^{p^i}) \in \operatorname{Ext}^2(BP_*)$$

detects the element $kfj \in \pi^S_{2p^{i+1}(p-1)-2}$. □

The statement in 6.4.1 that $\alpha_1\theta_i^p$ is nonzero modulo the indeterminacy is a corollary of the following result, which relies heavily on the results of the previous three sections.

6.4.4. Detection Theorem. *In the Adams–Novikov E_2-term for $p > 2$ let θ^I be a monomial in the θ_i. Then each θ^I and $\alpha_1\theta^I$ is nontrivial.* □

We are not asserting that these monomials are linearly independent, which indeed they are not. Certain relations among them will be used below to prove 6.4.1. Assuming 6.4.4, we have

PROOF OF 6.4.1. We begin with a computation in $\mathrm{Ext}(BP_*/(p))$. We use the symbol θ_i to denote the mod p reduction of the θ_i defined above in $\mathrm{Ext}(BP_*)$. We also let h_i denote the element $-[t_1^{p^i}]$. In the cobar construction we have

$$d[t_2] = -[t_1|t_1^p] + v_1 \sum_{0<j<p} \frac{1}{p}\binom{p}{j}[t_1^j|t_1^{p-j}]$$

so

$$v_1\theta_0 = -h_0h_1. \tag{6.4.5}$$

May [5] developed a general theory of Steenrod operations which is applicable to this Ext group (see A1.5). His operations are similar to the classical ones in ordinary cohomology, except for the fact that $P^0 \neq 1$. Rather we have $P^0(h_i) = h_{i+1}$ and $P^0(\theta_i) = \theta_{i+1}$. We also have $\beta P^0(h_i) = \theta_i$, $\beta P^0(\theta_i) = 0$, $\beta P^0(v_1) = 0$, $P^1(\theta_i) = \theta_i^p$ and the Cartan formula implies that $P^{p^j}(\theta_i^{p^j}) = \theta_i^{p^{j+1}}$. Applying βP^0 to (6.4.6) gives

$$0 = \theta_0h_2 - h_1\theta_1. \tag{6.4.6}$$

If we apply the operation $P^{p^{i-1}}P^{p^{i-2}}\cdots P^1$ to (6.4.5) we get

$$h_{1+i}\theta_1^{p^i} = h_{2+i}\theta_0^{p^i}. \tag{6.4.7}$$

Now associated with the short exact sequence

$$0 \to BP_* \xrightarrow{p} BP_* \to BP_*/(p) \to 0$$

there is a connecting homomorphism

$$\delta\colon\ \mathrm{Ext}^{s,*}(BP_*/(p)) \to \mathrm{Ext}^{s+1,*}(BP_*)$$

with $\delta(h_{i+1}) = \theta_i$. Applying δ to 6.4.7 gives

$$\theta_i\theta_1^{p^i} = \theta_{i+1}\theta_0^{p^i} \in \mathrm{Ext}_{(BP_*BP_*)}(BP_*, BP_*). \tag{6.4.8}$$

We can now prove the theorem by induction on i, using 4.4.22 to start the induction. We have for $i > 0$

$$\begin{aligned}
d_{2p-1}(\theta_{i+1})\theta_0^{p^i} &= d_{2p-1}(\theta_{i+1}\theta_0^{p^i})\\
&= d_{2p-1}(\theta_i\theta_1^{p^i})\\
&= d_{2p-1}(\theta_i)\theta_1^{p^i}\\
&\equiv h_0\theta_{i-1}^p\theta_1^{p^i} \mod \ker\theta_0^{a_{i-1}}\\
&\equiv h_0(\theta_{i-1}\theta_1^{p^{i-1}})^p\\
&\equiv h_0(\theta_i\theta_0^{p^{i-1}})^p\\
&\equiv h_0\theta_i^p\theta_0^{p^i}
\end{aligned}$$

so

$$d_{2p-1}(\theta_{i+1}) \equiv h_0\theta_i^p \mod \ker\theta_0^{a_i}. \qquad \square$$

We now turn to the proof of 6.4.4. We map $\mathrm{Ext}(BP_*)$ to $\mathrm{Ext}(v_n^{-1}BP_*/I_n)$ with $n = p-1$. By 6.1.1 this group is isomorphic to $\mathrm{Ext}_{\Sigma(n)}(K(n)_*, K(n)_*)$, which is essentially the cohomology of the profinite group S_n by 6.2.4. By 6.2.12 S_n has a subgroup of order p since the field K obtained by adjoining pth roots of unity to $\mathbf{Q}_p$ has degree $p-1$. We will show that the elements of 6.4.4 have nontrivial images under the resulting map to the cohomology of $\mathbf{Z}/(p)$. In other words, we will consider the composite

$$BP_*(BP) \to \Sigma(n) \to S(n) \otimes \mathbf{F}_{p^n} \to C,$$

where C is the linear dual of the group ring $\mathbf{F}_{p^n}[\mathbf{Z}/(p)]$.

6.4.9. Lemma. *Let C be as above. As a Hopf algebra*

$$C = \mathbf{F}_{p^n}[t]/(t^p - t) \quad \textit{with} \quad \Delta t = t \otimes 1 + 1 \otimes t.$$

Proof. As a Hopf algebra we have $\mathbf{F}_{p^n}[\mathbf{Z}/(p)] = \mathbf{F}_{p^n}[u]/(u^p - 1)$ with $\Delta u = u \otimes u$, where u corresponds to a generator of the group $\mathbf{Z}/(p)$. We define an element $t \in C$ by its Kronecker pairing $\langle u^i, t\rangle = i$. Since the product in C is dual to the coproduct in the group algebra, we have

$$\langle u^i, t^k\rangle = \langle \Delta(u^i), t \otimes t^{k-1}\rangle = \langle u^i, t\rangle\langle u^i, t^{k-1}\rangle$$

so by induction on k

$$\langle u^i, t^k\rangle = i^k. \tag{6.4.10}$$

We also have $\langle u^i, 1\rangle = 1$.

We show that $\{1, t, t^2, \ldots, t^{p-1}\}$ is a basis for C by relating it to the dual basis of the group algebra. Define $x_j \in C$ by

$$x_j = \sum_{0<k<p} (jt)^k$$

for $0 < j < p$ and $x_0 = 1 + \sum_{0<j<p} x_j$. Then

$$\begin{aligned}\langle u^i, x_j\rangle &= \left\langle u^i, \sum_{0<k<p} (jt)^k \right\rangle = \sum_{0<k<p} j^k i^k \\ &= \sum_{0<k<p} (ij)^k = \begin{cases} -1 & \text{if } ij = 1 \mod p \\ 0 & \text{otherwise} \end{cases}\end{aligned}$$

and

$$\langle u^i, x_0\rangle = \left\langle u^i, 1 + \sum_{0<j<p} x_j \right\rangle = \begin{cases} 1 & \text{if } i = 0 \\ 0 & i \neq 0 \end{cases}$$

so $\{x_0, -x_1, -x_2, \ldots, -x_{p-1}\}$ is the dual basis up to permutation.

Moreover, 6.4.10 implies that $t^p = t$ so C has the desired algebra structure.

For the coalgebra structure we use the fact that the coproduct in C is dual the product in the group algebra. We have

$$\langle u^i \otimes u^j, t \otimes 1 + 1 \otimes t\rangle = i + j$$

and

$$\langle u^i \otimes u^j, \Delta(t)\rangle = \langle u^{i+j}, t\rangle = i + j$$

so $\Delta t = t \otimes 1 + 1 \otimes t$. □

To proceed with the proof of 6.4.4; we now show that under the epimorphism

$$f\colon \Sigma(n)\otimes_{K(n)_*}\mathbf{F}_{p^n}\to C \quad (\text{where } n=p-1),\quad f(t_1)\neq 0.$$

From the proof of 6.2.3, t_1 can be regarded as a continuous function from S_n to $\mathbf{F}_{p^n}$. It follows then that the nontriviality of $f(t_1)$ is equivalent to the nonvanishing of the function t_1 on the nontrivial element of order p in S_n. Suppose $x\in S_{p-1}$ is such an element. We can write

$$x=1+\sum_{i>0}e_iS^i$$

with $e_i\in W(\mathbf{F}_{p^n})$ and $e_i^{p^n}=e_i$. Recalling that $S^{p-1}=p$, we compute

$$1=x^p\equiv 1+pe_1S+(e_1S)^p \mod (S)^{1+p}$$

and

$$(e_1S)^p\equiv e_1^{(p^p-1)/(p-1)}S^p \mod (S)^{1+p}$$

so it follows that

$$e_1+e_1^{(p^p-1)/(p-1)}\equiv 0 \mod (p).$$

[Remember that $t_1(x)$ is the mod (p) reduction of e_1.] Clearly, one solution to this equation is $e_1\equiv 0 \mod (p)$ and hence $e_1=0$. We exclude this possibility by showing that it implies that $x=1$. Suppose inductively that $e_i=0$ for $i<k$. Then $x\equiv 1+e_kS^k \mod (S^{k+1})$ and $x^p\equiv 1+pe_kS^k \mod (S^{k+p})$ so $e_k\equiv 0 \mod (p)$. Since $e_k^{p^n}-e_k=0$, this implies $e_k=0$.

Hence, f is a map of Hopf algebras, $f(t_1)$ primitive, so $f(t_1)=ct$ where $c\in\mathbf{F}_{p^n}$ is nonzero. Now recall that

$$\operatorname{Ext}_C(\mathbf{F}_{p^n},\mathbf{F}_{p^n})=H^*(\mathbf{Z}/(p);\mathbf{F}_{p^n})=E(h)\otimes P(b),$$

where $E()$ and $P()$ denote exterior and polynomial algebras over $\mathbf{F}_{p^n}$, respectively, $h=[t]\in H^1$, and

$$b=\sum_{0<j<p}\frac{1}{p}\binom{p}{j}[t^j|t^{p-j}]\in H^2.$$

Let f^* denote the composition

$$\begin{aligned}&\operatorname{Ext}(BP_*)\to\operatorname{Ext}(v_n^{-1}BP_*/I_n)\\ &\qquad\xrightarrow{\cong}\operatorname{Ext}_{\Sigma(n)}(K(n)_*,K(n))\to\operatorname{Ext}_C(\mathbf{F}_{p^n},\mathbf{F}_{p^n})\xrightarrow{\cong}H^*(\mathbf{Z}/(p);\mathbf{F}_{p^n}).\end{aligned}$$

Then it follows that $f^*(h_0)=-ch$ and $f^*(b_i)=-c^{p^{i+1}}b$ and 6.4.4 is proved.

Note that the scalar c must satisfy $1+c^{(p^p-p)/(p-1)}=0$. Since $c^{p^{p-1}-1}=1$, the equation is equivalent to $1+c^{(p^{p-1}-1)/(p-1)}=0$. It follows that $c=w^{(p-1)/2}$ for some generator w of $\mathbf{F}_{p^{p-1}}^\times$, so c is not contained in any proper subfield of $\mathbf{F}_{p^{p-1}}$. Hence tensoring with this field is essential to the construction of the detecting map f.

Now we examine the corresponding situation in the Adams spectral sequence. The relations used to prove 6.4.1 (apart from the assertion of nontriviality) are also valid here, but the machinery used to prove 6.4.4 is, of course, not available. Indeed the monomials vanish in some cases. The following result was first proved by May [**1**].

6.4.11. PROPOSITION. *For $p=3$, $h_0b_1^3=0$ in $\operatorname{Ext}_{A_3}(\mathbf{Z}/(3),\mathbf{Z}/(3))$; i.e., b_2 cannot support the expected nontrivial differential.*

Proof. We use a certain Massey product identity (A1.4.6) and very simple facts about $\mathrm{Ext}_{A_3}(\mathbf{Z}/(3), \mathbf{Z}/(3))$ to show $h_0 b_1^2 = 0$. We have

$$h_0 b_1^2 = -h_0\langle h_1, h_1, h_1\rangle b_1 = -\langle h_0, h_1, h_1\rangle h_1 b_1.$$

By (6.4.7) $h_1 b_1 = h_2 b_0$, so

$$h_0 b_1^2 = -\langle h_0, h_1, h_1\rangle h_2 b_0 = -\langle h_1, h_0, h_1\rangle h_2 b_0 = -h_1\langle h_0, h_1, h_2\rangle b_0.$$

The element $\langle h_0, h_1, h_2\rangle$ is represented in the cobar construction by $\xi_1^9|\xi_2 + \xi_2^3|\xi_1$, which is the coboundary of ξ_3, so $h_0 b_1^2 = 0$. □

The case of b_2 at $p = 3$ is rather peculiar. One can show in the Adams–Novikov spectral sequence that $d_5(\beta_7) = \pm\alpha_1\beta_{3/3}^3$. (This follows from the facts that $d_5(\beta_4) = \pm\alpha_1\beta_1^2\beta_{3/3}$, $\beta_4^2 = \pm\beta_1\beta_7$, $\beta_4\beta_{3/3} = \pm\beta_1\beta_{6/3}$, and $\beta_{3/3}^3 = \pm\beta_1^2\beta_{6/3}$. We leave the details to the reader.) Hence $\beta_{9/9} \pm \beta_7$ is a permanent cycle mapping to b_2. The elements β_7 and $\alpha_1\beta_{3/3}^3 = \pm\alpha_1\beta_1^2\beta_{6/3}$ correspond to Adams elements in filtrations 8 and 10 which are linked by a differential. We do not know the fate of the b_i at $p = 3$ for $i > 2$.

To prove 6.4.2 we will need two lemmas.

6.4.12. Lemma. *For $p \geq 3$*

(i) $\mathrm{Ext}^{2,qp^{i+2}}(BP_*)$ *is generated by the* $[(i+3)/2]$ *elements* $\beta_{a_{i,j}/p^{i+3-2j}}$, *where* $j = 1, 2, \ldots, [(i+3)/2]$, $a_{i,j} = (p^{i+2} + p^{i+3-2j})/(p+1)$, *and* $[(i+3)/2]$ *is the largest integer* $\leq (i+3)/2$. *Each of these elements has order p.*

(i) *Each of these elements except* $\beta_{p^{i+1}/p^{i+1}}$ *reduces to zero in*

$$\mathrm{Ext}^{2,qp^{i+2}}(BP^*/I_3). \qquad \square$$

6.4.13. Lemma. *For $p \geq 5$, any element of* $\mathrm{Ext}^{2,qp^{i+2}}(BP_*)$ *(for $i \geq 0$) which maps to b_{i+1} in the Adams E_2-term supports a nontrivial differential d_{2p-1}.* □

We have seen above that 6.4.13 is false for $p = 3$.

Theorem 6.4.2 follows immediately from 6.4.13 because a permanent cycle in the Adams spectral sequence of filtration 2 must correspond to one in the Adams–Novikov spectral sequence of filtration ≤ 2. By sparseness (4.4.2) the Novikov filtration must also be 2, but 6.4.13 says that no element in $\mathrm{Ext}^2(BP_*)$ mapping to b_i for $i \geq 1$ can be a permanent cycle.

Proof of 6.4.12. Part (i) can be read off from the description of $\mathrm{Ext}^{2,*}(BP_*)$ given in 5.4.5.

To prove (ii) we recall the definition of the elements in question. We have short exact sequences of $BP_*(BP)$-comodules

$$0 \to BP_* \to BP_* \xrightarrow{p} BP_*/(p) \to 0. \tag{6.4.14}$$

$$0 \to BP_*/(p) \xrightarrow{v_1^{p^{i+3-2j}}} BP_*/(p) \to BP_*/\big(p, v_1^{p^{i+3-2j}}\big) \to 0. \tag{6.4.15}$$

Let δ_0 and δ_1, denote the respective connecting homomorphisms. Then we have $v_2^{a_{i,j}} \in \mathrm{Ext}^0_{BP_*BP}\big(BP_*, BP_*/\big(p, v_1^{i+3-2j}\big)\big)$ and $\beta_{\alpha_{i,j}p^{i+3-2j}} = \delta_0\delta_1(v_2^{a_{1,j}})$. The element $\beta_{p^{i+1}/p^{i+1}}$ the above element for $j = 1$) can be shown to be b_{i+1} as follows. The right unit formula 4.3.21 gives

$$\eta_R(v_2) \equiv v_2 + v_1 t_1^p - v_1^p t_1 \mod (p), \tag{6.4.16}$$

$$\delta_1(v_2^{p^{i+1}}) = t_1^{p^{i+2}} - v_1^{p^{i+2}-p^{i+1}} t_1^{p^{i+1}}$$

and $\delta_0(t_1^{p^{i+2}}) = b_{i+1}$. Moreover 6.4.16 implies that in $\mathrm{Ext}(BP_*/(p))$,

$$v_1^{p^j} t_1^{pj+1} \cong v_1^{p^{j+1}} t_1^{p^j}, \quad \text{so} \quad v_1^{p^{i+2}-p^{i+1}} t_1^{p^{i+1}} \cong v_1^{p^{i+2}-1} t_1.$$

This element is the mod (p) reduction of $p^{-i-2}\delta_0(v_1^{p^{i+2}})$ and is therefore in $\ker \delta_0$. Hence $\delta_0\delta_1(v_2^{p^{i+1}}) = \delta_0(t_1^{p^{i+2}}) = b_{i+1}$.

This definition of $\beta_{p^{i+1}/p^{i+1}}$ differs from that of 5.4.5, where for $i > 0$ it is defined to be $\delta_0\delta_1(v_2^{p^2} - v_1^{p^2-1}v_2^{p^2-p+1})^{p^{i-1}}$.

In principle one can compute this element explicitly in the cobar complex (A1.2.11) and reduce mod I_3, but that would be very messy. A much easier method can be devised using Yoneda's interpretation of elements in Ext groups as equivalence classes of exact sequences (see, for example, Chapter IV of Hilton and Stammbach [**1**]) as in 5.1.20(b). Consider the following diagram.

(6.4.17)

$$\begin{tikzcd}
0 \arrow[r] & BP_* \arrow[r, "p"] \arrow[d, "p_1"] & BP_* \arrow[r, "v_1^{i+3-3j}"] \arrow[d, "p_2"] & BP_*/(p) \arrow[r, "\rho_2"] \arrow[d, "p_3"] & BP_*/(p, v_1^{p^{i+3-2j}}) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & BP_*/(p, v_1, v_2) \arrow[r] & M_1 \arrow[r] & M_2 \arrow[r] & BP_*/(p, v_1^{p^{i+3-2j}}) \arrow[r] & 0.
\end{tikzcd}$$

The top row is obtained by splicing 6.4.14 and 6.4.15 and it corresponds to an element in $\mathrm{Ext}^2(BP_*/(p, v_1^{p^{i+3-2j}}), BP_*)$. Composing this element with

$$v_2^{a_{i,j}} \in \mathrm{Ext}^0(BP_*/(p, v_1^{i+3-2j}))$$

gives $\beta_{a_{i,j}/p^{i+3-2j}}$.

We let p_1 be the standard surjection. It follows from Yoneda's result that if we choose BP_*BP-comodules M_1 and M_2, and comodule maps p_2 and p_3 such that the diagram commutes and the bottom row is exact, then the latter will determine the element of

$$\mathrm{Ext}^2_{BP_*BP}(BP_*/(p, v_1^{i+3-2j}), BP_*/(p, v_1, v_2))$$

which, when composed with $v_2^{a_{i,j}}$, will give the mod I_3 reduction of $\beta_{a_{i,j}/p^{i+3-2j}}$. We choose $M_1 = BP_*/(p^2, pv_1, v_1^2, pv_2)$ and $M_2 = BP_*/(p, v_1^{2+p^{i+3-2j}})$ and let p_2 and p_3 be the standard surjections. It is easy to check that M_1 and M_2 are comodules over $BP_*(BP)$, i.e., that the corresponding ideals in BP_* are invariant. (The ideal used to define M_1 is simply $I_2^2 + I_1 I_3$.) Moreover, the resulting diagram has the desired properties.

The resulting bottom row of 6.4.17 is the splice of the two following short exact sequences.

$$0 \to BP_*/(p, v_1, v_2) \xrightarrow{p} BP_*/(p^2, pv_1, pv_2, v_1^2) \to BP_*/(p, v_1^2) \to 0, \tag{6.4.18}$$

$$0 \to BP_*/(p, v_1^2) \xrightarrow{v_1^{p^{i+3-2j}}} BP_*/(p, v_1^{2+p^{i+3-2j}}) \to BP_*/(p, v_1^{p^{i+3-2j}}) \to 0. \tag{6.4.19}$$

Let δ_0', δ_1' denote the corresponding connecting homomorphisms. The elements we are interested in then are $\delta_0'\delta_1'(v_2^{a_{i,j}})$.

To compute $\delta_1'(v_2^{a_{i,j}})$ we use the formula $d(v_2^n) = (v_2 + v_1 v_1 t_1^p - v_1^p t_1)^n - v_2^n$, implied by 6.4.16, in the cobar construction for $BP_*/(p, v_1^{2+p^{i+3-2j}})$. Recall that

$$a_{i,j} = (p^{i+2} + p^{i+3-2j})/(p+1) \quad 1 \le j \le [(i+3)/2].$$

Hence $a_{i,j} = p^{i+3-2j} \mod (p^{i+4-2j})$ and $d(v_2^{a_{i,j}}) = v_2^{b_{i,j}} v_1^{p^{i+3-2j}} [t_1^{p^{i+4-2j}}]$, so

$$\delta_1'(v_2^{a_{i,j}}) = v_2^{b_{i,j}} [t_1^{p^{i+3-2j}}],$$

where $b_{i,j} = a_{i,j} - p^{i+3-2j} = (p^{i+2} - p^{i+4-2j})/(p+1)$.

For $j = 1$, $b_{i,1} = 0$ and

$$\delta_0' \delta_1'(v_2^{a_{i,1}}) = - \sum_{0<k<p} \frac{1}{p} \binom{p}{k} [t_1^{kp^i} | t_1^{(p-k)p^i}] = -b_{i+1}.$$

For $j > 1$, $b_{i,j}$ is divisible by p and $d(v_2^{b_{i,j}}) \equiv 0 \mod (p^2, pv_1, v_1^2)$ and

$$v_2^{b_{i,j}} d(t_1^{p^{i+4-2j}}) \equiv 0 \mod (pv_2),$$

so $\delta_1' v_2^{a_{i,j}} \in \mathrm{Ext}^1(BP_*/(p, v_1^2))$ pulls back in 6.4.17 to an element of

$$\mathrm{Ext}^1(BP_*/(p^2, pv_1, pv_2, v^2)) \quad \text{and} \quad \delta_0' \delta_1'(v_2^{a_{i,j}}) = 0,$$

completing the proof. □

Proof of 6.4.13. Any element of $\mathrm{Ext}^{2,qp^{i+2}}(BP_*)$ can be written uniquely as $cb_{i+1} + x$ where x is in the subgroup generated by the elements $\beta_{a_{i,j}/p^{i+3-2j}}$ for $j > 1$. In 5.4.6, we showed that x maps to zero in the classical Adams E_2-term. Hence it suffices to show that no such x can have the property

$$d_{2p-1}(x) = d_{2p-1}(b_{i+1})$$

By 5.5.2 for $p \geq 5$ there is an 8-cell spectrum $V(2) = M(p, v_1, v_2)$ with $BP_*(V(2)) = BP_*/(p, v_1, v_2)$, and a map $f\colon S^0 \to V(2)$ inducing a surjection in BP homology. f also induces the standard map

$$f_*\colon \mathrm{Ext}(BP_*) \to \mathrm{Ext}(BP_*/I_3).$$

Lemma 6.4.12 asserts that $f_*(\beta_{a_{i,j}/p^{i+3-2j}}) = 0$ for $j > 1$, so $f_*(d_{2p-1}(x)) = 0$ where x is as above. However, 6.4.1 and the proof of 6.4.4 show that

$$g_*(d_{2p-1}(b_{i+1})) \neq 0,$$

where g_* is induced by the obvious map

$$g\colon BP_* \to v_{p-1}^{-1} BP_*/I_{p-1}.$$

Since g factors through BP_*/I_3, this shows that $f_*(d_{2p-1}(b_{i+1})) \neq 0$, completing the proof. □

5. The Spectra $T(m)$

In this section will we construct certain spectra $T(m)$ and study the corresponding chromatic spectral sequence. $T(m)$ satisfies

$$BP_*(T(m)) = BP_*[t_1, t_2, \ldots, t_m] \subset BP_*(BP_*)$$

as a comodule algebra. These are used in Chapter 7 in a computation of the Adams–Novikov E_2-term. We will see there that the Adams–Novikov spectral sequence for $T(m)$ is easy to compute through a range of dimensions that grows rapidly with m, and here we will show that its chromatic spectral sequence is very regular.

To construct the $T(m)$ recall that $BU = \Omega SU$ by Bott periodicity, so we have maps $\Omega SU(k) \to BU$ for each k. Let $X(k)$ be the Thom spectrum of the corresponding vector bundle over $\Omega SU(k)$. An easy calculation shows that $H_*(X(k)) = \mathbf{Z}[b_1, b_2, \dots, b_{k-1}] \subset H_*(MU)$. Our first result is

6.5.1. SPLITTING THEOREM. *For any prime p, $X(k)_{(p)}$ is equivalent to a wedge of suspensions of $T(m)$ with m chosen so that $p^m \le k < p^{m+1}$, and $BP_*(T(m)) = BP_*[t_1, \dots, t_m] \subset BP_*(BP)$. Moreover $T(m)$ is a homotopy associative commutative ring spectrum.* $\square$

From this we get a diagram

$$S^0_{(p)} = T(0) \to T(1) \to T(2) \to \cdots \to BP.$$

In Ravenel [**8**, §3] we show that after p-adic completion there are no essential maps from $T(i)$ to $T(j)$ if $i > j$ or from BP to $T(i)$.

This theorem is an analog of 4.1.12, which says that $MU_{(p)}$ splits into a wedge of suspensions of BP, as is its proof. We start with the following generalization of 4.1.1.

6.5.2. DEFINITION. *Let E be an associative commutative ring spectrum. A complex orientation of degree k for E is a class $x_E \in \widetilde{E}^2(\mathbf{C}P^k)$ whose restriction to $\widetilde{E}^2(\mathbf{C}P^1) \cong \pi_0(E)$ is 1.* $\square$

A complex orientation as in 4.1.1 is of degree k for all $k > 0$. This notion is relevant in view of

6.5.3. LEMMA. *$X(k)$ admits a complex orientation of degree k.*

PROOF. $X(k)$ is a commutative associative ring spectrum (up to homotopy) because $\Omega SU(k)$ is a double loop space. The standard map $\mathbf{C}P^{k-1} \to BU$ lifts to $\Omega SU(k)$. Thomifying gives a stable map $\mathbf{C}P^k \to X(k)$ with the desired properties. $\square$

$X(k)$ plays the role of MU in the theory of spectra with orientation of degree k. The generalizations of lemmas 4.1.4, 4.1.7, 4.1.8, and 4.1.13 are straightforward. We have

6.5.4. PROPOSITION. *Let E be an associative commutative ring spectrum with a complex orientation $x_E \in \widetilde{E}_2(\mathbf{C}P^k)$ of degree k.*

(a) $E^*(\mathbf{C}P^k) = \pi_*(E)[x_E]/(x_E^{k+1})$.

(b) $E^*(\mathbf{C}P^k \times \mathbf{C}P^k) = \pi_*(E)[x_E \otimes 1, 1 \otimes x_E]/(x_E^{k+1} \otimes 1, 1 \otimes x_E^{k+1})$.

(c) *For $0 < i < k$ the map $t\colon \mathbf{C}P^i \times \mathbf{C}P^{k-i} \to \mathbf{C}P^k$ induces a formal group law k-chunk; i.e., the element $t^*(x_E)$ in the truncated power series ring*

$$\pi_*(E)[x_E \otimes 1, 1 \otimes x_E]/(x_E \otimes 1, 1 \otimes x_E)^{k+1}$$

has properties analogous to an formal group law (A2.1.1).

(d) $E_*(X(k)) = \pi_*(E)[b_1^E, \dots, b_{k-1}^E]$ *where $b_i^E \in E_{2i}(X(k))$ is defined as in* 4.1.7.

(e) *With notation as in* 4.1.8*, in $(E \wedge X(k))^2(\mathbf{C}P^k)$ we have*

$$\hat{x}_{X(k)} = \sum_{0 \le i \le k-1} b_i^E \hat{x}_E^{i+1} \quad \textit{where} \quad b_0 = 1.$$

This power series will be denoted by $g_E(\hat{x}_E)$.

(f) *There is a one-to-one correspondence between degree k orientations of E and multiplicative maps $X(k) \to E$ as in* 4.1.13. $\square$

We do not have a generalization of 4.1.15, i.e., a convenient way of detecting maps $X(k) \to X(k)$, but we can get by without it. By 6.5.4(f) a multiplicative map $g\colon X(k)_{(p)} \to X(k)_{(p)}$ is determined by a polynomial $f(x) = \sum_{0\le i\le k-1} f_i x^{i+1}$ with $f_0 = 1$ and $f_i \in \pi_{2i}(X(k)_{(p)})$. In this range of dimensions $\pi_*(X(k))$ is isomorphic to $\pi_*(MU)$, so we can take $f(x)$ to be the truncated form of the power series of A2.1.23. Then the calculations of 4.1.12 show that g induces an idempotent in ordinary or BP_*-homology. In the absence of 4.1.15 it does not follow that g itself is idempotent. Nevertheless we can define

$$T(m) = \lim_g X(k)_{(p)},$$

i.e., $T(m)$ is the mapping telescope of g. Then we can compose the map $X(k)_{(p)} \to T(m)$ with various self-maps of $X(k)_{(p)}$ to construct the desired splitting, thereby proving 6.5.1.

Now we consider the chromatic spectral sequence for $T(m)$. Using the change-of-rings isomorphism 6.1.1, the input needed for the machinery of Section 5.1 is $\mathrm{Ext}_{\Sigma(n)}(K(n)_*, K(n)_*(T(m)))$ where $K(n)_*(T(m)) = K(n)_*[t_1, \dots, t_m]$. Using notation as in 6.3.7, let $\Sigma(n, m+1) = \Sigma(n)/(t_1, \dots, t_m)$. Then we have

6.5.5. THEOREM. *With notation as above we have*

$$\begin{aligned}&\mathrm{Ext}_{\Sigma(n)}(K(n)_*, K(n)_*(T(m)))\\ &\qquad = K(n)_*[u_{n+1}, \dots, u_{n+m}] \otimes_{K(n)_*} \mathrm{Ext}_{\Sigma(n,m+1)}(K(n)_*, K(n)),\end{aligned}$$

where $\dim u_j = \dim v_j$. *Moreover* u_j *maps to* v_j *under the map to* $\mathrm{Ext}_{\Sigma(n)}(K(n)_*, K(n)_*(BP)) = B(n)_*$ (6.1.11) *induced by* $T(m) \to BP$. *In other words its image in* $K(n)_*(BP)$ *coincides with that of* $\eta_R(v_j) \in BP_*(BP)$ *under the map* $BP_*(BP) \to K(n)_*(BP)$. □

Applying 6.3.7 gives

6.5.6. COROLLARY. *If* $n < m+2$ *and* $n < 2(p-1)(m+1)/p$ *then*

$$\begin{aligned}&\mathrm{Ext}_{\Sigma(n)}(K(n)_*, K(n)_*(T(m)))\\ &\qquad = K(n)_*[u_{n+1}, \dots, u_{n+m}] \otimes E(h_{k,j}\colon m+1 \le k \le m+n, \quad j \in \mathbf{Z}/(n)).\end{aligned}$$

□

PROOF OF 6.5.5. The images of $\eta_R(v_{n+j})$ (for $1 \le j \le m$) in $K(n)_*(T(m))$ are primitive and give the u_{n+j}. The image of $BP_*(T(m)) \to BP_*(BP) \to \Sigma(n)$ is the subalgebra generated by $\{t_n\colon n \le m\}$. The result follows by a routine argument. □

Now we will use the chromatic spectral sequence to compute $\mathrm{Ext}^s(BP_*(T(m)))$ for $s = 0$ and 1. We assume $m > 0$ since $T(0) = S^0$, which was considered in 5.2.1 and 5.2.6. By 6.5.5 and 6.5.6 we have

$$\begin{aligned}&\mathrm{Ext}_{\Sigma(0)}(K(0)_*, K(0)_*(T(m))) = \mathbf{Q}[u_1, \dots, u_m] \quad \text{and}\\ &\mathrm{Ext}_{\Sigma(1)}(K(1)_*, K(1)_*(T(m))) = K(1)_*[u_2, \dots, u_{m+1}] \otimes E(h_{m+1,0}).\end{aligned} \tag{6.5.7}$$

The short exact sequence

$$0 \to M_1^0 \otimes BP_*(T(m)) \xrightarrow{i} M^1 \otimes BP_*(T(m)) \xrightarrow{p} M^1 \otimes BP_*(T(m)) \to 0 \tag{6.5.8}$$

induces a six-term exact sequence of Ext groups with connecting homomorphism δ. For $j \le m$, $\eta_R(v_j) \in BP_*(T(m)) \subset BP_*(BP)$, so if u is any monomial in these elements then $\delta(u/p^i) = 0$ for all $i > 0$ and $\mathrm{Ext}^0(M^1 \otimes BP_*(T(m)))$ has a corresponding summand isomorphic to $\mathbf{Q}/\mathbf{Z}(p)$. Hence in the chromatic spectral sequence, $E_1^{1,0}$ has a summand isomorphic to $\mathbf{Z}_{(p)}[u_1, \dots, u_m] \otimes \mathbf{Q}/\mathbf{Z}(p)$, which is precisely the image of $d_1 \colon E_1^{0,0} \to E_1^{1,0}$, giving

6.5.9. PROPOSITION.

$$\mathrm{Ext}^0(BP_*(T(m))) = \mathbf{Z}_{(p)}[u_1, \dots, u_m]. \qquad \square$$

Next we need to consider the divisibility of $u_{m+1}^t/p \in \mathrm{Ext}^0(M^1 \otimes BP_*(T(m)))$. Note that $\eta_R(v_{m+1})$ is not in $BP_*(T(m))$ but $\eta_R(v_{m+1}) - pt_{m+1}$ (where v_{m+1} is Hazewinkel's generator given by A2.2.1) is, so we call this element u_{m+1}. It follows that in the cobar complex $C(BP_*(T(m)))$ (A1.2.11) $d(u_{m+1}) = pt_{m+1}$ and

$$d(u_{m+1}^t) \equiv \mathrm{pt} u_{m+1}^{t-1} t_{m+1} + p^2 \binom{t}{2} u_{m+1}^{t-2} t_{m+1}^2 \mod (p^2 t), \tag{6.5.10}$$

where the second term is nonzero only when $p = 2$ and t is even. Thus the situation is similar to that for $m = 0$ where we have $v_1 = u_1$. Recall that in that case the presence of the second term caused Ext^1 to behave differently at $p = 2$. We will show that this does not happen for $m \ge 1$ and we have

6.5.11. THEOREM. *For $m \ge 1$ and all primes p*

$$\mathrm{Ext}^1(BP_*(T(m))) = \mathrm{Ext}^0(BP_*(T(m))) \otimes \{u_{m+1}^t/\mathrm{pt} \colon t > 0\}.$$

PROOF. For $p > 2$ the result follows from 6.5.10 as in 5.2.6. Now recall the situation for $m = 0$, $p = 2$. For $t = 2$, 6.5.10 gives $d(v_1^2) = 4(v_1 t_1 + t_1^2)$ and we have $d(4v_1^{-1} v_2) \equiv 4(v_2 t_1 + t_1^2) \mod (8)$, so we get a cocycle $(v_1^2 + 4v_1^{-1} v_2)/8$. The analogous cocycle for $m \ge 1$ would be something like

$$(u_{m+1}^2 + 4v_1^{-1} u_{m+2})/q$$

where u_{m+2} is related somehow to v_{m+2}. However, the relevant terms in $\eta_R(v_{m+2})$ mod (2) are $v_1 t_{m+1}^2 + v_2^{2^{m+1}} t_{m+1}$, which does not bear the resemblance to 6.5.10 for $m \ge 1$ that it does for $m = 0$. In other words $u_{m+1}^{t-2} t_{m+1}^2$ is not cohomologous mod (2) to $u_{m+1}^{t-1} t_{m+1}$, so the calculation for $p = 2$ can proceed as it does for $p > 2$. $\square$

Our last result is useful for computing the Adams–Novikov E_2-term for $T(m)$ by the method used in Section 4.4.

6.5.12. THEOREM. *For $t < 2(p^{2m+2} - 1)$*

$$\mathrm{Ext}(BP_*(T(m))/I_{m+1}) = \mathbf{Z}/(p)[u_{m+1}, u_{m+2}, \dots, u_{2m+1}] \otimes E(h_{i,j}) \otimes P(b_{i,j})$$

with $i \ge m+1$, $i + j \le 2m+2$, $h_{i,j} \in \mathrm{Ext}^{1, 2p^j(p^i - 1)}$ and $b_{i,j} \in \mathrm{Ext}^{2, 2p^{j+1}(p^i - 1)}$.

6.5.13. EXAMPLE. For $m = 1$ we have

$$\begin{aligned} \mathrm{Ext}(BP_*(T(1))/I_2) = \mathbf{Z}/(p)[u_2, u_3] &\otimes E(h_{2,0}, h_{2,1}, h_{2,2}, h_{3,0}, h_{3,1}) \\ &\otimes P(b_{2,0}, b_{2,1}, b_{3,0}) \end{aligned}$$

in 6.5.1 for $t \le 2(p^4 - 1)$

Proof of 6.5.12. By a routine change-of-rings argument (explained in Section 7.1) the Ext in question is the cohomology of $C_\Gamma(BP_*/I_{m+1})$ (A1.2.11) where $\Gamma = BP_*(BP)/(t_1, \dots, t_m)$. Then from 4.3.15 and 4.3.20 we can deduce that v_i and t_i are primitive for $m+1 \le i \le 2m+1$. $h_{i,j}$ corresponds to $t_i^{p^i}$ and $b_{i,j}$ to $-\sum_{0<k<p} p^{-1}\binom{p}{k} t_i^{kp^j} | t_i^{(p-k)p^j}$. The result follows by routine calculation. $\square$

CHAPTER 7

Computing Stable Homotopy Groups with the Adams–Novikov Spectral Sequence

In this chapter we apply the Adams–Novikov spectral sequence to the motivating problem of this book, the stable homotopy groups of spheres. Our main accomplishment is to find the first thousand stems for $p = 5$, the previous record being 760 by Aubry [**1**]. In Section 1 we describe the method of infinite descent for computing the Adams–Novikov spectral sequence E_2-term in a range of dimensions, namely to find it for the spectra $T(m)$ of Section 6.5 by downward induction on m. Recall $BP_*(T(m)) = BP_*[t_1, \ldots, t_m]$ as a comodule, so $T(m)$ is equivalent to BP in dimensions less than $|v_{m+1}|$. This starts our downward induction since we always restrict our attention to a finite range of dimensions.

In Section 2 we construct a resolution enabling us in theory to extract the Adams–Novikov E_2-term for S^0 from that for $T(1)$. In practice we must proceed more slowly, computing for skeleta $T(1)^{(p^i-1)q}$ by downward induction on i. In Section 3 we do this down to $i = 1$; see 7.5.1. $T(1)^{(p-1)q}$ is a complex with p cells, its Adams–Novikov spectral sequence collapses in our range, and its homotopy is surprisingly regular.

In Section 4 we take the final step from $T(1)^{(p-1)q}$ to S^0. We have a spectral sequence (7.1.16) for this calculation and a practical procedure (7.1.18) for the required bookkeeping. We illustrate this method for $p = 3$, but here our range of dimensions is not new; see Tangora [**6**] and Nakamura [**3**].

In Section 5 we describe the calculations for $p = 5$, giving a running account of the more difficult differentials in the spectral sequence of 7.1.16 for that case. The results are tabulated in Appendix 3 and range up to the 1000-stem.

In more detail, the method in question involves the connective p-local ring spectra $T(m)$ of 6.5, which satisfy

$$BP_*(T(m)) = BP_*[t_1, \ldots, t_m] \subset BP_*(BP).$$

$T(0)$ is the p-local sphere spectrum, and there are maps

$$S^0 = T(0) \to T(1) \to T(2) \to \cdots \to BP.$$

The map $T(m) \to BP$ is an equivalence below dimension $|v_{m+1}| - 1 = 2p^{m+1} - 3$.

To descend from $\pi_*(T(m))$ to $\pi_*(T(m-1))$ we need some spectra interpolating between $T(m-1)$ and $T(m)$. Note that $BP_*(T(m))$ is a free module over $BP_*(T(m-1))$ on the generators $\{t_m^j \colon j \geq 0\}$. In Lemma 7.1.11 we show that for each h there is a $T(m-1)$-module spectrum $T(m-1)_h$ with

$$BP_*(T(m-1)_h) = BP_*(T(m-1))\{t_m^j \colon 0 \leq j \leq h\}.$$

We will be most interested in the case where h is one less than a power of p, and we will denote $T(m)_{p^i-1}$ by $T(m)_{(i)}$.

We have inclusions

$$T(m-1) = T(m-1)_{(0)} \to T(m-1)_{(1)} \to T(m-1)_{(2)} \to \cdots T(m)$$

and the map $T(m-1)_{(i)} \to T(m)$ is an equivalence below dimension $p^i|t_m| - 1 = 2(h+1)(p^m-1) - 1$.

For example when $m = i = 0$, the spectrum $T(m)_{(i)}$ is S^0 while $T(m)_{p^{i+1}-1}$ is the p-cell complex

$$Y = S^0 \cup_{\alpha_1} e^q \cup_{\alpha_1} e^{2q} \cdots \cup_{\alpha_1} e^{(p-1)q},$$

where $q = 2p-2$.

In Theorem 7.1.16 we give a spectral sequence for computing $\pi_*(T(m-1)_{(i)})$ in terms of $\pi_*(T(m-1)_{(i+1)}$. Its E_1-term is

$$E(h_{m,i}) \otimes P(b_{m,i}) \otimes \pi_*(T(m-1)_{(i+1)}$$

where the elements

$$\begin{array}{rrcl} & h_{m,i} & \in & E_1^{1,2p^i(p^m-1)} \\ \text{and} & b_{m,i} & \in & E_1^{2,2p^{i+1}(p^m-1)} \end{array}$$

are permanent cycles.

In the case $m = i = 0$ cited above, the E_1-term of this spectral sequence is

$$E(h_{1,0}) \otimes P(b_{1,0}) \otimes \pi_*(Y).$$

where $h_{1,0}$ and $b_{1,0}$ represent the homotopy elements α_1 and β_1 (α_1^2 for $p = 2$) respectively.

Thus to compute $\pi_*(S^0)$ below dimension $p^3(2p-2)$ we could proceed as follows. In this range we have

$$BP \cong T(3) \cong T(2)_{(1)}.$$

We then use the spectral sequence of 7.1.16 to get down to $T(2)$, which is equivalent in this range to $T(1)_{(2)}$, then use it twice to get down to $T(1) \cong T(0)_{(3)}$, and so on. This would make for a total of six applications of 7.1.16. Fortunately we have some shortcuts that make this process easier.

The Adams–Novikov E_2-term for $T(m)$ is

$$\mathrm{Ext}_{BP_*(BP)}(BP_*, BP_*(T(m))).$$

From now on we will drop the first variable when writing such Ext groups, since we will never consider any value for it other than BP_*. There is a change-of-rings isomorphism that equates this group with

$$\mathrm{Ext}_{\Gamma(m+1)}(BP_*)$$

where

$$\Gamma(m+1) = BP_*(BP)/(t_1, \ldots, t_m) = BP_*[t_{m+1}, t_{m+2}, \ldots].$$

Using our knowledge of $\mathrm{Ext}^0_{\Gamma(m+1)}(BP_*)$ (Proposition 7.1.24) and $\mathrm{Ext}^1_{\Gamma(m+1)}(BP_*)$ (Theorem 7.1.31) in all dimensions, we will construct a 4-term exact sequence

$$0 \to BP_* \to D^0_{m+1} \to D^1_{m+1} \to E^2_{m+1} \to 0$$

of $\Gamma(m+1)$-comodules. The two D^i_{m+1} are weak injective, meaning that all of their higher Ext groups (above Ext^0) vanish (we study such comodules systematically at the end of Section 1), and below dimension $p^2|v_{m+1}|$

$$\mathrm{Ext}^0_{\Gamma(m+1)}(D^i_{m+1}) \cong \mathrm{Ext}^i_{\Gamma(m+1)}(BP_*).$$

It follows that in that range

$$\mathrm{Ext}^s_{\Gamma(m+1)}(E^2_{m+1}) \cong \mathrm{Ext}^{s+2}_{\Gamma(m+1)}(BP_*) \qquad \text{for all } s \geq 0.$$

The comodule E^2_{m+1} is $(2p^{m+2}-2p-1)$-connected. In Theorem 7.2.6 we determine its Ext groups (and hence those of BP_*) up to dimension $p^2|v_{m+1}|$. There are no Adams–Novikov differentials or nontrivial group extensions in this range (except in the case $m=0$ and $p=2$), so this also determines $\pi_*(T(m))$ in the same range.

Thus Theorem 7.2.6 gives us the homotopy of $T(0)_{(3)}$ in our range directly without any use of 7.1.16. In a future paper with Hirofume Nakai we will study the homotopy of $T(m)_{(2)}$ and the spectral sequence of 7.1.16 for the homotopy of $T(m)_{(1)}$ below dimension $p^3|v_{m+1}|$. There are still no room for Adams–Novikov differentials, so the homotopy and Ext calculations coincide. For $m=0$ this computation was the subject of Ravenel [**11**].

It is only when we pass from $T(m)_{(1)}$ to $T(m)_{(0)} = T(m)$ that we encounter Adams–Novikov differentials below dimension $p^3|v_{m+1}|$. For $m=0$ the first of these is the Toda differential

$$d_{2p-1}(\beta_{p/p}) = \alpha_1\beta_1^p$$

of Toda [**3**] and Toda [**2**].

1. The method of infinite descent

First we define some Hopf algebroids that we will need.

7.1.1. Definition. *$\Gamma(m+1)$ is the quotient $BP_*(BP)/(t_1, t_2, \ldots, t_m)$,*

$$A(m) = BP_*\square_{\Gamma(m+1)}BP_* = \mathbf{Z}_{(p)}[v_1, v_2, \ldots, v_m]$$

and

$$G(m+1, k-1) = \Gamma(m+1)\square_{\Gamma(m+k+1)}BP_* = A(m+k)[t_{m+1}, t_{m+1} \ldots, t_{m+k}]$$

We abbreviate $G(m+1,0)$ by $G(m+1)$, and is understood that $G(m+1,\infty) = \Gamma(m+1)$.

In particular, $\Gamma(1) = BP_*(BP)$.

7.1.2. Proposition. *$G(m+1,k-1) \to \Gamma(m+1) \to \Gamma(m+k+1)$ is a Hopf algebroid extension (A1.1.15). Given a left $\Gamma(m+1)$-comodule M there is a Cartan–Eilenberg spectral sequence (A1.3.14) converging to $\mathrm{Ext}_{\Gamma(m+1)}(BP_*, M)$ with*

$$\tilde{E}_2^{s,t} = \mathrm{Ext}^s_{G(m+1,k-1)}(A(m+k), \mathrm{Ext}^t_{\Gamma(m+k+1)}(BP_*, M))$$

and $d_r : E_r^{s,t} \to \tilde{E}_r^{s+r,t-r+1}$. (We use the notation $\tilde{E}_r^{s,t}$ to distinguish the Cartan–Eilenberg spectral sequence from the resolution spectral sequence.)

7.1.3. COROLLARY. *Let M be a $\Gamma(m+1)$-comodule concentrated in nonnegative dimensions. Then*

$$\operatorname{Ext}_{\Gamma(m+k+1)}(BP_*, M) = \operatorname{Ext}_{\Gamma(m+1)}(BP_*, G(m+1, k-1) \otimes_{A(m+k)} M).$$

In particular, $\operatorname{Ext}^{s,t}_{\Gamma(m+1)}(BP_*, M)$ *for* $t < 2(p^{m+1}-1)$ *is isomorphic to* M *for* $s = 0$ *and vanishes for* $s > 0$*. Moreover for the spectrum* $T(m)$ *constructed in 6.5 and having* $BP_*(T(m)) = BP_*[t_1, \ldots, t_m]$,

$$\operatorname{Ext}_{BP_*(BP)}(BP_*, BP_*(T(m))) = \operatorname{Ext}_{\Gamma(m+1)}(BP_*, BP_*).$$

The following characterization of the Cartan–Eilenberg spectral sequence is a special case of (A1.3.16).

7.1.4. LEMMA. *The Cartan–Eilenberg spectral sequence of 7.1.2 is the one associated with the decreasing filtration of the cobar complex* $C_{\Gamma(m+1)}(BP_*, M)$ *(see below) defined by saying that*

$$\gamma_1 \otimes \cdots \otimes \gamma_s \otimes m \in C^s_{\Gamma(m+1)}(BP_*, M)$$

is in F^i *if* i *of the* γ*'s project trivially to* $\Gamma(m+k+1)$.

The *method of infinite descent* for computing $\operatorname{Ext}_{BP_*(BP)}(BP_*, M)$ for a connective comodule M (e.g. the BP-homology of a connective spectrum) is to compute over Ext over $\Gamma(m+1)$ by downward induction on m. To calculate through a fixed range of dimensions k, we choose m so that $k \leq 2(p^{m+1}-1)$ and use 7.1.3 to start the induction. For the inductive step we could use the Cartan–Eilenberg spectral sequence of 7.1.2, but it is more efficient to use a different spectral sequence, which we now describe.

7.1.5. DEFINITION. *A comodule M over a Hopf algebroid (A, Γ) is* **weak injective** *(through a range of dimensions) if* $\operatorname{Ext}^s(M) = 0$ *for* $s > 0$ *(through the same range).*

We will study such comodules in the at the end of this section.

7.1.6. DEFINITION. *For a left $G(m+1, k-1)$-comodule M let*

$$\widehat{r}_j : M \to \Sigma^{j|t_{m+1}|} M$$

be the group homomorphism defined by

$$M \xrightarrow{\psi_M} G(m+1, k-1) \otimes M \xrightarrow{\rho_j \otimes M} \Sigma^{j|t_{m+1}|} M$$

where $\rho_j : G(m+1, k-1) \to A(m+k)$ *is the* $A(m+k)$*-linear map sending* t^j_{m+1} *to 1 and all other monomials in the* t_{m+i} *to 0.*

We will refer to this map as a **Quillen operation**. *When* $m = 0$ *we denote it simply by* r_j.

It follows that

$$\psi(x) = \sum_j t^j_{m+1} \otimes \widehat{r}_j(x) + \ldots,$$

where the missing terms involve t_ℓ for $\ell > m+1$.

The following is proved in Ravenel [**12**] as Lemma 1.10.

7.1.7. LEMMA. *The Quillen operation $\widehat{r}_j$ of 7.1.6 is a comodule map and for* $j > 0$ *it induces the trivial endomorphism in Ext.*

7.1.8. Definition. *Let $T_m^h \subset G(m+1, k-1)$ denote the sub-$A(m+k)$-module generated by $\{t_{m+1}^j\colon 0 \le j \le h\}$. We will denote $T_m^{p^i-1}$ by $T_m^{(i)}$. A $G(m+1, k-1)$-comodule M is i-**free** if the comodule tensor product $T_m^{(i)} \otimes_{A(m+k)} M$ is weak injective.*

We have suppressed the index k from the notation T_m^h because it will usually be clear from the context. In the case $k = \infty$ the Ext group has the topological interpretation given in Lemma 7.1.11 below. The following lemma is useful in dealing with such comodules. It is proved in Ravenel [**12**] as Lemma 1.12.

7.1.9. Lemma. *For a left $G(m+1)$-comodule M, the group*

$$\mathrm{Ext}^0_{G(m+1)}(A(m+1), T_m^{(i)} \otimes_{A(m+k)} M)$$

is isomorphic as an $A(m)$-module to

$$L = \bigcap_{j \ge p^i} \ker \widehat{r}_j \subset M.$$

The following is proved in Ravenel [**12**] as Lemma 1.14.

7.1.10. Lemma. *Let D be a weak injective comodule over $\Gamma(m+1)$. Then $T_m^{(i)} \otimes D$ is also weak injective with*

$$\mathrm{Ext}^0_{\Gamma(m+1)}(T_m^{(i)} \otimes D) \cong A(m)\left\{t_{m+1}^j\colon 0 \le j \le p^i - 1\right\} \otimes \mathrm{Ext}^0_{\Gamma(m+1)}(D).$$

Given $x_0 \in \mathrm{Ext}^0_{\Gamma(m+1)}(D)$, the element isomorphic to $t_{m+1}^j \otimes x_0$ is

$$\sum_{0 \le k \le j} (-1)^k \binom{j}{k} t_{m+1}^k \otimes x_{j-k} \in T_m^{(i)} \otimes D$$

where $x_j \in D$ satsifies

$$\psi(x_j) = \sum_{0 \le k \le j} \binom{j}{k} t_{m+1}^{j-k} \otimes x_k.$$

The following is proved in Ravenel [**12**] as Lemma 1.15. The only case of it that we will need here is for $m = 0$, where $T(0)_h$ is the $2(p-1)h$-skeleton of $T(1)$.

7.1.11. Lemma. *For each nonnegative m and h there is a spectrum $T(m)_h$ where $BP_*(T(m)_h) \subset BP_*(BP)$ is a free module over*

$$BP_*(T(m)) = BP_*[t_1, \ldots, t_m]$$

on generators $\{t_{m+1}^j\colon 0 \le j \le h\}$. Its Adams–Novikov E_2-term is

$$\mathrm{Ext}_{BP_*(BP)}(BP_*, BP_*(T(m)_h)) \cong \mathrm{Ext}_{\Gamma(m+1)}(BP_*, T_m^h).$$

We will denote $T(m)_{p^i-1}$ by $T(m)_{(i)}$.

To pass from $\mathrm{Ext}_{G(m+1,k-1)}(T_m^{(i+1)} \otimes M)$ to $\mathrm{Ext}_{G(m+1,k-1)}(T_m^{(i)} \otimes M)$ we can make use of the tensor product (over $A(m+k)$) of M with the long exact sequence

$$0 \longrightarrow T_m^{(i)} \xrightarrow{i} R^0 \xrightarrow{d^0} R^1 \xrightarrow{d^1} R^2 \xrightarrow{d^2} \cdots, \tag{7.1.12}$$

where

$$\begin{array}{rcll} R^{2s+e} & = & \Sigma^{(ps+e)2p^i(p^{m+1}-1)}T_m^{(i)} & \text{for } e=0,1 \\ \text{and} \qquad d^s & = & \begin{cases} \widehat{r}_{p^i} & \text{for } s \text{ even} \\ \widehat{r}_{(p-1)p^i} & \text{for } s \text{ odd,} \end{cases} & \end{array}$$

which leads to a spectral sequence as in (A1.3.2).

7.1.13. THEOREM. *For a $G(m+1,k-1)$-comodule M there is a spectral sequence converging to $\operatorname{Ext}_{G(m+1,k-1)}(M\otimes T_m^{(i)})$ with*

$$E_1^{*,t}=E(h_{m+1,i})\otimes P(b_{m+1,i})\otimes \operatorname{Ext}^t_{G(m+1,k-1)}(T_m^{(i)}\otimes M)$$

with $h_{m+1,i}\in E_1^{1,0}$, $b_{m+1,i}\in E_1^{2,0}$, and $d_r: E_r^{s,t}\to E_r^{s+r,t-r+1}$. If M is $(i+1)$-free in a range of dimensions, then the spectral sequence collapses from E_2 in the same range.

Moreover d_1 is induced by the action on M of $\widehat{r}_{p^i\Delta_{m+1}}$ for s even and $\widehat{r}_{(p-1)p^i}$ for s odd.

The action of d_1 is as follows. Let

$$x=\sum_{0\le j<p^{i+1}} t_{m+1}^j\otimes m_j\in T_m^{(i)}\otimes M$$

Then d_1 is induced by the endomorphism

$$x\mapsto\begin{cases} -\displaystyle\sum_{0\le k<p^i}\ \sum_{k\le j<p^{i+1}}\binom{j}{k}t_{m+1}^{j-k}\otimes\widehat{r}_{(p^i-k)}(m_j) & \textit{for } s \textit{ even} \\ -\displaystyle\sum_{0\le k<(p-1)p^i}\ \sum_{k\le j<p^{i+1}}\binom{j}{k}t_{m+1}^{j-k}\otimes\widehat{r}_{((p-1)p^i-k)}(m_j) & \textit{for } s \textit{ odd.}\end{cases}$$

We will refer to this as the **small descent spectral sequence***.*

PROOF. Additively this spectral sequence is a special case of the one in (A1.3.2) associated with M tensored with the long exact sequence (7.1.12), and the collapsing for $(i+1)$-free M follows from the fact that the spectral sequence is in that case concentrated on the horizontal axis.

For the identification of d_1, note that by (7.1.12) it is induced by the endomorphism

$$\begin{aligned} x \ \mapsto\ & \begin{cases}\displaystyle\sum_{0\le j<p^{i+1}}\widehat{r}_{p^i}(t_{m+1}^j)\otimes m_j & \text{for } s \text{ even}\\ \displaystyle\sum_{0\le j<p^{i+1}}\widehat{r}_{(p-1)p^i}(t_{m+1}^j)\otimes m_j & \text{for } s \text{ odd}\end{cases} \\ =\ & \begin{cases}\displaystyle\sum_{p^i\le j<p^{i+1}}\binom{j}{p^i}t_{m+1}^{j-p^i}\otimes m_j & \text{for } s \text{ even}\\ \displaystyle\sum_{(p-1)p^i\le j<p^{i+1}}\binom{j}{(p-1)p^i}t_{m+1}^{j-(p-1)p^i}\otimes m_j & \text{for } s \text{ odd.}\end{cases}\end{aligned}$$

It follows from Lemma 7.1.7 that $\widehat{r}_{p^i\Delta_{m+1}}$ and $\widehat{r}_{(p-1)p^i\Delta_{m+1}}$ each induce trivial endomorphisms in Ext, so d_1 is also induced by

$$x \quad\mapsto\quad \begin{cases} -\widehat{r}_{p^i}(x) + \displaystyle\sum_{0\le j<p^{i+1}} \widehat{r}_{p^i}(t_{m+1}^j)\otimes m_j & \text{for } s \text{ even} \\ -\widehat{r}_{(p_1)p^i}(x) + \displaystyle\sum_{0\le j<p^{i+1}} \widehat{r}_{(p-1)p^i}(t_{m+1}^j)\otimes m_j & \text{for } s \text{ odd,} \end{cases}$$

which leads to the stated formula.

The multiplicative structure requires some explanation. The elements $h_{m+1,i}$ and $b_{m+1,i}$ correspond under Yoneda's isomorphism Hilton and Stammbach [**1**, page 155] to the tensor product of M with the exact sequences

$$0\to T_m^{(i)}\to T_m^{2p^i-1}\to \Sigma^{p^i|t_{m+1}|}T_m^{(i)}\to 0$$
and
$$0\to T_m^{(i)}\to T_m^{(i+1)}\to \Sigma^{p^i|t_{m+1}|}T_m^{(i+1)}\to \Sigma^{p^{i+1}|t_{m+1}|}T_m^{(i)}\to 0$$

respectively. Products of these elements correspond to the splices of the these. It follows that these two elements are permanent cycles and that the spectral sequence is one of modules over the algebra $E(h_{m+1,i})\otimes P(b_{m+1,i})$. $\square$

In practice we will find higher differentials in this spectral sequence by computing in the cobar complex $C_{G(m+1,k-1)}(M\otimes T_m^{(i)})$ or its subcomplex $C_{G(m+1,k-1)}(M)$. As explained in the proof of (A1.3.2), it can be embedded by a quasi-isomorphism (i.e., a map inducing an isomorphism in cohomology) into the double complex $B=\oplus_{s,t\ge 0}B^{s,t}$ defined by

$$B^{s,t}=C^t_{G(m+1,k-1)}(M\otimes R^s)$$

with coboundary

$$\partial = d+(-1)^s d^s,$$

where d is the coboundary operator in the cobar complex. Our spectral sequence is obtained from the filtration of B by horizontal degree, i.e., the one defined by

$$F^rB=\bigoplus_{s\ge r,t\ge 0}B^{s,t}.$$

Theorem 7.1.13 also has a topological counterpart in the case $M=BP_*$. Before stating it we need to define topological analogs of the operations $\widehat{r}_{p^i}$ and $\widehat{r}_{(p-1)p^i}$. One can show that there are cofiber sequences

$$T(m)_{(i)}\to T(m)_{(i+1)}\to \Sigma^{p^i|t_{m+1}|}T(m)_{p^i(p-1)-1} \tag{7.1.14}$$

and

$$T(m)_{p^i(p-1)-1}\to T(m)_{(i+1)}\to \Sigma^{(p-1)p^i|t_{m+1}|}T(m)_{(i)}. \tag{7.1.15}$$

We define

$$T(m)_{(i+1)}\xrightarrow{\rho_{p^i}}\Sigma^{p^i|t_{m+1}|}T(m)_{(i+1)}$$

and

$$T(m)_{(i+1)}\xrightarrow{\rho_{p^i(p-1)}}\Sigma^{(p-1)p^i|t_{m+1}|}T(m)_{(i+1)}$$

to be the composites

$$T(m)_{(i+1)} \to \Sigma^{p^i|t_{m+1}|}T(m)_{p^i(p-1)-1} \to \Sigma^{p^i|t_{m+1}|}T(m)_{(i+1)}$$

and

$$T(m)_{(i+1)} \to \Sigma^{(p-1)p^i|t_{m+1}|}T(m)_{(i)} \to \Sigma^{(p-1)p^i|t_{m+1}|}T(m)_{(i+1)}.$$

7.1.16. THEOREM. *Let $T(m)_{(i)}$ be the spectrum of Lemma 7.1.11. There is a spectral sequence converging to $\pi_*(T(m)_{(i)})$ with*

$$E_1^{s,t} = E(h_{m+1,i}) \otimes P(b_{m+1,i}) \otimes \pi_*(T(m)_{(i+1)}) \qquad \text{and} \qquad d_r : E_r^{s,t} \to E_r^{s+r,t-r+1}$$

with $h_{m+1,i} \in E_1^{1,2p^i(p^{m+1}-1)}$ and $b_{m+1,i} \in E_1^{2,2p^{i+1}(p^{m+1}-1)}$. Moreover d_1 is ρ_{p^i} for s even and $\rho_{(p-1)p^i}$ for s odd. The elements $h_{m+1,i}$ and $b_{m+1,i}$ are permanent cycles, and the spectral sequence is one of modules over the ring

$$R = E(h_{m+1,i}) \otimes P(b_{m+1,i})$$

We will refer to this as the **topological small descent spectral sequence***.*

PROOF. This the spectral sequence based on the Adams diagram

$$\begin{tikzcd}
X & \Sigma^a X' \arrow[l] \arrow[d] & \Sigma^b X \arrow[l] \arrow[d] & \Sigma^{a+b} X' \arrow[l] \arrow[d] & \dots \arrow[l] \\
Y \arrow[from=u] & \Sigma^a Y & \Sigma^b Y & \Sigma^{a+b} Y &
\end{tikzcd}$$

where

$$\begin{aligned}
a &= 2p^i(p^{m+1}-1)-1,\\
b &= 2p^{i+1}(p^{m+1}-1)-2,\\
X &= T(m)_{(i)},\\
X' &= T(m)_{p^i(p-1)-1},\\
\text{and} \qquad Y &= T(m)_{(i+1)}.
\end{aligned}$$

We will show that the elements $h_{m+1,i}$ and $b_{m+1,i}$ can each be realized by maps of the form

$$S^0 \longrightarrow X \xrightarrow{f} \Sigma^{-?}X$$

For $h_{m+1,i}$, f is the boundary map for the cofiber sequence

$$T(m)^{(i)} \to T(m)^{2p^i-1} \to \Sigma^{h+1}T(m)^{(i)},$$

and for $b_{m+1,i}$ it is the composite (in either order) of the ones for (7.1.14) and (7.1.15). □

7.1.17. EXAMPLE. When $m = i = 0$, the spectrum $T(0)_{(0)}$ is S^0 while $T(0)_{(1)}$ is the p-cell complex

$$Y = S^0 \cup_{\alpha_1} e^q \cup_{\alpha_1} e^{2q} \cdots \cup_{\alpha_1} e^{(p-1)q},$$

where $q = 2p-2$. The E_1-term of the spectral sequence of Theorem 7.1.16 is

$$E(h_{1,0}) \otimes P(b_{1,0}) \otimes \pi_*(Y).$$

where $h_{1,0}$ and $b_{1,0}$ represent the homotopy elements α_1 and β_1 (α_1^2 for $p = 2$) respectively.

We will use this spectral sequence through a range of dimensions in the following way. For each spectrum $T(m)_{(i+1)}$ the elements of Adams–Novikov filtration 0 and 1 are all permanent cycles, so we ignore them, replacing $\pi_*(T(m)_{(i+1)})$ by an appropriate subquotient of $\mathrm{Ext}_{\Gamma(m+1)}(T_m^{(i)} \otimes E_{m+1}^2)$. Let $\mathbf{N}$ be a list of generators of this group arranged by dimension. When an element x has order greater than p, we also list its nontrivial multiples by powers of p. Thus

$$\mathbf{N} \otimes E(h_{m+1,i}) \otimes P(b_{m+1,i})$$

contains a list of generators of the E_1-term in our range. Rather than construct similar lists for each E_r term we use the following method.

7.1.18. Input/output procedure. *We make two lists* $\mathbf{I}$ *(input) and* $\mathbf{O}$ *(output).* $\mathbf{I}$ *is the subset of* $\mathbf{N} \otimes E(h_{m+1,i})$ *that includes all elements in our range. Then* $\mathbf{O}$ *is constructed by dimensional induction starting with the empty list as follows. Assuming* $\mathbf{O}$ *has been constructed through dimensions* $k-1$*, add to it the* k*-dimensional elements of* $\mathbf{I}$*. If any of them supports a nontrivial differential in the spectral sequence, remove both the source and target from* $\mathbf{O}$*. (It may be necessary to alter the list of* $(k-1)$*-dimensional elements by a linear transformation so that each nontrivial target is a "basis" element.) Then if* $k > |b_{m+1,i}|$*, we append the product of* $b_{m+1,i}$ *with each element of* $\mathbf{O}$ *in dimension* $k - |b_{m+1,i}|$*. This completes the inductive step.*

Note that each element in E_1 of filtration greater than 1 is divisible by $b_{m+1,i}$. Since the spectral sequence is one of R-modules, that same is true of each E_r. In 7.1.18 we compute the differentials originating in filtrations 0 and 1. If $d_r(x) = y$ is one of them, there is no chance that for some minimal $t > 0$

$$d_{r'}(x') = b_{m+1,i}^t y \qquad \text{with } r' < r$$

because such an x' would have to be divisible by $b_{m+1,i}$. This justifies the removal of $b_{m+1,i}^t x$ and $b_{m+1,i}^t y$ for all $t \geq 0$ from consideration.

We will consider various $\Gamma(m+1)$-comodules M and will abbreviate $\mathrm{Ext}_{\Gamma(m+1)}(BP_*, M)$ by $\mathrm{Ext}_{\Gamma(m+1)}(M)$ or simply $\mathrm{Ext}(M)$.

Excluding the case $m = 0$ and $p = 2$, we will construct a diagram of 4-term exact sequences of $\Gamma(m+1)$-comodules

$$\begin{tikzcd}
0 \arrow[r] & BP_* \arrow[r] \arrow[d, equal] & D_{m+1}^0 \arrow[r] \arrow[d, equal] & D_{m+1}^1 \arrow[r] \arrow[d] & E_{m+1}^2 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & BP_* \arrow[r] \arrow[d, equal] & D_{m+1}^0 \arrow[r] \arrow[d] & v_1^{-1}E_{m+1}^1 \arrow[r] \arrow[d] & E_{m+1}^1/(v_1^\infty) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & BP_* \arrow[r] & M^0 \arrow[r] & M^1 \arrow[r] & N^2 \arrow[r] & 0
\end{tikzcd} \tag{7.1.19}$$

where each vertical map is a monomorphism, M^i and N^2 are as in 5.1.5, the D_{m+1}^i are weak injectives with $\mathrm{Ext}^0(D_{m+1}^0) = \mathrm{Ext}^0(BP_*)$, $\mathrm{Ext}^0(D_{m+1}^1)$ contains $\mathrm{Ext}^1(BP_*)$ (with equality holding for $m = 0$ and p odd), and $E_{m+1}^1 = D_{m+1}^0/BP_*$. $\mathrm{Ext}^0(BP_*)$ and $\mathrm{Ext}^1(BP_*)$ are given in 7.1.24 and 7.1.31 respectively.

It follows that for $m = 0$ and p odd, there is an isomorphism

$$\mathrm{Ext}_{\Gamma(m+1)}^s(E_{m+1}^2) \cong \mathrm{Ext}_{\Gamma(m+1)}^{s+2}(BP_*),$$

and for $m > 0$ there is a similar isomorphism below dimension $p^2|v_{m+1}|$ for all primes. E_{m+1}^2 is locally finite and $(p|v_{m+1}| - 1)$-connected, which means that $\mathrm{Ext}^s_{\Gamma(m+1)}$ for $s > 1$ vanishes below dimension $p|v_{m+1}|$.

We will construct the map from BP_* to the weak injective D_{m+1}^0, inducing an isomorphism in Ext^0 , explicitly in Theorem 7.1.28. For $m > 0$ we cannot construct a similar map out of $E_{m+1}^1 = D_{m+1}^0/BP_*$. Instead we will construct a map to a weak injective D_{m+1}^1 which enlarges Ext^0 by as little as possible. We will do this by producing a comodule $E_{m+1}^2 \subset E_{m+1}^1/v_1^\infty$ and using the induced extension

$$\begin{tikzcd} 0 \arrow[r] & E_{m+1}^1 \arrow[r] & v_1^{-1}E_{m+1}^1 \arrow[r] & E_{m+1}^1/(v_1^\infty) \arrow[r] & 0 \\ 0 \arrow[r] & E_{m+1}^1 \arrow[r] \arrow[u, equal] & D_{m+1}^1 \arrow[r] \arrow[u] & E_{m+1}^2 \arrow[r] \arrow[u] & 0 \end{tikzcd} \tag{7.1.20}$$

The comodule E_{m+1}^2 for $m > 0$ will be described in the next section. For $m = 0$ and p odd, a map from E_1^1 to a weak injective D_1^1 inducing an isomorphism in Ext^0 will be constructed in below in Lemma 7.2.1.

We will use the following notations for $m > 0$. We put hats over the symbols in order to distinguish this notation from the usual one for elements in $\mathrm{Ext}_{BP_*(BP)}$. For $m = 0$ we will use similar notation without the hats.

$$\begin{cases} \widehat{v}_i = v_{m+i}, & \widehat{t}_i = t_{m+i}, \quad \omega = p^m, \\ \widehat{h}_{i,j} = h_{m+i,j}, & \text{and} \quad \widehat{b}_{i,j} = b_{m+i,j}. \end{cases} \tag{7.1.21}$$

We will show that in dimensions below $p^2|\widehat{v}_1|$, E_{m+1}^2 is the $A(m+1)$-module generated by the set of chromatic fractions

$$\left\{ \frac{\widehat{v}_2^{e_2}}{p^{e_0}v_1^{e_1}} : e_0, e_1 > 0,\ e_2 \geq e_0 + e_1 - 1 \right\}, \tag{7.1.22}$$

and its Ext group in this range is

$$A(m+1)/I_2 \otimes E(\widehat{h}_{1,0}) \otimes P(\widehat{b}_{1,0}) \otimes \left\{ \frac{\widehat{v}_2^{e_2}}{pv_1} : e_2 \geq 1 \right\}, \tag{7.1.23}$$

where $\widehat{h}_{1,0} \in \mathrm{Ext}^{1,2(p\omega-1)}$ corresponds to the primitive $\widehat{t}_1 \in \Gamma(m+1)$, and $\widehat{b}_{1,0} \in \mathrm{Ext}^{1,2p(p\omega-1)}$ is its transpotent. In both cases there is no power of v_1 in the numerator when $m = 0$. These statements will be proved below as Theorem 7.2.6.

An Adams–Novikov differential for $T(m)$ originating in the 2-line would have to land in filtration $2p + 1$, which is trivial in the is range of dimensions, so by proving 7.2.6 we have determined $\pi_*(T(m))$ in this range.

Our first goal here is to compute Ext^0 and Ext^1. The following generalization of the Morava-Landweber theorem (4.3.2) is straightforward.

7.1.24. PROPOSITION.

$$\mathrm{Ext}^0_{\Gamma(m+1)}(BP_*/I_n) = A(n+m)/I_n.$$

For $n = 0$ each of the generators is a permanent cycle.

PROOF. The indicated elements are easily seen to be invariant in $\Gamma(m+1)$. In dimensions less that $|\widehat{v}_1|-1$, $T(m)$ is homotopy equivalent to BP, so the generators v_i for $i \le m$ are permanent cycles as claimed. □

Now we will describe a map from BP_* to a weak injective D^0_{m+1} inducing an isomorphism in Ext^0. D^0_{m+1} is the sub-$A(m)$-algebra of $p^{-1}BP_*$ generated by certain elements $\widehat{\lambda}_i$ for $i>0$ congruent to $\widehat{v}_i/p$ modulo decomposables.

To describe them we need to recall the formula of Hazewinkel [**4**] (see A2.2.1) relating polynomial generators $v_i \in BP_*$ to the coefficients ℓ_i of the formal group law, namely

(7.1.25)
$$p\ell_i = \sum_{0\le j<i} \ell_j v_{i-j}^{p^j} \qquad \text{for } i>0.$$

This recursive formula expands to

$$\begin{aligned} \ell_1 &= \frac{v_1}{p} \\ \ell_2 &= \frac{v_2}{p} + \frac{v_1^{p+1}}{p^2} \\ \ell_3 &= \frac{v_3}{p} + \frac{v_1 v_2^p}{p^2} + \frac{v_2 v_1^{p^2}}{p^2} + \frac{v_1^{1+p+p^2}}{p^3} \\ &\ \vdots \end{aligned}$$

We need to define reduced log coefficients $\widehat{\ell}_i$ for $i>0$ obtained from the ℓ_{m+i} by subtracting the terms which are monomials in the v_j for $j\le m$. Thus for $m>0$ we have

$$\begin{aligned} \widehat{\ell}_1 &= \frac{\widehat{v}_1}{p} \\ \widehat{\ell}_2 &= \frac{\widehat{v}_2}{p} + \frac{v_1 \widehat{v}_1^p}{p^2} + \frac{\widehat{v}_1 v_1^{p\omega}}{p^2} \\ &\ \vdots \end{aligned}$$

The analog of Hazewinkel's formula for these elements is

(7.1.26)
$$p\widehat{\ell}_i = \sum_{0\le j<i} \ell_j \widehat{v}_{i-j}^{p^j} + \sum_{0<j<\min(i,m+1)} \widehat{\ell}_{i-j} v_j^{p^{i-j}\omega}.$$

We use these to define our generators $\widehat{\lambda}_i$ recursively for $i>0$ by

(7.1.27)
$$\widehat{\lambda}_i = \widehat{\ell}_i - \sum_{0<j<i} \ell_j \widehat{\lambda}_{i-j}^{p^j}.$$

For $m=0$ we will denote these by λ_i.

The following is proved as Theorem 3.12 and equation (3.15) in Ravenel [**12**].

7.1.28. THEOREM. *The BP_*-module $D^0_{m+1} \subset p^{-1}BP_*$ described above is a subcomodule over $\Gamma(m+1)$ that is weak injective (7.1.5) with $\mathrm{Ext}^0 = A(m)$. In it we have*

$$\eta_R(\widehat{\lambda}_i) \equiv \widehat{\lambda}_i + \widehat{t}_i \bmod \textit{decomposables.}$$

Before proceeding further we need the following technical tool.

7.1.29. DEFINITION. *Let H be a graded connected torsion abelian p-group of finite type, and let H_i have order p^{h_i}. Then the* **Poincaré series** *for H is*

$$g(H) = \Sigma h_i t^i.$$

7.1.30. EXAMPLE. Let $I \subset BP_*$ be the maximal ideal so that $BP_*/I = \mathbf{Z}/(p)$. Then the Poincaré series for $\Gamma(m+1)/I$ is

$$G_m(t) = \prod_{i>0}(1 - t^{|v_{m+i}|})^{-1}.$$

We will abbreviate $t^{|v_{m+i}|}$ by x_i and denote x_1 simply by x. When $m > 0$ we will denote $t^{|v_i|}$ for $i \leq m$ by y_i and $t^{|v_1|}$ simply by y.

For Ext^1 we have

7.1.31. THEOREM. *Unless $m = 0$ and $p = 2$ (which is handled in (5.2.6)), $\mathrm{Ext}^1_{\Gamma(m+1)}(BP_*, BP_*)$ is the $A(m)$-module generated by the set*

$$\left\{\delta_0\left(\frac{\widehat{v}_1^j}{jp}\right) : j > 0\right\},$$

where δ_0 is the connecting homomorphism for the short exact sequence

$$0 \to BP_* \to M^0 \to N^1 \to 0$$

as in (5.1.5). Its Poincaré series is

$$g_m(t) \sum_{i>0} \frac{x^{p^{i-1}}}{1 - x^{p^{i-1}}},$$

where $x = t^{|v_{m+1}|}$. Moroever each of these elements is a permanent cycle.

PROOF. The Ext calculation follows from (6.5.11) and (7.1.3). For the Poincaré series, note that the set of $A(m)$-module generators of order p^i is

$$\left\{\delta_0\left(\frac{\widehat{v}_1^{jp^{i-1}}}{p^i}\right) : j > 0\right\},$$

and its Poincaré series is

$$\frac{x^{p^{i-1}}}{1 - x^{p^{i-1}}}.$$

To show that each of these elements is a permanent cycle, we will study the Bockstein spectral sequence converging to $\pi_*(T(m))$ with

$$E_1 = \mathbf{Z}/(p)[v_0] \otimes \pi_*(V(0) \wedge T(m)).$$

$V(0) \wedge T(m)$ is a ring spectrum in all cases except $m = 0$ and $p = 2$. We know that $T(m)$ is a ring spectrum for all m and p and that $V(0)$ is one for p odd. The case $p = 2$ and $m > 0$ is dealt with in Lemma 3.18 of Ravenel [**12**].

Low dimensional calculations reveal that $\widehat{v}_1 \in \mathrm{Ext}^0(BP_*/p)$ is a homotopy element. The elements $\widehat{\alpha}_j = \frac{\widehat{v}_1^j}{p}$ can then be constructed in the usual way using the self-map of $V(0) \wedge T(m)$ inducing multiplictation by $\widehat{v}_1^j$ followed by the pinch map

$$V(0) \wedge T(m) \to \Sigma T(m).$$

In the Bockstein spectral sequence it follows that $\widehat{v}_1^{sp^i}$ survives to E_{i+1}, so $\widehat{\alpha}_{sp^i}$ is divisible (as a homotopy element) by p^i. $\square$

Now we will recall some results about weak injective comodules M over a general Hopf algebroid (A, Γ) over $\mathbf{Z}_{(p)}$. In most cases we will refer to Ravenel [**12**] for the proofs. We will abbreviate $\mathrm{Ext}_\Gamma(A, M)$ by $\mathrm{Ext}(M)$.

The definition 7.1.5 of a weak injective should be compared with other notions of injectivity. A comodule I (or more generally an object in an abelian category) is *injective* if any homomorphism to it extends over monomorphisms, i.e., if one can always fill in the following diagram.

$$\begin{tikzcd} & I & \\ 0 \arrow[r] & M \arrow[u] \arrow[r, "i"] & N \arrow[ul, dashed] \end{tikzcd}$$

This definition is rather limiting. For example if A is a free $\mathbf{Z}_{(p)}$-module, then an injective I must be p-divisible since a homomorphism $A \to I$ must extend over $A \otimes \mathbf{Q}$.

There is also a notion of relative injectivity (A1.2.7) requiring I to be a summand of $\Gamma \otimes_A I$, which implies that the diagram above can always be completed when i is split over A. This implies weak injectivity as we have defined it here (see (A1.2.8)(b)), but we do not know if the converse is true. In any case the requirements of our definition can be said to hold only through a range of dimensions. The following is proved in Ravenel [**12**] as Lemma 2.1.

7.1.32. Lemma. *A connective comodule M over (A, Γ) is weak injective in a range of dimensions iff $\mathrm{Ext}^1(M) = 0$ in the same range.*

The following is proved in Ravenel [**12**] as Lemma 2.2.

7.1.33. Lemma. *Let*

$$(D, \Phi) \to (A, \Gamma) \to (A, \Sigma)$$

be an extension (A1.1.15) of graded connected Hopf algebroids of finite type, and suppose that M is a weak injective comodule over Γ. Then M is also weak injective over Σ, and $\mathrm{Ext}^0_\Sigma(A, M)$ is weak injective over Φ with

$$\mathrm{Ext}^0_\Phi(D, \mathrm{Ext}^0_\Sigma(A, M)) \cong \mathrm{Ext}^0_\Gamma(A, M).$$

Here is a method of recognizing weak injectives without computing any higher Ext groups. The following is proved in Ravenel [**12**] as Theorem 2.6.

7.1.34. Theorem. *Let (A, Γ) be a graded connected Hopf algebroid over $\mathbf{Z}_{(p)}$, and let M be a connected torsion Γ-comodule of finite type. Let $I \subset A$ be the maximal ideal (so that $A/I = \mathbf{Z}/(p)$). Then*

$$g(M) \leq g(\mathrm{Ext}^0(M)) g(\Gamma/I),$$

meaning that each coefficient of the power series on the left is dominated by the corresponding one on the right, with equality holding if and only if M is a weak injective (7.1.5).

It would be nice if for any comodule M one could find a map $M \to W$ to a weak injective inducing an isomorphism in Ext^0, but this is not always possible. In Ravenel [**12**, Example 2.8] we showed that it fails when $(A, \Gamma) = (A(1), G(1))$ and $M = A/(p^2)$.

For future reference will need the Poincaré series of $E^1_{m+1} = D^0_{m+1}/BP_*$. The following is proved as Lemma 3.16 in Ravenel [**12**].

7.1.35. LEMMA. *Let*

$$\begin{aligned} g_m(t) &= \prod_{1\le i\le m} \frac{1}{1-y_i} \\ \textit{and} \qquad G_m(t) &= \prod_{i>0} \frac{1}{1-x_i}, \end{aligned}$$

(with x_i and y_i as in 7.1.30) the series for $A(m)/(p)$ and $\Gamma(m+1)/I$ respectively. Then the Poincaré series for $E^1_{m+1} = D^0_{m+1}/BP_*$ is

$$g_m(t)G_m(t)\sum_{i>0}\frac{x_i}{1-x_i}.$$

2. The comodule E^2_{m+1}

In this section we will describe the comodule E^2_{m+1} needed above in (7.1.20) below dimension $p^2|\widehat{v}_1|$. This will determine $\pi_*(T(m))$ below dimension $p^2|\widehat{v}_1| - 3$. For $m = 0$ and p odd we can construct D^1_1 in all dimensions directly as follows.

7.2.1. LEMMA. *For p odd there is a map $E^1_1 \to D^1_1$ to a weak injective inducing an isomorphism in* Ext^0.

PROOF. $M^1 = v_1^{-1}E^1_1$ is not a weak injective for $m = 0$ since $\mathrm{Ext}^1_{\Gamma(1)}(M^1) = \mathbf{Q}/\mathbf{Z}$ concentrated in degree 0.

We will construct D^1_1 as a union of submodules of M^1 as follows. Let $K_0 = E^1_1 \subset M^1$. For each $i \ge 0$ we will construct a diagram

$$\begin{tikzcd} & L_{i+1} \arrow[r, equal] & L_{i+1} \\ K_i \arrow[r] & M^1 \arrow[r] \arrow[u] & L_i \arrow[u] \\ K_i \arrow[r] \arrow[u, equal] & K_{i+1} \arrow[r] \arrow[u] & L'_i \arrow[u] \end{tikzcd}$$

in which each row and column is exact. L'_i will be the sub-BP_*-module of $L_i = M^1/K_i$ generated by the positive dimensional part of $\mathrm{Ext}^0(L_i)$. It is a subcomodule of L_i, K_{i+1} is defined to be the induced extension by K_i, and $L_{i+1} = M^1/K_{i+1}$. Hence K_i, K_{i+1}, and L'_i are connective while L_i and L_{i+1} are not.

We know that in positive dimensions $K_0 = E^1_1$ has the same Ext^0 as M^1. We will show by induction that the same is true for each K_i. In the long exact sequence of Ext groups associated with the right column, the map $\mathrm{Ext}^0(L'_i) \to \mathrm{Ext}^0(L_i)$ is an isomorphism in positive dimensions, so the positive dimensional part of $\mathrm{Ext}^0(L_{i+1})$ is contained in $\mathrm{Ext}^1(L'_i)$, which has higher connectivity than $\mathrm{Ext}^0(L_i)$.

It follows that the connectivity of L'_i increases with i, and therefore the limit K_∞ has finite type. The connectivity of the positive dimensional part of $\mathrm{Ext}^0(L_i)$

also increases with i, so $\mathrm{Ext}^0(L_\infty)$ is trivial in positive dimensions. From the long exact sequence of Ext groups for the short exact sequence

$$0 \to K_\infty \to M^1 \to L_\infty \to 0$$

we deduce that $\mathrm{Ext}^1(K_\infty) = 0$, so K_∞ is a weak injective by Lemma 7.1.32. It has the same Ext^0 as E^1_1, so it is our D^1_1. $\square$

Now we are ready to study the hypothetical comodule E^2_{m+1} of (7.1.19) for $m > 0$.

7.2.2. LEMMA. *The Poincaré series for E^2_{m+1} is at least*

$$g_m(t)G_m(t)\sum_{i>0}\frac{x^{p^i}(1-y_i)}{(1-x^{p^i})(1-x_{i+1})}$$

(where $g_m(t)$ and $G_m(t)$ are as in Lemma 7.1.35), with equality holding for $m = 0$ and $p > 2$. In dimensions less than $p^2|\widehat{v}_1|$ it simplifies to

$$g_{m+2}(t)\left(\frac{x^p(1-y)}{(1-x_2)(1-x^p)}\right),$$

where x, y, x_i and y_i are as in 7.1.30.

We will see in Theorem 7.2.6 below that equality also holds in dimensions less than $p^2|\widehat{v}_1|$.

PROOF OF 7.2.2. The relevant Poincaré series (excluding the case $m = 0$ and $p = 2$) are

$$\begin{aligned} g(E^1_{m+1}) &= g_m(t)G_m(t)\sum_{i>0}\frac{x_i}{(1-x_i)} \qquad \text{by 7.1.35} \\ &= g_m(t)G_m(t)\left(\frac{x}{1-x}+\sum_{i>0}\frac{x_{i+1}}{1-x_{i+1}}\right), \\ \text{and}\quad g(\mathrm{Ext}^0(E^1_{m+1})) &= g(\mathrm{Ext}^1(BP_*)) \\ &= g_m(t)\sum_{i>0}\frac{x^{p^{i-1}}}{1-x^{p^{i-1}}} \qquad \text{by 7.1.31} \\ &= g_m(t)\left(\frac{x}{1-x}+\sum_{i>0}\frac{x^{p^i}}{1-x^{p^i}}\right). \end{aligned}$$

If there were a map $E^1_{m+1} \to D^1_{m+1}$ to a weak injective inducing an isomorphism in Ext^0 (which there is for $m = 0$ and p odd by 7.2.1), we would have

$$\begin{aligned} g(D^1_{m+1}) &= G_m(t)g(\mathrm{Ext}^0(E^1_{m+1})) \qquad \text{by 7.1.34} \\ &= G_m(t)g(\mathrm{Ext}^1(BP_*)) \\ &= g_m(t)G_m(t)\left(\frac{x}{1-x}+\sum_{i>0}\frac{x^{p^i}}{1-x^{p^i}}\right). \end{aligned}$$

It follows that

$$\begin{aligned} g(E^2_{m+1}) &\geq g_m(t)G_m(t)\left(\frac{x}{1-x}+\sum_{i>0}\frac{x^{p^i}}{1-x^{p^i}}\right)-g(E^1_{m+1}) \\ &= g_m(t)G_m(t)\sum_{i>0}\left(\frac{x^{p^i}}{1-x^{p^i}}-\frac{x_{i+1}}{1-x_{i+1}}\right) \\ &= g_m(t)G_m(t)\sum_{i>0}\frac{x^{p^i}(1-y_i)}{(1-x^{p^i})(1-x_{i+1})}. \end{aligned}$$

In our range of dimensions we can replace $g_m(t)G_m(t)$ by $g_{m+2}(t)$, and only the first term of the last sum is relevant. Hence we have

$$g(E^2_{m+1}) \equiv g_{m+2}(t)\left(\frac{x^p(1-y)}{(1-x_2)(1-x^p)}\right) \qquad \mod (t^{p^2|\widehat{v}_1|}). \quad \square$$

7.2.3. COROLLARY. *For a locally finite bounded below subcomodule*

$$E \subset E^1_{m+1}/(v_1^\infty),$$

let D denote the induced (as in (7.1.20)) extension by E^1_{m+1} shown in the following commutative diagram with exact rows.

$$\begin{tikzcd} 0 \arrow[r] & E^1_{m+1} \arrow[r] & v_1^{-1}E^1_{m+1} \arrow[r] & E^1_{m+1}/(v_1^\infty) \arrow[r] & 0 \\ 0 \arrow[r] & E^1_{m+1} \arrow[r] \arrow[u, equal] & D \arrow[r] \arrow[u] & E \arrow[r] \arrow[u] & 0 \end{tikzcd}$$

Let K denote the kernel of the connecting homomorphism

$$\delta : \mathrm{Ext}^0(E) \to \mathrm{Ext}^1(E^1_{m+1}) = \mathrm{Ext}^2(BP_*).$$

Then D is weak injective if and only if the Poincaré series $g(E)$ is $g(K)G_m(t)$ plus the series specified in Lemma 7.2.2. In particular it is weak injective if δ is a monomorphism and $g(E)$ is the specifed series.

PROOF. The specified series is $G_m(t)g(\mathrm{Ext}^0(E^1_{m+1}))-g(E^1_{m+1})$, and

$$g(\mathrm{Ext}^0(D)) = g(\mathrm{Ext}^0(E^1_{m+1}))+g(K).$$

Hence our hypothesis implies

$$\begin{aligned} g(D) &= g(E^1_{m+1})+g(E) \\ &= g(E^1_{m+1})+G_m(t)g(\mathrm{Ext}^0(E^1_{m+1}))-g(E^1_{m+1})+g(K)G_m(t) \\ &= G_m(t)(g(\mathrm{Ext}^0(E^1_{m+1})+g(K)) \\ &= G_m(t)g(\mathrm{Ext}^0(D)), \end{aligned}$$

which is equivalent to the weak injectivity of D by Theorem 7.1.34. $\square$

Now we need to identify some elements in $E^1_{m+1}/(v_1^\infty)$.

7.2.4. LEMMA. *The comodule $E^1_{m+1}/(v_1^\infty)$ contains the sets*

(a)

$$\begin{cases} \left\{\dfrac{1}{p^{1+e_0}v_1^{e_1-e_0}} : e_1 > e_0 \geq 0\right\} & \textit{for } m = 0 \\ \left\{\dfrac{\widehat{v}_1^{1+e_0}}{p^{1+e_0}v_1^{1+e_1}} : e_0, e_1 \geq 0\right\} & \textit{for } m > 0 \end{cases}$$

(b)

$$\left\{\frac{\widehat{v}_2^{1+e_0+e_1}}{p^{1+e_0}v_1^{1+e_1}} : e_0, e_1 \geq 0\right\}.$$

These generators will be discussed further in Theorem 7.2.6 below.

PROOF. (i) The element in question is the image of $v_1^{-1-e_1}\widehat{\lambda}_1^{1+e_0}$.
(ii) In D_{m+1}^0 we have

$$\begin{aligned} \widehat{\lambda}_2 &= \widehat{\ell}_2 - \ell_1\widehat{\lambda}_1^p \\ &= \frac{\widehat{v}_2}{p} - \frac{v_1\widehat{\lambda}_1^p}{p} + \begin{cases} \dfrac{v_1\widehat{v}_1^p}{p^2} + \dfrac{v_1^{p\omega}\widehat{v}_1}{p^2} & \text{for } m = 0 \\ \dfrac{v_1^{p+1}}{p^2} & \text{for } m > 0 \end{cases} \\ &= \frac{\widehat{v}_2}{p} + \frac{v_1}{p}(1 - p^{p-1})\widehat{\lambda}_1^p + \begin{cases} 0 & \text{for } m = 0 \\ v_1^{p\omega}\widehat{\lambda}_1 & \text{for } m > 0 \end{cases} \end{aligned}$$

so

$$\frac{\widehat{v}_2}{p} = \widehat{\lambda}_2 + \frac{v_1}{p}\mu \tag{7.2.5}$$

where

$$\mu = (1 - p^{p-1})\widehat{\lambda}_1^p + \begin{cases} 0 & \text{for } m = 0 \\ v_1^{\omega-1}\widehat{\lambda}_1 & \text{for } m > 0. \end{cases}$$

Hence in $p^{-1}v_1^{-1}BP_*$ we have

$$\begin{aligned} \frac{\widehat{v}_2^{1+e_0+e_1}}{p^{1+e_0}v_1^{1+e_1}} &= \frac{p^{e_1}}{v_1^{1+e_1}}\left(\frac{\widehat{v}_2}{p}\right)^{1+e_0+e_1} \\ &= \frac{p^{e_1}}{v_1^{1+e_1}}\left(\widehat{\lambda}_2 + \frac{v_1}{p}\mu\right)^{1+e_0+e_1} \\ &= \frac{p^{e_1}}{v_1^{1+e_1}}\sum_{k\geq 0}\binom{1+e_0+e_1}{k}\widehat{\lambda}_2^{1+e_0+e_1-k}\frac{v_1^k}{p^k}\mu^k \\ &= \sum_{k\geq 0}\binom{1+e_0+e_1}{k}\frac{p^{e_1-k}}{v_1^{1+e_1-k}}\widehat{\lambda}_2^{1+e_0+e_1-k}\mu^k. \end{aligned}$$

The image of this element in $p^{-1}BP_*/(v_1^\infty)$ is

$$\sum_{0\leq k\leq e_1}\binom{1+e_0+e_1}{k}\frac{p^{e_1-k}}{v_1^{1+e_1-k}}\widehat{\lambda}_2^{1+e_0+e_1-k}\mu^k.$$

The coefficient of each term is an integer, so the expression lies in $D_{m+1}^0/(v_1^\infty)$, and its image in $E_{m+1}^1/(v_1^\infty)$ is the desired element.

□

We will now construct a comodule $E^2_{m+1} \subset E^1_{m+1}/(v_1^\infty)$ satisfying the conditions of Corollary 7.2.3 with δ monomorphic below dimension $p^2|\widehat{v}_1|$.

7.2.6. THEOREM. *Let $E^2_{m+1} \subset E^1_{m+1}/(v_1^\infty)$ be the $A(m+2)$-module generated by the set*

$$\left\{ \frac{\widehat{v}_2^{1+e_0+e_1}}{p^{1+e_0}v_1^{1+e_1}} : e_0, e_1 \geq 0 \right\}.$$

Below dimension $p^2|\widehat{v}_1|$ it has the Poincaré series specified in Lemma 7.2.2, it is a comodule, it is 1-free, and its Ext group is

$$A(m+1)/I_2 \otimes E(\widehat{h}_{1,0}) \otimes P(\widehat{b}_{1,0}) \otimes \left\{ \frac{\widehat{v}_2^{e_2}}{pv_1} : e_2 \geq 1 \right\}.$$

In particular Ext^0 *maps monomorphically to* $\mathrm{Ext}^2(BP_*)$ *in that range.*

PROOF. Recall that the Poincaré series specified in Lemma 7.2.2 in this range is

$$g_{m+2}(t)\left(\frac{x^p(1-y)}{(1-x_2)(1-x^p)}\right) = g(BP_*/I_2)\frac{x^p}{(1-x_2)(1-x^p)}.$$

Each generator of E^2_{m+1} can be written as

$$x_{e_0,e_1} = \frac{\widehat{v}_2^{1+e_0+e_1}}{p^{1+e_0}v_1^{1+e_1}} = \frac{\widehat{v}_2}{pv_1}\left(\frac{\widehat{v}_2}{p}\right)^{e_0}\left(\frac{\widehat{v}_2}{v_1}\right)^{e_1}$$

with $e_0, e_1 \geq 0$. Since $|\frac{\widehat{v}_2}{pv_1}| = p|\widehat{v}_1|$, the Poincaré series for this set of generators is

$$\frac{x^p}{(1-x_2)(1-x^p)}.$$

We can filter E^2_{m+1} by defining F_i to be the submodule generated by the x_{e_0,e_1} with $e_0+e_1 \leq i$. Then each subquotient is a direct sum of suspensions of BP_*/I_2, so the Poincaré series is as claimed.

To see that E^2_{m+1} is a comodule, we will use the I-adic valuation as defined in the proof of Lemma 7.1.35. In our our range the set of elements with valuation at least -1 is the $A(m)$-submodule M generated by

$$\left\{ \frac{\widehat{v}_1^i\widehat{v}_2^j}{p^{1+e_0}v_1^{1+e_1}} : e_0, e_1 \geq 0,\ i+j \geq 1+e_0+e_1 \right\},$$

while E^2_{m+1} is generated by a similar set with $j \geq 1+e_0+e_1$. Thus it suffices to show that the decreasing filtration on M defined by letting F^kM be the submodule generated by all such generators with $j-e_0-e_1 \geq k$ is a comodule filtration. For this observe that modulo $\Gamma(m+1) \otimes F^{1+j-e_0-e_1}M$ we have

$$\frac{\eta_R(\widehat{v}_1^i\widehat{v}_2^j)}{p^{1+e_0}v_1^{1+e_1}} \equiv \frac{\widehat{v}_1^i(\widehat{v}_2+v_1\widehat{t}_1^p+p\widehat{t}_2)^j}{p^{1+e_0}v_1^{1+e_1}} \in \Gamma(m+1) \otimes F^{j-e_0-e_1}M,$$

so $E^2_{m+1} = F^1M$ is a subcomodule.

We use the same filtration for the Ext computation. Assuming that $j \geq 1 + e_0 + e_1 > 1$ we have

$$\begin{aligned}
\frac{\eta_R(\widehat{v}_1^i\widehat{v}_2^j) - \widehat{v}_1^i\widehat{v}_2^j}{p^{1+e_0}v_1^{1+e_1}} &\equiv \frac{\widehat{v}_1^i(\widehat{v}_2 + v_1\widehat{t}_1^p + p\widehat{t}_2)^j - \widehat{v}_1^i\widehat{v}_2^j}{p^{1+e_0}v_1^{1+e_1}} \\
&\equiv \binom{j}{e_0+e_1}\frac{\widehat{v}_1^i\widehat{v}_2^{j-e_0-e_1}(v_1\widehat{t}_1^p + p\widehat{t}_2)^{e_0+e_1}}{p^{1+e_0}v_1^{1+e_1}} + \dots \\
&\equiv (e_0, e_1, j-e_0-e_1)\frac{\widehat{v}_1^i\widehat{v}_2^{j-e_0-e_1}\widehat{t}_1^{pe_1}\widehat{t}_2^{e_0}}{pv_1} + \dots
\end{aligned}$$

where the missing terms involve higher powers of $\widehat{v}_2$. The multinomial coefficient $(e_0, e_1, j-e_0-e_1)$ is always nonzero since $j < p$. This means no linear combination of such elements is invariant, and the only invariant generators are the ones with $e_0 = e_1 = 0$, so Ext^0 is as claimed.

We will use this to show that E^2_{m+1} is 1-free (as defined in 7.1.8), i.e., that $T_m^{p-1} \otimes_{BP_*} E^2_{m+1}$ is weak injective in this range. For $0 \leq k \leq p-1$ we have

$$\frac{\psi(\widehat{v}_1^i\widehat{v}_2^j\widehat{t}_1^k) - \widehat{v}_1^i\widehat{v}_2^j\widehat{t}_1^k}{p^{1+e_0}v_1^{1+e_1}} = (e_0, e_1, j-e_0-e_1)\widehat{t}_1^{pe_1+k}\widehat{t}_2^{e_0} \otimes \frac{\widehat{v}_1^i\widehat{v}_2^{j-e_0-e_1}}{pv_1} + \dots.$$

This means that

$$\mathrm{Ext}^0(T_m^{p-1} \otimes_{BP_*} E^2_{m+1}) = \mathrm{Ext}^0(E^2_{m+1}).$$

It follows that

$$\begin{aligned}
g(\mathrm{Ext}^0) &= g_{m+1}(t)(1-y)\frac{x^p}{1-x_2} \\
\text{so} \qquad g(E^2_{m+1}) &= g(\mathrm{Ext}^0)\frac{1}{(1-x^p)(1-x_2)}, \\
\text{and} \qquad g(T_m^{p-1} \otimes_{BP_*} E^2_{m+1}) &= g(\mathrm{Ext}^0)\frac{1}{(1-x)(1-x_2)} \\
&= g(\mathrm{Ext}^0)G_m(t)
\end{aligned}$$

This makes $T_m^{p-1} \otimes_{BP_*} E^2_{m+1}$ weak injective in this range by Theorem 7.1.34.

We can use the small descent spectral sequence of Theorem 7.1.13 to pass from $\mathrm{Ext}(T_m^{p-1} \otimes_{BP_*} E^2_{m+1})$ to $\mathrm{Ext}(E^2_{m+1})$. It collapses from E_1 since the two comodules have the same Ext^0. This means that $\mathrm{Ext}(E^2_{m+1})$ is as claimed.

To show that $\mathrm{Ext}^0(E^2_{m+1})$ maps monomorphically to $\mathrm{Ext}^2(BP_*)$, the chromatic method tells us that $\mathrm{Ext}^2(BP_*)$ is a certain subquotient of $\mathrm{Ext}^0(M^2)$, namely the kernel of the map to $\mathrm{Ext}^0(M^3)$ modulo the image of the map from $\mathrm{Ext}^0(M^1)$. We know that the latter is the $A(m)$-module generated by the elements $\dfrac{\widehat{v}_1^i}{pi}$, and the elements in question, the $A(m+1)$ multiples of $\dfrac{\widehat{v}_2^i}{pv_1}$ are not in the image. The latter map trivially to $\mathrm{Ext}^0(M^3)$ because they involve no negative powers of v_2. $\square$

7.2.7. COROLLARY. *Excluding the case $(p,m) = (2,0)$, below dimension $p^2|\widehat{v}_1|$,*

$$\mathrm{Ext}^s_{\Gamma(m+1)} = \begin{cases} A(m) & \text{for } s = 0 \\ A(m)\left\{\dfrac{\widehat{v}_1^j}{pj} : j > 0\right\} & \text{for } s = 1 \\ \mathrm{Ext}^{s-2}_{\Gamma(m+1)}(E^2_{m+1}) & \text{for } s \geq 2. \end{cases}$$

The Adams–Novikov spectral sequence collapses with no nontrivial extensions in this range, so $\pi_*(T(m))$ has a similar description below dimension $p^2|\widehat{v}_1| - 3$.

The group $\mathrm{Ext}_{\Gamma(m+1)}(E^2_{m+1})$ was described in Theorem 7.2.6.

We now specialize to the case $m = 0$ and p odd. Using Lemma 7.2.1 we get the 4-term exact sequence

$$0 \to BP_* \to D^0_1 \to D^1_1 \to E^2_1 \to 0, \tag{7.2.8}$$

for which the resolution spectral sequence (A1.3.2) collapses from E_1.

We could get at $\mathrm{Ext}_{\Gamma(1)}(E^2_1)$ via the Cartan–Eilenberg spectral sequence for the extension

$$(A(1), G(1)) \to (BP_*, \Gamma(1)) \to (BP_*, \Gamma(2))$$

if we knew the value of $\mathrm{Ext}_{\Gamma(2)}(E^2_1)$ as a $G(1)$-comodule. For this we need to consider (7.2.8) as an exact sequence of $\Gamma(2)$-comodules and study the resulting resolution spectral sequence. By Lemma 7.1.33 we know that D^0_1 and D^1_1 are weak injectives over $\Gamma(2)$. It follows that the resolution spectral sequence collapses from E_2 and that the connecting homomorphism

$$\delta : \mathrm{Ext}^s_{\Gamma(2)}(E^2_1) \to \mathrm{Ext}^{s+1}_{\Gamma(2)}(E^1_1) = \mathrm{Ext}^{s+2}_{\Gamma(2)}$$

is an isomorphism for $s > 0$. This implies that

$$\mathrm{Ext}^s_{\Gamma(2)}(E^2_1) \cong \mathrm{Ext}^{s+2}_{\Gamma(2)},$$

which is described in our range by Theorem 7.2.6.

For $s = 0$, the situation is only slightly more complicated. Recall that the 4-term exact sequence (7.2.8) is the splice of two short exact sequences,

$$0 \to BP_* \to D^0_1 \to E^1_1 \to 0$$

(where $E^1_1 = D^0_1/BP_*$) and

$$0 \to E^1_1 \to D^1_1 \to E^2_1 \to 0,$$

Thus we have a short exact sequence

$$0 \to \mathrm{Ext}^0_{\Gamma(2)}(E^1_1) \to \mathrm{Ext}^0_{\Gamma(2)}(D^1_1) \to L \to 0 \tag{7.2.9}$$

and this L is the kernel of δ for $s = 0$. Thus there is a short exact sequence

$$0 \to L \to \mathrm{Ext}^0_{\Gamma(2)}(E^2_1) \to U \to 0, \tag{7.2.10}$$

where $U = \mathrm{Ext}^2_{\Gamma(2)}$, which is described in our range by Theorem 7.2.6.

7.2.11. THEOREM. *The comodule L of (7.2.9) is the $A(1)$-submodule $B \subset N^2$ generated by the set*

$$\left\{\frac{v_2^i}{ipv_1^i} : i > 0\right\}.$$

We will denote the element $\frac{v_2^i}{ipv_1^i}$ by $\beta'_{i/i}$. Theorem 7.2.11 implies

7.2.12. THEOREM. *In the resolution spectral sequence for (7.2.8) we have*

$$E_1^{0,s} = E_\infty^{0,s} = \begin{cases} \mathbf{Z}_{(p)} & \text{for } s = 0 \\ 0 & \text{for } s > 0, \end{cases}$$

$$E_2^{1,s} = E_\infty^{1,s} = \begin{cases} \mathrm{Ext}^1_{\Gamma(1)} & \text{for } s = 0 \\ 0 & \text{for } s > 0, \end{cases}$$

and for

$$E_1^{2,*} = \mathrm{Ext}_{\Gamma(1)}(E_1^2).$$

In the Cartan–Eilenberg spectral sequence (A1.3.14) for this group we have

$$\tilde{E}_2^{s,t} = \mathrm{Ext}^s_{G(1)}(\mathrm{Ext}^t_{\Gamma(2)}(E_2^1)).$$

For $t > 0$,

$$\mathrm{Ext}^s_{G(1)}(\mathrm{Ext}^t_{\Gamma(2)}(E_2^1)) = \mathrm{Ext}^s_{G(1)}(\mathrm{Ext}^{t+2}_{\Gamma(2)})$$

and for $t = 0$ there is long exact sequence

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Ext}^0_{G(1)}(B) \arrow[r] & \tilde{E}_2^{0,0} \arrow[r] & \mathrm{Ext}^0_{G(1)}(U) \arrow[dll, rounded corners] \\ & \mathrm{Ext}^1_{G(1)}(B) \arrow[r] & \tilde{E}_2^{1,0} \arrow[r] & \mathrm{Ext}^1_{G(1)}(U) \arrow[dll, rounded corners] \\ & \mathrm{Ext}^2_{G(1)}(B) \arrow[r] & \cdots & \end{tikzcd}$$

associated with the short exact sequence (7.2.10).

We will also need to consider the tensor product of (7.2.8) with $T_0^{(j)}$, and we will denote the resulting resolution spectral sequence by $\left\{E_r^{s,t}(T_0^{(j)})\right\}$. Let $\left\{\tilde{E}_r^{s,t}(T_0^{(j)})\right\}$ denote the Cartan–Eilenberg spectral sequence for $\mathrm{Ext}_{\Gamma(1)}(T_0^{(j)} \otimes E_1^2)$. For a $\Gamma(1)$-comodule M, we have

$$\mathrm{Ext}_{\Gamma(2)}(T_0^{(j)} \otimes_{BP_*} M) \cong \overline{T}_0^{(j)} \otimes_{A(1)} \mathrm{Ext}_{\Gamma(2)}(M),$$

where $T_0^{(j)} \subset \Gamma(1)$ and $\overline{T}_0^{(j)} \subset G(1)$, since $T_0^{(j)}$ is isomorphic over $\Gamma(2)$ to a direct sum of p^j suspensions of BP_*. It follows that we have a short exact sequence

$$0 \to \mathrm{Ext}^0_{\Gamma(2)}(T_0^{(j)} \otimes E_1^1) \to \mathrm{Ext}^0_{\Gamma(2)}(T_0^{(j)} \otimes D_1^1) \to \overline{T}_0^{(j)} \otimes B \to 0$$

and the long exact sequence of Theorem 7.2.12 generalizes to

$$
\begin{tikzcd}
0 \arrow[r] & \mathrm{Ext}^0_{G(1)}(\overline{T}_0^{(j)} \otimes B) \arrow[r] & \tilde{E}_2^{0,0}(T_0^{(j)}) \arrow[r] & \mathrm{Ext}^0_{G(1)}(\overline{T}_0^{(j)} \otimes U) \arrow[dll, rounded corners] \\
& \mathrm{Ext}^1_{G(1)}(\overline{T}_0^{(j)} \otimes B) \arrow[r] & \tilde{E}_2^{1,0}(T_0^{(j)}) \arrow[r] & \mathrm{Ext}^1_{G(1)}(\overline{T}_0^{(j)} \otimes U) \arrow[dll, rounded corners] \\
& \mathrm{Ext}^2_{G(1)}(\overline{T}_0^{(j)} \otimes B) \arrow[r] & \cdots &
\end{tikzcd}
\tag{7.2.13}
$$

The following is helpful in proving Theorem 7.2.11.

7.2.14. LEMMA. *Let $M \subset \mathrm{Ext}^0_{\Gamma(2)}(E_1^1/(v_1^\infty))$ be a $G(1)$-subcomodule with trivial image (under the connecting homomorphism) in*

$$\mathrm{Ext}^1_{\Gamma(2)}(E_1^1) = \mathrm{Ext}^2_{\Gamma(2)};$$

equivalently let

$$M \subset E/(v_1^\infty).$$

where $E = \mathrm{Ext}^0_{\Gamma(2)}(E_1^1)$. Then it is a subcomodule of $\mathrm{Ext}^0_{\Gamma(2)}(E_1^2)$ if it has a preimage

$$\tilde{M} \subset \mathrm{Ext}^0_{\Gamma(2)}(v_1^{-1}E_1^1) \subset v_1^{-1}E_1^1$$

that is obtained from E by adjoining elements divisible by the ideal $J = (\widehat{\lambda}_2, \widehat{\lambda}_3, \dots)$.

PROOF. We have a diagram with exact rows

$$
\begin{tikzcd}
0 \arrow[r] & E \arrow[r] \arrow[d, equal] & \tilde{M} \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & E \arrow[r] & v_1^{-1}E \arrow[r] & E/v_1^\infty \arrow[r] & 0
\end{tikzcd}
$$

We need to verify that the monomorphism

$$\mathrm{Ext}^0_{\Gamma(1)}(E_1^1) = \mathrm{Ext}^0_{G(1)}(E) \to \mathrm{Ext}^0_{G(1)}(\tilde{M})$$

is an isomorphism. If an element $x \in \tilde{M}$ is invariant, then some v_1-multiple of it must lie in $\mathrm{Ext}^0_{\Gamma(1)}(E_1^1)$, which has no elements divisible by J. Hence x has trivial image in M and therefore lies in in E, and we have our isomorphism.

Now consider the diagram

$$
\begin{tikzcd}
0 \arrow[r] & E \arrow[r] \arrow[d, equal] \arrow[dd, bend left] & \tilde{M} \arrow[r] \arrow[d, dotted] \arrow[dd, bend left] & M \arrow[r] \arrow[d, dotted] \arrow[dd, bend left] & 0 \\
0 \arrow[r] & E_1^1 \arrow[r] \arrow[d, equal] & D_1^1 \arrow[r] \arrow[d] & E_1^2 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & E_1^1 \arrow[r] & v_1^{-1}E_1^1 \arrow[r] \arrow[d] & E_1^1/(v_1^\infty) \arrow[r] \arrow[d] & 0 \\
& & v_1^{-1}E_1^1/D_1^1 \arrow[r, equal] & v_1^{-1}E_1^1/D_1^1 &
\end{tikzcd}
$$

We have shown that the map $\tilde{M} \to v_1^{-1}E_1^1/D_1^1$ is trivial in Ext^0, so it is trivial. It follows that $\tilde{M}$ maps to D_1^1, so M maps to E_1^2. $\square$

7.2.15. LEMMA. *Let L be as in (7.2.9). Then*

$$g(L) = \frac{1}{1-x}\sum_{i\geq 0}\frac{x^{p^{i+1}}(1-x^{p^i})}{(1-x^{p^{i+1}})(1-x_2^{p^i})},$$

where $x = t^{|v_1|}$ and $x_2 = t^{|v_2|}$.

PROOF. Since D_1^0 is weak injective, applying the functor $\mathrm{Ext}_{\Gamma(2)}$ to the short exact sequence

$$0 \to BP_* \to D_1^0 \to E_1^1 \to 0$$

yields a 4-term exact sequence

$$0 \to A(1) \to \mathrm{Ext}^0_{\Gamma(2)}(D_1^0) \to \mathrm{Ext}^0_{\Gamma(2)}(E_1^1) \to \mathrm{Ext}^1_{\Gamma(2)} \to 0$$

and hence a short exact sequence

$$0 \to \mathrm{Ext}^0_{\Gamma(2)}(D_1^0)/A(1) \to \mathrm{Ext}^0_{\Gamma(2)}(E_1^1) \to \mathrm{Ext}^1_{\Gamma(2)} \to 0,$$

where

$$\mathrm{Ext}^0_{\Gamma(2)}(D_1^0) = A(1)[p^{-1}v_1].$$

A calculation similar to that of Lemma 7.1.35 shows that

$$g(\mathrm{Ext}^0_{\Gamma(2)}(D_1^0)/A(1)) = \frac{x}{(1-x)^2}$$

so

$$\mathrm{Ext}^0_{\Gamma(2)}(E_1^1) = \frac{x}{1-x}\left(\frac{x}{1-x} + \sum_{i\geq 0}\frac{x_2^{p^i}}{1-x_2^{p^i}}\right). \tag{7.2.16}$$

Now consider the short exact sequence

$$0 \to \mathrm{Ext}^0_{\Gamma(2)}(E_1^1) \to \mathrm{Ext}^0_{\Gamma(2)}(D_1^1) \to L \to 0. \tag{7.2.17}$$

Since D_1^1 is weak injective over $\Gamma(1)$, Lemma 7.1.33 tells us that $\mathrm{Ext}^0_{\Gamma(2)}(D_1^1)$ is weak injective over $G(1,0)$ with

$$\mathrm{Ext}^0_{G(1,0)}(\mathrm{Ext}^0_{\Gamma(2)}(D_1^1)) = \mathrm{Ext}^0_{\Gamma(1)}(D_1^1) = \mathrm{Ext}^1_{\Gamma(1)}$$

so

$$g(\mathrm{Ext}^0_{\Gamma(2)}(D_1^1)) = \frac{x}{1-x}\sum_{i\geq 0}\frac{x^{p^i}}{1-x^{p^i}}. \tag{7.2.18}$$

Combining (7.2.16), (7.2.17), and (7.2.18) gives

$$\begin{aligned}
g(L) &= g(\mathrm{Ext}^0_{\Gamma(2)}(D_1^1)) - g(\mathrm{Ext}^0_{\Gamma(2)}(E_1^1)) \\
&= \frac{x}{1-x}\left(\sum_{i\geq 0}\frac{x^{p^i}}{1-x^{p^i}}\frac{x}{1-x} - \sum_{i\geq 0}\frac{x_2^{p^i}}{1-x_2^{p^i}}\right) \\
&= \frac{x}{1-x}\sum_{i\geq 0}\left(\frac{x^{p^{i+1}}}{1-x^{p^{i+1}}} - \frac{x_2^{p^i}}{1-x_2^{p^i}}\right) \\
&= \frac{x}{1-x}\sum_{i\geq 0}\frac{x^{p^{i+1}}(1-x_1^{p^i})}{(1-x^{p^{i+1}})(1-x_2^{p^i})}.
\end{aligned}$$

□

7.2.19. Lemma. *Let B be as in Theorem 7.2.11. Its Poincaré series is the same as the one for L, as given in Lemma 7.2.15.*

Proof. Let $F_kB \subset B$ denote the submodule of exponent p^k, with $B_0 = \phi$. Then we find that

$$F_kB = F_{k-1}B + A(1)\left\{\beta_{ip^{k-1}/ip^{k-1},k} \colon i > 0\right\}$$

so

$$F_kB/F_{k-1}B = A(1)/I_1\left\{\beta_{ip^{k-1}/ip^{k-1},k} \colon i > 0\right\},$$

and

$$\begin{aligned}
F_kB &= F_{k-1}B + g(F_kB/F_{k-1}B) \\
&= g(A(1)/I_2)\sum_{i>0}\frac{x^{ip^k} - x_2^{ip^{k-1}}}{1-x} \\
&= \frac{x}{1-x}\sum_{i>0}\left(x^{ip^k} - x_2^{ip^{k-1}}\right) \\
&= \frac{x}{1-x}\left(\frac{x^{p^k}}{1-x^{p^k}} - \frac{x_2^{p^{k-1}}}{1-x_2^{p^{k-1}}}\right) \\
&= \frac{x}{1-x}\frac{x^{p^k}(1-x^{p^{k-1}})}{(1-x^{p^k})(1-x_2^{p^{k-1}})}.
\end{aligned}$$

Summing this for $k \geq 1$ gives the desired Poincaré series of B. □

Proof of Theorem 7.2.11. We first show that B is a $G(1)$-comodule by showing that it is invariant over $\Gamma(2)$. In $\Gamma(2)$ we have

$$\eta_R(v_2) = v_2 + pt_2,$$

so for each $i > 0$, the elements

$$\frac{v_2^i}{ip} \in N^1 \qquad \text{and hence} \qquad \frac{v_2^i}{ipv_1^i} \in N^2$$

are invariant.

Next we show that $B \subset E_1^1/(v_1^\infty)$. Note that

$$
\begin{aligned}
v_1^{-1}v_2 &= pv_1^{-1}\widehat{\lambda}_2 + (1-p^{p-1})\widehat{\lambda}_1^p \\
&= pv_1^{-1}(\widehat{\lambda}_2 + \widehat{\lambda}_1 w) \\
&\qquad \text{where } w = (1-p^{p-1})\widehat{\lambda}_1^p \\
\text{so} \quad \beta'_{i/i} &= \frac{p^i(\widehat{\lambda}_2 + \widehat{\lambda}_1 w)^i}{ipv_1^i} = \frac{p^{i-1}(\widehat{\lambda}_2 + \widehat{\lambda}_1 w)^i}{iv_1^i}.
\end{aligned}
\tag{7.2.20}
$$

The coefficient p^{i-1}/i in this expression is always a p-local integer, so

$$\beta'_{i/i} \in E_1^1/(v_1^\infty).$$

Let

$$\tilde{\beta}'_{i/i} = \frac{v_1^{-i}v_2^i - w^i}{pi}.$$

Then we have

$$
\begin{aligned}
\tilde{\beta}'_{i/i} &= \frac{v_1^{-i}(p\widehat{\lambda}_2 + v_1 w)^i - w^i}{ip} \\
&= \sum_{j>0} \binom{i}{j} \frac{(pv_1^{-1}\widehat{\lambda}_2)^j w^{i-j}}{ip} \\
&\in v_1^{-1}E_1^1,
\end{aligned}
$$

so $\tilde{\beta}'_{i/i}$ is a lifting of $\beta'_{i/i}$ to $v_1^{-1}E_1^1$. Let $\tilde{B} \subset \mathrm{Ext}^0_{\Gamma(2)}(v_1^{-1}E_1^1)$ be the $A(1)$-submodule obtained by ajoining the elements $\tilde{\beta}'_{i/i}$ to $\mathrm{Ext}^0_{\Gamma(2)}(E_1^1)$; it projects to $B \subset E_1^1/(v_1^\infty)$. Since each $\tilde{\beta}'_{i/i}$ is divisible by $\widehat{\lambda}_2$, it follows from Lemma 7.2.14 that $B \subset E_1^2$.

B and L have the same Poincaré series by 7.2.15 and 7.2.19, so they are equal. □

3. The homotopy of $T(0)_{(2)}$ and $T(0)_{(1)}$

In this section we will determine the Adams–Novikov E_2-term

$$\mathrm{Ext}_{\Gamma(1)}(BP_*(T(0)_{(2)}))$$

and $\pi_*(T(0)_{(2)})$ in dimensions less than $(p^3+p)|v_1| - 3$. This is lower than the range of the previous section for reasons that will be explained below in Lemma 7.3.5. All assertions about Ext groups and related objects will apply only in that range.

We will then state a theorem (7.3.15) about differentials in the spectral sequence of (7.2.13) for $j = 1$, which we will prove in the next section.

Our starting point is the determination in Corollary 7.2.7 of $\pi_*(T(1))$ and its Adams–Novikov E_2-term through a larger range, roughly $p^2|v_2|$. There is an equivalence

$$T(1) \cong T(0)_{p^3+p^2-1},$$

so we could use the small descent spectral sequence of Theorem 7.1.13 and the topological small descent spectral sequence 7.1.16 (which turn out to be the same up to regrading) to get what we want. It turns out that we can finesse this by standard algebra.

Theorem 7.2.12 gives a Cartan–Eilenberg spectral sequence converging to $\mathrm{Ext}_{\Gamma(1)}$ whose E_2-term is expressed in terms of $\mathrm{Ext}_{G(1)}(B)$ and $\mathrm{Ext}_{G(1)}(\mathrm{Ext}^s_{\Gamma(2)})$ for $s \geq 2$.

First we have the following partial result about $\mathrm{Ext}_{G(1)}(B)$.

7.3.1. LEMMA. *For each $j > 0$, the $G(1)$-comodule B of Theorem 7.2.11 is j-free below dimension $p^j|v_2|$, and $\mathrm{Ext}^0_{G(1)}(T_0^{(j)} \otimes B)$ is additively isomorphic in this range to the $A(1)$-submodule of $E_1^1/(v_1^\infty)$ generated by the set*

$$\left\{\beta'_{i/\min(i,p^{j-1})} : i > 0\right\} \cup \left\{\beta_{i/p^j} : p^j \leq i < p^j + p^{j-1}\right\}.$$

In particular it is 2-free in our range of dimensions.

PROOF. We will denote the indicated group by $H^0(B)$. Given a $G(1)$-comodule M, let $M' = T_0^{(j)} \otimes_{A(1)} M$. According to Theorem 7.1.34, M is j-free (i.e. M' is weak injective) if

$$g(M') = \frac{g(\mathrm{Ext}^0(M'))}{1-x},$$

where as before $x = t^{|v_1|}$. We also know that

$$g(M') = g(M)\frac{1-x^{p^j}}{1-x},$$

so the condition for weak injectivity can be rewritten as

$$g(M) = \frac{g(\mathrm{Ext}^0(M'))}{1-x^{p^j}}.$$

Now in B we have

$$(7.3.2)\qquad \begin{cases} r_{kp^j}(\beta'_{i/i}) &= \dbinom{i}{kp^{j-1}}\dfrac{v_2^{i-kp^{j-1}}}{ipv_1^{i-kp^{j-1}}} = \dbinom{i-1}{kp^{j-1}}\beta'_{i-kp^{j-1}/i-kp^{j-1}} \\ r_{kp^j}(\beta_{i/i}) &= \dbinom{i}{kp^{j-1}}\beta_{i-kp^{j-1}/i-kp^{j-1}}. \end{cases}$$

For $p^{j-1} < i < p^j + p^{j-1}$, choose k so that $0 < i - kp^{j-1} \leq p^{j-1}$. Then the coefficients of β and β' above are units in every case except for $r_{kp^j}(\beta_{p^j/p^j})$. It follows that for each element in B, applying r_{kp^j} for some k will yield an element in $H^0(B)$. This means that in our range we have

$$g(B) = \frac{g(H^0(B))}{1-x^{p^j}},$$

so B is j-free if $H^0(B)$ is additively isomorphic to $\mathrm{Ext}^0(B')$.

Each element in $H^0(B)$ is killed by r_i for $i \geq p^j$, so there is a corresponding invariant element in $T_0^{(j)} \otimes B'$ by Lemma 7.1.9. On the other hand, (7.3.2) implies that no element in B' outside of $T_0^{(j)} \otimes H^0(B)$ is invariant, so $\mathrm{Ext}^0(B')$ is as desired. □

The groups $\mathrm{Ext}^s_{\Gamma(2)}$ for $s \geq 2$ in our range were determined in Theorem 7.2.6. Translated to the present context, it reads as follows.

7.3.3. THEOREM. *Below dimension* $p^2|v_2|$, *the group* $\mathrm{Ext}^{2+*}_{\Gamma(2)}$ *is*

$$E(h_{2,0}) \otimes P(b_{2,0}) \otimes U$$

(where $U = \mathrm{Ext}^2_{\Gamma(2)}$*), or more explicitly*

$$A(1)/I_2 \otimes E(h_{2,0}) \otimes P(b_{2,0}) \otimes \left\{\delta_0\delta_1\left(\frac{v_3^i v_2^j}{pv_1}\right) : i > 0,\ j \geq 0\right\},$$

where δ_0 *and* δ_1 *are the connecting homomorphisms for the short exact sequences*

$$0 \to BP_* \to M^0 \to N^1 \to 0$$

and

$$0 \to N^1 \to M^1 \to N^2 \to 0$$

respectively.

7.3.4. THEOREM. *For* $i, j \geq 0$, *let*

$$u_{i,j} = v_2^j\left(\frac{v_3^i}{i!pv_1} - \frac{v_2^{i+ip}}{c_{i,j}pv_1^{1+ip}}\right) \in N^2$$

where

$$c_{i,j} \quad = \quad \prod_{1\leq k\leq i}\binom{i+j+kp}{p}.$$

Then $u_{i,j}$ *has the following properties.*

(i) $u_{i,j}$ *lies in* $E_1^1/(v_1^\infty)$ *and is invariant over* $\Gamma(2)$, *i.e., it lies in*

$$\mathrm{Ext}^0_{\Gamma(2)}(E_1^1/(v_1^\infty)).$$

(ii) *Its image in* U *is that of*

$$\frac{v_3^i v_2^j}{i!pv_1}.$$

(iii) *For* $i > 0$

$$r_{p^2}(u_{i,j}) = u_{i-1,j+1},$$

where $u_{0,j} = 0$.

(iv) *For* $j \geq 0$,

$$r_p(u_{1,j}) = -\frac{j+1}{(p,j)}\beta_{j+p/p}.$$

We will denote $u_{1,j}$ by u_j. The coefficients $i!$, $c_{i,j}$ and (p,j) are always nonzero modulo p in our range except in the case

$$u_{p^2-p-1} = \frac{v_2^{p^2-p-1}v_3}{pv_1} - \frac{v_2^{p^2}}{p^2v_1^{p+1}}.$$

Proof of Theorem 7.3.4. (i) Recall (7.2.5) that

$$\frac{v_2}{p} = \lambda_2 + (1 - p^{p-1})\lambda_1^{p+1},$$

while the definition of λ_3 implies that

$$\frac{v_3}{p} \equiv \lambda_3 \bmod (v_1).$$

Hence

$$\begin{aligned} \frac{v_2^j v_3^i}{pv_1} &= \frac{p^{i+j-1}\lambda_2^j\lambda_3^i}{v_1} \\ &\in E_1^1/(v_1^\infty), \\ \text{and} \qquad \frac{v_2^{j+i+ip}}{pv_1^{1+ip}} &= p^{j+i+ip-1}\frac{(\lambda_2+(1-p^{p-1})\lambda_1^{p+1})^{j+i+ip}}{v_1^{1+ip}} \\ &\in E_1^1/(v_1^\infty), \end{aligned}$$

so $u_{i,j} \in E_1^1/(v_1^\infty)$.

The invariance of $u_{i,j}$ over $\Gamma(2)$ follows from the fact (Proposition 7.1.24) that v_2 is invariant modulo (p) and v_3 is invariant modulo I_2.

(ii) We will show that the difference between the two elements maps trivially to U. It is a scalar multiple of

$$e = \frac{v_2^{i+ip}}{pv_1^{1+ip}},$$

which is the image of

$$\frac{v_1^{-1-ip}v_2^{i+ip}}{p} \in M^1.$$

This is invariant over $\Gamma(2)$, so our element $e \in \operatorname{Ext}^0_{\Gamma(2)}(N^2)$ is in the image of $\operatorname{Ext}^0_{\Gamma(2)}(M^1)$, so it maps trivally to $\operatorname{Ext}^2_{\Gamma(2)} = U$.

(iii) Since

$$\begin{aligned} \eta_R(v_3) &\equiv v_3 + v_2t_1^{p^2} - v_2^p t_1 \quad \operatorname{mod} I_2 \\ \text{and} \qquad \eta_R(v_2) &\equiv v_2 + v_1t_1^p - v_1^p t_1 \quad \operatorname{mod} I_1, \end{aligned}$$

we have

$$\begin{aligned} \eta_R\left(\frac{v_2^j v_3^i}{i!pv_1}\right) &= \frac{v_2^j(v_3^i + iv_2v_3^{i-1}t_1^{p^2} + \dots)}{i!pv_1}, \\ \text{so} \qquad r_{p^2}\left(\frac{v_2^j v_3^i}{i!pv_1}\right) &= \frac{v_2^{j+1}v_3^{i-1}}{(i-1)!pv_1}. \end{aligned}$$

For the second term we have

$$\begin{aligned} \eta_R\left(\frac{v_2^{j+i+ip}}{c_{i,j}pv_1^{1+ip}}\right) &= \frac{(v_2+v_1t_1^p - v_1^p t_1)^{j+i+ip}}{c_{i,j}pv_1^{1+ip}} \\ &= \sum_{0\le k\le ip}\binom{j+i+ip}{k}\frac{v_2^{j+i+ip-k}(t_1^p - v_1^{p-1}t_1)^k}{c_{i,j}pv_1^{1+ip-k}} \\ &= \sum_{0\le k\le ip}\binom{j+i+ip}{k}\sum_{0\le \ell\le k}(-1)^\ell\binom{k}{\ell}\frac{v_2^{j+i+ip-k}t_1^{p(k-\ell)+\ell}}{c_{i,j}pv_1^{1+ip-k-(p-1)\ell}}. \end{aligned}$$

We need to collect the terms in which the exponent of t_1 is p^2, i.e. for which $(p-1)\ell = p(k-p)$. Hence $k-p$ must be divisible by $p-1$, so we can write

$k = p + (p-1)k'$ and $\ell = pk'$. This gives

$$\begin{aligned}
&r_{p^2}\left(\frac{v_2^{j+i+ip}}{c_{i,j}pv_1^{1+ip}}\right)\\
&\qquad = \sum_{0\le k'\le p} (-1)^{pk'}\binom{j+i+ip}{p+(p-1)k'}\binom{p+(p-1)k'}{pk'}\frac{v_2^{j+i+ip-p-(p-1)k'}}{c_{i,j}pv_1^{1+ip-p-(p^2-1)k'}}\\
&\qquad = \binom{j+i+ip}{p}\frac{v_2^{j+i+ip-p}}{c_{i,j}pv_1^{1+ip-p}}\\
&\qquad = \frac{v_2^{j+i+(i-1)p}}{c_{i-1,j+1}pv_1^{1+(i-1)p}}
\end{aligned}$$

and the result follows.

(iv) Using the methods of (iii), we find that

$$\begin{aligned}
r_p(u_{1,j}) &= r_p\left(\frac{v_2^j v_3}{pv_1}\right) - r_p\left(\frac{v_2^{j+p+1}}{c_{1,j}pv_1^{p+1}}\right)\\
&= -\binom{j+p+1}{1}\frac{v_2^{j+p}}{c_{1,j}pv_1^{p}}\\
&= -\frac{j+1}{(p,j)}\frac{v_2^{j+p}}{c_{1,j}pv_1^{p}}.
\end{aligned}$$

□

In order to use the Cartan–Eilenberg spectral sequence of (7.2.13) we need to know $\operatorname{Ext}_{G(1)}(T_0^{(j)} \otimes U)$. We will compute it by downward induction on j using the small descent spectral sequence of Theorem 7.1.13. Recall (Theorem 7.3.3) that U in our range is generated as an $A(1)$-module by the elements

$$\delta_0\delta_1(u_{i,j}) = \delta_0\delta_1\left(\frac{v_3^i v_2^j}{pv_1}\right).$$

We start with the following.

7.3.5. LEMMA. *Let $U = \operatorname{Ext}^2_{\Gamma(2)}$ as before. In dimensions less than $(p^3+p)|v_1|$, there is a short exact sequence of $G(1)$-comodules*

$$0 \to U \to U_0 \to U_1 \to 0 \tag{7.3.6}$$

where $U_0 \subset v_2^{-1}U$ is the $A(2)$-submodule generated by

$$\left\{\delta_0\delta_1\left(\frac{v_2^{1-i}v_3^i}{pv_1}\right) : i > 0\right\}.$$

U_0 and U_1 are each 2-free (7.1.8) as $G(1)$-comodules, and we have

$$\begin{aligned}
\operatorname{Ext}^0_{G(1)}(T_0^{(2)} \otimes U_0) &= A(1)\left\{\delta_0\delta_1(u_{1,j}) \colon j \ge 0\right\}\\
\text{and}\quad \operatorname{Ext}^0_{G(1)}(T_0^{(2)} \otimes U_1) &= A(1)\left\{\delta_0\delta_1\delta_2\left(\frac{u_{i,j}}{v_2}\right) : i \ge 2,\ j \ge 0\right\}
\end{aligned}$$

(where δ_2 is the connecting homomorphism for 7.3.6) so

$$\operatorname{Ext}^s_{G(1)}(T_0^{(2)} \otimes U) = \begin{cases} A(1)\left\{\delta_0\delta_1(u_{1,j})\colon j \geq 0\right\} & \text{for } s = 0 \\ A(1)\left\{\gamma_i\colon i \geq 2\right\} & \text{for } s = 1 \\ 0 & \text{for } s > 1, \end{cases}$$

where

$$\gamma_i = \delta_0\delta_1\delta_2\left(\frac{u_{i,0}}{v_2}\right).$$

Note that we have reduced our range of dimensions from $(p^3+p^2)|v_1|$ to $(p^3+p)|v_1|$. A 2-free subcomodule of M^2 containing U must contain the element

$$x = \frac{v_2^{1-p}v_3^p}{p^2 v_1},$$

and $|x| = (p^3+p)|v_1|$. $v_2^{p-1}x$ is in $\operatorname{Ext}^2_{\Gamma(2)}$, but is out of the range of Theorems 7.2.6 and 7.3.3.

Proof. We will construct the desired extension of $\operatorname{Ext}^2_{\Gamma(2)}$ by inducing from one of $\operatorname{Ext}^0_{\Gamma(2)}(E_1^2)$ as in the following diagram.

$$\begin{tikzcd} 0 \arrow[r] & \operatorname{Ext}^0_{\Gamma(2)}(E_1^2) \arrow[r] \arrow[d, "\delta_0\delta_1"] & U_0' \arrow[r] \arrow[d] & U_1 \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & U \arrow[r] & U_0 \arrow[r] & U_1 \arrow[r] & 0 \end{tikzcd}$$

We can extend the defintion of $u_{i,j}$ to negative j and we have $u_{i,j-i} = v_2^{j-1}u_{i,0}$ for $1 \leq j \leq i$. U_0' is the $A(1)$-submodule of $v_2^{-1}\operatorname{Ext}^0_{\Gamma(2)}(E_1^2)$ obtained by adjoining the elements

$$\left\{u_{i,j-i}\colon i > 0, 1 \leq j \leq i\right\}$$

Theorem 7.3.4(v) implies that U_0' and hence U_1 and U_0 are comodules.

It follows that $U_0 \subset v_2^{-1}\operatorname{Ext}^2_{\Gamma(2)}$ is as claimed. The 2-freeness of U_0 and U_1 follows from Theorem 7.3.4(iii).

For the computation of $\operatorname{Ext}^0_{G(1)}(T_0^{(2)} \otimes U_k)$ for $k = 0$ and 1, the following pictures for $p = 3$ may be helpful. We denote $\delta_0\delta_1(u_{i,j})$ by $u_{i,j}'$ and each diagonal arrow represents the action of r_{p^2}. For U_0 (which is v_2-torsion free) we have

$$\begin{tikzcd}
\vdots & \vdots & \vdots \\
u'_{1,0} & u'_{2,0} \arrow[ul] & u'_{3,0} \arrow[ul] \\
 & u'_{2,-1} \arrow[ul] & u'_{3,-1} \arrow[ul] \\
 & & u'_{3,-2} \arrow[ul]
\end{tikzcd}$$

where the missing elements have higher second subscripts. For U_1 (which is all v_2-torsion) we denote $\delta_0\delta_1\delta_2(v_2^{-k}u_{i,j})$ by $\frac{u'_{i,j}}{v_2^k}$, and the picture is

$$\begin{tikzcd}
\dfrac{u'_{1,0}}{v_2} & \dfrac{u'_{2,0}}{v_2} & \dfrac{u'_{3,0}}{v_2} \\
 & \dfrac{u'_{2,0}}{v_2^2} \arrow[ul] & \dfrac{u'_{3,0}}{v_2^2} \arrow[ul] \\
 & & \dfrac{u'_{3,0}}{v_2^3} \arrow[ul]
\end{tikzcd}$$

In each case Ext^0 is generated by the elements not supporting an arrow, i.e., the ones in the left column of the first picture and the top row of the second. $\square$

Now consider the Cartan–Eilenberg spectral sequence of (7.2.13) for $j = 2$. For $t > 2$, $\mathrm{Ext}^t_{\Gamma(2)}$ is a suspension of $U = \mathrm{Ext}^2_{\Gamma(2)}$, so $\tilde{E}_r^{s,t}(T_0^{p^2-1}) = 0$ for $s > 1$. More precisely for $t = \varepsilon + 2t'$ with $\varepsilon = 0$ or 1,

$$\mathrm{Ext}^{t+2}_{\Gamma(2)} = h_{2,0}^{\varepsilon} b_{2,0}^{t'} U,$$

which we abbreviate by $\Sigma^{(t)}U$. Then we have

7.3.7. Corollary. *In the resolution spectral sequence we have the following short exact sequences for the groups $E_2^{2,t}(T_0^{(2)})$: for $t = 0$*

$$0 \longrightarrow \mathrm{Ext}^0(T_0^{(2)} \otimes B) \longrightarrow E_2^{2,0}(T_0^{(2)}) \longrightarrow \mathrm{Ext}^0(T_0^{(2)} \otimes U) \longrightarrow 0,$$

and for $t > 0$

$$0 \longrightarrow \mathrm{Ext}^1(T_0^{(2)} \otimes \Sigma^{(t-1)}U) \longrightarrow E_2^{2,t}(T_0^{(2)}) \longrightarrow \mathrm{Ext}^0(T_0^{(2)} \otimes \Sigma^{(t)}U) \longrightarrow 0,$$

where $\Sigma^{(t)}U$ is as above and the Ext groups are over $G(1)$.

The groups $\mathrm{Ext}^0_{G(1)}(T_0^{(2)} \otimes B)$ and $\mathrm{Ext}_{G(1)}(T_0^{(2)} \otimes U)$ are described in Lemmas 7.3.1 and 7.3.5 respectively.

PROOF. The long exact sequence of (7.2.13) and Lemma 7.3.1 imply that

$$\tilde{E}_2^{s,t}(T_0^{(2)}) = \mathrm{Ext}^s_{G(1)}(T_0^{(2)} \otimes \Sigma^{(t)} U) \qquad \text{for } t > 0.$$

For $t = 0$ there is a short exact sequence

$$0 \longrightarrow \mathrm{Ext}^0_{G(1)}(T_0^{(2)} \otimes B) \longrightarrow \tilde{E}_2^{0,0}(T_0^{p^2-1}) \longrightarrow \mathrm{Ext}^0_{G(1)}(T_0^{(2)} \otimes U) \longrightarrow 0$$

and

$$\tilde{E}_2^{s,0}(T_0^{p^2-1}) = \begin{cases} \mathrm{Ext}^1_{G(1)}(T_0^{(2)} \otimes \Sigma^{(t)} U) & \text{for } s = 1 \\ 0 & \text{for } s > 1. \end{cases}$$

Since$\tilde{E}_2^{s,t}$ vanishes for $s > 1$, this spectral sequence collapses from E_2 and reduces to the indicated collection of short exact sequences for the groups $E_2^{2,*}(T_0^{p^2-1})$ in the resolution spectral sequence. □

7.3.8. COROLLARY. *The Adams–Novikov spectral sequence for $\pi_*(T(0)_{(2)})$ collapses in our range of dimensions, i.e., below dimension $(p^2)|v_2| - 3$.*

PROOF. This will follow by a spareness argument if we can show that in this range $E_2^{s,*}$ (for the Adams–Novikov spectral sequence) vanishes for $s < 2p+1$. We can rull out differentials originating in filtrations 0 or 1 by the usual arguments, and by sparesness the each nontrivial differential d_r has $r \equiv 1 \bmod 2p-2$. Thus the shortest possible one is d_{2p-1}, for which the filtration of the target would be too high.

For the vanishing statement the first element in filtration $2p+1$ is $u_1 b_{2,0}^{p-1} h_{2,0}$, and we have

$$\begin{aligned} |u_1| = |b_{2,0}| &= p|v_2| - 2 \\ &= p(2p^2-2) - 2 = 2p^3 - 2p - 2 \\ \text{and} \qquad |h_{2,0}| &= |v_2| - 1 = 2p^2 - 3 \\ \text{so} \qquad |u_1 b_{2,0}^{p-1} h_{2,0}| &= p(2p^3 - 2p - 2) + 2p^2 - 3 \\ &= 2p^4 - 2p - 3 \\ &> p^2|v_2| - 3. \end{aligned}$$

□

Now we will analyze the Cartan–Eilenberg spectral sequence of (7.2.13) for $j = 1$. It has a rich pattern of differentials. This (in slightly different language) was the subject of Ravenel [**11**]. In order to use this spectral sequence we need to know $\mathrm{Ext}_{G(1)}(T_0^{(1)} \otimes B)$ and $\mathrm{Ext}_{G(1)}(T_0^{(1)} \otimes U)$. We will derive these from the corresponding Ext groups for $T_0^{(2)}$ given in Lemmas 7.3.1 and 7.3.5 using the the small descent spectral sequence of Theorem 7.1.13.

The former collapses from E_2 since $\mathrm{Ext}_{G(1)}(T_0^{(2)} \otimes B)$ is concentrated in degree 0. The action of r_p on $\mathrm{Ext}^0_{G(1)}(T_0^{(2)} \otimes B)$ is given by

$$\begin{aligned} r_p\left(\beta'_{i/e_1}\right) &= \beta_{i-1/e_1-1} \\ \text{and} \qquad r_p\left(\beta_{pi/e_1}\right) &= 0. \end{aligned}$$

In order to understand this, the following picture for $p = 3$ may be helpful.

$$\begin{tikzcd}[column sep=small]
\beta_1 & \beta_2 & \beta'_3 & \beta_3 & & \\
& \beta_{2/2} \arrow[ul] & \beta'_{3/2} \arrow[ul] & \beta_{3/2} & \beta_{4/2} \arrow[ul] & \\
& & \beta'_{3/3} \arrow[ul] & \beta_{3/3} & \beta_{4/3} \arrow[ul] & \beta_{5/3} \arrow[ul]
\end{tikzcd} \tag{7.3.9}$$

Each arrow represents the action of r_p up to unit scalar. Thought of as a graph, this picture has $2p$ components, two of which have maximal size. Each component corresponds to an $A(m)$-summand of our E_2-term, with the caveat that $p\beta'_{p/e_1} = \beta_{p/e_1}$ and $v_1\beta'_{i/e} = \beta'_{i/e-1}$.

In the summand containing β_1, the subset of E_1

$$\left\{\beta_1,\, \beta_{2/2},\, \beta'_{3/3}\right\} \otimes E(h_{1,1}) \otimes P(b_{1,1})$$

reduces on passage to E_2 to simply $\{\beta_1\}$. Similarly

$$\left\{\beta_2,\, \beta'_{3/2}\right\} \otimes E(h_{1,1}) \otimes P(b_{1,1})$$

reduces to

$$\left\{\beta_2,\, \beta'_{3/2}h_{1,1}\right\} \otimes P(b_{1,1}),$$

where

$$\begin{aligned} \beta'_{3/2}h_{1,1} &= \langle h_{1,1},\, h_{1,1},\, \beta_2\rangle \\ \text{and} \qquad h_{1,1}(\beta'_{3/2}h_{1,1}) &= h_{1,1}\langle h_{1,1},\, h_{1,1},\, \beta_2\rangle \\ &= \langle h_{1,1},\, h_{1,1},\, h_{1,1}\rangle\beta_2 \\ &= b_{1,1}\beta_2. \end{aligned}$$

The entire configuration is v_2^p-periodic. This leads to the following.

7.3.10. PROPOSITION. *In dimensions less than* $p^2|v_2|$, $\mathrm{Ext}_{G(1)}(T_0^{(1)} \otimes B)$ *has* $\mathbf{Z}/(p)$ *basis*

$$\left\{\beta_{1+pi}, \beta_{p+pi};\, \beta_{p^2/p^2-p+1}, \ldots, \beta_{p^2/p+1}\right\}$$
$$\bigoplus$$
$$P(b_{1,1}) \otimes \left\{ \begin{array}{c} \left\{\beta'_{2+pi}, \ldots, \beta'_{p+pi};\, \beta_{p+pi/p}, \ldots, \beta_{p+pi/2};\, \beta_{p^2/p^2}, \ldots, \beta_{p^2/p^2-p+2}\right\} \\ \oplus \\ h_{1,1}\left\{\beta'_{p+pi/p-1}, \ldots, \beta'_{p+pi};\, \beta_{p+pi/p}, \ldots, \beta_{2p-1+pi/p};\right. \\ \hfill \left.\beta_{p^2/p^2}, \ldots, \beta_{p^2+p-2/p^2}\right\}, \end{array} \right.$$

where $0 \le i < p$*, subject to the caveat that* $v_1\beta_{p/e} = \beta_{p/e-1}$ *and* $p\beta'_{p/e} = \beta_{p/e}$*. In particular* $\mathrm{Ext}^0_{G(1)}(T_0^{(1)} \otimes B)$ *has basis*

$$\left\{\beta'_{1+pi}, \ldots, \beta'_{p+pi};\, \beta_{p+pi/p}, \ldots, \beta_{p+pi/1};\, \beta_{p^2/p^2}, \ldots, \beta_{p^2/p+1}\right\}.$$

The action of r_p on U is trivial, so $E_1 = E_2$ in the small descent spectral sequence for $\mathrm{Ext}_{G(1)}(T_0^{(1)} \otimes U)$. In theory there could be a nontrivial differential

$$d_2 : E_2^{s,1} \to E_2^{s+2,0},$$

but this cannot happen since $E_2^{*,1}$ is v_2-torsion while $E_2^{*,0}$ is v_2-torsion free. Hence the spectral sequence collapses and we have

$$\mathrm{Ext}_{G(1)}(T_0^{(1)} \otimes U) = E(h_{1,1}) \otimes P(b_{1,1}) \otimes \mathrm{Ext}_{G(1)}(T_0^{(2)} \otimes U), \tag{7.3.11}$$

where $\mathrm{Ext}_{G(1)}(T_0^{(2)} \otimes U)$ is as in Lemma 7.3.5.

We now have the ingredients needed to study the Cartan–Eilenberg spectral sequence $\left\{\tilde{E}_r^{s,t}(T_0^{(1)})\right\}$ of (7.2.13). We first need to analyze the connecting homomorphism δ in the long exact sequence for $t = 0$. Since the target groups of it are v_2-torsion free, δ is trivial on the v_2-torsion module

$$E(h_{1,1}) \otimes P(b_{1,1}) \otimes \mathrm{Ext}^1_{G(1)}(T_0^{(2)} \otimes U).$$

For its behavior on

$$E(h_{1,1}) \otimes P(b_{1,1}) \otimes \mathrm{Ext}^0_{G(1)}(T_0^{(2)} \otimes U)$$

we have

7.3.12. LEMMA. *In the long exact sequence of (7.2.13) for* $j = 1$ *we have (up to unit scalar)*

$$\begin{aligned} \delta^{2k}(b_{1,1}^k u_i) &= (i+1)h_{1,1}b_{1,1}^k\beta_{i+p/p}, \\ \textit{and} \qquad \delta^{2k+1}(h_{1,1}b_{1,1}^k u_i) &= \binom{i+1}{p-1} b_{1,1}^{k+1}\beta_{i+2/2} \end{aligned}$$

for all $i, k \ge 0$*.*

This means that $\tilde{E}_2^{*,0}(T_0^{(1)})$ looks like the Ext group one would would have if the picture of (7.3.9) were replaced by

(7.3.13)

$$\begin{tikzcd}[row sep=large, column sep=small]
\beta_1 & \beta_2 & \beta'_3 & \beta_3 & & & \\
& \beta_{2/2} \arrow[ul] & \beta'_{3/2} \arrow[ul] & \beta_{3/2} & \beta_{4/2} \arrow[ul] & & \\
& & \beta'_{3/3} \arrow[ul] & \beta_{3/3} & \beta_{4/3} \arrow[ul] & \beta_{5/3} \arrow[ul] & \\
& & & & u_0 \arrow[ul] & u_1 \arrow[ul] & u_2.
\end{tikzcd}$$

The graph now has $2p + 1$ instead of $2p$ components, three of which are maximal.

Proof of Lemma 7.3.12. It suffices to show that $r_p(u_i)$ is as indicated in the picture above. We have (using Theorem 7.3.4)

$$u_i = u_{1,i} = v_2^i\left(\frac{v_3}{pv_1} - \frac{v_2^{p+1}}{pc_{1,i}v_1^{p+1}}\right)$$

so

$$r_p(u_i) = -(i+p+1)\frac{v_2^{i+p}}{pc_{1,i}v_1^p} = -\frac{i+1}{(p,i)}\beta_{i+p/p}.$$

□

7.3.14. COROLLARY. *In the Cartan–Eilenberg spectral sequence of (7.2.13), $\tilde{E}_2(T_0^{(1)})$ has $\mathbf{Z}/(p)$-basis*

$$\left\{\beta_{1+pi}, \beta_{p+pi}, \beta_{p+pi/2};\ \beta_{p^2/p^2-p+1}, \dots, \beta_{p^2/p+1}\right\}$$
$$\bigoplus$$
$$P(b_{1,1})\otimes\left\{\begin{array}{c}\begin{array}{l}\{\beta'_{2+pi},\dots,\beta'_{p+pi};\ \beta_{p+pi/p},\dots,\beta_{p+pi/2};\\ \qquad u_{pi+p-1};\ \beta_{p^2/p^2},\dots,\beta_{p^2/p^2-p+2}\}\end{array}\\ \oplus \\ h_{1,1}\left\{\begin{array}{l}\beta'_{p+pi/p-1},\dots,\beta'_{p+pi};\ u_{pi},\dots,u_{pi+p-2};\\ \qquad \beta_{p^2/p^2},\dots,\beta_{p^2+p-2/p^2}\end{array}\right\}\end{array}\right.$$
$$\bigoplus$$
$$E(h_{1,1})\otimes P(b_{1,1},b_{2,0})\otimes\{h_{2,0}u_j,\ b_{2,0}u_j : j\ge 0\}$$
$$\bigoplus$$
$$E(h_{1,1},\ h_{2,0})\otimes P(b_{1,1},b_{2,0})\otimes\{\gamma_2,\gamma_3,\dots\},$$

where $0 \le i < p$, (omitting unnecessary subscripts)

$$u,\ v,\ \beta, \in \tilde{E}_2^{0,0} \qquad \text{and} \qquad \gamma\in\tilde{E}_2^{0,1},$$

and the operators $h_{i,j}$, $b_{i,j}$, etc. behave as if they had the following bidegrees.

$$\begin{aligned} h_{2,0} &\in \tilde{E}_2^{0,1}, & h_{1,1} &\in \tilde{E}_2^{1,0},\\ b_{2,0} &\in \tilde{E}_2^{0,2}, \quad \text{and} & b_{1,1} &\in \tilde{E}_2^{2,0}.\end{aligned}$$

Now we need to study higher differentials.

7.3.15. THEOREM. *The Cartan–Eilenberg spectral sequence of (7.2.13) for $j = 1$ has the following differentials and no others in dimensions less than $(p^3+p)|v_1|$.*

(i)

$$d_2(h_{2,0}u_i) = b_{1,1}\beta'_{i+2}.$$

(ii)

$$d_3(h_{2,0}^{\varepsilon}b_{2,0}^k u_i) = (k+i+1)h_{1,1}h_{2,0}^{\varepsilon}b_{1,1}b_{2,0}^{k-1}u_i \qquad \text{for } k>0 \text{ and } \varepsilon = 0 \text{ or } 1.$$

(iii)

$$d_{2k+2}(h_{1,1}h_{2,0}b_{2,0}^k u_{pi'-2-k}) = h_{1,1}b_{1,1}^{k+1}\beta'_{pi'/k+1} \qquad \text{for } k<p-1.$$

(iv)

$$d_{2k+1}(h_{1,1}b_{2,0}^k u_{pi'-2-k}) = b_{1,1}^{k+1}\beta_{pi'/k+2} \qquad \text{for } k>0.$$

(v)
$$d_3(h_{2,0}^{\varepsilon}b_{2,0}^{k}\gamma_j) = kh_{1,1}h_{2,0}^{\varepsilon}b_{1,1}b_{2,0}^{k-1}\gamma_j.$$

We will prove Theorem 7.3.15 in the next section. For a more explicit description of the resulting Ext group, see Theorem 7.5.1. An illustration of it for $p = 5$ can be found in Figure 7.3.17. There are no Adams–Novikov differentials in this range. In the figure

- Ext^0 and Ext^1 are not shown.
- Short vertical and horizontal lines indicate multiplication by p and v_1.
- Diagonal lines indicate multiplication by $h_{1,1}$, $h_{2,0}$ and the Massey product operations $\underline{5i}$ of 7.4.12.

Now that we have computed $\mathrm{Ext}_{\Gamma(1)}(T_0^{(1)} \otimes E_1^2)$, it is a simple matter to get to $\mathrm{Ext}_{\Gamma(1)}(T_0^{(1)})$ itself. We have the 4-term exact sequence

$$0 \to T_0^{(1)} \to T_0^{(1)} \otimes D_1^0 \to T_0^{(1)} \otimes D_1^1 \to T_0^{(1)} \otimes E_1^2 \to 0 \qquad (7.3.16)$$

in which the two middle terms are weak injectives by Lemma 7.1.10 with

$$\begin{aligned}\mathrm{Ext}^0_{\Gamma(1)}(T_0^{(1)} \otimes D_1^i) &\cong \mathbf{Z}_{(p)}\left\{t_1^j\colon 0 \le j < p\right\} \otimes \mathrm{Ext}^0_{\Gamma(1)}(D_1^i)\\ &\cong \mathbf{Z}_{(p)}\left\{t_1^j\colon 0 \le j < p\right\} \otimes \mathrm{Ext}^i_{\Gamma(1)}(BP_*).\end{aligned}$$

We will compute Ext^0 of the middle map of (7.3.16) using the description of the groups given in 7.1.10. Recall that D_1^0 contains all powers of $\lambda_1 = p^{-1}v_1$. Then $\mathrm{Ext}^0_{\Gamma(1)}(T_0^{(1)} \otimes D_1^0)$ is the free $\mathbf{Z}_{(p)}$-module on the set $\{z_j\colon 0 \le j < p\}$ where

$$z_j = \sum_{0 \le k \le j} (-1)^k \binom{j}{k} t_1^k \otimes \lambda_1^{j-k} = t_1^j \otimes 1 + \dots.$$

The image of

$$\begin{aligned}p^t z_j &= \sum_{0 \le k \le j} (-1)^k p^t \binom{j}{k} t_1^k \otimes \lambda_1^{j-k}\\ &= \sum_{0 \le k \le j} (-1)^k \binom{j}{k} t_1^k \otimes p^{t+k-j} v_1^{j-k}\end{aligned}$$

in $\mathrm{Ext}^0_{\Gamma(1)}(T_0^{(1)} \otimes D_1^1)$ is

$$\begin{aligned}\sum_{0 \le k \le j} (-1)^k \binom{j}{k} t_1^k \otimes \frac{v_1^{j-k}}{p^{j-k-t}} &= (-1)^{j-1-t}\binom{j}{t+1} t_1^{j-1-t} \otimes \alpha_{t+1} + \dots\\ &= 0 \qquad \text{if } t \ge j.\end{aligned}$$

From this we deduce that

$$\mathrm{Ext}^0(T_0^{(1)}) = \mathbf{Z}_{(p)}\left\{p^j z_j\colon 0 \le j < p\right\},$$

and Ext^0 of the third map of (7.3.16) sends

$$t_1^{(1)} \otimes \alpha_1 + \dots \mapsto 1 \otimes \beta_1.$$

Thus the map

$$\mathrm{Ext}^0(T_0^{(1)} \otimes E_1^2) \to \mathrm{Ext}^2(T_0^{(1)})$$

has a kernel, namely the $\mathbf{Z}_{(p)}$-summand generated by β_1, and for $s > 2$,

$$\mathrm{Ext}^s_{\Gamma(1)}(T_0^{(1)}) \cong \mathrm{Ext}^{s-2}_{\Gamma(1)}(T_0^{(1)} \otimes E_1^2),$$

which can be read off from Theorem 7.3.15.

4. The proof of Theorem 7.3.15

Recall that our range of dimensions is now $(p^3 + p)|v_1|$.

It is easy to see that all of the elements in Corollary 7.3.14 save those involving u_j or $b_{2,0}$ are permanent cycles. Establishing the indicated differentials will ultimately be reduced to computing Ext groups for certain comodules over the Hopf algebra

$$P(1)_* = \mathbf{Z}/(p)[c(t_1), c(t_2)]/(c(t_1^{p^2}), c(t_2)^p)$$

with coproduct inherited from that of $BP_*(BP)$, i.e., with

$$\begin{aligned} \Delta(c(t_1)) &= c(t_1) \otimes 1 + 1 \otimes c(t_1) \\ \text{and} \qquad \Delta(c(t_2)) &= c(t_2) \otimes 1 + c(t_1)^p \otimes c(t_1) + 1 \otimes c(t_2). \end{aligned}$$

It is dual to the subalgebra $P(1)$ of the Steenrod algebra generated by the reduced power operations P^1 and P^p. For a $P(1)_*$-comodule M, we will abbreviate $\mathrm{Ext}_{P(1)_*}(\mathbf{Z}/(p), M)$ by $\mathrm{Ext}_{P(1)_*}(M)$, or, when $M = \mathbf{Z}/(p)$, by simply $\mathrm{Ext}_{P(1)_*}$

In principle one could get at $\mathrm{Ext}_{\Gamma(1)}(T_0^{(1)} \otimes E_1^2)$ in our range of dimensions (i.e., below dimension $p^3|v_1|$) by finding $\mathrm{Ext}_{\Gamma(3)}(T_0^{(1)} \otimes E_1^2)$ and using the Cartan–Eilenberg spectral sequence for the extension

$$G(1,1) \to \Gamma(1) \to \Gamma(3).$$

(Recall that $G(1,1) = A(2)[t_1, t_2]$.)

Consider our 4-term exact sequence

$$0 \to BP_* \to D_1^0 \to D_1^1 \to E_1^2 \to 0$$

The two middle terms are weak injective over $\Gamma(1)$ and hence over $\Gamma(3)$. For the last term we have,

$$\mathrm{Ext}^s_{\Gamma(3)}(E_1^2) = \mathrm{Ext}^{s+2}_{\Gamma(3)} \qquad \text{for } s > 0.$$

The first generator for $s = 1$ is $\frac{v_4 t_3}{p v_1}$, which is out of our range. This means that the fourth term is also weak injective over $\Gamma(3)$ in our range.

For a $\Gamma(1)$-comodule M, we will denote the $G(1,1)$-comodule $\mathrm{Ext}^0_{\Gamma(3)}(M)$ by $\tilde{M}$. Applying $\mathrm{Ext}^0_{\Gamma(3)}(\cdot)$ to our 4-term exact sequence yields a 4-term exact (in our range) sequence of $G(1,1)$-comodules

$$0 \to A(2) \to \tilde{D}_1^0 \to \tilde{D}_1^1 \to \tilde{E}_1^2 \to 0.$$

Let $\tilde{D}_1^2$ be the $A(2)$-submodule of $\tilde{M}^2$ (where M^2 is the chromatic comodule) obtained by adjoining the elements

$$\left\{ \frac{v_2^{-j} v_3^k}{p v_1^i} : i, j > 0,\, k \geq i + j \right\}$$

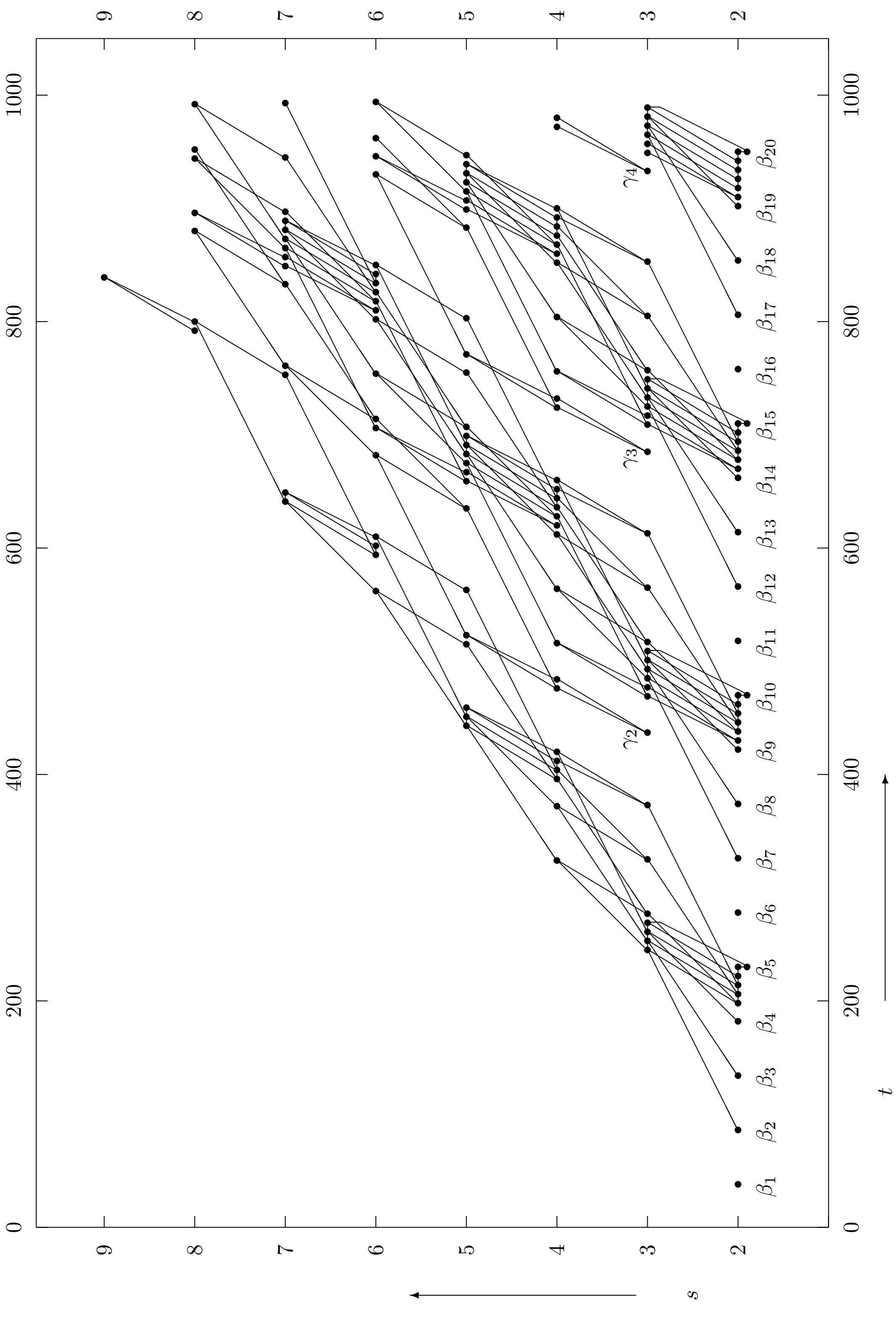


FIGURE 7.3.17. $\mathrm{Ext}(T_0^{(1)})$ for $p = 5$ in dimensions below 998.

to $\tilde{E}_1^2$, so we have a short exact sequence of $G(1,1)$-comodules

$$0 \to \tilde{E}_1^2 \to \tilde{D}_1^2 \to \tilde{E}_1^3 \to 0, \tag{7.4.1}$$

where $\tilde{E}_1^3$ is the $A(2)$-submodule of $\tilde{N}^3$ generated by

$$\left\{\frac{v_3^{2+e_1+e_2+e_3}}{pv_1^{1+e_1}v_2^{1+e}} : e_1, e_2, e_3 \geq 0\right\}.$$

Its Poincaré series is

$$g(\tilde{E}_1^3) = \frac{x^{2p^2+p}}{(1-x^{p^2})(1-x_2^p)(1-x_3)}. \tag{7.4.2}$$

7.4.3. Definition. *Let P be the left $G(1,1)$-comodule*

$$\begin{aligned} P &= A(2)\left\{c(t_1^i t_2^j) \colon 0 \leq i, pj < p^2\right\} \\ &= A(2)\left\{t_1^i(t_2 - t_1^{p+1})^j \colon 0 \leq i, pj < p^2\right\} \subset G(1,1). \end{aligned}$$

*A $G(1,1)$-comodule M is P-**free** (in a range of dimensions) if $P \otimes_{A(2)} M$ is weak injective (in the same range).*

7.4.4. Lemma. *$\tilde{D}_1^2$ and $\tilde{E}_1^3$ are P-free in our range, i.e. below dimension $p^2|v_2|$.*

Proof. For $\tilde{E}_1^3$ we can show this by direct calculation. Up to unit scalar we have

$$r_{(j-1)p^2\Delta_1+(i-1)p\Delta_2}\left(\frac{v_3^k}{pv_1^i v_2^j}\right) = \frac{v_3^{k+2-i-j}}{pv_1v_2} = \gamma_{k+2-i-j},$$

so these elements form a basis for $\mathrm{Ext}^0_{G(1,1)}(\tilde{E}_1^3)$ and for $\mathrm{Ext}^0_{G(1,1)}(P \otimes \tilde{E}_1^3)$. (Here $r_{a,b}$ denotes the Quillen operation dual to $t_1^a t_2^b$.) The Poincaré series for this Ext^0 is

$$\frac{x^{2p^2+p}}{1-x_3}.$$

Meanwhile we have

$$\begin{aligned} g(P \otimes_{A(2)} \tilde{E}_1^3) &= g(P(1))g(\tilde{E}_1^3) \\ &= \frac{(1-x^{p^2})(1-x_2^p)}{(1-x)(1-x_2)} \frac{x^{2p^2+p}}{(1-x^{p^2})(1-x_2^p)(1-x_3)} \\ &= \frac{x^{2p^2+p}}{(1-x)(1-x_2)(1-x_3)} \\ &= g_2(t)g(\mathrm{Ext}^0_{G(1,1)}(\tilde{E}_1^3)) \end{aligned}$$

so $\tilde{E}_1^3$ is P-free as claimed.

For $\tilde{D}_1^2$ we will first show that $P \otimes \tilde{D}_1^2$ is weak injective over $G(2)$. Then it will suffice to show that

$$\mathrm{Ext}^0_{G(2)}(P \otimes \tilde{D}_1^2)$$

is weak injective over $G(1)$, i.e. that $\mathrm{Ext}^0_{G(2)}(\tilde{D}_1^2)$ is 2-free.

As a $G(2)$-comodule, P is isomorphic to a direct sum of certain suspensions of $T_1^{(1)}$. We know by Theorem 7.2.6 that $T_1^{(1)} \otimes E_2^2$ is weak injective over $\Gamma(2)$ in our range. The same is true of $T_1^{(1)} \otimes E_1^2$ since it has the same positively graded Ext groups over $\Gamma(2)$. Thus the same goes for $T_1^{(1)} \otimes \tilde{E}_1^2$ and $P \otimes \tilde{E}_1^2$ over $G(2)$. Since we already know that $P \otimes \tilde{E}_1^3$ is weak injective over $G(1,1)$ and hence over $G(2)$, this implies that $P \otimes \tilde{D}_1^2$ is weak injective over $G(2)$.

This means that it suffices to show that $\mathrm{Ext}^0_{G(2)}(\tilde{D}_1^2)$ is 2-free. For this we have the following diagram with exact rows and columns.

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & B \arrow[r] \arrow[d, equal] & \mathrm{Ext}^0_{G(2)}(\tilde{E}_1^2) \arrow[r] \arrow[d] & U \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & B \arrow[r] & \mathrm{Ext}^0_{G(2)}(\tilde{D}_1^2) \arrow[r] \arrow[d] & U_0 \arrow[r] \arrow[d] & 0 \\
 & & \mathrm{Ext}^0_{G(2)}(\tilde{E}_1^3) \arrow[r, equal] \arrow[d] & U_1 \arrow[d] & \\
 & & 0 & 0, &
\end{tikzcd}$$

where B is as in Theorem 7.2.11 and the column on the right is as in Lemma 7.3.5. Since B and U_0 are both 2-free in our range, so is $\mathrm{Ext}^0_{G(2)}(\tilde{E}_1^2)$. $\square$

We will show that $\mathrm{Ext}^0_{G(1,1)}(P \otimes \tilde{D}_1^2)$ and $\mathrm{Ext}^0_{G(1,1)}(P \otimes \tilde{E}_1^3)$ each admit filtrations whose associated bigraded objects are comodules over $P(1)_*$, and analyzing them will lead to a proof of Theorem 7.3.15.

As in the above lemma, $\tilde{E}_1^3$ is easier to handle. We have

$$\mathrm{Ext}^0_{G(1,1)}(P \otimes \tilde{E}_1^3) = \mathrm{Ext}^0_{G(1,1)}(\tilde{E}_1^3) = \mathbf{Z}/(p)\left\{\gamma_k \colon k \geq 2\right\}. \tag{7.4.5}$$

No filtration is necessary here since it is annihilated by I_2, and we have

$$\mathrm{Ext}_{G(1,1)}(\tilde{E}_1^3) = \mathbf{Z}/(p)\left\{\gamma_k \colon k \geq 2\right\} \otimes \mathrm{Ext}_{P(1)_*}(\mathbf{Z}/(p)).$$

The case of $\tilde{D}_1^2$ is more complicated.

7.4.6. Lemma. *Let*

$$M = \mathrm{Ext}^0_{G(1,1)}(P \otimes \tilde{D}_1^2).$$

In our range it is generated by the following set.

$$\begin{gathered}
\left\{\beta_{i/j,k} \colon 1 \leq j, k \leq p,\ i \geq j+k-1\right\} \\
\cup \left\{\tilde{\beta}_{i/j,\min(p+1,i+2-j)} \colon 1 \leq j \leq p,\ i \geq p\right\} \\
\cup \left\{\tilde{\beta}_{i/p+1} \colon i \geq p+1\right\} \cup \left\{\beta_{p^2/p^2-j} \colon 0 \leq j < p\right\}
\end{gathered}$$

Here

$$
\begin{aligned}
\tilde{\beta}_{i/j,k} &= \frac{v_2^i}{p^k v_1^j}(1+x)\left(1-\binom{i}{p}y+\binom{i}{2p}y^2\right) \\
&\qquad \text{where } x = p^p v_1^{-1-p}v_2 \text{ and } y = v_1^p v_2^{-1-p}v_3 \\
&= \frac{v_2^i}{p^k v_1^j} + \begin{cases} 0 & \text{for } j,k<p+1 \\ -\binom{i}{p}\dfrac{v_2^{i-p-1}v_3}{pv_1} & \text{for } (j,k)=(p+1,1) \\ \dfrac{v_2^{i+1}}{pv_1^{j+p+1}} - \binom{i}{p}\dfrac{v_2^{i-p}v_3}{pv_1^{j+1}} + \binom{i}{2p}\dfrac{v_2^{i-2p-1}v_3^2}{pv_1^{j+1-p}} & \\ & \text{for } k=p+1. \end{cases}
\end{aligned}
$$

It has a decreasing filtration defined by

$$||\tilde{\beta}_{i/j,k}|| = i + [i/p] - j - k.$$

The above set is a $\mathbf{Z}/(p)$*-basis for the associated bigraded object, which is a* $P(1)_*$*-comodule. Its structure as a* $P(1)$*-module is given by*

$$
\begin{aligned}
r_1(\tilde{\beta}_{i/j,k}) &= j\tilde{\beta}_{i/j+1,k-1} \\
r_p(\tilde{\beta}_{i/j,k}) &= \begin{cases} \binom{i}{p}\tilde{\beta}_{i-1/j-1,k-1} & \text{for } p|i \\ \begin{cases} i\tilde{\beta}_{i-1/j-1,k} & \text{for } j>1 \\ i\tilde{\beta}_{i/j+p,1} & \text{for } (j,k)=(1,p+1) \\ 0 & \text{for } j=1 \text{ and } k<p+1. \end{cases} & \text{for } p\nmid i. \end{cases}
\end{aligned}
$$

Note that $\tilde{\beta}_{i/p+1}$ is a unit multiple of the element u_{i-p-1} of Theorem 7.3.4.

PROOF. Recall that $g(E_1^2)$ was determined in Lemma 7.2.2, which implies that in our range,

$$
\begin{aligned}
g(\tilde{E}_1^2) &= g_2(t)\sum_{i\geq 1}\frac{x^{p^i}(1-x_i)}{(1-x^{p^i})(1-x_{i+1})} \\
&= \frac{1}{(1-x)(1-x_2)}\left(\frac{x^p(1-x)}{(1-x^p)(1-x_2)} + \frac{x^{p^2}(1-x_2)}{(1-x^{p^2})(1-x_3)}\right. \\
&\qquad \left. + \frac{x^{p^3}(1-x_3)}{(1-x^{p^3})(1-x_4)}\right) \\
&= \frac{x^p}{(1-x^p)(1-x_2)^2} + \frac{x^{p^2}}{(1-x)(1-x^{p^2})(1-x_3)} + \frac{x^{p^3}}{1-x},
\end{aligned}
$$

so we have

$$\begin{aligned}
g(M) &= \frac{g(P(1))}{g_2(t)}\left(g(\tilde{E}_1^2)+g(\tilde{E}_1^3)\right)\\
&= (1-x^{p^2})(1-x_2^p)\left(\frac{x^p}{(1-x^p)(1-x_2)^2}+\frac{x^{p^2}}{(1-x)(1-x^{p^2})(1-x_3)}\right.\\
&\qquad\left.+\frac{x^{2p^2+p}}{(1-x^{p^2})(1-x_2^p)(1-x_3)}+\frac{x^{p^3}}{1-x}\right)\\
&= \frac{x^p(1-x^{p^2})(1-x_2^p)}{(1-x^p)(1-x_2)^2}+\frac{x^{p^2}}{1-x}+\frac{x^{p^3}}{1-x}\\
&= \sum_{1\le j\le p} x^{jp}\frac{1-x_2^p}{(1-x_2)^2}+\sum_{1\le j\le p}\frac{x^{p^2+p-j}}{1-x_2}+\frac{x_2^p}{1-x_2}+\frac{x^{p^3}}{1-x}.
\end{aligned}$$

The four indicated subsets correspond to these four terms.

In order to show that we have the right elements, we need to show that for each indicated generator z, the invariant element

$$\tilde{z} = \sum_{a,b\ge 0} \bar{t}_1^a\bar{t}_2^b \otimes r_{a,b}(z) \in G(1,1)\otimes M$$

actually lies in $P\otimes M$. For dimensional reasons we need only consider the cases where $a<p^3$ and $b<p^2$. Then if $a\ge p^2$ or $b\ge p$, $r_{a,b}(z)$ vanishes if both $r_{p^2}(z)$ and $r_{0,p}(z)$ do. But for each of our generators, the correcting terms (i.e. $\tilde{\beta}_{i/j,k}-\beta_{i/j,k}$) are chosen to insure that r_{p^2} and $r_{0,p}$ act trivially.

Our putative filtration is similar to the I-adic one, which is given by

$$||\tilde{\beta}_{i/j,k}|| = i-j-k.$$

Note that we are not assigning a filtration to each chromatic monomial, but to each of the generators listed in Lemma 7.4.6.

Roughly speaking, it suffices to show that an operation $r_{a,b}$ raises this filtration by the amount by which it lowers the value of $[i/p]$. Since r_{p^2} and $r_{0,p}$ act trivially, it suffices to consider the action of r_1 and r_p. The actions of r_1 on v_2 and v_3, and the action of r_p on v_3 raise I-adic filtration by at least $p-1$ and can therefore be ignored. It follows that modulo such terms, we have

$$\begin{aligned}
r_1\left(\tilde{\beta}_{i/j,k}\right) &= -j\frac{v_2^i}{p^{k-1}v_1^{j+1}} = j\tilde{\beta}_{i/j+1,k-1} \\
&= \begin{cases} 0 & \text{for } p|j \\ -j\tilde{\beta}_{i/j+1,k-1} & \text{for } p\nmid j \end{cases} \\
r_p\left(\tilde{\beta}_{i/j,k}\right) &= i\frac{v_2^{i-1}}{p^k v_1^{j-1}} + \binom{-j}{p}\frac{v_2^i}{p^{k-p}v_1^{j+p}} \\
&\qquad + \begin{cases} 0 & \text{for } k < p+1 \\ (i+1)\dfrac{v_2^i}{pv_1^{j+p}} - i\dbinom{i}{p}\dfrac{v_2^{i-p-1}v_3}{pv_1^j} & \text{for } k = p+1 \end{cases} \\
&= i\left(\frac{v_2^{i-1}}{p^k v_1^{j-1}} + \left\{\begin{array}{ll} 0 & \text{for } k < p+1 \\ \dfrac{v_2^i}{pv_1^{j+p}} - \dbinom{i}{p}\dfrac{v_2^{i-p-1}v_3}{pv_1^j} & \text{for } k = p+1 \end{array}\right\}\right) \\
&= \begin{cases} \dbinom{i}{p}\tilde{\beta}_{i-1/j-1,k-1} & \text{for } p|i \\ \begin{cases} i\tilde{\beta}_{i-1/j-1,k} & \text{for } j > 1 \\ i\tilde{\beta}_{i/j+p,1} & \text{for } (j,k) = (1,p+1) \\ 0 & \text{for } j = 1 \text{ and } k < p+1. \end{cases} & \text{for } p\nmid i. \end{cases}
\end{aligned}$$

Note that r_1 never changes the value of i or the I-adic filtration, while r_p raises the latter by 1 precisely when lowers the value of $[i/p]$ by 1. It follows that the indicated filtration is preserved by r_1 and r_p.

The associated bigraded is killed by I_2 because multiplication by it always raises filtration. $\square$

In what follows we will ignore the elements

$$\left\{\beta_{p^2/p^2-j} : 0 \le j < p\right\}.$$

They are clearly permanent cycles and will thus have no bearing on the proof of Theorem 7.3.15. *From now on, M will denote the quotient of M (as defined previously) by the subspace spanned by these elements.*

To explore the structure of E_0M further, we need to introduce some auxiliary $P(1)_*$-comodules. For $0 \le i < p$ let

$$C_i = \mathbf{Z}/(p)\left\{t_1^{pj} : 0 \le j \le i\right\},$$

and let $C_{-1} = 0$. Let

$$H = P(1)_* \square_{P(0)_*} \mathbf{Z}/(p).$$

7.4.7. LEMMA. (i) *For $i \ge 0$, let $c(i) = p\left[\frac{i+p}{p}\right] - i - 1$. There is a 4-term exact sequence*

$$0 \longrightarrow \Sigma^e C_{c(i)-1} \longrightarrow \Sigma^e H \longrightarrow E_0^{||u_i||} M \longrightarrow \Sigma^{|u_i|} C_{c(i)} \longrightarrow 0,$$

where

$$e = \begin{cases} |\beta_{i+2}| & \text{for } i \equiv -1 \bmod (p) \\ |\beta_{i+1}| = |u_i| - |b_{1,1}| & \text{otherwise.} \end{cases}$$

When i is congruent to -1 modulo p, then $c(i) = 0$ so the first term is trivial. The sequence splits in that case, i.e. for $j > 0$

$$E_0^{(p+1)j-1} M = \Sigma^{|\beta_{pj+1}|} H \oplus \Sigma^{|u_{pj-1}|} \mathbf{Z}/(p).$$

The value of $||u_i||$ is never congruent to -2 modulo $p+1$, and for $j > 0$

$$E_0^{(p+1)j-2} M = \Sigma^{|\beta_{pj}|} H.$$

(ii) *For i not congruent to -1 modulo p, there is are maps of 4-term sequences*

$$\begin{tikzcd}
0 \arrow[r] & \Sigma^e C_{c(i)-1} \arrow[r] & \Sigma^e H \arrow[r] & E_0^{||u_i||} M \arrow[r] & \Sigma^{|u_i|} C_{c(i)} \arrow[r] & 0 \\
0 \arrow[r] & \Sigma^e C_{c(i)-1} \arrow[r] \arrow[u, equal] & \Sigma^e C_{p-1} \arrow[r, "r_{pc(i)}"] \arrow[u] & \Sigma^{e+|t_1^{pc(i)}|} C_{p-1} \arrow[r] \arrow[u] & \Sigma^{|u_i|} C_{c(i)-1} \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & \Sigma^e \mathbf{Z}/(p) \arrow[r] \arrow[u] & \Sigma^e C_{p-1} \arrow[r, "r_p"] \arrow[u, equal] & \Sigma^{e+|t_1^p|} C_{p-1} \arrow[r] \arrow[u, "r_{p(c(i)-1)}"'] & \Sigma^{|u_i|} \mathbf{Z}/(p) \arrow[r] \arrow[u] & 0
\end{tikzcd}$$

in which each vertical map is a monomorphism. The bottom sequence is a Yoneda representative for the class $b_{1,1} \in \mathrm{Ext}^2_{P(1)_*}$.

PROOF. (i) Let $r_{0,1} = r_p r_1 - r_1 r_p \in P(1)$. It generates a truncated polynomial algebra of height p which we denote by $T(r_{0,1})$. It follows from 7.4.6 that

$$r_{0,1}(\tilde{\beta}_{i/j,k}) = i\tilde{\beta}_{i-1/j,k-1}.$$

The element on the right is nonzero modulo higher filtration when i/p^{k-1} is not a $p$local integer, i.e. when k is not too small. Thus up to unit scalar we get

$$\begin{aligned} r_{0,1}^{p-1}(\tilde{\beta}_{i+p-1/j,p+1}) &= \binom{i+p-1}{p-1}\tilde{\beta}_{i/j,2} \\ &= \begin{cases} \tilde{\beta}_{i/j} & \text{for } p \nmid i \\ \tilde{\beta}_{i/j,2} & \text{for } p|i \end{cases} \\ r_{0,1}^{p-1}(\tilde{\beta}_{pi+p-1/j,p}) &= \tilde{\beta}_{pi/j} \end{aligned}$$

This means that each element in the first two subsets in Lemma 7.4.6 is part of free module over $T(r_{0,1})$.

In $P(1)$, r_p commmutes with $r_{0,1}$, and H is free as a module over $T(r_p, r_{0,1})$ on its top element x. It is characterized as a cyclic $P(1)$-module by $r_1(x) = 0$ and $r_{p(p-1),p}(x) \neq 0$.

In $E_0^{(p+1)j-2}M$, the top element is $\tilde{\beta}_{pj+2p-2/p,p+1}$. It is killed by r_1, and up to unit scalar,

$$r_{p(p-1),p}(\tilde{\beta}_{pj+2p-2/p,p+1}) = \beta_{pj},$$

so $E_0^{(p+1)j-2}M$ has the indicated structure.

In $E_0^{(p+1)j-2}M$ for $j > 0$, u_{pj-1} is killed by both r_1 and r_p and generates a $P(1)$-summand. It is not present for $j = 0$. For $j \geq 0$, the class $\tilde{\beta}_{pj+2p-1/p,p+1}$ generates a summand isomorphic to a suspension of H as claimed.

In $E_0^{(p+1)j}M$ consider the sub-$P(1)$-module generated by the element $x = \beta_{pj+2p-1/p,p}$. Up to unit scalar we have

$$\begin{aligned} r_1(x) &= 0 \\ r_{0,p-1}(x) &= \beta_{pj+p/p} \\ r_{(p-1)p,p-2}(x) &= \beta_{pj+2} \\ r_{p,p-1}(x) &= 0. \end{aligned}$$

Thus there is a homomorphism from the indicated suspension of H to $E_0^{(p+1)j}M$ sending the top element to x with kernel isomorphic to C_{p-2}. Its cokernel is a copy of C_{p-1} in which top element is the image of $\tilde{\beta}_{pj+2p-1/p-1,p+1}$ and the bottom element is the image of $\tilde{\beta}_{pj+p+1/p+1}$.

The remaining cases, $E_0^{(p+1)j+k}M$ for $1 \le k \le p-2$, are similar. The top element in the image of H is $\tilde{\beta}_{pj+k+2p-1/p,p+1}$, and the top and bottom elements in the cokernel are the images of $\tilde{\beta}_{pj+k+2p-1/p-1,p+1}$ and $\tilde{\beta}_{pj+p+1+k/p+1}$ respectively.

(ii) The existence of the map of follows by inspection. Consider the case $p = 3$ and $i = 0$. Then the diagram is

$$\begin{tikzcd} 0 \arrow[r] & \Sigma^{12}C_1 \arrow[r] & \Sigma^{12}H \arrow[r] & E_0^0M \arrow[r] & \Sigma^{48}C \arrow[r] & 0 \\ 0 \arrow[r] & \Sigma^{12}C_0 \arrow[r] \arrow[u] & \Sigma^{12}C_2 \arrow[r] \arrow[u] & \Sigma^{24}C_2 \arrow[r] \arrow[u] & \Sigma^{48}C_0 \arrow[r] \arrow[u] & 0 \end{tikzcd}$$

The following diagram may be helpful in understanding the vertical maps.

$$\begin{tikzcd} * & * \arrow[l] & \beta_{3/3} \arrow[l] & \tilde{\beta}_{4/4} \arrow[l] & \\ & \beta_2 \arrow[u] & \beta_{3/2,2} \arrow[l] \arrow[u] & \beta_{4/3,2} \arrow[l] & \\ & & \beta_{3/1,3} \arrow[u] & \beta_{4/2,3} \arrow[l] \arrow[u] & \beta_{5/3,3} \arrow[l] \\ & & & \tilde{\beta}_{4/1,4} \arrow[u] \arrow[uuu, bend right] & \tilde{\beta}_{5/2,4} \arrow[l] \arrow[u] \end{tikzcd}$$

Here the short vertical arrows represent the action of r_1, and the longer arrows represent r_3. The named elements form a basis of E_0^0M and the asterisks are elements in $\Sigma^{12}H$ which map trivially to E_0^0M. H consists of all elements in the first three rows except $\tilde{\beta}_{4/4}$. $\square$

We will use Lemma 7.4.7 to determine $\mathrm{Ext}_{P(1)_*}(E_0M)$ in the following way. We regard the 4-term sequence of 7.4.7(i) as a resolution of 0, apply the functor $\mathrm{Ext}_{P(1)_*}(\overline{T}_0^{(1)} \otimes \cdot)$, and get a 4-column spectral sequence converging to 0. It turns out to have a d_3 that is determined by 7.4.7(ii), and this information will determine our Ext group.

In order to proceed further we need to know

$$\mathrm{Ext}_{P(1)_*}(\overline{T}_0^{(1)} \otimes H) \qquad \text{and} \qquad \mathrm{Ext}_{P(1)_*}(\overline{T}_0^{(1)} \otimes C_i),$$

where

$$\overline{T}_0^h = T_0^h \otimes_{BP_*} \mathbf{Z}/(p) \qquad \text{with} \qquad \overline{T}_0^{(i)} = \overline{T}_0^{p^i-1}.$$

This is a comodule over $P(1)_*$.

We will abbreviate $\mathrm{Ext}^s_{P(1)_*}(\overline{T}_0^{(1)} \otimes N)$ by $F^s(N)$.

Since $\overline{T}_0^{(1)} \otimes H = P(1)_*$, we have

$$F^{s,t}(H) = \begin{cases} \mathbf{Z}/(p) & \text{for } (s,t) = (0,0) \\ 0 & \text{otherwise.} \end{cases} \tag{7.4.8}$$

Next we compute $F^*(\mathbf{Z}/(p))$. There is a Hopf algebra extension

$$\mathbf{Z}/(p)[t_1]/(t_1^{p^2}) \to P(1)_* \to \mathbf{Z}/(p)[t_2]/(t_2^p) \tag{7.4.9}$$

and we have

$$\mathrm{Ext}_{\mathbf{Z}/(p)[t_2]/(t_2^p)} = E(h_{2,0}) \otimes P(b_{2,0})$$

where

$$h_{2,0} \in \mathrm{Ext}^{1,2(p^2-1)} \qquad \text{and} \qquad b_{2,0} \in \mathrm{Ext}^{2,2p(p^2-1)}.$$

In particular $\overline{T}_0^{(2)} = \mathbf{Z}/(p)[t_1]/(t_1^{p^2})$, so

$$\mathrm{Ext}_{P(1)_*}(\overline{T}_0^{(2)}) = \mathrm{Ext}_{\mathbf{Z}/(p)[t_2]/(t_2^p)} = E(h_{2,0}) \otimes P(b_{2,0}) \tag{7.4.10}$$

where $h_{2,0} \in \mathrm{Ext}^{1,2p^2-2}$ and $b_{2,0} \in \mathrm{Ext}^{2,2p^3-2p}$.

To compute $F^*(\mathbf{Z}/(p))$, we will use the long exact sequence

$$0 \longrightarrow \overline{T}_0^{(1)} \longrightarrow \overline{T}_0^{(2)} \xrightarrow{r_p} \Sigma^{p|v_1|}\overline{T}_0^{(2)} \xrightarrow{r_{p^2-p}} \Sigma^{p^2|v_1|}\overline{T}_0^{(2)} \xrightarrow{r_p} \cdots. \tag{7.4.11}$$

This leads to a resolution spectral sequence converging to $\mathrm{Ext}_{P(1)_*}(\overline{T}_0^{(1)})$ with

$$E_1^{s,t} = E(h_{1,1}, h_{2,0}) \otimes P(b_{1,1}, b_{2,0}),$$

where

$$\begin{aligned} h_{1,1} &\in E_1^{1,0}, & h_{2,0} &\in E_1^{0,1}, \\ b_{1,1} &\in E_1^{2,0}, \quad \text{and} & b_{2,0} &\in E_1^{0,2}. \end{aligned}$$

Alternatively, one could use the same resolution to show that

$$\mathrm{Ext}_{\mathbf{Z}/(p)[t_1]/(t_1^{p^2})}(\overline{T}_0^{(1)}) = E(h_{1,1}) \otimes P(b_{1,1})$$

and then use the Cartan–Eilenberg spectral sequence for (7.4.9). It is isomorphic to the resolution spectral sequence above.

Before describing this spectral sequence we need some notation for certain Massey products.

7.4.12. DEFINITION. *Let i be an integer with $0 < i < p$. Then $\underline{i}x$ denotes the Massey product (when it is defined)*

$$\langle h_{1,0}, \ldots, h_{1,0}, x\rangle$$

with i factors $h_{1,0}$, and $\underline{pi}x$ denotes the Massey product (when it is defined)

$$\langle h_{1,1}, \ldots, h_{1,1}, x\rangle$$

with i factors $h_{1,1}$.

Under suitable hypotheses we have $b_{1,0}x \in \underline{p-i} \cdot \underline{i}x$ and $b_{1,1}x \in \underline{p(p-i)} \cdot \underline{pi}x$.

7.4.13. THEOREM. *The differentials in the above spectral sequence are as follows:*

(a) $d_3(h_{2,0}^{\varepsilon} b_{2,0}^{i}) = ih_{1,1}h_{2,0}^{\varepsilon} b_{11} b_{2,0}^{i-1}$;

(b) $d_{2p-1}(h_{2,0}^{\varepsilon} h_{1,1} b_{2,0}^{pi+p-1}) = h_{2,0}^{\varepsilon} b_{1,1}^{p} b_{2,0}^{pi}$,

where $\varepsilon = 0$ or 1. These differentials commute with multiplication by $h_{2,0}$, $h_{1,1}$, and $b_{1,1}$, and all other differentials are trivial. Consequently $\operatorname{Ext}_{P(1)_*}(\overline{T}_0^{(1)})$ is a free module over

$$P(b_{2,0}^{p}) \otimes E(h_{2,0})$$

on the set

$$\{b_{1,1}^{i} \colon 0 \le i \le p-1\} \cup \{h_{1,1} b_{2,0}^{i} \colon 0 \le i \le p-2\}.$$

There are Massey product relations

$$h_{1,1}b_{2,0}^{i} \in \underline{p(i+1)}b_{1,1}^{i} \qquad \textit{and} \qquad b_{1,1}^{i+1} \in \underline{p(p-i-1)}h_{1,1}b_{2,0}^{i}$$

for $0 \le i \le p-2$. We will denote this object by R.

PROOF. In the Cartan–Eilenberg spectral sequence for (7.4.9) one has

$$d_2(h_{2,0}) = \pm h_{1,0}h_{1,1}$$

since the reduced diagonal on t_2 is $t_1 \otimes t_1^{p}$. Now we use the theory of algebraic Steenrod operations of A1.5 and the Kudo transgression theorem A1.5.7. Up to sign we have $\beta P^0(h_{2,0}) = b_{2,0}$, so

$$d_3(b_{2,0}) = \beta P^0(h_{1,0}h_{1,1}) = \beta(h_{1,1}h_{1,2}) = h_{1,1}b_{1,1}$$

as claimed in (a). Then A1.5.7 implies that

$$d_{2p-1}(h_{1,1}b_{1,1}b_{2,0}^{p-1}) = \beta(h_{1,2}b_{1,1}^{p}) = b_{1,1}^{p+1},$$

so $d_{2p-1}(h_{1,1}b_{2,0}^{p-1}) = b_{1,1}^{p}$ as claimed in (b). The stated Massey product relations follow easily from (a) and (b). $\square$

To compute $F^*(C_i)$ for $0 < i < p$, we use the spectral sequence associated with the skeletal filtration of C_i. In it we have

$$E_1^{j,k} = F^k(\Sigma^{pj|t_1|}\mathbf{Z}/(p)) \quad \text{for } 0 \le j \le i \qquad \text{and} \qquad d_r \colon E_r^{j,k} \to E_r^{j-r,k+1}.$$

We will denote the generator of $E_1^{j,0}$ by x_{pj} and write x_0 as 1. Since

$$\overline{T}_0^{(1)} \otimes C_{p-1} = \overline{T}_0^{(2)},$$

its Ext group is given by (7.4.10). There is a pattern of differentials implied by the Massey product relations of Theorem 7.4.13.

7.4.14. PROPOSITION. *In the skeletal filtration spectral sequence for*

$$F^*(C_i) = \mathrm{Ext}^*_{P(1)_*}(\overline{T}_0^{(1)} \otimes C_i)$$

we have the following differentials and no others.

$$\begin{aligned} & d_{k+1}(x_{pj}h_{2,0}^{\varepsilon}b_{1,1}^{k}) \\ & \qquad = \quad x_{p(j-k-1)}h_{2,0}^{\varepsilon}h_{1,1}b_{2,0}^{k} \quad && \textit{for } 0 \le k < j \le i \\ & d_{p-1-k}(x_{pj}h_{2,0}^{\varepsilon}h_{1,1}b_{2,0}^{k}) \\ & \qquad = \quad x_{p(j+k+1-p)}h_{2,0}^{\varepsilon}b_{1,1}^{k+1} \quad && \textit{for } p-1-j \le k \le p-1-j+i \\ & && \textit{and } 0 \le j \le i, \end{aligned}$$

where $\varepsilon = 0$ *or 1.*

The following diagram illustrates this for $p = 5$.

$$\begin{tikzcd}[column sep=small]
1 & x_5 \arrow[dl, "d_1"'] & x_{10} \arrow[dl, "d_1"'] & x_{15} \arrow[dl, "d_1"'] & x_{20} \arrow[dl, "d_1"'] \\
h_{1,1} & x_5h_{1,1} & x_{10}h_{1,1} & x_{15}h_{1,1} & x_{20}h_{1,1} \arrow[dllll, "d_4"'] \\
b_{1,1} & x_5b_{1,1} & x_{10}b_{1,1} \arrow[dll, "d_2"'] & x_{15}b_{1,1} \arrow[dll, "d_2"'] & x_{20}b_{1,1} \arrow[dll, "d_2"'] \\
h_{1,1}b_{2,0} & x_5h_{1,1}b_{2,0} & x_{10}h_{1,1}b_{2,0} & x_{15}h_{1,1}b_{2,0} \arrow[dlll, "d_3"'] & x_{20}h_{1,1}b_{2,0} \arrow[dlll, "d_3"'] \\
b_{1,1}^2 & x_5b_{1,1}^2 & x_{10}b_{1,1}^2 & x_{15}b_{1,1}^2 \arrow[dlll, "d_3"'] & x_{20}b_{1,1}^2 \arrow[dlll, "d_3"'] \\
h_{1,1}b_{2,0}^2 & x_5h_{1,1}b_{2,0}^2 & x_{10}h_{1,1}b_{2,0}^2 \arrow[dll, "d_2"'] & x_{15}h_{1,1}b_{2,0}^2 \arrow[dll, "d_2"'] & x_{20}h_{1,1}b_{2,0}^2 \arrow[dll, "d_2"'] \\
b_{1,1}^3 & x_5b_{1,1}^3 & x_{10}b_{1,1}^3 & x_{15}b_{1,1}^3 & x_{20}b_{1,1}^3 \arrow[dllll, "d_4"'] \\
h_{1,1}b_{2,0}^3 & x_5h_{1,1}b_{2,0}^3 \arrow[dl, "d_1"'] & x_{10}h_{1,1}b_{2,0}^3 \arrow[dl, "d_1"'] & x_{15}h_{1,1}b_{2,0}^3 \arrow[dl, "d_1"'] & x_{20}h_{1,1}b_{2,0}^3 \arrow[dl, "d_1"'] \\
b_{1,1}^4 & x_5b_{1,1}^4 & x_{10}b_{1,1}^4 & x_{15}b_{1,1}^4 & x_{20}b_{1,1}^4
\end{tikzcd} \tag{7.4.15}$$

Each row and column corresponds to a different value of k and j respectively. The skeletal filtration spectral sequence for C_{p-1} is obtained by tensoring the pattern indicated above with $E(h_{2,0}) \otimes P(b_{2,0}^p)$. Note that the only element in the jth column not on either end of a differential is $x_{pj}b_{1,1}^j$, which represents $b_{2,0}^j$.

The skeletal filtration spectral sequence for C_i is obtained from that for C_{p-1} by looking only at the first $i+1$ columns.

Now we consider the resolution spectral sequence converging to 0 associated the 4-term exact sequence of Lemma 7.4.7(i). In it we have

$$\begin{aligned} E_1^{0,s} &= F^s(\Sigma^e C_{c(i)-1}) \\ E_1^{1,s} &= F^s(\Sigma^e H) \\ E_1^{2,s} &= F^s(\Sigma^e E_0^{b(i)} M) \\ E_1^{3,s} &= F^s(\Sigma^{|u_i|} C_{c(i)}); \end{aligned}$$

each of these groups is graded by dimension. The last differential here is

$$d_3 : E_3^{0,s} \to E_3^{3,s-2}.$$

It is an isomorphism and hence has an inverse since the spectral sequence converges to 0. The bottom dimension is $e = |\beta_{b(i)+1}|$. By (7.4.8) we have

$$E_1^{1,s} = \begin{cases} \mathbf{Z}/(p) & \text{concentrated in dimension } e \text{ for } s = 0 \\ 0 & \text{for } s > 0. \end{cases}$$

The bottom class here is killed by a d_1 coming from the one in $E_1^{1,s}$. Above the bottom dimension, the only differentials in addition to the d_3 above are

$$d_2 : E_2^{0,s} \to E_2^{2,s-1} \qquad \text{and} \qquad d_1 : E_1^{2,s} \to E_1^{3,s}.$$

It follows that above dimension e there is a short exact sequence

$$0 \longrightarrow \operatorname{coker} \tilde{d}_3^{-1} \xrightarrow{d_2} F^s(\Sigma^e E_0^{b(i)} M) \xrightarrow{d_1} \ker \tilde{d}_3^{-1} \longrightarrow 0, \tag{7.4.16}$$

where $\tilde{d}_3^{-1}$ denotes the composite

$$F^s(\Sigma^{b_{1,1}} C_{c(i)}) \longrightarrow E_3^{3,s} \xrightarrow[\cong]{d_3^{-1}} E_3^{0,s+2} \longrightarrow F^{s+2}(C_{c(i)-1})$$

Here $\operatorname{coker} \tilde{d}_3^{-1}$ is a quotient of $F^{s+1}(\Sigma^e C_{c(i)-1})$ and $\ker \tilde{d}_3^{-1}$ is a subgroup of $F^s(\Sigma^{|u_i|} C_{c(i)})$. Note that $|u_i| - e = |b_{1,1}|$ in all cases. Lemma 7.4.7(ii) implies that $\tilde{d}_3^{-1}$, roughly speaking, multiplication by $b_{1,1}$.

We illustrate this for the case $p = 5$ and $i = 0$. The 4-term sequence is

$$0 \longrightarrow \Sigma^{40} C_3 \longrightarrow \Sigma^{40} H \longrightarrow E_0^0 M \longrightarrow \Sigma^{240} C_4 \longrightarrow 0.$$

Referring to (7.4.15) we see that the product of $b_{1,1}$ with any element in $F^*(C_4)$ (except $x_{20} b_{1,1}^4$, which is out of our range) is killed by a differential originating in the last column, which means that it is alive in $F^*(C_3)$. Thus $\tilde{d}_3^{-1}$ is a monomorphism in our range, so its kernel is trivial and the d_2 in (7.4.16) is an isomorphism. The cokernel of $\tilde{d}_3^{-1}$ is the quotient of

$$\Sigma^{40} E(h_{2,0}) \otimes \left\{ 1,\, x_5 h_{1,1},\, x_5 b_{1,1},\, x_{10} h_{1,1} b_{2,0},\, x_{10} b_{1,1}^2,\, x_5 h_{1,1} b_{2,0}^2,\, x_{15} b_{1,1}^3,\, h_{1,1} b_{2,0}^3 \right\}$$

obtained by killing the bottom class. The classes $h_{2,0}$ and $x_{15} h_{11}$ map to β_2 and $\beta_{5/5}$. By inspection this leads to the desired value of $F^*(E_0^0 M)$.

For $i = 1$, the 4-term sequence is

$$0 \longrightarrow \Sigma^{88} C_2 \longrightarrow \Sigma^{88} H \longrightarrow E_0^1 M \longrightarrow \Sigma^{288} C_3 \longrightarrow 0.$$

Again $\tilde{d}_3^{-1}$ is a monomorphism in our range. The cokernel of $\tilde{d}_3^{-1}$ is the quotient of

$$\Sigma^{88}E(h_{2,0})\otimes\left\{1,\, x_{10}h_{1,1},\, x_5b_{1,1},\, x_5h_{1,1}b_{2,0},\, x_{10}b_{1,1}^2,\, h_{1,1}b_{2,0}^2\right\}$$

obtained by killing the bottom class. The classes $h_{2,0}$ and $x_{10}h_{11}$ map to β_3 and $\beta_{5/4}$. By inspection this leads to the desired value of $F^*(E_0^1M)$.

In order to see that this works in general it is useful to compare the comodules $\overline{T}_0^{(1)}\otimes E_0^iM$ with certain others with known Ext groups. Let

$$0\longrightarrow\overline{T}_0^{(1)}\longrightarrow F_0\xrightarrow{d_0}F_1\xrightarrow{d_1}\cdots \tag{7.4.17}$$

be a minimal free resolution of $\overline{T}_0^{(1)}$. Its structure is as follows.

7.4.18. PROPOSITION. *The free $P(1)_*$-comodule F_i above is*

$$F_i=\begin{cases}P(1)_* & \text{for } i=0\\ \Sigma^{i'|b_{1,1}|}P(1)_*\oplus\Sigma^{(i'-1)|b_{2,0}|+|t_1^pt_2|}P(1)_* & \text{for } i=2i' \text{ and } 0<i'<p\\ \Sigma^{i'|b_{2,0}|+|t_1^p|}P(1)_*\oplus\Sigma^{i'|b_{1,1}|+|t_2|}P(1)_* & \text{for } i=2i'+1 \text{ and}\\ & \qquad 0<i'<p-1\\ \Sigma^{|t_2|+(p-1)|b_{1,1}|}P(1)_* & \text{for } i=2p-1\\ \Sigma^{p|b_{2,0}|}F_{i-2p} & \text{for } i\geq 2p.\end{cases}$$

In $P(1)$ let $x=P^1$, $y=P^p$, $z=yx-xy$. Then there are relations

$$x^p=0,\ [x,z]=0,\ [y,z]=0,\ \text{ and } y^p=xz^{p-1},$$

(corresponding to the four generators of $\mathrm{Ext}^2_{P(1)_*}$) which imply that $z^p=0$. Then d_i is represented (via left multiplication) by a matrix M_i over $P(1)$ as follows.

$$M_i=\begin{cases}\begin{bmatrix} y\\ z\end{bmatrix} & \text{for } i=0\\[2mm] \begin{bmatrix} y^{p-i'} & -xz^{p-2}\\ z & -y^{i'}\end{bmatrix} & \text{for } i=2i'-1 \text{ with } 0<i'<p\\[2mm] \begin{bmatrix} y^{i'+1} & -(xy+(i'+1)z)z^{p-2}\\ z & -y^{p-i'}\end{bmatrix} & \text{for } i=2i' \text{ with } 0<i'<p-1\\[2mm] \begin{bmatrix} z & -y\end{bmatrix} & \text{for } i=2p-2\\ \begin{bmatrix} y^{p-1}z^{p-1}\end{bmatrix} & \text{for } i=2p-1\\ M_{i-2p} & \text{for } i\geq 2p.\end{cases}$$

Let K_i denote the kernel of d_i, and consider the following diagram with exact rows and columns for $0<i<p$.

(7.4.19)
$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & & \\
0 \arrow[r] & X_i \arrow[r] \arrow[d] & \Sigma^{a_i} P(1)_* \arrow[r] \arrow[d] & K_{2i} \arrow[r] \arrow[d, equal] & Y_i \arrow[r] & 0 \\
0 \arrow[r] & K_{2i-1} \arrow[r, "d_{2i-2}"] \arrow[d] & F_{2i-1} \arrow[r, "d_{2i-1}"] \arrow[d] & K_{2i} \arrow[r] & 0 & \\
 & \Sigma^{b_i} P(1)_* \arrow[r, equal] \arrow[d] & \Sigma^{b_i} P(1)_* \arrow[d] & & & \\
 & Y_i \arrow[d] & 0 & & & \\
 & 0 & & & &
\end{tikzcd}$$

where $a_i = (i-1)|b_{2,0}| + |t_1^p|$, $b_i = (i-1)|b_{1,1}| + |t_2|$, and the middle column is split. We will see that the top row (up to reindexing and suspension) is the 4-term sequence of Lemma 7.4.7(i) tensored with $\overline{T}_0^{(1)}$. For this we need to identify X_i and Y_i.

X_i is the kernel of the map represented by the first column of M_{2i-1}, namely

$$\begin{bmatrix} y^{p-i} \\ z \end{bmatrix}.$$

This kernel is the ideal generated by $y^i z^{p-1}$, which is

$$\Sigma^{a_i}\overline{T}_0^{p(p-i)-1} = \Sigma^{i|b_{2,0}|-|b_{1,1}|}\overline{T}_0^{(1)} \otimes C_{p-1-i}.$$

Y_i is the cokernel of the map to $\Sigma^{b_i}P(1)_*$ represented by the bottom row of M_{2i-2}, namely

$$\begin{cases} \begin{bmatrix} z \end{bmatrix} & \text{for } i = 1 \\ \begin{bmatrix} z & -y^{p+1-i} \end{bmatrix} & \text{for } 1 < i < p. \end{cases}$$

This cokernel is

$$\Sigma^{b_i+|y^{i-1}z^{p-1}|}\overline{T}_0^{(p+1-i)p-1} = \Sigma^{i|b_{2,0}|}\overline{T}_0^{(1)} \otimes C_{p-i}.$$

This enables us to prove the following analog of Lemma 7.4.7.

7.4.20. Lemma. *For $0 < i < p$ there are maps of 4-term exact sequences*

$$\begin{tikzcd}
0 \arrow[r] & \Sigma^{a_i}\overline{T}_0^{k} \arrow[r] & \Sigma^{a_i}P(1)_* \arrow[r] & K_{2i} \arrow[r] & \Sigma^{a_i+|b_{1,1}|}\overline{T}_0^{k+p} \arrow[r] & 0 \\
0 \arrow[r] & \Sigma^{a_i}\overline{T}_0^{k} \arrow[r] \arrow[u, equal] & \Sigma^{a_i}\overline{T}_0^{(2)} \arrow[r, "r_{k+1}"] \arrow[u] & \Sigma^{i|b_{1,1}|}\overline{T}_0^{(2)} \arrow[r] \arrow[u] & \Sigma^{a_i+|b_{1,1}|}\overline{T}_0^{k} \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & \Sigma^{a_i}\overline{T}_0^{(1)} \arrow[r] \arrow[u] & \Sigma^{a_i}\overline{T}_0^{(2)} \arrow[r, "r_p"] \arrow[u, equal] & \Sigma^{a_i+|t_1^p|}\overline{T}_0^{(2)} \arrow[r] \arrow[u, "r_{k+1-p}"] & \Sigma^{a_i+|b_{1,1}|}\overline{T}_0^{(1)} \arrow[r] \arrow[u] & 0
\end{tikzcd}$$

where $k = p(p-i)-1$, the top row is the same as that in (7.4.19), and each vertical map is a monomorphism.

PROOF. The statement about the top row is a reformulation of our determination of X_i and Y_i above. Each vertical map is obvious except the one to K_{2i}. K_{2i} is the kernel of the map d_{2i} from

$$F_{2i} = \Sigma^{i|b_{1,1}|}P(1)_* \oplus \Sigma^{(i-1)|b_{2,0}|+|t_1^p t_2|}P(1)_*$$

(note that $i|b_{1,1}| = a_i + |t_1^{p(p-i)}|$) to

$$F_{2i+1} = \Sigma^{i|b_{2,0}|+|t_1^p|}P(1)_* \oplus \Sigma^{i|b_{1,1}|+|t_2|}P(1)_*$$

represented by the matrix

$$M_{2i} = \left[\begin{array}{cc} y^{i+1} & -(xy+(i+1)z)z^{p-2} \\ z & -y^{p-i} \end{array}\right].$$

The map $\Sigma^{a_i}P(1)_*$ is the restriction of d_{2i-1}, under which we have

$$t_1^{p(p-i)} \mapsto \left[\begin{array}{c} 1 \\ 0 \end{array}\right],$$

so this is the image of the bottom element in $\Sigma^{i|b_{1,1}|}\overline{T}_0^{(2)}$ in K_{2i}. This means that the top element in $\Sigma^{i|b_{1,1}|}\overline{T}_0^{(2)}$ must map to an element of the form

$$\left[\begin{array}{c} t_1^{p(p-1)} + \varepsilon_1 \\ \varepsilon_2 \end{array}\right]$$

where ε_1 and ε_2 are each killed by y^{p-1}. We also need this element to be in K_{2i}, so it must satisfy

$$\begin{aligned}\left[\begin{array}{c} 0 \\ 0 \end{array}\right] &= M_{2i}\left[\begin{array}{c} t_1^{p(p-1)} + \varepsilon_1 \\ \varepsilon_2 \end{array}\right] \\ &= \left[\begin{array}{c} ct_1^{p(p-2-i)} + y^{i+1}(\varepsilon_1) - (xy+(i+1)z)z^{p-2}(\varepsilon_2) \\ z(\varepsilon_1) - y^{p-i}(\varepsilon_2) \end{array}\right]\end{aligned}$$

for a certain unit scalar c. We can get this by setting $\varepsilon_1 = 0$ and making ε_2 a linear combination of $t_1^{p(p-2-i)}t_2^{p-1}$ and $t_1^{1+p(p-1-i)}t_2^{p-2}$ chosen to make the element in the top row vanish. Such an ε_2 will be killed by y^{p-i}, so the element in the bottom row will vanish as well. □

This means that the Ext computation for the K_{2i} is essentially identical to that $\overline{T}_0^{(1)} \otimes E_0M$ described above. It follows from the way the K_i were constructed that for all i and s,

$$Ext^s_{P(1)_*}(K_i) = \mathrm{Ext}^{s+i}_{P(1)_*}(K_0) = \mathrm{Ext}^{s+i}_{P(1)_*}(\overline{T}_0^{(1)}). \tag{7.4.21}$$

These groups are known by Theorem 7.4.13.

We need the following analog of Theorem 7.4.13 for these comodules, whose proof we leave as an exercise for the reader.

7.4.22. THEOREM. *In the Cartan–Eilenberg spectral sequence converging to* $\mathrm{Ext}_{P(1)_*}(K_{2i})$ *based on the extension (7.4.9) for* $0<i<p$, $\tilde{E}_2$ *is a subquotient (determined by the* d_1 *indicated below) of*

$$P(b_{1,1})\otimes\left\{\begin{array}{c}\left\{b_{2,0}^i\right\}\otimes E(h_{2,0},h_{1,1})\otimes P(b_{2,0})\\ \oplus\\ \left\{h_{1,1}h_{2,0}b_{2,0}^{i-1},\ \underline{p(p-i)}(h_{1,1}h_{2,0}b_{2,0}^{i-1}),\ b_{1,1}^i,\ \underline{pi}b_{1,1}^i\right\}.\end{array}\right.$$

Here we are using the isomorphism of (7.4.21) to name the generators in the two indicated sets. Thus we have

$$\begin{array}{rcl}b_{2,0}^i,\ h_{1,1}h_{2,0}b_{2,0}^{i-1},\ b_{1,1}^i & \in & \tilde{E}_2^{0,0}\\ \underline{p(p-i)}h_{1,1}h_{2,0}b_{2,0}^{i-1},\ \underline{pi}b_{1,1}^i & \in & \tilde{E}_2^{1,0}\\ h_{1,1}\in E_2^{1,0}\qquad\qquad b_{1,1} & \in & E_2^{2,0}\\ h_{2,0}\in E_2^{0,1}\qquad\qquad b_{2,0} & \in & E_2^{0,2},\end{array}$$

and the differentials are (up to unit scalar)

$$\begin{array}{rcl}d_1(b_{2,0}^i) & = & \underline{pi}b_{1,1}^i\\ d_2(h_{2,0}b_{2,0}^i) & = & b_{1,1}\cdot h_{1,1}h_{2,0}b_{2,0}^{i-1}\\ d_3(h_{2,0}^{\varepsilon}b_{2,0}^k\cdot b_{2,0}^i) & = & (i+k)h_{1,1}h_{2,0}^{\varepsilon}b_{1,1}b_{2,0}^{k-1}\cdot b_{2,0}^i\\ & & \qquad\text{for } k>0 \text{ and } \varepsilon=0 \text{ or } 1\\ d_{2p-2i-1}(h_{1,1}b_{2,0}^{p-1-i}\cdot b_{2,0}^i) & = & b_{1,1}^{p-i}\cdot b_{1,1}^i\\ d_{2p-2i-2}(h_{1,1}h_{2,0}b_{2,0}^{p-1-i}\cdot b_{2,0}^i) & = & b_{1,1}^{p-i}\cdot\underline{p(p-i)}h_{1,1}h_{2,0}b_{2,0}^{i-1}.\end{array}$$

The last four differentials listed above should be compared with the first four listed in Theorem 7.3.15. The first differential of Theorem 7.4.13 corresponds to the last one of 7.3.15, while the second differential of 7.4.13 would correspond to one in 7.3.15 that is out of our range.

Thus Theorem 7.3.15 is a consequence of the relation between the K_{2i} and E_0M.

5. Computing $\pi_*(S^0)$ for $p=3$

We begin by recalling the results of the previous sections. We are considering groups $\mathrm{Ext}^{s,t}=0$ for $t<p^3|v_1|$ (where $|v_1|=2p-2$) with $p>2$. For each odd prime p, we have the 4-term exact sequence (7.1.19) of comdules over $BP_*(BP)$

$$0\longrightarrow BP_*\longrightarrow D_1^0\longrightarrow D_1^1\longrightarrow E_1^2\longrightarrow 0$$

in which D_1^0 and D_1^1 are weak injective (meaning that their higher Ext groups vanish, see 7.1.5) and the maps

$$\mathrm{Ext}^0(D_1^0)\longrightarrow \mathrm{Ext}^0(D_1^1)\longrightarrow \mathrm{Ext}^0(E_1^2)$$

are trivial. This means that the resolution spectral sequence collapses from E_1 and we have isomorphisms

$$\mathrm{Ext}^s=\begin{cases}\mathrm{Ext}^0(D_1^0) & \text{for } s=0\\ \mathrm{Ext}^0(D_1^1) & \text{for } s=1\\ \mathrm{Ext}^{s-2}(E_1^2) & \text{for } s\geq 2\end{cases}$$

We have determined $\operatorname{Ext}(T_0^{(1)} \otimes E_1^2)$ in Theorem 7.3.15, which can be reformulated as follows.

7.5.1. ABC Theorem. *For $p > 2$ and $t < (p^3 + p)|v_1|$*

$$\operatorname{Ext}(T_0^{(1)} \otimes E_1^2) = A \oplus B \oplus C$$

where A is the $\mathbf{Z}/(p)$-vector space spanned by

$$\left\{ \beta_i = \frac{v_2^i}{pv_1} : i > 0 \text{ and } i \equiv 0, 1 \mod (p) \right\} \cup \left\{ \beta_{p^2/p^2-j} : 0 \le j < p \right\},$$

$$B = R \otimes \left\{ \gamma_k \in \operatorname{Ext}^{1, 2k(p^3-1)-2(p^2+p-2)} : k \ge 2 \right\}$$

and

$$C^{s,t} = \bigoplus_{i \ge 0} R^{2+s+2i, t+i(p^2-1)q}.$$

Here $R = \operatorname{Ext}_{P(1)_*}(\mathbf{Z}/(p), \overline{T}_0^{(1)})$ as described in Theorem 7.4.13. □

This result is illustrated for $p = 5$ in Figure 7.3.17. Each dot represents a basis element. Vertical lines represent multiplication by 5 and horizontal lines represent the Massey product operation $\langle -, 5, \alpha_1 \rangle$, corresponding to multiplication by v_1. The diagonal lines correspond either to mulitplication by $h_{2,0}$ or to Massey product operations $\langle -, h_{11}, h_{11}, \ldots, h_{11} \rangle$.

The next step is to pass from this group to $\operatorname{Ext}(E_1^2)$ using the small descent spectral sequence of Theorem 7.1.13. Alternatively one could observe that $E_1^2 = BP_*(\operatorname{coker} J)$ and that the Adams–Novikov spectral sequence for $\pi_*(T(0)_{(1)} \wedge \operatorname{coker} J)$ collapses for dimensional reasons. We can then use the topological small descent spectral sequence of Theorem 7.1.16 to pass from this group to $\pi_*(\operatorname{coker} J)$. We will do this using the input/output procedure of 7.1.18.

We give a basis for $\mathbf{N}$. Recall the the input $\mathbf{I}$ in this case is $\mathbf{N} \otimes E(h_{1,0})$.

7.5.2. Proposition. *For $p = 3$, $\mathbf{N}$ as in 7.1.18 has basis elements in dimensions indicated below.*

10 β_1
26 β_2
34 $\beta_{3/3} = b_{1,1}$
38 $\beta_{3/2}$
42 $\beta_3, \beta_{3/1,2}$
49 $h_{2,0} b_{1,1}$
53 $h_{11} \beta_{3/1,2}$
57 $\eta_1 = h_{11} u_0 = \underline{6} \beta_{3/3}$
58 β_4
68 $b_{1,1}^2$
72 $b_{2,0} \beta_2$
74 β_5
78 u_2
81 γ_2
82 $\beta_{6/3}$
83 $\beta_2 \eta_1 = h_{2,0} b_{1,1}^2$
86 $\beta_{6/2}$
89 $\eta_3 = h_{11} u_2$
90 $\beta_6, \beta_{6/1,2}$
92 $h_{11} \gamma_2$
93 $h_{2,0} u_2$
96 $h_{2,0} \gamma_2$
97 $h_{2,0} \beta_{6/3}$
97 $h_{2,0} \beta_{6/3}$
101 $h_{11} \beta_{6/1,2}$
104 $\beta_2 u_2$
105 $\eta_4 = h_{11} u_3 = \underline{6} \beta_{6/3}$
106 $\beta_7, \beta_{9/9}$
107 $\gamma_2 \beta_2$ □

The notation $\underline{n}x$ for an integer n denotes a certain Massey product involving x as in 7.4.12. $\underline{3}x$ and $\underline{6}x$ denote $h_{11}x$ and $\langle h_{11}, h_{11}, x\rangle$, respectively.

Now we turn to the list **O** of 7.1.18, shown in 7.5.3. Elements from **N** are underlined. A differential is indicated by enclosing the target in square brackets and indicating the source on the right. Hence such pairs are to be ommited from the final output. The computation of the differentials will be described below.

7.5.3. THEOREM. *With notation as above the list* **O** *of* 7.1.18 *for* $p = 3$ *is as follows.*

10 $\underline{\beta_1}$

13 $\alpha_1\beta_1$

20 β_1^2

23 $\alpha_1\beta_1^2$

26 $\underline{\beta_2}$

29 $\alpha_1\beta_2$

30 β_1^3

33 $[\alpha_1\beta_1^3]\underline{\beta_{3/3}}$

36 $\beta_1\beta_2$

37 $\underline{2}\beta_1^3$

38 $\underline{\beta_{3/2}}$

39 $\alpha_1\beta_1\beta_2$

40 β_1^4

41 $[\alpha_1\beta_{3/2}]\underline{\beta_{3/1,2}}$

42 $\underline{\beta_3}$

45 $\underline{2}\beta_{3/2}$

$\alpha_1\beta_3$

46 $\beta_1^2\beta_2$

47 $\underline{2}\beta_1^4$

48 $[\beta_1\beta_{3/2}]\underline{h_{2,0}b_{1,1}}$

49 $\alpha_1\beta_1^2\beta_2$

50 β_1^5

52 β_2^2

$[\beta_1\beta_3]\underline{h_{11}\beta_{3/1,2}}$

55 $\alpha_1\beta_2^2$

$[\alpha_1\beta_1\beta_3]\alpha_1h_{11}\beta_{3/1,2}$

56 $[\beta_1^3\beta_2]\eta_1$

57 $[\underline{2}\beta_1^5]\underline{\beta_4}$

59 $[\alpha_1\beta_1^3\beta_2]\alpha_1\eta_1$

60 $[\beta_1^6]\alpha_1\beta_4$

62 $\beta_1\beta_2^2$

65 $\alpha_1\beta_1\beta_2^2$

68 $\underline{\beta_{3/3}^2} \pm \beta_4\beta_1 = x_{68}$

71 $[\alpha_1x_{68}]\underline{b_{2,0}\beta_2}$

72 $\beta_1^2\beta_2^2$

74 $\underline{\beta_5}$

75 $\underline{2}x_{68}$

$\alpha_1\beta_1^2\beta_2^2$

77 $[\alpha_1\beta_5]\underline{u_2}$

78 $\beta_2^3 = \beta_1x_{68}$

81 $\underline{\gamma_2}$

$\underline{2}\beta_5$

82 $\underline{\beta_{6/3}}$

$[\beta_1^3\beta_2^2]\underline{\beta_2\eta_1}$

84 $\alpha_1\gamma_2$

$\beta_1\beta_5$

85 $\underline{2}\beta_2^3$

$\alpha_1\beta_{6/3}$

$[\alpha_1\beta_1^3\beta_2^2]\alpha_1\beta_2\eta_1$

86 $\underline{\beta_{6/2}}$

88 $[\beta_1\beta_2^3]\underline{\eta_3}$

89 $[\alpha_1\beta_{6/2}]\underline{\beta_{6/1,2}}$

90 $\underline{\beta_6}$

91 $\beta_1\gamma_2$

$\underline{2}\beta_1\beta_5$

92 $\beta_1\beta_{6/3}$

$\underline{h_{11}\gamma_2}$

$[\alpha_1u_3 + \beta_1\beta_{6/3}]\underline{h_{2,0}u_2}$

93 $\underline{2}\beta_{6/2}$

$\alpha_1\beta_6$

94 $\alpha_1\beta_1\gamma_2$

$\beta_1^2\beta_5$

95 $[\alpha_1\underline{3}\gamma_2]\underline{h_{2,0}\gamma_2}$

$\alpha_1\beta_1\beta_{6/3}$

$[\underline{2}\beta_1\beta_2^3]\alpha_1h_{2,0}u_2$

96 $[\beta_1\beta_{6/2}]\underline{h_{2,0}\beta_{6/3}}$

99 $\underline{2}\cdot\underline{3}\gamma_2$

100 $\beta_2\beta_5 = \underline{2}\cdot\underline{2}\beta_{6/2}$

$[\beta_1\beta_6]\underline{h_{11}\beta_{6/1,2}}$

$\underline{2}\beta_1^2\beta_5$

101 $\underline{2}\beta_1^2\beta_5$

102 $\beta_1^2\beta_{6/3} = \pm\beta_{3/3}^3$

$\underline{3}\beta_1\gamma_2$

103 $[\alpha_1\beta_2\beta_5]\underline{\beta_2u_2}$

$[\alpha_1\beta_1\beta_6]\alpha_1h_{11}\beta_{6/1,2}$

104 $\alpha_1\beta_1^2\gamma_2$

$[\beta_1^3\beta_5]\underline{\eta_4}$

105 $[\alpha_1\beta_{3/3}^3]\underline{\beta_{9/9}}$

106 $\underline{\beta_7} \pm \beta_{9/9}$

7.5.4. REMARK. In the calculations below we shall make use of *Toda brackets* (first defined by Toda [**6**]) and their relation to Massey products. Suppose we have spaces (or spectra) and maps $W \xrightarrow{f} X \xrightarrow{g} Y \xrightarrow{h} Z$ with gf and hg null-homotopic. Let $\bar{f}\colon CW \to Y$ and $\bar{g}\colon CX \to Z$ be null homotopies. Define a map $k\colon \Sigma W \to Z$ by regarding ΣW as the union of two copies of CW, and letting the restrictions of k be $h\bar{f}$ and $\bar{g}(Cf)$. k is not unique up to homotopy as it depends on the choice of the null homotopies $\bar{f}$ and $\bar{g}$. Two choices of $\bar{f}$ differ by a map $\Sigma W \to Y$ and similarly for $\bar{g}$. Hence we get a certain coset of $[\Sigma W, Z]$ denoted in Toda [**6**] by $\{f, g, h\}$, but here by $\langle f, g, h\rangle$. Alternatively, let C_g be the cofiber of g, $\bar{h}\colon C_g \to Z$ an extension of h and $\bar{f}\colon \Sigma W \to C_g$ a lifting of Σf. Then k is the composite $\overline{hf}$.

Recall (A1.4.1) that for a differential algebra C with $a, b, c \in H^*C$ satisfying $ab = bc = 0$ the Massey product $\langle a, b, c\rangle$ is defined in a similar way. The interested reader can formulate the definition of higher matric Toda brackets, but any such map can be given as the composite of two maps to and from a suitable auxiliary spectrum (such as C_g). For example, given

$$X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} \cdots \xrightarrow{f_n} X_n$$

satisfying suitable conditions with each X_i a sphere, the resulting n-fold Toda bracket is a composite $\Sigma^{n-2}X_0 \to Y \to X_n$, where Y is a complex with $(n-1)$ cells.

The relation between Toda brackets and Massey products and their behavior in the Adams spectral sequence is studied by Kochman [**2, 4, 5**]. The basic idea of Kochman [**4**] is to show that the Adams spectral sequence arises from a filtered complex, so the spectral sequence results of A1.4 apply. Given Kochman's work we will use Toda brackets and Massey products interchangeably.

7.5.5. REMARK. In the following discussions we will not attempt to keep track of nonzero scalars mod (p). For $p = 3$ this means that a $\pm$ should appear in front of every symbol in an equation. The reader does not have the right to sue for improper coefficients.

Now we provide a running commentary on this list. The notation $\underline{2}x$ denotes the Massey product $\langle \alpha_1, \alpha_1, x\rangle$. If $d_r(y) = \alpha_1 x$ then $\alpha_1 y$ represents $\underline{2}x$. Also note that $\alpha_1 \underline{2}x = \pm\beta_1 x$.

In the 33-stem we have the Toda differential of 4.4.22. The element $\alpha_1\beta_{3/2}$ is a permanent cycle giving $\underline{2}\beta_1^3$. The coboundary of $\frac{v_1 v_2^3}{9v_1^3}$ gives

$$\langle \alpha_1, \alpha_1, \beta_1^3\rangle\langle \beta_2, 3, \beta_1\rangle. \tag{7.5.6}$$

The differentials shown in the 41-, 48-, 52-, and 55-stems can be computed algebraically; i.e., they correspond to relations in Ext. The elements $\alpha_1\beta_{3/2}$, $\beta_1\beta_{3/2}$, and $\beta_1\beta_3$ are the coboundaries of

$$\frac{v_2^3}{9v_1}, \quad \frac{v_2^3 t_1^2}{9v_1} + \frac{v_2^3 t_2}{3v_1^3}, \quad \text{and} \quad \frac{v_1^2 v_2^3 t_1^3}{9v_1^3} - \frac{v_2^3 t_2}{3v_1^2},$$

respectively. We also have $3(\underline{2}\beta_{3/2}) = \alpha_1\beta_3$, i.e., $\pi_{45}(S^0) = \mathbf{Z}/(9)$.

For the differential in the 56-stem we claim $\alpha_1\eta_1 = \pm\beta_2\beta_{3/3}$, forcing $d_5(\eta_1) = \pm\beta_1^3\beta_2$ in the Adams–Novikov spectral sequence. The claim could be verified by direct calculation, but the following indirect argument is easier. $\beta_2\beta_{3/3}$ must be nonzero and hence a multiple of $\alpha_1\eta_1$ because $\alpha_1\beta_1^3\beta_2 \neq 0$ in Ext and must be killed by a differential.

However, we will need the direct calculation in the future, so we record it now for general p. Consider the element

$$\frac{v_2^{i-1}v_3(t_2 - t_1^{1+p}) - v_2^i(t_3 - t_1t_2^p - t_2t_1^{p^2} + t_1^{1+p+p^2}) + v_2^{i+p-1}(t_1^{p+2} - t_1t_2)}{pv_1}$$
$$+\frac{2v_2^{i+p}}{p^2v_1^p(i+p)}\sum_{0<j<p}\frac{(-1)^j}{j}p^{j-1}v_1^{p-j}t_1^j \quad \text{with} \quad i > 0.$$

Straightforward calculation shows the coboundary is

$$\frac{v_2^{i+1}b_{1,1}}{pv_1} + \frac{2p^{p-3}v_2^{i+p}b_{1,0}}{(i+p)v_1^p} - \frac{v_2^{i-1}(v_2t_2^p + v_2^pt_2 - v_2t_1^{p+p^2} - v_3t_1^p)|t_1}{pv_1}, \tag{7.5.7}$$

which gives the desired result since the third term represents η_i. The second term is nonzero in our range only in the case $i = p = 3$, where we have $\pm\alpha_1\eta_3 = \beta_4b_{1,1} + \beta_{6/3}b_{1,0}$. This element is also the coboundary of $\frac{v_2^6t_1^3}{9v_1^3} + \frac{v_2^5t_1^6}{3v_1^2}$, so $\alpha_3\eta_3 = 0$.

For the differentials in the 57- and 60-stems we claim $\beta_2^3 = \pm\beta_1^2\beta_4 \pm \beta_{3/3}^2\beta_1$ in Ext. This must be a permanent cycle since β_2 is. It is straightforward that $d_5(\beta_{3/3}^2) = \pm\underline{2}\beta_1^6$ in the Adams–Novikov spectral sequence, so we get $d_5(\beta_4) = \pm\underline{2}\beta_1^5$. Then $\beta_1^6 = \alpha_1\underline{2}\beta_1^5 = 0$ in $\pi_*(S^0)$, so $d_9(\alpha_1\beta_4) = \beta_1^6$.

To verify our claim that $\beta_2^3 = \pm\beta_1^2\beta_4 \pm \beta_1\beta_{3/3}^2$, it suffices to compute in $\mathrm{Ext}(BP_*/I_2)$. The mod I_2 reductions of β_2, β_4, and $\beta_{3/3}$ are $v_2b_{1,0} \pm k_0$, $v_2^3b_{1,0}$, and $b_{1,1}$, respectively, where $k_0 = \langle h_{10}, h_{11}, h_{11}\rangle$. A Massey product manipulation shows $k_0^3 = b_{1,0}b_{1,1}^2$ and the result follows.

Now we will show

$$x_{68} = \langle\alpha_1, \beta_{3/2}, \beta_2\rangle. \tag{7.5.8}$$

We can do this calculation in Ext and work mod I, i.e., in Ext_P, and it suffices to show that the indicate product is nonzero. We have

$$\begin{aligned}\langle h_{10}, h_{10}h_{12}, \langle h_{11}, h_{11}, h_{10}\rangle\rangle &= \langle h_{10}, h_{10}, h_{12}\langle h_{11}, h_{11}, h_{10}\rangle\rangle\\ &= \langle h_{10}, h_{10}, \langle h_{12}, h_{11}, h_{11}\rangle h_{10}\rangle\\ &= b_{1,0}\langle h_{12}, h_{11}, h_{11}\rangle\\ &= \langle b_{1,0}h_{12}, h_{11}, h_{11}\rangle = \langle b_{1,1}h_{11}, h_{11}, h_{11}\rangle\\ &= b_{1,1}^2 \neq 0.\end{aligned}$$

This element satisfies $\beta_1x_{68} = \beta_2^3$. To show $\alpha_1x_{68} = 0$, consider the coboundary of

$$\frac{v_2^2b_{2,0} \pm v_2v_3b_{1,0}}{3v_1} \pm \frac{v_1v_2^3b_{1,0}}{9v_1^3}.$$

Next we show that there is a nontrivial extension in the 75-stem. We have

$$\begin{aligned}\beta_1^3\langle\alpha_1, \alpha_1, \beta_{3/2}\rangle &= \langle\beta_1^3, \alpha_1, \alpha_1\rangle\beta_{3/2}\\ &= \langle\beta_2, 3, \beta_1\rangle\beta_{3/2} \quad \text{by 7.5.6}\\ &= 3\langle\beta_1, \beta_{3/2}, \beta_2\rangle = 3\langle\langle\alpha_1, \alpha_1, \alpha_1\rangle, \beta_{3/2}, \beta_2\rangle\\ &= 3\underline{2}\langle\alpha_1, \beta_{3/2}, \beta_2\rangle = 3\underline{2}x_{68}.\end{aligned}$$

For the differential in the 77-stem note that $\alpha_1\beta_5$ is the coboundary of $u_2 = \frac{v_2^6}{3v_1^4} + \frac{v_2^2 v_3}{3v_1}$.

This brings us to the 88-stem, where we need to show $\beta_1^2 x_{68} = 0$. Since $x_{68} = \langle \alpha_1, \alpha_1, \beta_1^5, \beta_1 \rangle$ we can show $\beta_1^3 x_{68} = 0$. There is no element in the 99-stem other than $\beta_1\eta_3$ to kill it, so the differential follows.

The differential in the 89-stem is similar to that in the 41-stem. The one in the 92-stem follows from 7.5.7.

In the 95-stem $\alpha_1 h_{11}\gamma_2$ the coboundary of $\frac{v_3^2 t_2}{3v_1v_2}$. The differential in the 105-stem is a special case of 6.4.1. The others are straightforward. The resulting homotopy groups are shown in Table A3.4.

6. Computations for $p = 5$

We will apply the results and techniques of Section 9 to compute up to the 1000-stem for $p = 5$. Naturally the lists **I** and **O** are quite long. The length of **O**, i.e., the number of additive generators in coker J through dimension k, appears to be roughly a quadratic function of k in our range. The conventions of 7.5.4 and 7.5.5 are still in effect.

The highlight of the 5-primary calculation is the following result

7.6.1. THEOREM. *For $p{=}5$, $\beta_1^{17} \neq 0$ and there are Adams–Novikov differentials $d_{33}(\gamma_3) = \beta_1^{18}$. Consequently the Smith–Toda complex $V(3)$ does not exist, and $V(2)$ is not a ring spectrum.* □

7.6.2. CONJECTURE. *For $p \geq 7$, $\beta_1^{p^2-p} \neq 0$ and $\beta_1^{p^2-p+1} = 0$. Moreover $\langle \gamma_3, \gamma_2, \ldots, \gamma_2 \rangle = \beta_1^{(2p-1)(p-1)/2}$ where γ_2 appears in the bracket $(p-5)/2$ times.* □

We will prove 7.6.1 modulo certain calculations to be carried out below. First we give a classical argument due to Toda for $\beta_1^{p^2-p+1} = 0$. We know $\alpha_1\beta_1^p = 0$ from Toda [**2, 3**]. It follows by bracket manipulations that $w_i = \langle \alpha_1, \alpha_1, \ldots, \alpha_1, \beta_1^{ip} \rangle$ is defined with $(i+1)$ factors α_1 and $1 \leq i \leq p-2$. The corresponding ANSS element is $\alpha_1\beta_{p/p}^i$. Now since $\beta_1 = \langle \alpha_1, \ldots, \alpha_1 \rangle$ with p factors we have [using A1.4.6(c)]

$$\alpha_1 w_{p-2} = \langle \alpha_1, \ldots, \alpha_1 \rangle \beta_1^{p^2-2p} = \beta_1^{p^2-2p+1}.$$

Hence $\beta_1^{p^2-p+1}$ is divisible by $\alpha_1\beta_1^p$ and is therefore zero. The corresponding Adams–Novikov differential is $d_r(\alpha_1\beta_{p/p}^{p-1}) = \beta_1^{p^2-p+1}$ with $r = 2p^2 - 4p + 3$.

We will give a more geometric translation of this argument for $p = 5$. Let $X_i = T(1)^{iq} = S^0 \bigcup_{\alpha_1} e^q \bigcup_{\alpha_1} \cdots \bigcup_{\alpha_1} e^{iq}$. The Toda bracket definition of β_1 means there is a diagram

(7.6.3)
$$\begin{tikzcd} S^{31} \arrow[r] & X_3 \\ S^{38} \arrow[u, "\alpha_1"] \arrow[r, "\beta_1"] & S^0 \arrow[u] \end{tikzcd}$$

where the cofiber of the top map is X_4. From $\alpha_1\beta_1^5 = 0$ we get a diagram

$$\begin{tikzcd}
\Sigma^{190}X_1 \arrow[dr, "f"] & \\
S^{190} \arrow[u] \arrow[r, "\beta_1^5"] & S^0.
\end{tikzcd}$$

We smash this with itself three times and use the fact that X_3 is a retract of X_1^3 to get

$$\begin{tikzcd}
\Sigma^{570}X_3 \arrow[dr, "g"] & \\
S^{570} \arrow[u] \arrow[r, "\beta_1^{15}"] & S^0.
\end{tikzcd}$$

Combining this with 7.6.3 we get

$$\begin{tikzcd}
& S^{601} \arrow[r] & \Sigma^{570}X_3 \arrow[dr, "g"] & \\
S^{79q} \arrow[ur, "0"] \arrow[r, "\beta_1^5"] & S^{608} \arrow[u, "\alpha_1"] \arrow[r, "\beta_1"] & S^{570} \arrow[u] \arrow[r, "\beta_1^{15}"] & S^0
\end{tikzcd}$$

so $\beta_1^{21} = 0$.

The calculation below shows that $\alpha_1\beta_{5/5}^4$ is a linear combination of $\beta_1^3\gamma_3$, $\underline{3}\beta_1^3\beta_{14}$, and $\beta_1 x_{761}$, where

$$x_{761} = \langle \alpha_1\beta_3, \beta_4, \gamma_2\rangle \in \mathrm{Ext}^{7,768}.$$

Each factor of x_{761} is a permanent cycle, so x_{761} can fail to be one only if one of the products $\alpha_1\beta_3\beta_4$ and $\beta_4\gamma_2$ is nonzero in homotopy. But these products lie in stems 323 and 619 which are trivial, so x_{761} is a permanent cycle, as is $\underline{3}\beta_1^3\beta_{14}$. Since $d_{33}(\alpha_1\beta_{5/5}^4) = \beta_1^{21}$, we must have $d_{33}(\gamma_3) = \beta_1^{18}$ as claimed.

The nonexistence of γ_3 as a homotopy element shows the Smith–Toda complex $V(3)$ [satisfying $BP_*(V(3)) = BP_*/I_4$] cannot exist for $p = 5$. If one computes the Adams–Novikov spectral sequence for $V(2)$ through dimension 248, one finds that $v_3 \in \mathrm{Ext}^0$ is a permanent cycle; i.e., v_3 is realized by a map $S^{248} \xrightarrow{f} V(2)$. If $V(2)$ were a ring spectrum we could use the multiplication to extend f to a self-map with cofiber $V(3)$, giving a contradiction.

Now we proceed with the calculation for $p = 5$.

7.6.4. THEOREM. *For $p = 5$ $\mathbf{N}$ as in 7.1.18 has basis elements in dimensions indicated below, with notation as in 7.5.2. η_i denotes $h_{1,1}u_{i-1}$.*

38 β_1

86 β_2

134 β_3

182 β_4

198 $\beta_{5/5} = b_{1,1}$

206 $\beta_{5/4}$

214 $\beta_{5/3}$

222 $\beta_{5/2}$

230 β_5

$\beta_{5/1,2}$

245 $h_{2,0}b_{1,1}$

253 $h_{2,0}\beta_{5/4}$

261 $h_{2,0}\beta_{5/3}$

269 $h_{11}\beta_{5/1,2}$

277 η_1

278 β_6

324 $b_{2,0}\beta_2$

325 η_2

326 β_7

372 $b_{2,0}\beta_3$

373 η_3

374 β_8
396 $\beta_{5/5}^2$
404 $\beta_{5/5}\beta_{5/4}$
412 $\beta_{5/4}^2$
420 $b_{2,0}\beta_4$
422 β_9
430 u_4
437 γ_2
438 $\beta_{10/5}$
443 $h_{2,0}b_{1,1}^2$
446 $\beta_{10/4}$
451 $h_{2,0}b_{1,1}\beta_{5/4}$
454 $\beta_{10/3}$
459 $h_{2,0}b_{1,1}\beta_{5/3}$
462 $\beta_{10/2}$
469 η_5
470 β_{10}
$\beta_{10/1,2}$
477 $h_{2,0}u_4$
476 $h_{11}\gamma_2$
484 $h_{2,0}\gamma_2$
485 $h_{2,0}\beta_{10/5}$
493 $h_{2,0}\beta_{10/4}$
501 $h_{2,0}\beta_{10/3}$
509 $h_{11}\beta_{10/1,2}$
515 $b_{2,0}\eta_1$
516 $h_{2,0}\eta_5$
517 η_6
518 β_{11}
523 $\beta_2\gamma_2$
562 $b_{2,0}^2\beta_2$
563 $b_{2,0}\eta_2$
564 $b_{2,0}\beta_7$
565 η_7
566 β_{12}
594 $\beta_{5/5}^3$
602 $\beta_{5/5}^2\beta_{5/4}^2$
610 $b_{2,0}^2\beta_3$
612 $b_{2,0}\beta_8$
613 η_8
614 β_{13}
620 $b_{2,0}u_3$
628 $b_{1,1}u_4$
635 $b_{1,1}\gamma_2$
636 $b_{1,1}\beta_{10/5}$
641 $h_{2,0}b_{1,1}^3$
644 $\beta_{5/4}\beta_{10/5}$
649 $h_{2,0}b_{1,1}^2\beta_{5/4}$
652 $\beta_{10/5}\beta_{5/3}$
659 $h_{2,0}b_{2,0}\beta_8$
660 $b_{2,0}\beta_9$
662 β_{14}
667 $h_{2,0}b_{2,0}u_3$
670 u_9
675 $h_{2,0}b_{1,1}u_4$
678 $\beta_{15/5}$
682 $h_{2,0}b_{1,1}\gamma_2$
683 $h_{2,0}b_{1,1}\beta_{10/5}$
685 γ_3
686 $\beta_{15/4}$
691 $h_{2,0}b_{1,1}\beta_{10/4}$
694 $\beta_{15/3}$
699 $\eta_1\beta_9$
702 $\beta_{15/2}$
706 $h_{2,0}b_{1,1}b_{2,0}u_3$
707 $b_{2,0}\eta_5$
709 η_{10}
710 β_{15}
$\beta_{15/1,2}$
714 $h_{11}b_{2,0}\gamma_2$
717 $h_{2,0}u_9$
724 $h_{11}\gamma_3$
725 $h_{2,0}\beta_{15/5}$
732 $h_{2,0}\gamma_3$
733 $h_{2,0}\beta_{15/4}$
741 $h_{2,0}\beta_{15/3}$
749 $h_{11}\beta_{15/1,2}$
753 $b_{2,0}^2\eta_1$
754 $h_{2,0}b_{2,0}\eta_5$
755 $b_{2,0}\eta_6$
756 $h_{2,0}\eta_{10}$
757 η_{11}
758 β_{16}
761 $b_{2,0}\beta_2\gamma_2$
771 $\beta_2\gamma_3$
792 $\beta_{5/5}^4$
800 $b_{2,0}^3\beta_2$
802 $b_{2,0}^2\beta_7$
803 $b_{2,0}\eta_7$
804 $b_{2,0}\beta_{12}$
805 η_{12}
806 β_{17}
810 $b_{2,0}^2u_2$
818 $b_{1,1}b_{2,0}u_3$
826 $b_{1,1}^2u_4$
833 $b_{1,1}^2\gamma_2$
834 $b_{1,1}^2\beta_{10/5}$
839 $h_{2,0}b_{1,1}^4$
842 $b_{1,1}^2\beta_{10/4}$
849 $h_{2,0}b_{2,0}^2\beta_7$
850 $b_{2,0}^2\beta_8$
852 $b_{2,0}\beta_{13}$
873 η_{13}
854 β_{18}
857 $h_{2,0}b_{2,0}^2u_2$

860 $b_{2,0}u_8$

865 $h_{2,0}b_{1,1}b_{2,0}u_3$

868 $b_{1,1}u_9$

873 $h_{2,0}b_{1,1}^2u_4$

876 $\beta_{10/5}^2$

880 $h_{2,0}b_{1,1}^2\gamma_2$

881 $h_{2,0}b_{1,1}^2\beta_{10/5}$

884 $\beta_{5/4}\beta_{15/5}$

889 $h_{2,0}b_{1,1}^2\beta_{10/4}$

892 $\beta_{10/4}^2$

896 $h_{2,0}h_{11}b_{2,0}^2u_2$

897 $h_{11}b_{2,0}^2u_3$

899 $h_{2,0}b_{2,0}\beta_{13}$

900 $b_{2,0}\beta_{14}$

902 β_{19}

907 $h_{2,0}b_{2,0}u_8$

910 u_{14}

915 $h_{2,0}b_{1,1}u_9$

918 $\beta_{20/5}$

923 $h_{2,0}b_{1,1}\beta_{15/5}$

926 $\beta_{20/4}$

930 $h_{2,0}b_{1,1}\gamma_3$

931 $h_{2,0}b_{1,1}\beta_{15/4}$

933 γ_4

934 $\beta_{20/3}$

939 $\eta_1\beta_{14}$

942 $\beta_{20/2}$

944 $h_{2,0}h_{11}b_{2,0}^2u_3$

945 $b_{2,0}^2\eta_5$

946 $h_{2,0}h_{11}b_{2,0}u_8$

947 $b_{2,0}\eta_{10}$

949 η_{15}

950 β_{20}

$\beta_{20/1,2}$

952 $h_{11}b_{2,0}^2\gamma_2$

957 $h_{2,0}u_{14}$

962 $h_{11}b_{2,0}\gamma_3$

965 $h_{2,0}\beta_{20/5}$

972 $h_{11}\gamma_4$

980 $h_{2,0}\gamma_4$

981 $h_{2,0}\beta_{20/3}$

989 $h_{11}\beta_{10/1,2}$

992 $h_{2,0}b_{2,0}^2\eta_5$

993 $b_{2,0}^2\eta_6$

994 $h_{2,0}b_{2,0}\eta_{10}$

995 $b_{2,0}\eta_{11}$

996 $h_{2,0}\eta_{15}$

997 η_{16}

998 β_{21}

$\beta_{25/25}$

999 $b_{2,0}^2\beta_2\gamma_2$

1000 $b_{2,0}^2u_1$ □

Now we will describe the list **O**, i.e., the analog of 7.5.3. The notation of that result is still in force, and we assume the reader is familiar with techniques used there. We will not comment on differentials with an obvious 3-primary analog, in particular on those following from 7.5.7. Many differentials we encounter are periodic under v_2 or v_2^p.

Since the list **O** is quite long, we will give it in six installments, pausing for comments and proofs when appropriate.

7.6.5. THEOREM. *For $p = 5$ the list* **O** (7.1.18) *is as follows.* (*First installment*)

38 $\underline{\beta_1}$

45 $\alpha_1\beta_1$

76 β_1^2

83 $\alpha_1\beta_1^2$

86 $\underline{\beta_2}$

93 $\alpha_1\beta_2$

114 β_1^3

121 $\alpha_1\beta_1^3$

124 $\beta_1\beta_2$

131 $\alpha_1\beta_1\beta_2$

134 $\underline{\beta_3}$

141 $\alpha_1\beta_3$

152 β_1^4

159 $\alpha_1\beta_1^4$

162 $\beta_1^2\beta_2$

169 $\alpha_1\beta_1^2\beta_2$

172 $\beta_1\beta_3$

179 $\alpha_1\beta_1\beta_3$

182 $\underline{\beta_4}$

189 $\alpha_1\beta_4$

190 β_1^5

197 $[\alpha_1\beta_1^5]\underline{\beta_{5/5}}$

200 $\beta_1^3\beta_2$

205 $\underline{2}\beta_1^5$

206 $\underline{\beta_{5/4}}$

207 $\alpha_1\beta_1^3\beta_2$

210 $\beta_1^2\beta_3$

213 $\alpha_1\beta_{5/4}$
214 $\underline{\beta_{5/3}}$
217 $\alpha_1\beta_1^2\beta_3$
220 $\beta_1\beta_4$
221 $\alpha_1\beta_{5/3}$
222 $\underline{\beta_{5/2}}$
227 $\alpha_1\beta_1\beta_4$
228 $\underline{\beta_1^6}$
229 $[\alpha_1\beta_{5/2}]\underline{\beta_{5/1,2}}$
230 $\underline{\beta_5}$
237 $\underline{2}\beta_{5/2}$
$\alpha_1\beta_5$
238 $\beta_1^4\beta_2$
243 $\underline{2}\beta_1^6$
244 $[\beta_1\beta_{5/4}]\underline{h_{2,0}b_{1,1}}$
245 $\alpha_1\beta_1^4\beta_2$
248 $\beta_1^3\beta_3$
251 $[\alpha_1\beta_1\beta_{5/4}]\alpha_1 h_{2,0}b_{1,1}$
252 $[\beta_1\beta_{5/3}]\underline{h_{2,0}\beta_{5/4}}$
255 $\alpha_1\beta_1^3\beta_3$
258 $\beta_1^2\beta_4$
259 $[\alpha_1\beta_1\beta_{5/3}]\alpha_1 h_{2,0}\beta_{5/4}$
260 $[\beta_1\beta_{5/2}]\underline{h_{2,0}\beta_{5/3}}$
265 $\alpha_1\beta_1^2\beta_4$
266 β_1^7
268 $[\beta_1\beta_5]\underline{h_{11}\beta_{5/1,2}}$
$\beta_2\beta_4$
275 $[\alpha_1\beta_1\beta_5]\alpha_1 h_{11}\beta_{5/1,2}$
276 $[\beta_1^5\beta_2]\underline{\eta_1}$
278 $\underline{\beta_6}$
281 $\underline{2}\beta_1^7$
283 $[\alpha_1\beta_1^4\beta_2]\alpha_1\eta_1$
285 $\alpha_1\beta_6$
286 $\beta_1^4\beta_3$
293 $\alpha_1\beta_1^4\beta_3$
296 $\beta_1^3\beta_4$
303 $\alpha_1\beta_1^3\beta_4$
304 β_1^8
306 $\beta_1\beta_2\beta_4$
313 $\alpha_1\beta_1\beta_2\beta_4$
316 $\beta_1\beta_6$
319 $\underline{2}\beta_1^8$
323 $[\alpha_1\beta_1\beta_6]\underline{b_{2,0}\beta_2}$
324 $[\beta_1^5\beta_3]\underline{\eta_2}$
326 β_7
331 $[\alpha_1\beta_1^5\beta_3]\alpha_1\eta_2$
$\underline{2}\beta_1\beta_6$
333 $\alpha_1\beta_7$
334 $\beta_1^4\beta_4$
341 $\alpha_1\beta_1^4\beta_4$
342 β_1^9
344 $\beta_1^2\beta_2\beta_4$
351 $\alpha_1\beta_1^2\beta_2\beta_4$
354 $\beta_1^2\beta_6$
357 $\underline{2}\beta_1^9$
364 $\beta_1\beta_7$
369 $\underline{2}\beta_1^2\beta_6$
371 $[\alpha_1\beta_1\beta_7]\underline{b_{2,0}\beta_3}$
372 $[\beta_1^5\beta_4]\underline{\eta_3}$
374 $\underline{\beta_8}$
379 $[\alpha_1\beta_1^5\beta_4]\alpha_1\eta_3$
$\underline{2}\beta_1\beta_7$
380 β_1^{10}
381 $\alpha_1\beta_8$
382 $\beta_1^3\beta_2\beta_4$
389 $\alpha_1\beta_1^3\beta_2\beta_4$
392 $\beta_1^3\beta_6$
395 $[\underline{2}\beta_1^{10}]\underline{\beta_{5/5}^2}$
402 $\beta_1^2\beta_7$
403 $\underline{3}\beta_1^{10}$
404 $\underline{\beta_{5/5}\beta_{5/4}}$
$= \langle \alpha_1\beta_1^4, \beta_1, \beta_{5/4}\rangle$
$= x_{404}$
407 $\underline{2}\beta_1^3\beta_6$
411 $\alpha_1 x_{404} = \beta_{5/4}\underline{2}\beta_1^5$
412 $\beta_1\beta_8$
$\beta_1\beta_8 + \underline{\beta_{5/4}^2} = x_{412}$
417 $\underline{2}\beta_1^2\beta_7$
418 β_1^{11}
419 $[\alpha_1 x_{412}]\underline{b_{2,0}\beta_4}$
$\alpha_1\beta_1\beta_8$
420 $\beta_1^4\beta_2\beta_4$
422 β_9
427 $\underline{2}x_{412}$
$\alpha_1\beta_1^4\beta_2\beta_4$
429 $[\alpha_1\beta_9]\underline{u_4}$ □

7.6.6. REMARK. The small descent spectral sequences of 7.1.13 and 7.1.16 have some useful multiplicative structure even though $T(0)_{(1)}$ (the complex with p cells) is not a ring spectrum and its BP-homology is not a comodule algebra. Recall that $T(0)_i$ is the iq-skeleton of $T(1)$. Then $\pi_*(T(0)_{(1)})$ is filtered by the images of $\pi_*(T(0)_i)$ for $i \le p-1$. One has maps $T(0)_i \wedge T(0)_j \to T(0)_{i+j}$ inducing pairings $F_i \otimes F_j \to F_{i+j}$ for $i+j \le p-1$. spectral sequence differentials always lower this filtration degree and respect this pairing. The filtration can be dualized as follows.

A map $S^m \to T(0)_i$ is dual to a map $\Sigma^{m-iq}T(0)_i \to S^0$ since $DT(0)_i = \Sigma^{-iq}T(0)_i$ for $i \le p-1$. An element in $\pi_m(T(0)_{p-1})$ is in F_i iff the diagram

$$\begin{tikzcd} S^m \arrow[r] & T(0)_{p-1} \\ \Sigma^{m-qi}T(0)_i \arrow[u] \arrow[r, dashed] & S^0 \arrow[u] \end{tikzcd}$$

can be completed. The pairing $T(0)_i \wedge T(0)_j \to T(0)_{i+j}$ dualizes to $DT(0)_{i+j} \to DT(0)_i \wedge DT(0)_j$. If $\alpha \in \pi_m(T(0)_{p-1})$ is in F_i and $\beta \in \pi_n(T(0)_{p-1})$ is in F_j with $i + j = p$, then we get a map $\Sigma^{m+n}DT(0)_p \to S^0$. If this map is trivial on the bottom cell then it factors through $\Sigma^{m+n}DT(0)_{p-1} = \Sigma^{m+n-q(p-1)}T(0)_{p-1}$. This factorization will often lead to a differential in our spectral sequence.

For the differentials in dimensions 323, 371, and 419 recall (4.3.22) that there is an element $b_{2,0} \in C(BP_*/I_2)$ with $d(b_{2,0}) = (b_{1,0}|t_1^{p^2}) - (t_1^9|b_{1,1})$. Since $b_{1,0}$ and $t_1^{p^2}$ are both cycles there is a $y \in C(BP_*/I_2)$ such that $d(y) = (b_{1,0}|t_1^{p^2}) - (t_1^{p^2}|b_{1,0})$. Hence the coboundary of

$$\frac{v_2^{i-p}v_1b_{1,0} + v_2^{i+p-1}(y - b_{2,0})}{pv_1} - \frac{v_2^{i+2-p}b_{1,1}}{(1+2-p)pv_1^2} \quad \text{for } i \ge p$$

is

$$\frac{v_2^i t_1|b_{1,0}}{pv_1} + \frac{2v_2^{i+2-p}|b_{1,1}}{(i+2-p)v_1^3},$$

where the second term is nonzero only if $i \equiv -2 \mod (p)$. This gives

$$\alpha_1\beta_1\beta_i = \begin{cases} 0 & \text{for } i \ge p,\ i \not\equiv -2 \mod (p) \\ \alpha_1\beta_{(i+2-p)/4}\beta_{p/p-1} & \text{for } i \equiv -2. \end{cases} \tag{7.6.7}$$

Remember (7.5.5) we are not keeping track of nonzero scalar coefficients. The differentials in question follow.

Next we show that there is a nontrivial group extension in the 427-stem, similar to that for $p = 3$ in the 75-stem. We want to prove $\alpha_1\beta_1^4\beta_2\beta_4 = 5\underline{2}x_{412}$. Since $\alpha_1\beta_2\beta_4 = \beta_1\underline{2}\beta_{5/2}$ we need to look at $\beta_1^5\underline{2}\beta_{5/2}$. We have

$$\begin{aligned} \beta_1^5\underline{2}\beta_{5/2} &= \beta_1^5\langle\alpha_1, \alpha_1, \beta_{5/2}\rangle = \beta_1^5\langle\alpha_1, \alpha_3, \beta_{5/4}\rangle = \langle\beta_1^5, \alpha_1, \alpha_3\rangle\beta_{5/4} \\ &= \alpha_1\beta_{5/3}\beta_{5/4} = \alpha_1\langle\alpha_1, 5, \beta_{5/4}\rangle\beta_{5/4} = \alpha_1\langle\alpha_1, 5, \beta_{5/4}^2\rangle \\ &= \alpha_1\langle\alpha_1, 5, x_{412}\rangle = x_{412}\langle\alpha_1, \alpha_1, 5\rangle = 5\underline{2}x_{412}. \end{aligned}$$

More generally one has

$$\beta_1^p\underline{2}\beta_{p/2} = p\underline{2}(\beta_{p/4}\beta_{p/p-1} + \beta_1\beta_{2p-2}). \tag{7.6.8}$$

Since

$$\alpha_1\langle\alpha_1, 5, x_{412}\rangle = \beta_1^5\underline{2}\beta_{5/2} = \alpha_1\beta_1^4\beta_2\beta_4$$

we have

$$\langle\alpha_1, 5, x_{412}\rangle = \beta_1^4\beta_2\beta_4$$

7.6.5 (*Second installment*)

430 $\beta_1^4\beta_6$

437 $\underline{2}\beta_9$

$\underline{\gamma_2}$

438 $\underline{\beta_{10/5}}$

440 $\underline{3}\beta_1^4$

442 $[\beta_1 x_{404}]h_{2,0}b_{1,1}^2$

444 $\alpha_1\gamma_2$

445 $\alpha_1\beta_{10/5}$

$\underline{2}\beta_1^4\beta_6$

446 $\underline{\beta_{10/4}}$

449 $[\alpha_1\beta_1 x_{404}]\alpha_1 h_{2,0}b_{1,1}^2$

450 $\beta_1^2\beta_8$

$[\beta_1\beta_{5/4}^2]\underline{h_{2,0}b_{1,1}\beta_{5/4}}$

453 $\alpha_1\beta_{10/4}$

454 $\underline{\beta_{10/3}}$

455 $\underline{2}\beta_1^3\beta_7$

456 β_1^{12}

457 $[\alpha_1\beta_1^2\beta_8]\alpha_1 h_{2,0}b_{1,1}\beta_{5/4}$

458 $[\beta_1^5\beta_2\beta_4]\underline{h_{2,0}b_{1,1}\beta_{5/3}}$

460 $\beta_1\beta_9$

461 $\alpha_1\beta_{10/3}$

462 $\underline{\beta_{10/2}}$

465 $\underline{2}\beta_1^2\beta_8$

$[\alpha_1\beta_1^5\beta_2\beta_4]\alpha_1 h_{2,0}b_{1,1}\beta_{5/3}$

468 $[\beta_1^5\beta_6]\underline{\eta_5}$

469 $[\alpha_1\beta_{10/2}]\underline{\beta_{10/1,2}}$

470 $\underline{\beta_{10}}$

475 $\beta_1\gamma_2$

$\underline{2}\beta_1\beta_9$

476 $[\alpha_1\eta_5 + \beta_1\beta_{10/5}]h_{2,0}u_4$

$\beta_1\beta_{10/5}$

$\underline{5\gamma_2}$

477 $\underline{2}\beta_{10/2}$

$\alpha_1\beta_{10}$

478 $\beta_1^4\beta_7$

479 $\underline{3}\beta_1^{12}$

482 $\alpha_1\beta_1\gamma_2$

483 $\alpha_1\beta_1\beta_{10/5}$

$[\underline{2}\beta_1^5\beta_6]\alpha_1 h_{2,0}u_4$

483 $[\alpha_1\underline{5}\gamma_2]\underline{h_{2,0}\gamma_2}$

484 $[\beta_1\beta_{10/4}]\underline{h_{2,0}\beta_{10/5}}$

488 $\beta_1^3\beta_8$

491 $\underline{2}\,\underline{5}\gamma_2$

$[\alpha_1\beta_1\beta_{10/4}]\alpha_1 h_{2,0}\beta_{10/5}$

492 $[\beta_1\beta_{10/3}]\underline{h_{2,0}\beta_{5/4}}$

493 $\underline{2}\beta_1^4\beta_7$

494 β_1^{13}

498 $\beta_1^2\beta_9$

499 $[\alpha_1\beta_1\beta_{10/3}]\alpha_1 h_{2,0}\beta_{5/4}$

500 $[\beta_1\beta_{10/2}]\underline{h_{2,0}\beta_{10/3}}$

503 $\underline{2}\beta_1^3\beta_8$

508 $\alpha_1 h_{2,0}\beta_{10/3} = \beta_2\beta_9$

$[\beta_1\beta_{10}]\underline{h_{11}\beta_{10/1,2}}$

513 $\beta_1^2\gamma_2$

$\underline{2}\beta_1^2\beta_9$

514 $\beta_1^2\beta_{10/5}$

515 $[\alpha_1\beta_2\beta_9]\underline{h_{2,0}\eta_5}$

$[\beta_1^2\beta_{10/5} + \beta_1\underline{5}\gamma_2]\underline{b_{2,0}\eta_1}$

$[\alpha_1\beta_1\beta_{10}]\alpha_1 h_{11}\beta_{10/1,2}$

516 $[\beta_1^5\beta_7]\eta_6$

517 $\underline{3}\beta_1^{13}$

518 $\underline{\beta_{11}}$

520 $\alpha_1\beta_1^2\gamma_2$

521 $[\alpha_1\beta_1^2\beta_{10/5}]\alpha_1 b_{2,0}\eta_1$

523 $\underline{2}\beta_2\beta_9$

$\underline{\beta_2\gamma_2}$

524 $\alpha_1\eta_6 = \beta_2\beta_{10/5}$

525 $\alpha_1\beta_{11}$

526 $\beta_1^4\beta_8$

529 $\underline{2}\,\underline{5}\beta_1\gamma_2$

530 $\alpha_1\beta_2\gamma_2$

531 $\underline{2}\beta_1^5\beta_7 = \alpha_1\beta_2\beta_{10/5}$

532 β_1^{14}

536 $\beta_1^3\beta_9$

541 $\underline{2}\beta_1^4\beta_8$

546 $\beta_1\beta_2\beta_9$

551 $\beta_1^3\gamma_2$

$\underline{2}\beta_1^3\beta_9$

552 $\underline{5}\beta_1^2\gamma_2$

555 $\underline{3}\beta_1^{14}$

558 $\alpha_1\beta_1^3\gamma_2$

556 $\beta_1\beta_{11}$

561 $\underline{2}\beta_1\beta_2\beta_9$

$[\beta_1\beta_2\gamma_2 + \underline{2}\beta_1\beta_2\beta_9]\underline{b_{2,0}^2\beta_2}$

562 $[\beta_1\beta_2\beta_{10/5}]\underline{b_{2,0}\eta_2}$

563 $[\alpha_1\beta_1\beta_{11}]\underline{b_{2,0}\beta_7}$

564 $[\beta_1^5\beta_8]\underline{\eta_7}$

566 $\underline{\beta}_{12}$

567 $\underline{2}\,\underline{5}\beta_1^2\gamma_2$

568 $[\alpha_1\beta_1\beta_2\gamma_2]\alpha_1 b_{2,0}^2\beta_2$

569 $[\alpha_1\beta_1\beta_2\beta_{10/5}]\alpha_1 b_{2,0}\eta_2$ □

For the differential in the 514-stem, note that in the corresponding spectral sequence for $\mathrm{Ext}_P(\mathbf{Z}/(p), \mathbf{Z}/(p))$ the image of $b_{2,0}\eta_1$ kills that of $\beta_1\underline{5}\gamma_2$, so the target in our spectral sequence of $b_{2,0}\eta_1$ is $\beta_1\underline{5}\gamma_2$ plus some multiple of $\beta_1^2\beta_{10/5}$. On the other hand, we have

$$\begin{aligned}\alpha_1\beta_1^2\beta_{10/5} &= \alpha_1\beta_1\beta_6\beta_{5/5}\\ &= \alpha_1\beta_1\langle\beta_6, \alpha_1\beta_1, \beta_1^4\rangle\\ &= \langle\alpha_1\beta_1\beta_6, \alpha_1\beta_1, \beta_1^4\rangle\\ &= 0\end{aligned}$$

and the result follows.

The relation in the 524-stem follows from 7.5.7. The differential in dimension 561 is $h_{2,0}$ times that in the 514-stem. The one in dimension 562 comes from a relation in Ext, i.e., $\beta_1^2\beta_2\beta_{10/5} = \beta_1^2\beta_7\beta_{5/5} = \beta_1\beta_6\alpha_1\eta_1 = 0$ since $\alpha_1\beta_1\beta_6 = 0$. Theorem 7.6.4 shows that there is no element in dimension 601 to give this relation, so we must have $\beta_1\beta_2\beta_{10/5}$ as indicated.

More generally, we have in Ext for $1 < i < p$ and $j > 1$

$$\begin{aligned}\beta_1^2\beta_i\beta_{pi/p} &= \beta_1\beta_{i+p}\beta_{i+pj-2p}\beta_{p/p}\\ &= \beta_1\beta_{1+p}\alpha_1\eta_{i+pj-1-2p} && \text{by 7.5.7}\\ &= 0 && \text{by 7.6.7.}\end{aligned} \tag{7.6.9}$$

In some cases this result along with inspection of $\underline{I}$ implies $\beta_1\beta_i\beta_{pj/p} = 0$.

7.6.5 (*Third installment*)

570 β_1^{15}

571 $\underline{2}\beta_1\beta_{11}$

572 $\alpha_1\eta_7 = \beta_3\beta_{10/5}$

573 $\alpha_1\beta_{12}$

574 $\beta_1^4\beta_9$

579 $\underline{2}\beta_1^5\beta_8 = \alpha_1\beta_3\beta_{10/5}$

584 $\beta_1^2\beta_2\beta_9$

589 $\beta_1^4\gamma_2$

$\underline{2}\beta_1^4\beta_9$

590 $\underline{5}\beta_1^3\gamma_2$

593 $[\underline{3}\beta_1^{15}]\underline{\beta_{5/5}^3}$

594 $\beta_1^2\beta_{11}$

596 $\alpha_1\beta_1^4\gamma_2$

599 $\underline{2}\beta_1^2\beta_2\beta_9$

601 $\underline{4}\beta_1^{15}$

602 $\underline{\beta_{5/5}^2\beta_{5/4}} = \langle \underline{2}\beta_1^9, \beta_1, \beta_{5/4}\rangle = x_{602}$

604 $\beta_1\beta_{12}$

605 $\underline{2}\,\underline{5}\beta_1^3\gamma_2$

608 β_1^{16}

609 $\alpha_1 x_{602}$

$[\underline{2}\beta_1^2\beta_{11} + \alpha_1 x_{602}]b_{2,0}^2\beta_3$

610 $\beta_1\beta_3\beta_{10/5} = \langle \alpha_1, 5, x_{602}\rangle$

611 $[\alpha_1\beta_1\beta_{12}]\underline{b_{2,0}\beta_8}$

612 $[\beta_1^5\beta_9]\underline{\eta_8}$

614 $\underline{\beta}_{13}$

617 $x_{617} = \alpha_1\beta_3 b_{2,0}^2$

$\alpha_1\beta_1\beta_3\beta_{10/5}$

619 $[2\beta_1\beta_{12}]\underline{b_{2,0}u_3}$

620 $\alpha_1\eta_8 = \beta_4\beta_{10/5}$

621 $\alpha_1\beta_{13}$

622 $\beta_1^3\beta_2\beta_9$

627 $\underline{3}\beta_1\beta_{12}$

$\beta_1^5\gamma_2$

$[\alpha_1\beta_4\beta_{10/5}]\underline{b_{1,1}u_4}$

628 $\underline{5}\beta_1^4\gamma_2$

632 $\beta_1^3\beta_{11}$

634 $[\alpha_1\beta_1^5\gamma_2]\underline{b_{1,1}\gamma_2}$

□

The differential in the 609-stem is an Ext relation derived as follows. Since x_{602} is divisible in Ext by $\beta_{5/4}$ we have $d(h_{2,0}b_{1,1}^3) = \beta_1 x_{602}$, so $d(\alpha_1 h_{2,0}b_{1,1}^3) = \alpha_1\beta_1 x_{602}$. On the other hand, whenever $\alpha_1 x = \alpha_1 y = 0$, $\underline{2}xy = 0$, e.g., $\underline{2}(\beta_1\beta_6)^2 = \underline{2}\beta_1^3\beta_{11} = 0$, forcing the image of $b_{2,0}^2\beta_3$ to contain a nonzero multiple of $\underline{2}\beta_1^2\beta_{11}$. Similary

(7.6.10) $$\underline{2}\beta_1^3\beta_k = 0 \quad \text{for all} \quad k \geq 2p+1.$$

In many cases (such as $k = 12$) inspection of $\underline{N}$ (7.6.4) shows $\underline{2}\beta_1^2\beta_k = 0$. To get the other term we compute modulo filtration 2 in our spectral sequence (7.1.16), i.e., mod β_{10}. Then we get $\langle h_{11}, h_{11}, b_{1,1}^2\rangle$ is killed by $b_{2,0}^2$ in $\mathrm{Ext}(BP_*/I_2)$, so $\beta_3 b_{2,0}^2$ kills $\langle \beta_3, h_{11}, h_{11}\rangle b_{1,1}^2$ and the coboundary of $\frac{v_2^5 v_1^2}{25 v_1^5}$ shows $\langle \beta_3, h_{11}, h_{11}\rangle = \beta_{5/4}\alpha_1$.

There is a nontrivial group extension in dimension 617 similar to the one in the 427-stem. We have

$$x_{617} = \left\langle \alpha_1, (\alpha_1\underline{2}\beta_1\beta_6), \begin{pmatrix} x_{602} \\ \beta_6 \end{pmatrix} \right\rangle$$

so

$$\begin{aligned} 5x_{617} &= \langle 5, \alpha_1, (\alpha_1\underline{2}\beta_1\beta_6)\rangle \begin{pmatrix} x_{602} \\ \beta_6 \end{pmatrix} \\ &= \langle 5, \alpha_1, \alpha_1\rangle x_{602} = \alpha_1 x_{602} \\ &= \alpha_2\langle \alpha_1, \beta_1^5, x_{404}\rangle \\ &= \langle \alpha_2, \alpha_1, \beta_1^5\rangle x_{404}. \end{aligned}$$

On the other hand

$$\begin{aligned} \beta_1^5\langle \alpha_1, \alpha_1, x_{412}\rangle &= \beta_1^5 \left\langle \alpha_1, (\alpha_2\alpha_1\beta_2), \begin{pmatrix} x_{404} \\ \beta_7 \end{pmatrix} \right\rangle \\ &= \langle \beta_1^5, \alpha_1, (\alpha_2\alpha_1\beta_2)\rangle \begin{pmatrix} x_{404} \\ \beta_7 \end{pmatrix} \\ &= \langle \beta_1^5, \alpha_1, \alpha_2\rangle x_{404} \end{aligned}$$

so the result follows. We also have $\alpha_2 x_{602} = \alpha_1\langle\alpha_1, 5, x_{602}\rangle$ so $\langle\alpha_1, 5, x_{602}\rangle = \beta_1\beta_3\beta_{10/5}$.

In the 627-stem we have an Ext relation

$$\alpha_1\beta_4\beta_{10/5} = \alpha_1\beta_9\beta_{5/5} = 0.$$

7.6.5 (*Fourth installment*)

635 $\underline{2}\beta_4\beta_{10/5} = \langle\beta_9, \alpha_1, \underline{2}\beta_1^5\rangle$

636 $\underline{\beta_{5/5}\beta_{10/5}} = \langle\beta_1^3, \alpha_1\beta_1^2, \beta_{10/5}\rangle = x_{636}$

637 $\underline{2}\beta_1^3\beta_2\beta_9$

639 $\underline{4}\beta_1^{16}$

640 $[\beta_1 x_{602}]\underline{h_{2,0}b_{1,1}^3}$

642 $\beta_1^2\beta_{12}$

643 $\underline{2}\,\underline{5}\beta_1^4\gamma_2 = \beta_{5/4}\gamma_2$

$\alpha_1 x_{636}$

644 $\underline{\beta_{5/4}\beta_{10/5}}$

646 β_1^{17}

647 $[\underline{2}\beta_1^3\beta_{11}]\alpha_1 h_{2,0}b_{1,1}^3$

648 $[\beta_1^2\beta_3\beta_{10/5}]\underline{h_{2,0}b_{1,1}^2\beta_{5/4}}$

651 $\alpha_1\beta_{5/4}\beta_{10/5}$

652 $\beta_1\beta_{13}$

$\underline{\beta_{5/3}\beta_{10/5}} + \beta_1\beta_{13} = x_{652}$

655 $[\alpha_1\beta_1^2\beta_3\beta_{10/5}]\alpha_1 h_{2,0}b_{1,1}^2\beta_{5/4}$

$\beta_1 x_{617}$

658 $[\beta_1\beta_4\beta_{10/5}]\underline{h_{2,0}b_{2,0}\beta_8}$

659 $[\alpha_1 x_{652}]b_{2,0}\beta_9$

660 $\beta_1^4\beta_2\beta_9$

662 $\underline{\beta}_{14}$

665 $\underline{3}\beta_1^2\beta_{12}$

$\beta_1^6\gamma_2$

666 $\underline{5}\beta_1^5\gamma_2$

$[\alpha_1 h_{2,0}b_{2,0}\beta_8]\underline{h_{2,0}b_{2,0}u_3}$

$\underline{2}\beta_1^5\gamma_2$

667 $\underline{2}x_{652}$

669 $[\alpha_1\beta_{14}]\underline{u_9}$

670 $\beta_1^4\beta_{11}$

673 $[\underline{2}\beta_1\beta_4\beta_{10/5}]\alpha_1 h_{2,0}b_{2,0}u_3$

674 $[\beta_1 x_{636}]\underline{h_{2,0}b_{1,1}u_4}$

675 $\underline{2}\beta_1^4\beta_2\beta_9$

677 $\underline{2}\beta_{14}$

$\underline{4}\beta_1^{17}$

678 $\underline{\beta}_{15/5}$

680 $\beta_1^3\beta_{12}$

$\underline{2}\beta_1^6\gamma_2$

681 $[\alpha_1\beta_1 x_{636}]\alpha_1 h_{2,0}b_{1,1}u_4$

$[\underline{2}\,\underline{5}\beta_1^5\gamma_2]\underline{h_{2,0}b_{1,1}\gamma_2}$

682 $[\beta_1\beta_{5/4}\beta_{10/5}]\underline{h_{2,0}b_{1,1}\beta_{10/5}}$

684 $[\beta_1^{18}]\underline{\gamma_3}$

$\alpha_1\beta_1\beta_{13}$

685 $\alpha_1\beta_{15/5}$

686 $\underline{\beta}_{15/4}$

689 $\underline{3}\,\underline{5}\beta_1^6\gamma_2$

$[\alpha_1\beta_1\beta_{5/4}\beta_{10/5}]\alpha_1 h_{2,0}b_{1,1}\beta_{10/5}$

690 $\beta_1^2\beta_{13}$

$[\beta_1\beta_{5/3}\beta_{10/5}]\underline{h_{2,0}b_{1,1}\beta_{10/4}}$

692 $\alpha_1\gamma_3 = \langle\alpha_1, \beta_1^5, \beta_1^{13}\rangle = x_{692}$

693 $\alpha_1\beta_{15/4}$

$\underline{3}\beta_1^4\beta_{11}$

694 $\underline{\beta}_{15/3}$

697 $[\alpha_1\beta_1\beta_{5/3}\beta_{10/5}]\alpha_1 h_{2,0}b_{1,1}\beta_{10/4}$

698 $[\beta_1^5\beta_2\beta_9]\eta_1\beta_9$

700 $\beta_1\beta_{14}$

701 $\alpha_1\beta_{15/3}$

702 $\underline{\beta}_{15/2}$

703 $\underline{3}\beta_1^3\beta_{12}$

$\beta_1^7\gamma_2$

704 $\underline{5}\beta_1^6\gamma_2$

705 $[\underline{2}\beta_1^2\beta_{13}]\underline{h_{2,0}h_{11}b_{2,0}u_3}$

706 $[\alpha_1\eta_1\beta_9]\underline{b_{2,0}\eta_5}$

708 $[\beta_1^5\beta_{11}]\underline{\eta}_{10}$

709 $[\alpha_1\beta_{15/2}]\underline{\beta}_{15/1,2}$

710 $\underline{\beta_{15}}$

713 $\underline{3}\beta_1^2\beta_{13}$

$[\underline{2}\beta_1^5\beta_2\beta_9]\alpha_1 b_{2,0}\beta_5$

714 $\underline{\eta_1\gamma_2} = \langle \beta_1^4, \beta_1\beta_2, \gamma_2 + \underline{2}\beta_9\rangle = x_{714}$

715 $\underline{3}x_{692}$

$\underline{2}\beta_1\beta_{14}$

716 $\beta_1\beta_{15/5}$

$[\alpha_1\eta_{10} + \beta_1\beta_{15/5}]\underline{h_{2,0}u_9}$

717 $\underline{2}\beta_{15/2}$

$\alpha_1\beta_{15}$

718 $\beta_1^4\beta_{12}$

$\underline{2}\beta_1^7\gamma_2$

721 $\alpha_1 x_{714}$

723 $\alpha_1\beta_1\beta_{15/5}$

724 $\underline{5}\underline{\gamma_3} = \langle \beta_1, 5, \beta_1, \beta_1^{17}\rangle = x_{724}$

$\alpha_1 h_{2,0}u_9 = \langle \beta_1^2, \beta_1^3\beta_{11}, \alpha_1, \alpha_1\rangle$

$= x'_{724}$

$[\beta_1\beta_{15/4}]\underline{h_{2,0}\beta_{15/5}}$

727 $\underline{3}\,\underline{5}\beta_1^6\gamma_2$

728 $\beta_1^3\beta_{13}$

730 $\beta_1 x_{692}$

731 $\underline{3}\beta_1^5\beta_{11}$

$[\alpha_1\beta_1\beta_{15/4}]\alpha_1 h_{2,0}\beta_{15/5}$

$[\alpha_1 x_{724}]\underline{h_{2,0}\gamma_3}$

732 $[\beta_1\beta_{15/3}]\underline{h_{2,0}\beta_{15/4}}$

738 $\beta_1^2\beta_{14}$

739 $\underline{2}x_{724}$

$[\alpha_1\beta_1\beta_{15/3}]\alpha_1 h_{2,0}\beta_{15/4}$

740 $[\beta_1\beta_{15/2}]\underline{h_{2,0}\beta_{15/3}}$

741 $\underline{3}\beta_1^4\beta_{12}$

$\beta_1^8\gamma_2$

742 $\underline{5}\beta_1^7\gamma_2$

748 $\beta_2\beta_{14}$

$[\beta_1\beta_{15}]\underline{h_{11}\beta_{15/1,2}}$

751 $\underline{3}\beta_1^3\beta_{13}$

752 $[\beta_1 x_{714}]\underline{b_{2,0}^2\eta_1}$

753 $[\underline{2}\beta_1^2\beta_{14}]\underline{h_{2,0}b_{2,0}\eta_5}$

$\underline{3}\beta_1 x_{692}$

754 $[\beta_1^2\beta_{15/5}]\underline{b_{2,0}\eta_6}$

755 $[\alpha_1\beta_2\beta_{14}]\underline{h_{2,0}\eta_{10}}$

$[\alpha_1\beta_1\beta_{15}]\alpha_1 h_{11}\beta_{15/1,2}$

756 $\underline{2}\beta_1^8\gamma_2$

$[\beta_1^5\beta_{12}]\underline{\eta_{11}}$

758 $\underline{\beta}_{16}$

759 $[\alpha_1\beta_1 x_{714}]\alpha_1 b_{2,0}^2\eta_1$

761 $[\alpha_1\beta_1^2\beta_{15/5}]\alpha_1 b_{2,0}\eta_6$

$\underline{b_{2,0}\beta_2\gamma_2} = \langle \alpha_1\beta_1, \beta_6, \gamma_2\rangle = x_{761}$

$\underline{3}\beta_1^2\beta_{14}$

762 $\beta_1 x_{724}$

763 $\underline{2}\beta_2\beta_{14}$

$\beta_1 x'_{724}$

764 $\alpha_1\eta_{11} = \beta_2\beta_{15/5}$

765 $\alpha_1\beta_{16}$

$\underline{3}\,\underline{5}\beta_1^7\gamma_2$

766 $\beta_1^4\beta_{13}$

768 $\alpha_1 x_{761} = \gamma_2\underline{2}\beta_1\beta_6$

$\beta_1^2 x_{692}$

769 $\alpha_1\beta_1 x'_{724}$

771 $\underline{\beta_2\gamma_3} = \langle \beta_2, \beta_1^5, \beta_1^{13}\rangle = x_{771}$

776 $\beta_1^3\beta_{14}$

777 $\underline{2}\beta_1 x_{724}$

778 $\alpha_1 x_{771}$

779 $\beta_1^9\gamma_2$

$\underline{2}\beta_2\beta_{15/5}$

780 $\underline{5}\beta_1^8\gamma_2$

786 $\beta_1\beta_2\beta_{14}$

789 $\underline{3}\beta_1^4\beta_{13}$

791 $[\underline{3}\beta_1^3 x_{692}]\beta_{5/5}^4$

796 $\beta_1\beta_{16}$

799 $\underline{3}\beta_1^3\beta_{14}$

$\beta_1 x_{761}$

$[\underline{4}\beta_1^2 x_{692}+?]\underline{b_{2,0}^3\beta_2}$

800 $\beta_1^2 x_{724}$

$\beta_1^2 x'_{724}$

801 $[\underline{2}\beta_1\beta_2\beta_{14}]\underline{b_{2,0}^2\beta_7}$

802 $[\beta_1\beta_2\beta_{15/5}]\underline{b_{2,0}\eta_7}$

803 $\underline{3}\,\underline{5}\beta_1^8\gamma_2$

$[\alpha_1\beta_1\beta_{16}]\underline{b_{2,0}\beta_{12}}$

804 $[\beta_1^5\beta_{13}]\underline{\eta_{12}}$

806 $\underline{\beta_{17}}$

$[\beta_1^3 x_{692} + \alpha_1\beta_1 x_{761}]\alpha_1 b_{2,0}^3\beta_2$

$\alpha_1\beta_1 x_{761}$

807 $\alpha_1\beta_1^2 x'_{724}$

809 $[\underline{3}\beta_1\beta_2\beta_{14}]\underline{b_{2,0}^2 u_2}$

794 $\underline{2}\beta_1^9\gamma_2$ $\beta_1 x_{771}$

810 $\alpha_1 b_{2,0}\eta_7 =$

$\langle\alpha_1, \alpha_1, \beta_1\beta_{12}, \beta_1^5\rangle = \beta_2 x'_{724}$

811 $\underline{2}\beta_1\beta_{16}$

812 $\alpha_1\eta_{12} = \beta_3\beta_{15/5}$

813 $\alpha_1\beta_{17}$

814 $\beta_1^4\beta_{14}$

815 $\underline{2}\beta_1^2 x_{724}$

816 $\alpha_1\beta_1 x_{771}$

817 $\beta_1^{10}\gamma_2$

$\underline{4}\beta_1\beta_2\beta_{14}$

$[\underline{2}\beta_1\beta_2\beta_{15/5}]\underline{b_{1,1}b_{2,0}u_3}$ $\square$

For the relation in the 643-stem we have

$$\begin{aligned}
\beta_{p/p-1} &= \langle\alpha_1\beta_1^{p-1}, \beta_1, p, \alpha_1\rangle \quad \text{and} \\
\underline{2}\,\underline{p}\gamma_2 &= \langle\alpha_1, \alpha_1\beta_1, p, \gamma_2\rangle \quad \text{so} \\
\beta_1^{p-1}\underline{2}\,\underline{p}\gamma_2 &= \langle\alpha_1\beta_1^{p-1}, \alpha_1\beta_1, p, \gamma_2\rangle \\
&= \alpha_1\langle\beta_1^{p-1}, \alpha_1\beta_1, p, \gamma_2\rangle \\
&= \alpha_1\langle\alpha_1\beta_1^{p-1}, \beta_1, p, \gamma_2\rangle \\
&= \alpha_1\langle\alpha_1\beta_1^{p-1}, \beta_1, p, \alpha_1\rangle\gamma_2 \\
&= \beta_{p/p-1}\gamma_2.
\end{aligned}$$

This generalizes immediately to

7.6.11. PROPOSITION. *Let x be an element satisfying $px = 0$, $\langle\alpha_1\beta_1, p, x\rangle = 0$, and $\alpha_1 x \neq 0$. Then $\beta_{p/p-1}x = \beta_1^{p-1}\underline{2}\,\underline{p}x$.* $\square$

For the differentials in dimensions 666 and 673 it suffices to show $\beta_1^2\underline{2}\beta_4\beta_{10/5} = 0$. We have $\beta_4\beta_{10/5} = \beta_9\beta_{5/5} = \langle\beta_9, \alpha_1, \beta_1^5\rangle$ so $\underline{2}\beta_4\beta_{10/5} = \langle\beta_9, \alpha_1, \underline{2}\beta_1^5\rangle$. Then

$$\begin{aligned}
\beta_1^2\underline{2}\beta_4\beta_{10/5} &= \langle\beta_1^2\beta_9, \alpha_1, \underline{2}\beta_1^5\rangle \\
&= \langle\alpha_1\beta_1\beta_6, \beta_4, \underline{2}\beta_1^5\rangle = 0.
\end{aligned}$$

The differential on γ_3 is explained in 7.6.1. Recall that the key point was that $\alpha_1\beta_{5/5}^4$ in Ext is a linear combination of the three elements $\underline{3}\beta_1^3 k_{14}$, $\beta_1 x_{761}$ and $\beta_1^3\gamma_3$. In our setting this relation is given by the differential on $b_{2,0}^3\beta_2$, whose target is some linear combination of the four elements (including $\alpha_1\beta_{5/5}^4$) in question. This target is difficult to compute precisely, but it suffices to show that it includes a nontrivial multiple of $\alpha_1\beta_{5/5}^4$. Knowing then that $\underline{3}\beta_1^3\beta_{14}$ and $\beta_1 x_{761}$ are permanent cycles and $\alpha_1\beta_{5/5}^4$ is not, we can conclude that the linear combination also includes $\beta_1^3\gamma_3$ and that the latter is not a permanent cycle in the Adams–Novikov spectral sequence.

To make this calculation we map to the spectral sequence going from

$$\mathrm{Ext}_{P(1)}(\mathbf{Z}/(p), P(0))$$

(this is the R of 7.5.1 and 7.4.13) to $\mathrm{Ext}_{P(1)}(\mathbf{Z}/(p), \mathbf{Z}/(p))$. The elements $\underline{3}\beta_1^3\beta_{14}$, γ_3 and x_{761} all have trivial images, while $b_{2,0}^3\beta_2$ and $\alpha_1\beta_{5/5}^4$ do not, and it suffices to show that $\alpha_1\beta_{5/5}^4 = h_{10}b_{1,1}^4$ vanishes in $\mathrm{Ext}_{P(1)}$. $h_{11}b_{1,1}$ is killed by $b_{2,0}$, so $\langle b_{1,1}^3, h_{11}, h_{11}, h_{11}\rangle$ is killed by $b_{2,0}^3$ so we have

$$\begin{aligned} 0 &= \langle b_{1,1}^3, h_{11}, h_{11}, h_{11}\rangle\langle h_{11}, h_{11}, h_{10}\rangle \\ &= b_{1,1}^3\langle h_{11}, h_{11}, h_{11}, \langle h_{11}, h_{11}, h_{10}\rangle\rangle \\ &= b_{1,1}^3\langle h_{11}, h_{11}, h_{11}, h_{11}, h_{11}\rangle h_{10} \\ &= b_{1,1}^4 h_{10}. \end{aligned}$$

Given this situation the target of the differential from $\beta_{5/5}$, $\underline{4}\beta_1^{20}$, is the same as $\underline{3}\beta_1^2 x_{692}$, and $\alpha_1\beta_{5/5}^4$ is $\underline{4}\beta_1^2 x_{692}$ which accounts for the indicated differentials in dimensions 791 and 799.

The differential in the 752-stem can be recovered from the corresponding spectral sequence for Ext_P. The images of η_1 and γ_2 are the Massey products $\langle h_{11}, \underline{h}, \underline{b}\rangle$ and $\langle h_{12}, \underline{h}, \underline{b}\rangle$ where $\underline{h}$ and $\underline{b}$ denote the matrices

$$\begin{pmatrix} h_{11} & h_{12}\end{pmatrix} \quad \text{and} \quad \begin{pmatrix} b_{1,1} \\ b_{1,0}\end{pmatrix},$$

respectively. Then we have $\beta_1\eta_1\gamma_2 = \langle h_{12}b_{1,0}\eta_1, \underline{h}, \underline{b}\rangle = \langle h_{11}b_{1,1}\eta_1, \underline{h}, \underline{b}\rangle = 0$ since $h_{11}\eta_1 = 0$.

7.6.5 (*Fifth installment*)

818 $\beta_1^{10}\beta_{10/5}$

824 $\beta_1^2\beta_2\beta_{14}$

825 $\alpha_1 b_{1,1}b_{2,0}u_3 = \underline{2}\beta_2 x'_{724}$

826 $\underline{b_{1,1}^4}u_4 = \langle\alpha_1, \beta_1^5, \alpha_1\beta_4, \beta_{10/5}\rangle = x_{826}$

827 $\underline{2}\beta_3\beta_{15/5}$

832 $[\underline{2}\beta_1^{10}\gamma_2]\underline{b_{1,1}^2\gamma_2}$

833 $\alpha_1 x_{826}$

834 $\beta_1^2\beta_{16}$

$\underline{b_{1,1}^2\beta_{10/5}} = \langle\beta_1^4, \underline{2}\beta_1^6, \beta_{10/5}\rangle = x_{834}$

837 $\underline{3}\beta_1^4\beta_{14}$

$\beta_1^2 x_{761}$

838 $\beta_1^3 x_{724}$

$[\beta_1^3 x'_{724}]\underline{h_{2,0}b_{1,1}^4}$

840 $\underline{3}\beta_1^{10}\gamma_2$

841 $\alpha_1 x_{834}$

$\underline{3}\beta^{10}\beta_{10/5}$

842 $\underline{b_{1,1}^2\beta_{10/4}} = \langle\underline{2}\beta_1^9, \beta_1, \beta_{10/4}\rangle = x_{842}$

844 $\alpha_1\beta_1^2 x_{761}$

$\beta_1\beta_{17}$

845 $[\alpha_1\beta_1^3 x'_{724}]\alpha_1 h_{2,0}b_{1,1}^4$

847 $\beta_1^2 x_{771}$

848 $[\beta_1 x_{810}]\underline{h_{2,0}b_{2,0}^2\beta_7}$

849 $[\underline{2}\beta_1^2\beta_{16} + \alpha_1 x_{842}]b_{2,0}^2\beta_8$

$2\beta_1^2\beta_{16}$

850 $\beta_1\beta_3\beta_{15/3}$

851 $[\alpha_1\beta_1\beta_{17}]\underline{b_{2,0}\beta_{13}}$

852 $[\beta_1^5\beta_{14}]\underline{\eta_{13}}$

853 $\underline{2}\beta_1^3 x_{724}$

854 $\underline{\beta_{18}}$

$\alpha_1\beta_1^2x_{771}$

855 $\beta_1^{11}\gamma_2$

$\underline{4}\beta_1^2\beta_2\beta_{14}$

856 $[\underline{4}\,\underline{2}x_{810}]\underline{h_{2,0}b_{2,0}^2u_2}$

856 $\beta_1^{11}\beta_{10/5}$

857 $\alpha_1b_{2,0}^2\beta_8 = x_{857}$

859 $[\underline{2}\beta_1\beta_{17}]\underline{b_{2,0}u_8}$

860 $\alpha_1\eta_{13} = \beta_4\beta_{15/5}$

861 $\alpha_1\beta_{18}$

862 $\beta_1^3\beta_2\beta_{14}$

853 $[\underline{2}\beta_1x_{810}]\alpha_1h_{2,0}b_{2,0}^2u_2$

864 $[\beta_1x_{826}]\underline{h_{2,0}b_{1,1}b_{2,0}u_3}$

865 $\underline{2}\beta_1\beta_2\beta_{15/5}$

867 $\underline{3}\beta_1\beta_{17}$ □

For the differential in the 838-stem we use the method of 7.6.6. We have maps $f\colon \Sigma^{190}T(0)_1 \to S^0$ and $g\colon \Sigma^{609}T(0)_4 \to S^0$ where f is β_1^5 on the bottom cell, and g is α_1x_{602} on the bottom cell and x_{617} on the second cell. The smash product vanishes on the bottom cell so we have a map $\Sigma^{807}T(0)_4 \to S^0$ which is $\beta_1^5x_{617} + 2\beta_1^5x_{602}$ on the bottom cell. The second term vanishes because $\beta_1^5x_{602} \in \pi_{792} = 0$. We have

$$x_{617} = \left\langle \alpha_1, (\underline{2}\beta_1\beta_{11}\alpha_1), \begin{pmatrix} \beta_1 \\ x_{602} \end{pmatrix} \right\rangle.$$

A routine calculation gives $\beta_1x_{617} = \underline{3}\beta_1^3\beta_{11}$ and $\beta_1^3x_{617} = \alpha_1x'_{724}$. Our map gives $0 = \underline{4}\beta_1^5x_{617} = \beta_1^3x'_{724}$, hence the desired differential.

We use a similar argument in the 848-stem. We start with the maps

$$\Sigma^{316}T(0)_1 \to S^0 \qquad \text{and} \qquad \Sigma^{531}T(0)_4 \to S^0$$

carrying $\beta_1\beta_6$, and $\alpha_1\beta_2\beta_{10/5}$ on the bottom cells. The resulting relation is $\beta_1^2x_{810} = 0$. From 7.6.4 we see that $\mathbf{N}$ is vacuous in dimensions 887 and 856, so the indicated differential is the only one which can give this relation.

The argument in dimension 849 is similar to that in dimension 609.

In dimension 864 we use 7.6.6 again starting with the extensions of β_1^5 and $\underline{2}\beta_4\beta_{10/5}$ to $T(0)_1$ and $T(0)_4$.

7.6.5 (*Sixth installment*)

868 $\underline{b_{1,1}u_9} = x_{868}$

871 $[\alpha_1\beta_1x_{826}]\alpha_1h_{2,0}b_{1,1}b_{2,0}u_3$

872 $\beta_1^3\beta_{16}$

$[\beta_1x_{834}]\underline{h_{2,0}b_{1,1}^2u_4}$

875 α_1x_{868}

$\underline{2}\beta_4\beta_{15/5}$

$\beta_1^3x_{761} = \beta_{10/5}\gamma_2$

876 $\underline{\beta_{10/5}^2}$

$\beta_1^4x_{724}$

878 $\underline{3}\beta_1^{11}\gamma_2$

879 $[\alpha_1\beta_1x_{834}]\alpha_1h_{2,0}b_{1,1}^2u_4$

$[\underline{3}\beta_1^{11}\beta_{10/5}]\underline{h_{2,0}b_{1,1}^2\gamma_2}$

880 $[\beta_1x_{842}]\underline{h_{2,0}b_{1,1}^2\gamma_2}$

882 $[\alpha_1\beta_1^3x_{761}]\underline{b_{1,1}\gamma_3}$

$\beta_1^2\beta_{17}$

883 $\alpha_1\beta_{10/5}^2$

884 $\underline{\beta_{5/4}\beta_{15/5}} = \beta_{10/4}\beta_{10/5}$

885 $\beta_1^3x_{771}$

887 $\underline{4}\beta_1^{11}\beta_{10/5}$

$[\underline{2}\beta_1^3\beta_{16}]\alpha_1h_{2,0}b_{1,1}^2\gamma_2$

888 $[\beta_1^2\beta_3\beta_{15/5}]\underline{h_{2,0}h_{11}^2\beta_{10/4}}$

890 $\alpha_1 b_{1,1}\gamma_3 = \langle \alpha_1,\alpha_1,\beta_1^{10},\beta_1^{13}\rangle \underline{2}\beta_1^3 x_{761}$

891 $\underline{2}\beta_1^4 x_{724}$

$\alpha_1\beta_{5/4}\beta_{15/5}$

892 $\beta_1\beta_{18}$

$\alpha_1\beta_1^3 x_{771}$

$\underline{\beta_{10/4}^2} + \beta_1\beta_{18} = x_{892}$

893 $\beta_1^{12}\gamma_2$

$\underline{4}\beta_1^3\beta_2\beta_{14}$

894 $\beta_1^{12}\beta_{10/5}$

895 $[\beta_1 x_{857}]\underline{h_{2,0}h_{11}b_{2,0}^2 u_2}$

896 $[\underline{4}\,\underline{2}\beta_1\beta_3\beta_{15/5}]\underline{h_{11}b_{2,0}^2u_3}$

898 $[\beta_1\beta_4\beta_{15/5}]\underline{h_{2,0}b_{1,0}\beta_{13}}$

899 $[\alpha_1 x_{892}]\underline{h_{2,0}\beta_{14}}$

$\underline{\alpha_1\beta_1\beta_{18}}$

900 $\beta_1^4\beta_2\beta_{14}$

902 $\underline{\beta_{19}}$

903 $\underline{4}\beta_1^3\beta_{16}$

$[\underline{2}\beta_1^2\beta_3\beta_{15/5}]\alpha_1 h_{11}b_{2,0}^2u_3$

905 $\underline{3}\beta_1^2\beta_{17}$

906 $[\alpha_1 h_{2,0}b_{2,0}\beta_{13}]\underline{h_{2,0}b_{2,0}u_8}$

$\beta_1 x_{868}$

907 $\underline{2}x_{892}$

909 $[\alpha_1\beta_{19}]u_{14}$

910 $\beta_1^4\beta_{16}$

913 $\alpha_1\beta_1x_{868}$

$\beta_1^4x_{761}$

$[\underline{2}\beta_1\beta_4\beta_{15/5}]\alpha_1h_{2,0}b_{2,0}u_8$

914 $\beta_1^5x_{724}$

$[\beta_1\beta_{10/5}^2]\underline{h_{2,0}b_{1,1}u_9}$

916 $\underline{3}\beta_1^{12}\gamma_2$

917 $\underline{2}\beta_{19}$

918 $\underline{\beta_{20/5}}$

920 $\beta_1^3\beta_{17}$

921 $[\alpha_1\beta_1\beta_{10/5}^2]\alpha_1\underline{h_{2,0}b_{1,1}u_9}$

922 $[\beta_1\beta_{5/4}\beta_{15/5}]\underline{h_{2,0}b_{1,1}\beta_{15/5}}$

923 $\beta_1^4x_{771}$

925 $\alpha_1\beta_{20/5}$

$\underline{4}\beta_1^{12}\beta_{10/5}$

926 $\underline{\beta_{20/4}}$

928 $\underline{2}\beta_1^4x_{761}$

929 $[\underline{2}\beta_1^5x_{724}]\underline{h_{2,0}b_{1,1}\gamma_3}$

$[\alpha_1\beta_1\beta_{5/4}\beta_{15/5}]\alpha_1h_{2,0}b_{1,1}\beta_{15/5}$

930 $\alpha_1\beta_1^4x_{771}$

$\beta_1^2\beta_{18}$

$[\beta_1x_{892}]\underline{h_{2,0}b_{1,1}\beta_{15/4}}$

931 $\beta_1^{13}\gamma_2$

$\underline{4}\beta_1^4\beta_2\beta_{14}$

932 $\beta_1^{13}\beta_{10/5}?$

933 $\underline{\gamma_4}?$

$\alpha_1\beta_{20/4}$

934 $\underline{\beta_{20/3}}$

937 $\underline{3}\beta_1^5x_{724}$

$[\alpha_1\beta_1\beta_{10/4}^2]\alpha_1h_{2,0}b_{1,1}\beta_{15/4}$

938 $[\beta_1^5\beta_2\beta_{14}]\underline{\eta_1\beta_{14}}$

940 $\alpha_1\gamma_4$

$\beta_1\beta_{19}$

941 $\alpha_1\beta_{20/3}$

$\underline{4}\beta_1^4\beta_{16}$

942 $\underline{\beta_{20/2}}$

943 $[\underline{3}\beta_1^3\beta_{17}]\underline{h_{2,0}h_{11}b_{2,0}^2u_3}$

944 $[\beta_1^2x_{868}]\underline{b_{2,0}^2\eta_5}$

945 $[\underline{2}\beta_1^2\beta_{18}]\underline{h_{2,0}h_{11}b_{2,0}u_8}$

946 $[\beta_2\beta_4\beta_{15/5}]\underline{b_{2,0}\eta_{10}}$

948 $[\beta_1^5\beta_{16}]\underline{\eta_{15}}$

949 $[\alpha_1\beta_{20/2}]\underline{\beta_{20/1,2}}$

950 $\underline{\beta_{20}}$

951 $\beta_1^5x_{761}$

$\underline{4}\beta_1^3\beta_{17}$

$[\alpha_1\beta_1^2x_{868}]\alpha_1b_{2,0}^2\eta_5$

952 $\beta_1^6 x_{724}$

$\underline{b_{2,0}\eta_1\gamma_2} = \langle \beta_1, \beta_1\beta_{10/5} + \underline{5}\gamma_2, \gamma_2\rangle$

$= x_{952}$

953 $\underline{3}\beta_1^2\beta_{18}$

954 $\underline{3}\beta_1^{13}\gamma_2$

$\underline{3}\,\underline{4}\beta_1^4\beta_2\beta_{14} +$

$\langle\alpha_1, \alpha_1, \beta_1\beta_2\beta_{14}, \beta_1^4\rangle = x_{954}$

955 $\underline{2}\beta_1\beta_{19}$

956 $\beta_1\beta_{20/5}$

$[\alpha_1\eta_{15} + \beta_1\beta_{20/5}]\underline{h_{2,0}u_{14}}$

957 $\underline{2}\beta_{20/2}$

$\alpha_1\beta_{20}$

958 $\beta_1^4\beta_{17}$

959 $\alpha_1 x_{952}$

961 $[\beta_1^5 x_{771}]\underline{h_{11}b_{2,0}\gamma_3}$

963 $\alpha_1\beta_1\beta_{20/5}$

$\underline{4}\beta_1^{13}\beta_{10/5}$

964 $\alpha_1 h_{2,0}u_{14} = \underline{3}\,\underline{4}\beta_1^4\beta_{16} = x_{964}$

$[\beta_1\beta_{20/4}]\underline{h_{2,0}\beta_{20/5}}$

966 $\underline{2}\beta_1^5 x_{761}$

968 $\beta_1^3\beta_{18}$

$[\alpha_1\beta_1^5 x_{771}]\alpha_1 h_{11}b_{2,0}\gamma_3$

969 $\beta_1^{14}\gamma_2$

$\underline{2}x_{954}$

970 $\beta_1^{14}\beta_{10/5}$?

971 $\beta_1\gamma_4$?

$[\alpha_1\beta_1\beta_{20/4}]\alpha_1 h_{2,0}\beta_{20/5}$

972 $\underline{5\gamma_4}$

$[\beta_1\beta_{20/3}]\underline{h_{2,0}\beta_{20/4}}$

975 $\underline{3}\beta_1^5 x_{724}$

978 $\alpha_1\beta_1\gamma_4$

$\beta_1^2\beta_{19}$

979 $\underline{2}x_{964}$

$[\alpha_1\underline{5}\gamma_4]\underline{h_{2,0}\gamma_4}$

$[\alpha_1\beta_1\beta_{20/3}]\alpha_1 h_{2,0}\beta_{20/4}$

980 $[\beta_1\beta_{20/2}]\underline{h_{2,0}\beta_{20/3}}$

987 $\underline{2}\cdot\underline{5}\gamma_4$

988 $\beta_2\beta_{19}$

$[\beta_1\beta_{20}]\underline{h_{11}\beta_{20/1,2}}$

989 $\beta_1^6 x_{761}$

$\underline{4}\beta_1^4\beta_{17}$

990 $\beta_1^7 x_{724}$

$\beta_1 x_{952} = \beta_{5/5}^5$

991 $[\underline{3}\beta_1^3\beta_{18}]\underline{h_{2,0}b_{2,0}^2\eta_5}$

992 $\underline{3}\beta_1^{14}\gamma_2$

$[\beta_1 x_{954}]\underline{b_{2,0}^2\eta_6}$

993 $[\underline{2}\beta_1^2\beta_9]h_{2,0}b_{2,0}\eta_{10}$

994 $[\beta_1^2\beta_{20/5}]\underline{b_{2,0}\eta_{11}}$

995 $[\alpha_1\beta_2\beta_{19}]\underline{h_{2,0}\eta_{15}}$

$[\alpha_1\beta_1\beta_{20}]\alpha_1 h_{11}\beta_{20/1,2}$

996 $[\beta_1^5\beta_{17}]\underline{\eta_{16}}$

997 $[\alpha_1\beta_1 x_{952}]\underline{\beta_{25/25}}$

998 $\underline{\beta_{21}}$

999 $\underline{b_{2,0}^2\beta_2\gamma_2} = \langle\beta_1\beta_2, \gamma_1, \gamma_2\rangle = x_{999}$

$[\underline{4}\beta_1^3\beta_{18}]\underline{b_{2,0}^3 u_1}$

1000 $[\alpha_1\beta_1^2\beta_{20/5}]\alpha_1 b_{2,0}\eta_{11}$ □

The element x_{868} is constracted as follows. There is a commutative diagram

$$\begin{tikzcd} S^{860} \arrow[r, "f"] \arrow[d, "\beta_q"] & \Sigma^{837}T(0)_2 \arrow[d, "g"] \\ S^{438} \arrow[r, "\beta_{10/5}"] & S^0 \end{tikzcd}$$

where the cofiber of f is $\Sigma^{837}T(0)_3$ and g is an extension of $\underline{3}\beta_1^4\beta_{14}$. Both f and β_9 extend to $\Sigma^{860}T(0)_1$. The difference of the composite extensions of $\beta_{10/5}\beta_9$ and gf gives x_{868} on the top cell. In other words x_{868} iis the Toda bracket for

$$S^{867} \xrightarrow{\alpha_1} S^{860} \to S^{438} \vee \Sigma^{837}T(0)_2 \to S^0.$$

We will see below that $\beta_1^2 g = 0$ and $\beta_1\beta_{10/5}\beta_9 = 0$ so it follows that $\beta_1^2 x_{868}$ is divisible by α_1 and hence trivial.

For the relation in dimension 875 we have $x_{761} = \langle \alpha_1\beta_1, \beta_6, \gamma_2\rangle$ and $\beta_1\beta_{10/5} = \langle \beta_1^4, \alpha_1\beta_1, \beta_6\rangle$ so $\beta_1^4 x_{761} = \gamma_2\beta_1\beta_{10/5}$.

For dimension 879 we have, using 7.6.11,

$$\begin{aligned}\gamma_2 x_{404} &= \gamma_2\langle \beta_{5/4}, \beta_1, \alpha_1\beta_1^4\rangle = \langle \gamma_2\beta_{5/4}, \beta_1, \alpha_1\beta_1^4\rangle \\ &= \langle \underline{2}\cdot\underline{5}\gamma_2\beta_1^4, \beta_1, \alpha_1\beta_1^4\rangle = \underline{3}\cdot\underline{5}\beta_1\gamma_2 = \underline{3}\beta_1^{10}\beta_{10/5}\end{aligned}$$

so

$$\underline{3}\beta_1^{11}\beta_{10/5} = \gamma_2\beta_1 x_{404} = 0.$$

In dimension 888 we have

$$\beta_1^2\beta_3\beta_{15/5} = \beta_1^2\beta_8\beta_{10/5} = \beta_8\beta_1\underline{5}\gamma_2 = \beta_2\beta_7\underline{5}\gamma_2 = 0.$$

For the 896 stem we have

$$\beta_1\underline{4}\,\underline{2}\beta_1\beta_3\beta_{15/5} = \beta_1\beta_2\,\underline{42}\beta_2\beta_{15/5} = 0,$$

which (by inspection 7.6.4) implies $\underline{4}\,\underline{2}\beta_1\beta_3\beta_{15/5} = 0$.

We are not sure about γ_4. A possible approach to it is this. Extrapolating 7.6.4 slightly we see that $\mathrm{Ext}^{7,1016}$ has two generators, $\beta_1^2\gamma_4$ and $\langle \gamma_3, \gamma_1, \beta_3\rangle$. The latter supports a differential hitting $\beta_1^{15}\beta_{10/5} = \langle \beta_1^{18}, \gamma_1, \beta_2\rangle$. The same Ext group contains $\langle \gamma_2, \gamma_2, \beta_3\rangle$, which is a permanent cycle. Hence if it is nonzero it is neither $\beta_1^2\gamma_4$, in which case γ_4 is a permanent cycle, or $\beta_1^2\gamma_4 + \langle \gamma_3, \gamma_1, \beta_3\rangle$, in which case $d_{25}(\gamma_4) = \beta_1^{13}\beta_{10/5}$.

In the 992-stem we have $\beta_1 x_{954} = \beta_4 x_{810}$ so $\beta_1^2 x_{954} = \beta_1\beta_4 x_{810} = 0$. Extrapolating the pattern in 7.6.4 we find that the only element in the appropriate dimension is $b_{1,1}^3\gamma_2$, which kills $\underline{3}\beta_1^{15}\gamma_2$.

APPENDIX A1

Hopf Algebras and Hopf Algebroids

Commutative, noncocommutative Hopf algebras, such as the dual of the Steenrod algebra A (3.1.1), are familiar objects in algebraic topology and the importance of studying them is obvious. Computations with the Adams spectral sequence require the extensive use of homological algebra in the category of A-modules or, equivalently, in the category of A_*-comodules. In particular there are several change-of-rings theorems (A1.1.18, A1.1.20, and A1.3.13) which are major labor-saving devices. These results are well known, but detailed proofs (which are provided here) are hard to find.

The use of generalized homology theories such as MU- and BP-theory requires a generalization of the definition of a Hopf algebra to that of a *Hopf algebroid*. This term is due to Haynes Miller and its rationale will be explained below. The dual Steenrod algebra A_* is defined over $\mathbf{Z}/(p)$ and has a coproduct $\Delta\colon A_* \to A_* \otimes_{\mathbf{Z}/(p)} A_*$ dual to the product on A. The BP-theoretic analog $BP_*(BP)$ has a coproduct $\Delta\colon BP_*(BP) \to BP_*(BP) \otimes_{\pi_*(BP)} BP_*(BP)$, but the tensor product is defined with respect to a $\pi_*(BP)$-*bimodule* structure on $BP_*(BP)$; i.e., $\pi_*(BP)$ acts differently on the two factors. These actions are defined by two different $\mathbb{Z}_{(p)}$-algebra maps $\eta_L, \eta_R\colon \pi_*(BP) \to BP_*(BP)$, known as the left and right units. In the case of the Steenrod algebra one just has a single unit $\eta\colon \mathbf{Z}/(p) \to A_*$. Hence $BP_*(BP)$ is not a Hopf algebra, but a more general sort of object of which a Hopf algebra is a special case.

The definition of a Hopf algebroid A1.1.1 would seem rather awkward and unnatural were it not for the following category theoretic observation, due to Miller. A Hopf algebra such as A_* is a *cogroup object* in the category of graded $\mathbf{Z}/(p)$-algebras. In other words, given any such algebra R, the coproduct $\Delta\colon A_* \to A_* \otimes A_*$ induces a set map $\mathrm{Hom}(A_*, R) \times \mathrm{Hom}(A_*, R) \to \mathrm{Hom}(A_*, R)$ which makes $\mathrm{Hom}(A_*, R)$ into a group. Now the generalization of Hopf algebras to Hopf algebroids corresponds precisely to that from groups to *groupoids*. Recall that a group can be thought of as a category with a single object in which every morphism is invertible; the elements in the group are identified with the morphisms in the category. A groupoid is a small category in which every morphism is invertible and a Hopf algebroid is a cogroupoid object in the category of commutative algebras over a commutative ground ring K [$\mathbf{Z}_{(p)}$ in the case of $BP_*(BP)$]. The relation between the axioms of a groupoid and the structure of a Hopf algebroid is explained in A1.1.1.

The purpose of this appendix is to generalize the standard tools used in homological computations over a Hopf algebra to the category of comodules over a Hopf algebroid. It also serves as a self-contained (except for Sections 4 and 5) account of the Hopf algebra theory itself. These standard tools include basic definitions (Section 1), some of which are far from obvious; resolutions and homological functors such as Ext and Cotor (Section 2); spectral sequences of various sorts (Section 3), including that of Cartan and Eilenberg [**1**, p. 349]; Massey products (Section 4); and algebraic Steenrod operations (Section 5). We will now describe these five sections in more detail.

In Section 1 we start by defining Hopf algebroids (A1.1.1), comodules and primitives (A1.1.2), cotensor products (A1.1.4), and maps of Hopf algebroids (A1.1.7). The category of comodules is shown to be abelian (A1.1.3), so we can do homological algebra over it in Section 2. Three special types of groupoid give three corresponding types of Hopf algebroid. If the groupoid has a single object (or if all morphisms have the same source and target) we get an ordinary Hopf algebra, as remarked above. The opposite extreme is a groupoid with many objects but at most a single morphism between any pair of them. From such groupoids we get unicursal Hopf algebroids (A1.1.11). A third type of groupoid can be constructed from a group action on a set, and a corresponding Hopf algebroid is said to be split (A1.1.22).

The most difficult definition of Section 1 (which took us quite a while to formulate) is that of an extension of Hopf algebroids (A1.1.15). An extension of Hopf algebras corresponds to an extension of groups, for which one needs to know what a normal subgroup is. We are indebted to Higgins [**1**] for the definition of a normal subgroupoid. A groupoid C_0 is normal in C_1 if

(i) the objects of C_0 are the same as those of C_1,

(ii) the morphisms in C_0 form a subset of those in C_1, and

(iii) if $g\colon X \to Y$ and $h\colon Y \to Y$ are morphisms in C_1 and C_0, respectively, then $g^{-1}hg\colon X \to X$ is a morphism in C_0.

This translates to the definition of a normal map of Hopf algebroids (A1.1.10). The quotient groupoid $C = C_1/C_0$ is the one

(i) whose objects are equivalence classes of objects in C_1, where two objects are equivalent if there is a morphism between them in C_0, and

(ii) whose morphisms are equivalence classes of morphisms in C_1, where two morphisms g and g' are equivalent if $g' = h_1gh_2$ where h_1 and h_2 are morphisms in C_0.

The other major result of Section 1 is the comodule algebra structure theorem (A1.1.17) and its corollaries, which says that a comodule algebra (i.e., a comodule with a multiplication) which maps surjectively to the Hopf algebroid Σ over which it is defined is isomorphic to the tensor product of its primitives with Σ. This applies in particular to a Hopf algebroid Γ mapping onto Σ (A1.1.19). The special case when Σ is a Hopf algebra over a field was first proved by Milnor and Moore [**3**].

In Section 2 we begin our study of homological algebra in the category of comodules over a Hopf algebroid. We show (A1.2.2) that there are enough injectives and define Ext and Cotor (A1.2.3). For our purposes Ext can be regarded as a special case of Cotor (A1.1.6). We find it more convenient here to state and prove our results in terms of Cotor, although no use of it is made in the text. In most cases the translation from Cotor to Ext is obvious and is omitted. After defining these functors we discuss resolutions (A1.2.4, A1.2.10) that can be used to compute them, especially the cobar resolution (A1.2.11). We also define the cup product in Cotor (A1.2.14).

In Section 3 we construct some spectral sequences for computing the Cotor and Ext groups we are interested in. First we have the spectral sequence associated with an LES of comodules (A1.3.2); the example we have in mind is the chromatic spectral sequence of Chapter 5. Next we have the spectral sequence associated with a (decreasing or increasing) filtration of a Hopf algebroid (A1.3.9); examples include the classical May spectral sequence (3.2.9), the spectral sequence of 3.5.2, and the so-called algebraic Novikov spectral sequence (4.4.4).

In A1.3.11 we have a spectral sequence associated with a map of Hopf algebroids which computes Cotor over the target in terms of Cotor over the source. When the map is surjective the spectral sequence collapses and we get a change-of-rings isomorphism (A1.3.12). We also use this spectral sequence to construct a Cartan–Eilenberg spectral sequence (A1.3.14 and A1.3.15) for an extension of Hopf algebroids.

In Section 4 we discuss Massey products, an essential tool in some of the more intricate calculations in the text. The definitive reference is May [**3**] and this section is little more than an introduction to that paper. We refer to it for all the proofs and we describe several examples designed to motivate the more complicated statements therein. The basic definitions of Massey products are given as A1.4.1, A1.4.2, and A1.4.3. The rules for manipulating them are the juggling theorems A1.4.6, A1.4.8, and A1.4.9. Then we discuss the behavior of Massey products in spectral sequences. Theorem A1.4.10 addresses the problem of convergence; A1.4.11 is a Leibnitz formula for differentials on Massey products; and A1.4.12 describes the relation between differentials and extensions.

Section 5 treats algebraic Steenrod operations in suitable Cotor groups. These are defined in the cohomology of any cochain complex having certain additional structure and a general account of them is given by May [**5**]. Our main result (A1.5.1) here (which is also obtained by Bruner *et al.* [**1**]) is that the cobar complex (A1.2.11) has the required structure. Then the theory of May [**5**] gives the operations described in A1.5.2. Our grading of these operations differs from that of other authors including May [**5**] and Bruner *et al.* [**1**]; our $\mathcal{P}^i$ raises cohomological (as opposed to topological) degree by $2i(p-1)$.

1. Basic Definitions

A1.1.1. Definition. *A* Hopf algebroid *over a commutative ring K is a cogroupoid object in the category of (graded or bigraded) commutative K-algebras, i.e., a pair (A,Γ) of commutative K-algebras with structure maps such that for any other commutative K-algebra B, the sets* $\mathrm{Hom}(A,B)$ *and* $\mathrm{Hom}(\Gamma,B)$ *are the objects and morphisms of a groupoid (a small category in which every morphism is an equivalence). The structure maps are*

$\eta_L\colon A\to\Gamma$	*left unit or source,*
$\eta_R\colon A\to\Gamma$	*right unit or target,*
$\Delta\colon \Gamma\to\Gamma\otimes_A\Gamma$	*coproduct or composition,*
$\varepsilon\colon\Gamma\to A,$	*counit or identity,*
$c\colon\Gamma\to\Gamma$	*conjugation or inverse.*

Here Γ is a left A-module via η_L and a right A-module via η_R, $\Gamma\otimes_A\Gamma$ is the usual tensor product of bimodules, and Δ and ε are A-bimodule maps. The defining properties of a groupoid correspond to the following relations among the structure maps:

(a) $\varepsilon\eta_L = \varepsilon\eta_R = 1_A$, *the identity map on* A. *(The source and target of an identity morphism are the object on which it is defined.)*

(b) $(\Gamma\otimes\varepsilon)\Delta = (\varepsilon\otimes\Gamma)\Delta = 1_\Gamma$. *(Composition with the identity leaves a morphism unchanged.)*

(c) $(\Gamma\otimes\Delta)\Delta = (\Delta\otimes\Gamma)\Delta$. *(Composition of morphisms is associative.)*

(d) $c\eta_R = \eta_L$ *and* $c\eta_L = \eta_R$. *(Inverting a morphism interchanges source and target.)*

(e) $cc = 1_\Gamma$. *(The inverse of the inverse is the original morphism.)*

(f) Maps exist which make the following commute

$$\begin{tikzcd}
\Gamma & \Gamma\otimes_K\Gamma \arrow[l, "c\cdot\Gamma"'] \arrow[r, "\Gamma\cdot c"] \arrow[d] & \Gamma \\
 & \Gamma\otimes_A\Gamma \arrow[ul, dashed] \arrow[ur, dashed] & \\
A \arrow[uu, "\eta_R"] & \Gamma \arrow[l, "\varepsilon"'] \arrow[r, "\varepsilon"] \arrow[u, "\Delta"] & A \arrow[uu, "\eta_L"']
\end{tikzcd}$$

where $c\cdot\Gamma(\gamma_1\otimes\gamma_2) = c(\gamma_1)\gamma_2$ *and* $\Gamma\cdot c(\gamma_1\otimes\gamma_2) = \gamma_1 c(\gamma_2)$. *(Composition of a morphism with its inverse on either side gives an identity morphism.)*

If our algebras are graded the usual sign conventions are assumed; i.e., commutativity means $xy = (-1)^{|x||y|}yx$, where $|x|$ and $|y|$ are the degrees or dimensions of x and y, respectively.

A graded Hopf algebroid is *connected* if the right and left sub-A-modules generated by Γ_0 are both isomorphic to A.

In most cases the algebra A will be understood and the Hopf algebroid will be denoted simply by Γ.

Note that if $\eta_R = \eta_L$, then Γ is a commutative Hopf algebra over A, which is to say a cogroup object in the category of commutative A-algebras. This is the origin of the term Hopf algebroid. More generally if $D\subset A$ is the subalgebra on which $\eta_R = \eta_L$, then Γ is also a Hopf algebroid over D.

The motivating example of a Hopf algebroid is $(\pi_*(E), E_*(E))$ for a suitable spectrum E (see Section 2.2).

A1.1.2. Definition. *A* left Γ-comodule M *is a left* A*-module* M *together with a left* A*-linear map* $\psi\colon M\to\Gamma\otimes_A M$ *which is counitary and coassociative, i.e., such that* $(\varepsilon\otimes M)\psi = M$ *(i.e., the identity on* M*) and* $(\Delta\otimes M)\psi = (\Gamma\otimes\psi)\psi$. *A* right Γ-comodule *is similarly defined. An element* $m\in M$ *is* primitive *if* $\psi(m) = 1\otimes m$.

A comodule algebra M *is a comodule which is also a commutative associative* A*-algebra such that the structure map* ψ *is an algebra map. If* M *and* N *are left* Γ*-comodules, their* comodule tensor product *is* $M\otimes_A N$ *with structure map being the composite*

$$M\otimes N \xrightarrow{\psi_M\otimes\psi_N} \Gamma\otimes M\otimes\Gamma\otimes N \to \Gamma\otimes\Gamma\otimes M\otimes N \to \Gamma\otimes M\otimes N,$$

where the second map interchanges the second and third factors and the third map is the multiplication on Γ. *All tensor products are over* A *using only the left* A*-module structure on* A. *A* differential comodule C^* *is a cochain complex in which each* C^s *is a comodule and the coboundary operator is a comodule map.*

A1.1.3. THEOREM. *If* Γ *is flat as an* A*-module then the category of left* Γ*-comodules is abelian* (see Hilton and Stammbach [1]).

PROOF. If $0 \to M' \to M \to M'' \to 0$ is a short exact sequence of A-modules, then since Γ is flat over A,

$$0 \to \Gamma \otimes_A M' \to \Gamma \otimes_A M \to \Gamma \otimes_A M'' \to 0$$

is also exact. If M is a left Γ-comodule then a comodule structure on either M' or M'' will determine such a structure on the other one. From this fact it follows easily that the kernel or cokemel (as an A-module) of a map of comodules has a unique comodule structure, i.e., that the category has kernels and cokernels. The other defining properties of an abelian category are easily verified. $\square$

In view of the above, *we assume from now on that* Γ *is flat over* A.

A1.1.4. DEFINITION. *Let* M *and* N *be right and left* Γ*-comodules, respectively. Their* cotensor product over Γ *is the* K*-module defined by the exact sequence*

$$0 \to M \,\square_\Gamma\, N \to M \otimes_A N \xrightarrow{\psi\otimes N - M\otimes\psi} M \otimes_A \Gamma \otimes_A N,$$

where ψ *denotes the comodule structure maps for both* M *and* N.

Note that $M \,\square_\Gamma\, N$ is not a comodule or even an A-module but merely a K-module.

A left comodule M can be given the structure of a right comodule by the composition

$$M \xrightarrow{\psi} \Gamma \otimes M \xrightarrow{T} M \otimes \Gamma \xrightarrow{M\otimes c} M \otimes \Gamma,$$

where T interchanges the two factors and c is the conjugation map (see A1.1.1). A right comodule can be converted to a left comodule by a similar device. With this in mind we have

A1.1.5. PROPOSITION. $M \,\square_\Gamma\, N = N \,\square_\Gamma\, M$.

The following relates the cotensor product to Hom.

A1.1.6. LEMMA. *Let* M *and* N *be left* Γ*-comodules with* M *projective over* A. *Then*

(a) $\mathrm{Hom}_A(M, A)$ *is a right* Γ*-comodule and*

(b) $\mathrm{Hom}_\Gamma(M, N) = \mathrm{Hom}_A(M, A) \,\square_\Gamma\, N$, *e.g.,* $\mathrm{Hom}_\Gamma(A, N) = A \,\square_\Gamma\, N$.

PROOF. Let $\psi_M\colon M \to \Gamma \otimes_A M$ and $\psi_N\colon N \to \Gamma \otimes_A N$ be the comodule structure maps. Define

$$\psi_M^*, \psi_N^*\colon \mathrm{Hom}_A(M, N) \to \mathrm{Hom}_A(M, \Gamma \otimes_A N)$$

by

$$\psi_M^*(f) = (\Gamma \otimes f)\psi_M \quad \text{and} \quad \psi_N^*(f) = \psi_N f$$

for $f \in \mathrm{Hom}_A(M, N)$. Since M is projective we have a canonical isomorphism,

$$\mathrm{Hom}_A(M, A) \otimes_A N \approx \mathrm{Hom}_A(M, N).$$

Hence for $N = A$ we have

$$\psi_M^*\colon \mathrm{Hom}_A(M, A) \to \mathrm{Hom}_A(M, A) \otimes_A \Gamma.$$

To show that this is a right Γ-comodule structure we need to show that the following diagram commutes

$$\begin{tikzcd}
\mathrm{Hom}_A(M,A) \arrow[r, "\psi_M^*"] \arrow[d, "\psi_M^*"'] & \mathrm{Hom}_A(M,\Gamma) \arrow[d, "\mathrm{Hom}(M,\Delta)"] \\
\mathrm{Hom}_A(M,\Gamma) \arrow[r, "\psi_M^*"] & \mathrm{Hom}_A(M,\Gamma\otimes\Gamma),
\end{tikzcd}$$

i.e., that ψ_M^* is coassociative.

We have a straightforward calculation

$$\begin{aligned}
\psi_M^*\psi_M^*(f) &= (\Gamma\otimes\psi_M^*(f))\psi_M \\
&= (\Gamma\otimes((\Gamma\otimes f)\psi_m))\psi_M \\
&= (\Gamma\otimes\Gamma\otimes f)(\Gamma\otimes\psi_M)\psi_M \\
&= (\Gamma\otimes\Gamma\otimes f)(\Delta\otimes M)\psi_M \\
&= (\Delta\otimes A)(\Gamma\otimes f)\psi_M \\
&= (\Delta\otimes A)\psi_M^* f
\end{aligned}$$

so the diagram commutes and (a) follows.

For (b) note that by definition

$$\mathrm{Hom}(M,N) = \ker(\psi_M^* - \psi_M^*) \subset \mathrm{Hom}_A(M,N)$$

while

$$\begin{aligned}
\mathrm{Hom}_A(M,A)\,\square_\Gamma\, N &= \ker(\psi_M^*\otimes N - \mathrm{Hom}_A(M,A)\otimes\psi_N) \\
&\subset \mathrm{Hom}_A(M,A)\otimes_A N
\end{aligned}$$

and the following diagram commutes

$$\begin{tikzcd}
\mathrm{Hom}(M,A)\otimes N \arrow[r, "\simeq"] \arrow[d, shift right, "\psi_M^*\otimes N"'] \arrow[d, shift left, "\mathrm{Hom}(M,A)\otimes\psi_N"] & \mathrm{Hom}_A(M,N) \arrow[d, shift right, "\psi_M^*"'] \arrow[d, shift left, "\psi_N^*"] \\
\mathrm{Hom}(M,A)\otimes\Gamma\otimes N \arrow[r, "\simeq"] & \mathrm{Hom}_A(M,\Gamma\otimes_A N)
\end{tikzcd}$$

□

The next few definitions and lemmas lead up to that of an extension of Hopf algebroids given in A1.1.15. In A1.3.14 we will derive a corresponding Cartan–Eilenberg spectral sequence.

A1.1.7. DEFINITION. *A map of Hopf algebroids $f\colon (A,\Gamma)\to(B,\Sigma)$ is a pair of K-algebra maps $f_1\colon A\to B$, $f_2\colon \Gamma\to\Sigma$ such that*

$$\begin{gathered}
f_1\varepsilon = \varepsilon f_2, \quad f_2\eta_R = \eta_R f_1, \quad f_2\eta_L = \eta_L f_1, \\
f_2 c = c f_2, \quad \text{and} \quad \Delta f_2 = (f_2\otimes f_2)\Delta.
\end{gathered}$$

A1.1.8. LEMMA. *Let $f\colon (A,\Gamma)\to(B,\Sigma)$ be a map of Hopf algebroids. Then $\Gamma\otimes_A B$ is a right Σ-comodule and for any left Σ-comodule N, $(\Gamma\otimes_A B)\,\square_\Sigma\, N$ is a sub-left Γ-comodule of $\Gamma\otimes_A N$, where the structure map for the latter is $\Delta\otimes N$.*

PROOF. The map $(\Gamma\otimes f_2)\Delta\colon \Gamma\to\Gamma\otimes_A\Sigma = (\Gamma\otimes_A B)\otimes_B\Sigma$ extends uniquely to $\Gamma\otimes_A B$, making it a right Σ-comodule. By definition $(\Gamma\otimes_A B)\,\square_\Sigma\, N$ is the kernel in the exact sequence

$$0\to(\Gamma\otimes_A B)\,\square_\Sigma\, N\to\Gamma\otimes_A N\to\Gamma\otimes_A\Sigma\otimes_B N$$

where the right-hand arrow is the difference between $(\Gamma \otimes f_2)\Delta \otimes N$ and $\Gamma \otimes \psi$. Since $\Gamma \otimes_A N$ and $\Gamma \otimes_A \Sigma \otimes_B N$ are left Γ-comodules it suffices to show that the two maps respect the comodule structure. This is clear for $\Gamma \otimes \psi$, and for $(\Gamma \otimes f)\Delta \otimes N$ we need the commutativity of the following diagram, tensored over B with N.

$$\begin{tikzcd}
\Gamma \otimes_A B \arrow[r, "(\Gamma\otimes f_2)\Delta\otimes B"] \arrow[d, "\Delta\otimes B"'] & \Gamma \otimes_A \Sigma \arrow[d, "\Delta\otimes\Sigma"] \\
\Gamma \otimes_A \Gamma \otimes_A B \arrow[r, "\Gamma\otimes(\Gamma\otimes f_2)\Delta\otimes B"] & \Gamma \otimes_A \Gamma \otimes_A \Sigma
\end{tikzcd}$$

It follows from the fact that f is a Hopf algebroid map. $\square$

A1.1.9. DEFINITION. *If (A, Γ) is a Hopf algebroid the* associated Hopf algebra *(A, Γ') is defined by $\Gamma' = \Gamma/(\eta_L(a) - \eta_R(a) \mid a \in A)$. (The easy verification that a Hopf algebra structure is induced on Γ' is left to the reader.)*

Note that Γ' may not be flat over A even though Γ is.

A1.1.10. DEFINITION. *A map of Hopf algebroids $f: (A, \Gamma) \to (A, \Sigma)$ is* normal *if $f_2: \Gamma \to \Sigma$ is surjective, $f_1: A \to A$ is the identity, and $\Gamma \,\square_{\Sigma'} A = A \,\square_{\Sigma'} \Gamma$ in Γ.*

A1.1.11. DEFINITION. *A Hopf algebroid (A, U) is* unicursal *if it is generated as an algebra by the images of η_L and η_R, i.e., if $U = A \otimes_D A$ where $D = A \,\square_U A$ is a subalgebra of A. (The reader can verify that the Hopf algebroid structure of U is unique.)* $\square$

This term was taken from page 9 of Higgins [1].

A1.1.12. LEMMA. *Let M be a right comodule over a unicursal Hopf algebroid (A, U). Then*

(a) *M is isomorphic as a comodule to $M \otimes_A A$ with structure map $M \otimes \eta_R$ and*

(b) *$M = (M \,\square_U A) \otimes_D A$ as A-modules.*

PROOF. For $m \in M$ let $\psi(m) = m' \otimes m''$. Since U is unicursal we can assume that each m'' is in the image of η_R. It follows that

$$(\psi \otimes U)\psi(m) = (M \otimes \Delta)\psi(m) = m' \otimes 1 \otimes m''$$

so each m' is primitive. Let $\widetilde{m} = m'\varepsilon(m'')$. Then $\psi(\widetilde{m}) = m' \otimes m'' = \psi(m)$, so $m = \widetilde{m}$ since ψ is a monomorphism; Hence M is generated as an A-module by primitive elements and (a) follows. For (b) we have, using (a),

$$(M \,\square_U A) \otimes_D A = M \otimes_A (A \,\square_U A) \otimes_D A = M \otimes_A D \otimes_D A = M. \qquad \square$$

A1.1.13. LEMMA. *Let (A, Σ) be a Hopf algebroid, (A, Σ') the associated Hopf algebra (A1.1.7) $D = A \,\square_\Sigma A$, and (A, U) the unicursal Hopf algebroid (A1.1.9) with $U = A \otimes_D A$. Then*

(a) *$U = \Sigma \,\square_{\Sigma'} A$ and*

(b) *for a left Σ-comodule M, $A \,\square_{\Sigma'} M$ is a left U-comodule and $A \,\square_\Sigma M = A \,\square_U (A \,\square_{\Sigma'} M)$.*

PROOF. By definition, $\Sigma' = A \otimes_U \Sigma$, where the U-module structure on A is given by $\varepsilon: U \to A$, so we have

$$\Sigma \otimes_A \Sigma' = \Sigma \otimes_A A \otimes_U \Sigma = \Sigma \otimes_U \Sigma.$$

By A1.1.3, there is a short exact sequence

$$0 \to \Sigma \,\square_{\Sigma'}\, A \to \Sigma \otimes_U \Sigma$$

where the last map is induced by $\Delta - \Sigma \otimes \eta_L$. An element $\sigma \in \Sigma$ has $\Delta(\sigma) = \sigma \otimes 1$ in $\Sigma \otimes_U \Sigma$ iff $\sigma \in U$, so (a) follows.

For (b) we have

$$A \,\square_\Sigma\, M = A \,\square_U\, (U \,\square_\Sigma\, M)$$

and

$$U \,\square_\Sigma\, M = (A \,\square_{\Sigma'}\, \Sigma) \,\square_\Sigma\, M = A \,\square_{\Sigma'}\, M. \qquad \square$$

The following example may be helpful. Let $(A, \Gamma) = (\pi_*(BP), BP_*(BP))$ (4.1.19), i.e., $A = \mathbf{Z}_{(p)}[v_1, v_2, \dots]$ and $\Gamma = A[t_1, t_2, \dots]$ where $\dim v_i = \dim t_i = 2(p^i - 1)$. Let $\Sigma = A[t_{n+1}, t_{n+2}, \dots]$ for some $n \geq 0$. The Hopf algebroid structure on Σ is that of the quotient $\Gamma/(t_1, \dots, t_n)$. The evident map $(A, \Gamma) \to (A, \Sigma)$ is normal (A1.1.10). $D = A \,\square_\Sigma\, A$ is $\mathbf{Z}_{(p)}[v_1, \dots, v_n]$ and $\Phi = A \,\square_\Sigma\, \Gamma \,\square_\Sigma\, A$ is $D[t_1, \dots, t_n]$. (D, Φ) is a sub-Hopf algebroid of (A, Γ) and $(D, \Phi) \to (A, \Gamma) \to (A, \Sigma)$ is an extension (A1.1.15 below).

A1.1.14. THEOREM. *Let $f\colon (A, \Gamma) \to (A, \Sigma)$ be a normal map of Hopf algebroids and let $D = A \,\square_\Sigma\, A$ and $\Phi = A \,\square_\Sigma\, \Gamma \,\square_\Sigma\, A$. Then (D, Φ) is a sub-Hopf algebroid of (A, Γ).*

(Note that by A1.1.8, $A \,\square_\Sigma\, \Gamma$ and $\Gamma \,\square_\Sigma\, A$ are right and left Γ-comodules, respectively, so the expressions $(A \,\square_\Sigma\, \Gamma) \,\square_\Sigma\, A$ and $A \,\square_\Sigma\, (\Gamma \,\square_\Sigma\, A)$ make sense. It is easy to check, without using the normality of f, that they are equal, so Φ is well defined.)

PROOF. By definition an element $a \in A$ is in D iff $f_2\eta_L(a) = f_2\eta_R(a)$ and is in Φ iff $(f_2 \otimes \Gamma \otimes f_2)\,\Delta^2(\gamma) = 1 \otimes \gamma \otimes 1$. To see that η_R sends D to Φ, we have for $d \in D$

$$\begin{aligned}(f_2 \otimes \Gamma \otimes f_2)\Delta^2\eta_R(d) &= 1 \otimes 1 \otimes f_2\eta_R(d)\\ &= 1 \otimes 1 \otimes f_2\eta_L(d) = 1 \otimes \eta_R(d) \otimes 1.\end{aligned}$$

The argument for η_L is similar. It is clear that Φ is invariant under the conjugation c. To show that ε sends Φ to D we need to show $f_2\eta_R\varepsilon(\phi) = f_2\eta_L\varepsilon(\phi)$ for $\phi \in \Phi$. But $f_2\eta_R\varepsilon(\phi) = \eta_R\varepsilon f_2(\phi)$ and since $\Delta^2 f_2(\phi) = 1 \otimes f_2(\phi) \otimes 1$ we have $\Delta f_2(\phi) = 1 \otimes f_2(\phi) = f_2(\phi) \otimes 1$ so $f_2(\phi) \in D$, and $(\eta_R - \eta_L)\varepsilon f_2(\phi) = 0$.

To define a coproduct on Φ we first show that the natural map from $\Phi \otimes_D \Phi$ to $\Gamma \otimes_A \Gamma$ is monomorphic. This amounts to showing that $a\phi \in \Phi$ iff $a \in D$. Now by definition $a\phi \in \Phi$ iff

$$f_2(a\phi') \otimes \phi'' \otimes f_2(\phi''') = 1 \otimes a\phi \otimes 1 = f_2\eta_R(a) \otimes \phi \otimes 1.$$

Since $\phi \in \Phi$ we have

$$f_2(\phi') \otimes \phi'' \otimes f_2(\phi''') = 1 \otimes \phi \otimes 1,$$

so the criterion is

$$f_2(a) \otimes 1 \otimes 1 = f_2\eta_R(a) \otimes 1 \otimes 1,$$

i.e., $a \in D$.

Now consider the commutative diagram

$$\begin{tikzcd}
(D,\Phi) \arrow[r] \arrow[d, equal] & (A,\widetilde{\Phi}) \arrow[r, "g"] \arrow[d] & (A,U) \arrow[d] \\
(D,\Phi) \arrow[r] & (A,\Gamma) \arrow[r, "f"] \arrow[d, "f'"] & (A,\Sigma) \arrow[d] \\
 & (A,\Sigma') \arrow[r, equal] & (A,\Sigma')
\end{tikzcd}$$

where Σ' is the Hopf algebra associated to Σ (A1.1.9), f' is the induced map, U is the unicursal Hopf algebroid (A1.1.11) $A\otimes_D A$, $\widetilde{\Phi} = A\,\square_{\Sigma'}\,\Gamma\,\square_{\Sigma'}\,A$, and g will be constructed below. We will see that Φ and $\widetilde{\Phi}$ are both Hopf algebroids.

Now the map f' is normal since f is and $A\,\square_{\Sigma'}\,A = A$, so the statement that $\widetilde{\Phi}$ is a Hopf algebroid is a special case of the theorem. Hence we have already shown that it has all of the required structure but the coproduct. Since $\Gamma\,\square_{\Sigma'}\,A = A\,\square_{\Sigma'}\,\Gamma$, we have $\widetilde{\Phi} = A\,\square_{\Sigma'}\,\Gamma\,\square_{\Sigma'}\,A = A\,\square_{\Sigma'}\,A\,\square_{\Sigma'}\,\Gamma = A\,\square_{\Sigma'}\,\Gamma$. One easily verifies that the image of $\Delta\colon \Gamma\to\Gamma\otimes_A\Gamma$ is contained in $\Gamma\,\square_\Gamma\,\Gamma$ and hence in $\Gamma\,\square_{\Sigma'}\,\Gamma$. There Δ sends $\widetilde{\Phi} = A\,\square_{\Sigma'}\,\Gamma\,\square_{\Sigma'}\,A$ to $A\,\square_{\Sigma'}\,\Gamma\,\square_{\Sigma'}\,\Gamma\,\square_{\Sigma'}\,A = \widetilde{\Phi}\,\square_{\Sigma'}\,\widetilde{\Phi}\subset\widetilde{\Phi}\otimes_A\widetilde{\Phi}$, so $\widetilde{\Phi}$ is a Hopf algebroid.

Since $\widetilde{\Phi} = \Gamma\,\square_{\Sigma'}\,A$ and $U = \Sigma\,\square_{\Sigma'}\,A$ [A1.1.13(a)] we can define g to be $f_2\,\square\,A$. It follows from A1.1.13(b) that

$$\begin{aligned}
\Phi &= A\,\square_\Sigma\,\Gamma\,\square_\Sigma\,A = A\,\square_U\,(A\,\square_{\Sigma'}\,\Gamma\,\square_{\Sigma'}\,A)\,\square_U\,A\\
&= A\,\square_U\,\widetilde{\Phi}\,\square_U\,A.
\end{aligned}$$

By A1.1.12(b) we have $\widetilde{\Phi} = A\otimes_D\Phi\otimes_D A$, so $\widetilde{\Phi}\otimes_A\widetilde{\Phi} = A\otimes_D\widetilde{\Phi}\otimes_D A\otimes_D\Phi\otimes_D A$. The coproduct Δ sends $\widetilde{\Phi}$ to $\widetilde{\Phi}\,\square_U\,\widetilde{\Phi}\subset\Phi\otimes_A\widetilde{\Phi}$ and we have

$$\begin{aligned}
\widetilde{\Phi}\,\square_U\,\widetilde{\Phi} &= \widetilde{\Phi}\otimes_A(A\,\square_U\,A)\otimes_A\widetilde{\Phi} \quad \text{by A1.1.12(a)}\\
&= A\otimes_D\Phi\otimes_D(A\,\square_U\,A)\otimes_D\Phi\otimes_D A\\
&= A\otimes_D\Phi\otimes_D D\otimes_D\Phi\otimes_D A\\
&= A\otimes_D\Phi D\otimes\Phi\otimes_D A.
\end{aligned}$$

Since Δ is A-bilinear it sends Φ to $\Phi\otimes_D\Phi$ and Φ is a Hopf algebroid. $\square$

A1.1.15. Definition. *An* extension *of Hopf algebroids is a diagram*

$$(D,\Phi)\xrightarrow{i}(A,\Gamma)\xrightarrow{f}(A,\Sigma)$$

where f is normal (A1.1.10) and (D,Φ) is as in A1.1.14.

The extension is cocentral *if the diagram*

$$\begin{tikzcd}
 & \Gamma\otimes\Sigma \arrow[dd, "t"] \\
\Gamma \arrow[ur, "(\Gamma\otimes f_2)\Delta"] \arrow[dr, "(f_2\otimes\Gamma)\Delta"'] & \\
 & \Sigma\otimes\Gamma
\end{tikzcd}$$

(*where* t *interchanges factors*) *commutes up to the usual sign. In particular* Σ *must be cocommutative.*

A nice theory of Hopf algebra extensions is developed by Singer [**5**] and in Section II 3 of Singer [**6**].

Note that (as shown in the proof of A1.1.14) if Σ is a Hopf algebra then $\Phi = A \,\square_\Sigma\, \Gamma = \Gamma \,\square_\Sigma\, A$. More generally we have

A1.1.16. LEMMA. *With notation as above,* $A \,\square_\Sigma\, \Gamma = \Phi \otimes_D A$ *as right* Γ-*comodules.*

PROOF. Using A1.1.12 and A1.1.13 we have

$$\begin{aligned}
\Phi \otimes_D A &= A \,\square_\Sigma\, \Gamma \,\square_\Sigma\, A \otimes_D A \\
&= A \,\square_\Sigma\, \Gamma \,\square_{\Sigma'}\, A \,\square_I\, A \otimes_D A \\
&= A \,\square_\Sigma\, \Gamma \,\square_{\Sigma'}\, A \\
&= A \,\square_\Sigma\, A \,\square_{\Sigma'}\, \Gamma \\
&= A \,\square_U\, A \,\square_{\Sigma'}\, A \,\square_{\Sigma'}\, \Gamma \\
&= A \,\square_U\, A \,\square_{\Sigma'}\, \Gamma \\
&= A \,\square_\Sigma\, \Gamma.
\end{aligned}$$ $\square$

A1.1.17. COMODULE ALGEBRA STRUCTURE THEOREM. *Let* (B, Σ) *be a graded connected Hopf algebroid,* M *a graded connected right* Σ-*comodule algebra, and* $C = M \,\square_\Sigma\, B$. *Suppose*

(i) *there is a surjective comodule algebra map* $f \colon M \to \Sigma$ *and*

(ii) C *is a* B-*module and as such it is a direct summand of* M.

Then M *is isomorphic to* $C \otimes_B \Sigma$ *simultaneously as a left* C-*module and a right* Σ-*comodule.* $\square$

We will prove this after listing some corollaries. If Σ is a Hopf algebra over a field K then the second hypothesis is trivial so we have the following result, first proved as Theorem 4.7 of Milnor and Moore [**3**].

A1.1.18. COROLLARY. *Let* (K, Σ) *be a commutative graded connected Hopf algebra over a field* K. *Let* M *be a* K-*algebra and a right* Σ-*comodule and let* $C = M \,\square_\Sigma\, K$. *If there is a surjection* $f \colon M \to \Sigma$ *which is a homomorphism of algebras and* Σ-*comodules, then* M *is isomorphic to* $C \otimes \Sigma$ *simultaneously as a left* C-*module and as a right* Σ-*comodule.* $\square$

A1.1.19. COROLLARY. *Let* $f \colon (A, \Gamma) \to (B, \Sigma)$ *be a map of graded connected Hopf algebroids* (A1.1.7) *and let* $\Gamma' = \Gamma \otimes_A B$ *and* $C = \Gamma' \,\square_\Sigma\, B$. *Suppose*

(i) $f_2' \colon \Gamma' \to \Sigma$ *is onto and*

(ii) C *is a* B-*module and there is a* B-*linear map* $g \colon \Gamma' \to C$ *split by the inclusion of* C *in* Γ'.

Then there is a map $\tilde{g} \colon \Gamma' \to C \otimes_B \Sigma$ *defined by* $\tilde{g}(\gamma) = g(\gamma') \otimes f_2'(\gamma'')$ *which is an isomorphism of* C-*modules and* Σ-*comodules.* $\square$

A1.1.20. COROLLARY. *Let* K *be a field and* $f \colon (K, \Gamma) \to (K, \Sigma)$ *a map of graded connected commutative Hopf algebras and let* $C = \Gamma \,\square_\Sigma\, K$. *If* f *is surjective then* Γ *is isomorphic to* $C \otimes \Sigma$ *simultaneously as a left* C-*module and as a right* Σ-*comodule.* $\square$

In A1.3.12 and A1.3.13 we will give some change-of-rings isomorphisms of Ext groups relevant to the maps in the previous two corollaries.

PROOF OF A1.1.17. Let $i\colon C \to M$ be the natural inclusion and let $g\colon M \to C$ be a B-linear map such that gi is the identity. Define $\tilde{g}\colon M \to C \otimes_B \Sigma$ to be $(g \otimes \Sigma)\psi$; it is a map of Σ-comodules but not necessarily of C-modules and we will show below that it is an isomorphism.

Next observe that $f \,\square\, B\colon C \to B$ is onto. In dimension zero it is simply f, which is onto by assumption, and it is B-linear and therefore surjective. Let $j\colon B \to C$ be a B-linear splitting of $f \,\square\, B$. Then $h = \tilde{g}^{-1}(j \otimes \Sigma)\colon \Sigma \to M$ is a comodule splitting of f.

Define $\tilde{h}\colon C \otimes_B \Sigma \to M$ by $\tilde{h}(c \otimes \sigma) = i(c)h(\sigma)$ for $c \in C$ and $\sigma \in \Sigma$. It is clearly a C-linear comodule map and we will show that it is the desired isomorphism. We have

$$\tilde{g}\tilde{h}(c \otimes \sigma) = \tilde{g}(i(c)h(\sigma)) = g(i(c)h(\sigma')) \otimes \sigma'' = c \otimes \sigma$$

where the second equality holds because $i(c)$ is primitive in M and the congruence is modulo elements of lower degree with respect to the following increasing filtration (A1.2.7) on $C \otimes_B \Sigma$. Define $F_n(C \otimes_B \Sigma) \subset C \otimes_B \Sigma$ to be the sub-K-module generated by elements of the form $c \otimes \sigma$ with $\dim \sigma \leq n$. It follows that $\tilde{g}\tilde{h}$ and hence h are isomorphisms.

We still need to show that $\tilde{g}$ is an isomorphism. To show that it is 1-1, let $\widetilde{m} \otimes \sigma$ be the leading term (with respect to the above filtration of $M \otimes \Sigma$) of $\psi(m)$. It follows from coassociativity that $\widetilde{m}$ is primitive, so $g(\widetilde{m}) \neq 0$ if $m \neq 0$ and $\ker \tilde{g} = 0$. To show that $\tilde{g}$ is onto, note that for any $c \otimes \sigma \in C \otimes_B \Sigma$ we can choose $m \in f^{-1}(\sigma)$ and we have

$$\tilde{g}(i(c)m) = g(i(c)m') \otimes m'' = gi(c) \otimes \sigma = c \otimes \sigma$$

so $\operatorname{coker} \tilde{g} = 0$ by standard arguments. □

A1.1.21. DEFINITION. *An ideal $I \subset A$ is* invariant *if it is a sub-Γ-comodule, or equivalently if $\eta_R(I) \subset I\Gamma$.*

The following definition is intended to mimic that of a split groupoid, which is derived from the action of a group G acting on a set X. Here the set of objects is X and the set of morphism is $G \times X$, where (g, x) is a morphism from the object $x \in X$ to the object $g(x)$.

A1.1.22. DEFINITION. *A Hopf algebroid (A, Γ) is* split *if there is a Hopf algebroid map $i\colon (K, \Sigma) \to (A, \Gamma)$ (A1.1.19) such that $i_2'\colon \Sigma \otimes A \to \Gamma$ is an isomorphism of K-algebras.*

Note that composing $\eta_R\colon A \to \Gamma$ with the inverse of i_2' defines a left Σ-comodule structure on A.

2. Homological Algebra

Recall (A1.1.3) that the category of comodules over a Hopf algebroid (A, Γ) is abelian provided Γ is flat over A, which means that we can do homological algebra in it. We want to study the derived functors of Hom and cotensor product (A1.1.4). Derived functors are discussed in most books on homological algebra, e.g., Cartan and Eilenberg [**1**], Hilton and Stammbach [**1**], and Mac Lane [**1**]. In order to define them we must be sure that our category has enough injectives, i.e., that each Γ-comodule can be embedded in an injective one. This can be seen as follows.

A1.2.1. DEFINITION. *Given an A-module N, define a comodule structure on $\Gamma \otimes_A N$ by $\psi = \Delta \otimes N$. Then for any comodule M,*

$$\theta\colon \operatorname{Hom}_A(M, N) \to \operatorname{Hom}_\Gamma(M, \Gamma \otimes_A N)$$

is the isomorphism given by $\theta(f) = (\Gamma \otimes f)\psi_M$ for $f \in \operatorname{Hom}_A(M, N)$. For $g \in \operatorname{Hom}_\Gamma(M, \Gamma \otimes_A N)$, $\theta^{-1}(g)$ is given by $\theta^{-1}(g) = (\varepsilon \otimes N)g$.

A1.2.2. LEMMA. *If I is an injective A-module then $\Gamma \otimes_A I$ is an injective Γ-comodule. Hence the category of Γ-comodules has enough injectives.*

PROOF. To show that $\Gamma \otimes_A I$ is injective we must show that if M is a subcomodule of N, then a comodule map from M to $\Gamma \otimes_A I$ extends to N. But $\operatorname{Hom}_\Gamma(M, \Gamma \otimes_A I) = \operatorname{Hom}_A(M, I)$ which is a subgroup of $\operatorname{Hom}_A(N, I) = \operatorname{Hom}_\Gamma(N, \Gamma \otimes_A I)$ since I is injective as an A-module. Hence the existence of enough injectives in the category of A-modules implies the same in the category of Γ-comodules. $\square$

This result allows us to make

A1.2.3. DEFINITION. *For left Γ-comodules M and N, $\operatorname{Ext}^i_\Gamma(M, N)$ is the ith right derived functor of $\operatorname{Hom}_\Gamma(M, N)$, regarded as a functor of N. For M a right Γ-comodule, $\operatorname{Cotor}^i_\Gamma(M, N)$, is the ith right derived functor of $M \square_\Gamma N$ (A1.1.4), also regarded as a functor of N. The corresponding graded groups will be denoted simply by $\operatorname{Ext}_\Gamma(M, N)$ and $\operatorname{Cotor}_\Gamma(M, N)$, respectively.*

In practice we shall only be concerned with computing these functors when the first variable is projective over A. In that case the two functors are essentially the same by A1.1.6. We shall therefore make most of our arguments in terms of Cotor and list the corresponding statements about Ext as corollaries without proof.

Recall that the zeroth right derived functor is naturally equivalent to the functor itself if the latter is left exact. The cotensor product is left exact in the second variable if the first variable is flat as an A-comodule.

One knows that right derived functors can be computed using an injective resolution of the second variable. In fact the resolution need only satisfy a weaker condition.

A1.2.4. LEMMA. *Let*

$$0 \to N \to R^0 \to R^1 \to \cdots$$

be a long exact sequence of left Γ-comodules such that $\operatorname{Cotor}^n_\Gamma(M, R^i) = 0$ for $n > 0$. Then $\operatorname{Cotor}_\Gamma(M, N)$ is the cohomology of the complex

$$\operatorname{Cotor}^0_\Gamma(M, R^0) \xrightarrow{\delta_0} \operatorname{Cotor}^0_\Gamma(M, R^1) \xrightarrow{\delta_1} \cdots . \tag{A1.2.5}$$

PROOF. Define comodules N^i inductively by $N^0 = N$ and N^{i+1} is the quotient in the short exact sequence

$$0 \to N^i \to R^i \to N^{i+1} \to 0.$$

These give long exact sequences of Cotor groups which, because of the behavior of $\operatorname{Cotor}_\Gamma(M, R^i)$, reduce to four-term sequences

$$\begin{aligned} 0 \to \operatorname{Cotor}^0_\Gamma(M, N^i) &\to \operatorname{Cotor}^0_\Gamma(M, R^i) \\ &\to \operatorname{Cotor}^0_\Gamma(M, N^{i+1}) \to \operatorname{Cotor}^1_\Gamma(M, N^i) \to 0 \end{aligned}$$

and isomorphisms

$$\text{(A1.2.6)} \qquad \operatorname{Cotor}_\Gamma^n(M, N^{i+1}) \approx \operatorname{Cotor}_\Gamma^{n+1}(M.N^i) \quad \text{for} \quad n > 0.$$

Hence in A1.2.5, $\ker \delta_i = \operatorname{Cotor}_\Gamma^0(M, N^i)$ while $\operatorname{im} \delta_i$ is the image of $\operatorname{Cotor}_\Gamma^0(M, R^i)$ in $\operatorname{Cotor}_\Gamma^0(M, N^{i+1})$ so

$$\ker \delta_i / \operatorname{im} \delta_{i-1} = \operatorname{Cotor}_\Gamma^1(M, N^{i-1}) = \operatorname{Cotor}_\Gamma^i(M, N)$$

by repeated use of A1.2.6. This quotient by definition is H^i of A1.2.5. □

For another proof see A1.3.2.

We now introduce a class of comodules which satisfy the Ext condition of A1.2.4 when M is projective over A.

A1.2.7. Definition. *An* extended Γ-comodule *is one of the form* $\Gamma \otimes_A N$ *where* N *is an* A*-module. A* relatively injective Γ-comodule *is a direct summand of an extended one.*

This terminology comes from relative homological algebra, for which the standard references are Eilenberg and Moore [**1**] and Chapter IX of Mac Lane [**1**]. Our situation is dual to theirs in the following sense. We have the category $\mathbf{\Gamma}$ of left (or right) Γ-comodules, the category $\mathbf{A}$ of A-modules, the forgetful functor G from $\mathbf{\Gamma}$ to $\mathbf{A}$, and a functor $F\colon \mathbf{A} \to \mathbf{\Gamma}$ given by $F(M) = \Gamma \otimes_A M$ (A1.2.1). Mac Lane [**1**] then defines a resolvent pair to be the above data along with a natural transformation from GF to the identity on $\mathbf{A}$, i.e., natural maps $M \to \Gamma \otimes_A M$ with a certain universal property. We have instead maps $\varepsilon \otimes M\colon \Gamma \otimes_A M \to M$ such that for any A-homomorphism $\mu\colon C \to M$ where C is a Γ-comodule there is a unique comodule map $\alpha\colon C \to \Gamma \otimes_A M$ such that $\mu = (\varepsilon \otimes M)\alpha$. Thus we have what Mac Lane might call a coresolvent pair. Our F produces relative injectives while his produces relative projectives. This duality is to be expected because the example he had in mind was the category of modules over an algebra, while our category $\mathbf{\Gamma}$ is more like that of comodules over a coalgebra. The following lemma is comparable to Theorem IX.6.1 of Mac Lane [**1**].

A1.2.8. Lemma.

(a) *If* $i\colon M \to N$ *is a monomorphism of comodules which is split over* A*, then any map* f *from* M *to a relatively injective comodule* S *extends to* N*.* (*If* i *is not assumed to be split, then this property would make* S *injective.*)

(b) *If* M *is projective as an* A*-module and* S *is a relatively injective comodule, then* $\operatorname{Cotor}_\Gamma^i(M, S) = 0$ *for* $i > 0$ *and if* $S = \Gamma \otimes_A N$ *then* $\operatorname{Cotor}_\Gamma^0(M, S) = M \otimes_A N$*.*

Proof. (a) Let $j\colon N \to M$ be a splitting of i. Then $(\Gamma \otimes f)(\Gamma \otimes j)\psi = g$ is a comodule map from N to $\Gamma \otimes_A S$ such that $gi = \psi f\colon M \to \Gamma \otimes_A S$. It suffices then to show that S is a direct summand of $\Gamma \otimes_A S$, for then g followed by the projection of $\Gamma \otimes_A S$ onto S will be the desired extension of f. By definition S is a direct summand of $\Gamma \otimes_A T$ for some A-module T. Let $k\colon S \to \Gamma \otimes_A T$ and $k^{-1}\colon \Gamma \otimes_A T \to S$ be the splitting maps. Then $k^{-1}(\Gamma \otimes \varepsilon \otimes T)(\Gamma \otimes k)$ is the projection of $\Gamma \otimes_A S$ onto S.

(b) One has an isomorphism $\phi\colon M \otimes_A N \to M \,\square_\Gamma (\Gamma \otimes_A N)$ given by $\phi(m \otimes n) = \psi(m) \otimes n$. Since S is a direct summand of $\Gamma \otimes_A N$, it suffices to replace the former by the latter. Let

$$0 \to N \to I^0 \to I^1 \to \cdots$$

be a resolution of N by injective A-modules. Tensoring over A with Γ gives a resolution of $\Gamma \otimes_A N$ by injective Γ-comodules. $\operatorname{Cotor}_\Gamma(M, \Gamma \otimes_A N)$ is the cohomology of the resolution cotensored with M, which is isomorphic to

$$M \otimes_A I^0 \to M \otimes_A I^1 \to \cdots .$$

This complex is acyclic since M is projective over A. □

Compare the following with Theorem IX.4.3 of Mac Lane [**1**].

A1.2.9. Lemma. (a) *Let*

$$0 \to M \xrightarrow{d_{-1}} P^0 \xrightarrow{d_0} P^1 \xrightarrow{d_1} \cdots$$

and

$$0 \to N \xrightarrow{d_{-1}} R^0 \xrightarrow{d_0} R^1 \xrightarrow{d_1} \cdots$$

be long exact sequences of Γ-comodules in which each P^i and R^i is relatively injective and the image of each map is a direct summand over A. Then a comodule map $f\colon M \to N$ extends to a map of long exact sequences.

(b) *Applying $L \,\square_\Gamma\, (\cdot)$ (where L is a right Γ-comodule projective over A) to the two sequences and taking cohomology gives* $\operatorname{Cotor}_\Gamma(L, M)$ *and* $\operatorname{Cotor}_\Gamma(L, N)$, *respectively. The induced map from the former to the latter depends only on f.*

Proof. That the cohomology indicated in (b) is Cotor follows from A1.2.4 and A1.2.8(b). The proof of the other assertions is similar to that of the analogous statements about injective resolutions. Define comodules M^i and N^i inductively by $M^0 = M$, $N^0 = N$, and M^{i+1} and N^{i+1} are the quotients in the short exact sequences

$$0 \to M^i \to P^i \to M^{i+1} \to 0$$

and

$$0 \to N^i \to R^i \to N^{i+1} \to 0.$$

These sequences are split over A. Assume inductively that we have a suitable map from M^i to N^i. Then A1.2.8(a) gives us $f_i\colon P^i \to R^i$, and this induces a map from M^{i+1} to N^{i+1}, thereby proving (a).

For (b) it suffices to show that the map of long exact sequences is unique up to chain homotopy, i.e., given two sets of maps $f_i, f_i'\colon P^i \to R^i$ we need to construct $h_i\colon P^i \to R^{i-1}$ (with $h_0 = 0$) such that $h_{i+1}d_i + d_{i-1}h_i = f_i - f_i'$. Consider the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & M^i \arrow[r, "d_{i-1}"] \arrow[d, "g_{i-1}"'] & P^i \arrow[r, "d_i"] \arrow[d, "g_i"'] & M^{i+1} \arrow[r] \arrow[d, "g_i"'] & 0 \\
0 \arrow[r] & N^i d_{i-1} \arrow[r] & R^i d_i \arrow[r] & N^{i+1} \arrow[r] & 0
\end{tikzcd}$$

where $g_i = f_i - f_i'\colon P^i \to R^i$ and we use the same notation for the map induced from the quotient M^{i+1}. Assume inductively that $h_i\colon P^i \to R^{i-1}$ has been constructed. Projecting it to N^i we get $h_i\colon P^i \to N^i$ with $h_i d_{i-1} = g_{i-1}$. Now we want a map $hath_{i+1}\colon M^{i+1} \to R^i$ such that $hath_{i+1}d_i = g_i - d_{i-1}h_i$. By the exactness of the top row, $hath_{i+1}$ exists iff $(g_i - d_{i-1}h_i)d_{i-1} = 0$. But we have $g_i d_{i-1} - d_{i-1}(h_i d_{i-1}) = g_i d_{i-1} - d_i g_{i-1} = 0$, so $whath_i$ exists. By A1.2.8(a) it extends from M^{i+1} to P^{i+1} giving the desired h_{i+1}. □

Resolution of the above type serve as a substitute for injective resolutions. Hence we have

A1.2.10. DEFINITION. *A* resolution by relative injectives *of a comodule M is a long exact sequence*

$$0 \to M \to R^0 \to R^1 \to \dots$$

in which each R^i is a relatively injective and the image of each map is a direct summand over A. We now give an important example of such a resolution.

A1.2.11. DEFINITION. *Let M be a left Γ-comodule. The* cobar resolution $D_\Gamma^*(M)$ *is defined by $D_\Gamma^s(M) = \Gamma \otimes_A \overline{\Gamma}^{\otimes s} \otimes_A M$, where $\overline{\Gamma}$ is the unit coideal (the cokernel of η_L), with coboundary $d_s\colon D_\Gamma^s(M) \to D_\Gamma^{s+1}(M)$ given by*

$$\begin{aligned} d_s(\gamma_0 \otimes \gamma_1 \otimes \cdots \gamma_s \otimes m) = &\sum_{i=0}^{s} (-1)^i \gamma_0 \otimes \cdots \gamma_{i-1} \otimes \Delta(\gamma_i) \otimes \gamma_{i+1} \otimes \cdots m \\ &+ (-1)^{s+1} \gamma_0 \otimes \cdots \gamma_s \otimes \psi(m) \end{aligned}$$

for $\gamma_0 \in \Gamma$, $\gamma_1, \dots, \gamma_s \in \overline{\Gamma}$, and $m \in M$. For a right Γ-comodule L which is projective over A, the cobar complex *$C_\Gamma^*(L, M)$ is $L \,\square_\Gamma\, D_\Gamma^*(M)$, so $C_\Gamma^s(L, M) = L \otimes_A \overline{\Gamma}^{\otimes s} \otimes_A M$, where $\Gamma^{\otimes s}$ denotes the s-fold tensor product of Γ over A. Whenever possible the subscript Γ will be omitted, and $C_\Gamma^*(A, M)$ will be abbreviated to $C_\Gamma^*(M)$. The element $a \otimes \gamma_1 \otimes \cdots \otimes \gamma_n \otimes m \in C_\Gamma(L, M)$, where $a \in L$, will be denoted by $a\gamma_1|\gamma_2|\cdots|\gamma_n m$. If $a = 1$ or $m = 1$, they will be omitted from this notation.*

A1.2.12. COROLLARY. *$H(C_\Gamma^*(L, M)) = \operatorname{Cotor}_\Gamma(L, M)$ if L is projective over A, and $H(C_\Gamma^*(M)) = \operatorname{Ext}_\Gamma(A, M)$.*

PROOF. It suffices by A1.2.9 to show that $D_\Gamma(M) = C_\Gamma(\Gamma, M)$ is a resolution of M by relative injectives. It is clear that $D_\Gamma^s(M)$ is a relative injective and that d^s is a comodule map. To show that $D_\Gamma(M)$ is acyclic we use a contacting homotopy $S\colon D_\Gamma^s(M) \to D_\Gamma^{s-1}(M)$ defined by $S(\gamma\gamma_1|\cdots|\gamma_s m) = \varepsilon(\gamma)\gamma_1\gamma_2|\cdots|\gamma_s m$ for $s > 0$ and $S(\gamma m) = 0$. Then $Sd + dS$ is the identity on $D_\Gamma^s(M)$ for $s > 0$, and $1 - \phi$ on $D_\Gamma^0(M)$, where $\phi(\gamma m) = \varepsilon(\gamma)m'm''$. Hence

$$H^s(D_\Gamma(M)) = \begin{cases} 0 & \text{for } s > 0, \\ \operatorname{im}\phi = M & \text{for } s = 0. \end{cases} \qquad \square$$

Our next job is to define the external cup product in Cotor, which is a map $\operatorname{Cotor}_\Gamma(M_1, N_1) \otimes \operatorname{Cotor}_\Gamma(M_2, N_2) \to \operatorname{Cotor}_\Gamma(M_1 \otimes_A M_2, N_1 \otimes_A N_2)$ (see A1.1.2 for the definition of the comodule tensor product). If $M_1 = M_2 = M$ and $N_1 = N_2 = N$ are comodule algebras (A1.1.2) then composing the above with the map in Cotor induced by $M \otimes_A M \to M$ and $N \otimes_A N \to N$ gives a product on $\operatorname{Cotor}_\Gamma(M, N)$. Let P_1^* and P_2^* denote relative injective resolutions of N_1 and N_2, respectively. Then $P_1^* \otimes_A P_2^*$ is a resolution of $N_1 \otimes_A N_2$. We have canonical maps

$$\operatorname{Cotor}_\Gamma(M_1, N_1) \otimes \operatorname{Cotor}_\Gamma(M_2, N_2) \to H(M_1 \,\square_\Gamma\, P_1^* \otimes M_2 \,\square_\Gamma\, P_2^*)$$

(with tensor products over K) and

$$M_1 \,\square_\Gamma\, P_1^* \otimes M_2 \,\square_\Gamma\, P_2^* \to (M_1 \otimes_A M_2) \,\square_\Gamma\, (P_1^* \otimes_A P_2^*).$$

A1.2.13. DEFINITION. *The external cup product*

$$\mathrm{Cotor}_\Gamma(M_1, N_1) \otimes \mathrm{Cotor}_\Gamma(M_2, N_2) \to \mathrm{Cotor}_\Gamma(M_1 \otimes_A M_2, N_1 \otimes_A N_2)$$

and the internal cup product on $\mathrm{Cotor}_\Gamma(M, N)$ *for comodule algebras* M *and* N *are induced by the maps described above.*

Note that A1.2.9(b) implies that these products are independent of the choices made. Since the internal product is the composition of the external product with the products on M and N and since the latter are commutative and associative we have

A1.2.14. COROLLARY. *If* M *and* N *are comodule algebras then* $\mathrm{Cotor}_\Gamma(M, N)$ *is a commutative (in the graded sense) associative algebra.* □

It is useful to have an explicit pairing on cobar complexes

$$C_\Gamma(M_1, N_1) \otimes C_\Gamma(M_2, N_2) \to C_\Gamma(M_1 \otimes M_2, N_1 \otimes N_2).$$

This can be derived from the definitions by tedious straightforward calculation. To express the result we need some notation. For $m_2 \in M_2$ (a right Γ-comodule) and $n_1 \in N_1$ (a left Γ-comodule) let

$$m_2^{(0)} \otimes \cdots \otimes m_2^{(s)} \in M_2 \otimes_A \Gamma^{\otimes s},$$

with $m_2^{(0)} \in M_2$ and $m_2^{(i)} \in \Gamma$ for $1 \le i \le s$, and

$$n_1^{(1)} \otimes \cdots \otimes n_1^{(t+1)} \in \Gamma^{\otimes t} \otimes_A N_1,$$

for $n_1^{(j)} \in \Gamma$ for $1 \le j \le t$ and $n_1^{(1+t)} \in N_1$, denote the iterated coproducts. Each of these expressions is shorthand for a certain sum of monomials in the indicated tensor product. Then the pairing is given by

(A1.2.15) $\quad (m_1\gamma_1|\cdots|\gamma_s n_1) \otimes (m_2\gamma_{s+1}|\cdots|\gamma_{s+t}n_2)$

$$\mapsto (-1)^\tau \left(m_1 \otimes m_2^{(0)}\right)\left(\gamma_1 m_2^{(1)}\right)|\cdots|\gamma_s m_2^{(s)}|n_1^{(1)}\gamma_{s+1}|\cdots|\left(n_1^{(t)}\gamma_{s+t}\right)\left(n_1^{(1+t)} \otimes n_2\right)$$

where

$$\begin{aligned}\tau = \deg m_2 \deg n_1 + \sum_{i=0}^{s} \deg m_2^{(i)}\left(s - i + \sum_{j=i+1}^{s} \deg \gamma_j\right)& \\ + \sum_{i=1}^{t+1} \deg n_1^{(i)}\left(i - 1 + \sum_{j=1}^{i-1} \deg \gamma_{j+s}\right)&.\end{aligned}$$

Note that this is natural in all variables in sight.

Finally, we have two easy miscellaneous results.

A1.2.16. PROPOSITION. (a) *If* $I \subset A$ *is invariant* (A1.2.12) *then* $(A/I, \Gamma/I\Gamma)$ *is a Hopf algebroid.*

(b) *If* M *is a left* Γ*-comodule annihilated by* I *as above, then*

$$\mathrm{Ext}_\Gamma(A, M) = \mathrm{Ext}_{\Gamma/I\Gamma}(A/I, M).$$

PROOF. Part (a) is straightforward. For (b) observe that the complexes $C_\Gamma(M)$ and $C_{\Gamma/I\Gamma}(M)$ are identical. □

A1.2.17. PROPOSITION. *If* (A, Γ) *is split* (A1.1.22) *then* $\mathrm{Ext}_\Gamma(A, M) = \mathrm{Ext}_\Sigma(K, M)$ *where the left* Σ*-comodule structure on the left* Γ*-comodule* M *comes from the isomorphism* $\Gamma \otimes_A M = \Sigma \otimes M$.

PROOF. $C_\Gamma(M) = C_\Sigma(M)$. □

3. Some Spectral Sequences

In this section we describe several spectral sequences useful for computing Ext over a Hopf algebroid. The reader is assumed to be familiar with the notion of a spectral sequence; the subject is treated in each of the standard references for homological algebra (Cartan and Eilenberg [**1**], Mac Lane [**1**] and Hilton and Stammbach [**1**]) and in Spanier [**1**]. The reader is warned that most spectral sequences can be indexed in more than one way. With luck the indexing used in this section will be consistent with that used in the text, but it may differ from that appearing elsewhere in the literature and from that used in the next two sections.

Suppose we have a long exact sequence of Γ-comodules

$$0 \to M \to R^0 \xrightarrow{d^0} R^1 \xrightarrow{d^1} R^2 \to \cdots \tag{A1.3.1}$$

Let $S^{i+1} = \operatorname{im} d^i$ and $S^0 = M$ so we have short exact sequences

$$0 \to S^i \xrightarrow{a^i} R^i \xrightarrow{b^i} S^{i+1} \to 0$$

for all $i \geq 0$. Each of these gives us a connecting homomorphism

$$\delta^i \colon \mathrm{Cotor}_\Gamma^{s,t}(L, S^i) \to \mathrm{Cotor}_\Gamma^{s+1,t}(L, S^{i-1}).$$

Let $\delta_{(i)} \colon \mathrm{Cotor}_\Gamma^{s,t}(L, S^i) \to \mathrm{Cotor}_\Gamma^{s+i,t}(L, S^0)$ be the composition $\delta^1\delta^2 \cdots \delta^i$. Define a decreasing filtration on $\mathrm{Cotor}_\Gamma^{s,*}(L, M)$ by $F^i = \operatorname{im} \delta_{(i)}$ for $i \leq s$, where $\delta_{(0)}$ is the identity and $F^i = 0$ for $i \leq 0$.

A1.3.2. THEOREM. *Given a long exact sequence of* Γ*-comodules* A1.3.1 *there is a natural trigraded spectral sequence* (E_*^{***}) *(the resolution spectral sequence) such that*

(a) $E_1^{n,s,t} = \mathrm{Cotor}_\Gamma^{s,t}(L, R^n)$;
(b) $d_r \colon E_r^{n,s,t} \to E_r^{n+r,s-r+1,t}$ *and* d_1 *is the map induced by* d^* *in* A1.3.1 *and*
(c) $E_\infty^{n,s,t}$ *is the subquotient* F^n/F^{n+1} *of* $\mathrm{Cotor}_\Gamma^{n+s,t}(L, M)$ *defined above.*

PROOF. We will give two constructions of this spectral sequence. For the first define an exact couple (2.1.6) by

$$E_1^{s,t} = \mathrm{Cotor}_\Gamma^{-t,*}(L, R^s),$$
$$D_1^{s,t} = \mathrm{Cotor}_\Gamma^{-t,*}(L, S^s),$$

$i_1 = \delta^*$, $j_1 = a^*$, and $k_1 = b^*$. Then the associated spectral sequence is the one we want.

The second construction applies when L is projective over A and is more explicit and helpful in practice; we get the spectral sequence from a double complex as described in Cartan and Eilenberg [**1**], Section XV.6 or Mac Lane [**1**], Section XI.6. We will use the terminology of the former. Let

$$B^{n,s,*} = C_\Gamma^s(L, R^n) \qquad \text{(A1.2.11)},$$
$$\partial_1^{n,s,*} = (-1)^n C_\Gamma^s(d^n) \colon B^{n,s,*} \to B^{n+1,s,*},$$

and

$$\partial_2^{n,s,*} = d^s \colon B^{n,s,*} \to B^{n,s+1,*}.$$

(Our ∂_1, ∂_2 correspond to the d_1, d_2 in Cartan and Eilenberg [1], IV.4].) Then $\partial_2^{n+1,s,*}\partial_1^{n,s,*} + \partial_1^{n,s+1,*}\partial_2^{n,s,*} = 0$ since d^s commutes with $C_\Gamma^s(d^n)$. The associated complex $(B^{p,*}, \partial)$ is defined by

(A1.3.3)
$$B^{p,*} = \bigoplus_{n+s=p} B^{n,s,*} = \bigoplus_{n+s=p} C_\Gamma^s(L, R^n)$$

with $\partial = \partial_1 + \partial_2 \colon B^{p,*} \to B^{p+1,*}$.

This complex can be filtered in two ways, i.e.,

$$F_I^p B = \bigoplus_{r \geq p} \bigoplus_q B^{r,q,*},$$

$$F_{II}^q B = \bigoplus_{s \geq q} \bigoplus_p B^{p,s,*}$$

and each of these filtrations leads to a spectral sequence. In our case the functor $C_\Gamma^s(L, \cdot)$ is exact since Γ is flat over A, so $H^{s,*}(F_{II}B) = C_\Gamma^s(L, M)$. Hence in the second spectral sequence

$$E_1^{n,s,*} = \begin{cases} C_\Gamma^s(L, M) & \text{if } n = 0 \\ 0 & \text{otherwise} \end{cases}$$

and

$$E_2^{n,s,*} = E_\infty^{n,s,*} = \begin{cases} \operatorname{Cotor}_\Gamma^{s,*}(L, M) & \text{if } n = 0 \\ 0 & \text{otherwise.} \end{cases}$$

The two spectral sequences converge to the same thing, so the first one, which is the one we want, has the desired properties. $\square$

A1.3.4. Corollary. *The cohomology of the complex* B^{**} *of* A1.3.3 *is* $\operatorname{Cotor}_\Gamma^{**}(L, M)$. $\square$

Note that A1.2.4 is a special case of A1.3.3 in which the spectral sequence collapses.

Next we discuss spectral sequences arising from increasing and decreasing filtration of Γ.

A1.3.5. Definition. *An* increasing filtration *on a Hopf algebroid* (A, Γ) *is an increasing sequaence of sub-*K*-modules*

$$K = F_0\Gamma \subset F_1\Gamma \subset F_2\Gamma \subset \cdots$$

with $\Gamma = \bigcup F_s\Gamma$ *such that*

(a) $F_s\Gamma \cdot F_t\Gamma \subset F_{s+t}\Gamma$,
(b) $c(F_s\Gamma) \subset F_s\Gamma$, *and*
(c) $\Delta F_s\Gamma \subset \bigoplus_{p+q=s} F_p\Gamma \otimes_A F_q\Gamma$.

A decreasing filtration *on* (A, Γ) *is a decreasing sequaence of sub-*K*-modules*

$$\Gamma = F^0\Gamma \supset F^1\Gamma \supset F^2\Gamma \supset \cdots$$

with $0 = \bigcap F^s\Gamma$ *such that conditions similar to (a), (b), and (c) above (with the inclusion signs reversed) are satisfied. A* filtered Hopf algebroid (A, Γ) *is one equipped with a filtration. Note that a filtration on* Γ *induces one on* A*, e.g.,*

$$F_sA = \eta_L(A) \cap F_s\Gamma = \eta_R(A) \cap F_s\Gamma = \varepsilon(F_s\Gamma).$$

A1.3.6. Definition. *Let* (A, Γ) *be filtered as above. The associated graded object* $E^0\Gamma$ *(or* $E_0\Gamma$*) is defined by*

$$E^0_s\Gamma = F_s\Gamma/F_{s-1}\Gamma$$

or

$$E^s_0\Gamma = F^s\Gamma/F^{s-1}\Gamma.$$

The graded object E^0_*A *(or* E^*_0A*) is defined similarly.*

A1.3.7. Definition. *Let* M *be a* Γ*-comodule. An increasing filtration on* M *is an increasing sequence of sub-*K*-modules*

$$0 = F_1M \subset F_2M \subset \cdots$$

such that $M = \bigcup F_sM$, $F_sA \cdot F_tM \subset F_{s+t}M$*, and*

$$\psi(F_sM) \subset \bigoplus_{p+q=s} F_p\Gamma \otimes F_qM.$$

A decreasing filtration on M *is similarly defined, as is the associated graded object* E^0_*M *or* E^*_0M*. A* filtered comodule M *is a comodule equipped with a filtration.*

A1.3.8. Proposition. $(E^0A, E^0\Gamma)$ *or* $(E_0A, E_0\Gamma)$ *is a graded Hopf algebroid and* E^0M *or* E_0M *is a comodule over it.* $\square$

Note that if (A, Γ) and M are themselves graded than $(E^0A, E^0\Gamma)$ and E^0M are bigraded.

We assume from now on that $E^0\Gamma$ or $E_0\Gamma$ is flat over E^0A or E_0A.

A1.3.9. Theorem. *Let* L *and* M *be right and left filtered comodules, respectively, over a filtered Hopf algebroid* (A, Γ)*. Then there is a natural spectral sequence converging to* $\operatorname{Cotor}_\Gamma(L, M)$ *such that*

(a) *in the increasing case*

$$E_1^{s,*} = \operatorname{Cotor}^s_{E^0\Gamma}(E^0L, E^0M)$$

where the second grading comes from the filtration and

$$d_r \colon E_r^{s,t} \to E_r^{s+1,t-r};$$

(b) *in the decreasing case*

$$E_1^{s,*} = \operatorname{Cotor}^s_{E_0\Gamma}(E_0L, E_0M)$$

and

$$d_r \colon E_r^{s,t} \to E_r^{s+1,t+r}.$$

Note that our indexing differs from that of Cartan and Eilenberg [**1**] and Mac Lane [**1**].

PROOF. The filtrations on Γ and M induce one on the cobar complex (A1.2.11) $C_\Gamma M$ and we have $E_0C_\Gamma(L,M) = C_{E_0\Gamma}(E_0L, E_0M)$ or $E^0C_\Gamma(L,M) = C_{E^0\Gamma}(E^0L, E^0M)$. The associated spectral sequence is the one we want. □

The following is an important example of an increasing filtration.

A1.3.10. EXAMPLE. Let (K,Γ) be a Hopf algebra. Let $\overline{\Gamma}$ be the unit coideal, i.e., the quotient in the short exact sequence

$$0 \to K \xrightarrow{\eta} \Gamma \to \overline{\Gamma} \to 0,$$

The coproduct map Δ can be iterated by coassociativity to a map $\Delta^s\colon \Gamma \to \Gamma^{\otimes s+1}$. Let $F_s\Gamma$ be the kernel of the composition

$$\Gamma \xrightarrow{\Delta^s} \Gamma^{\otimes s+1} \to \overline{\Gamma}^{\otimes s+1}.$$

This is the *filtration of* $\mathbf{\Gamma}$ *by powers of the unit coideal.*

Next we treat the spectral sequence associated with a map of Hopf algebroids.

A1.3.11. THEOREM. *Let $f\colon (A,\Gamma) \to (B,\Sigma)$ be a map of Hopf algebroids* (A1.1.18)*, M a right Γ-comodule and N a left Σ-comodule.*

(a) *$C_\Sigma(\Gamma \otimes_A B, N)$ is a complex of left Γ-comodules, so $\mathrm{Cotor}_\Sigma(\Gamma \otimes_A B, N)$ is a left Γ-comodule.*

(b) *If M is flat over A, there is a natural spectral sequence converging to $\mathrm{Cotor}_\Sigma(M \otimes_A B, N)$ with*

$$E_2^{s,t} = \mathrm{Cotor}_\Gamma^s(M, \mathrm{Cotor}_\Sigma^t(\Gamma \otimes_A B, N))$$

and $d_r\colon E_r^{s,t} \to E_r^{s+r,t-r+1}$.

(c) *If N is a comodule algebra then so is $\mathrm{Cotor}_\Sigma(\Gamma \otimes_A B, N)$. If M is also a comodule algebra, then the spectral sequence is one of algebras.*

PROOF. For (a) we have $C_\Sigma^s(\Gamma \otimes_A B, N) = \Gamma \otimes_A \overline{\Sigma}^{\otimes s} \otimes_B N$ with the coboundary d_s as given in A1.2.11. We must show that d_s commutes with the coproduct on Γ. For all terms other than the first in the formula for d_s this commutativity is clear. For the first term consider the diagram

$$\begin{tikzcd}
\Gamma \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & \Gamma \otimes_A \Gamma \arrow[r, "\Gamma\otimes f"] \arrow[d, "\Delta\otimes\Gamma"'] & \Gamma \otimes_A \Sigma \arrow[d, "\Delta\otimes\Sigma"'] \\
\Gamma \otimes_A \Gamma \arrow[r, "\Gamma\otimes\Delta"] & \Gamma \otimes_A \Gamma \otimes_A \Gamma \arrow[r, "\Gamma\otimes\Gamma\otimes f"] & \Gamma \otimes_A \Gamma \otimes_A \Sigma
\end{tikzcd}$$

The left-hand square commutes by coassociativity and other square commutes trivially. The top composition when tensored over B with $\Sigma^{\otimes s} \otimes_B N$ is the first term in d_s. Hence the commutativity of the diagram shows that d_s is a map of left Γ-comodules.

For (b) consider the double complex

$$C_\Gamma^*(M, C_\Sigma^*(\Gamma \otimes_A B, N)),$$

which is well defined because of (a). We compare the spectral sequences obtained by filtering by the two degrees. Filtering by the first gives

$$E_1 = C_\Gamma^*(M, \mathrm{Cotor}_\Sigma(\Gamma \otimes_A B, N))$$

so

$$E_2 = \mathrm{Cotor}_\Gamma(M, \mathrm{Cotor}_\Sigma(\Gamma \otimes_A B, N))$$

which is the desired spectral sequence. Filtering by the second degree gives a spectral sequence with

$$\begin{aligned} E_1^{s,t} &= \mathrm{Cotor}_\Gamma^s(M, C_\Sigma^t(\Gamma \otimes_A B, N)) \\ &= \mathrm{Cotor}_\Gamma(M, \Gamma \otimes_A \overline{\Sigma}^{\otimes t} \otimes_B N) \\ &= M \otimes_A \overline{\Sigma}^{\otimes t} \otimes_B N \quad \text{by A1.2.8(b)} \\ &= C_\Sigma^t(M \otimes_A B, N) \end{aligned}$$

so $E_2 = E_\infty = \mathrm{Cotor}_\Sigma(M \otimes_A B, N)$.

For (c) note that $\Gamma \otimes_A B$ as well as N is a Σ-comodule algebra. The Γ-coaction on $C_\Sigma(\Gamma \otimes_A B, N)$ is induced by the map

$$\begin{aligned} C(\Delta \otimes B, N)\colon C_\Sigma(\Gamma \otimes_A B, N) &\to C_\Sigma(\Gamma \otimes_A \Gamma \otimes_A B, N) \\ &= \Gamma \otimes_A C_\Sigma(\Gamma \otimes_A B, N). \end{aligned}$$

Since the algebra structure on $C_\Sigma(\ ,\ )$ is functorial, $C(\Delta \otimes B, N)$ induces an algebra map in cohomology and $\mathrm{Cotor}_\Sigma(\Gamma \otimes_A B, N)$ is a Γ-comodule algebra.

To show that we have a spectral sequence of algebras we must define an algebra structure on the double complex used in the proof of (b), which is $M \square_\Gamma D_\Gamma(\Gamma \otimes_A B \square_\Sigma D_\Sigma(N))$. Let $\widetilde{N} = \Gamma \otimes_A B \square_\Sigma D_\Sigma(N)$. We have just seen that it is a Γ-comodule algebra. Then this algebra structure extends to one on $D_\Gamma(\widetilde{N})$ by A1.2.9 since $D_\Gamma(\widetilde{N}) \otimes_A D_\Gamma(\widetilde{N})$ is a relatively injective resolution of $\widetilde{N} \otimes_A \widetilde{N}$. Hence we have maps

$$\begin{aligned} M \square_\Gamma D_\Gamma(\widetilde{N}) \otimes M \square_\Gamma D_\Gamma(\widetilde{N}) &\to M \otimes_A M \square_\Gamma D_\Gamma(\widetilde{N}) \otimes_A D_\Gamma(\widetilde{N}) \\ &\to M \square_\Gamma D_\Gamma(\widetilde{N}) \otimes_A D_\Gamma(\widetilde{N}) \to M \square_\Gamma D_\Gamma(\widetilde{N}), \end{aligned}$$

which is the desired algebra structure. □

Our first application of this spectral sequence is a change-of-rings isomorphism that occurs when it collapses.

A1.3.12. Change-of-Rings Isomorphism Theorem. *Let $f\colon (A, \Gamma) \to (B, \Sigma)$ be a map of graded connected Hopf algebroids (A1.1.7) satisfying the hypotheses of A1.1.19; let M be a right Γ-comodule and let N be a left Σ-comodule which is flat over B. Then*

$$\mathrm{Cotor}_\Gamma(M, (\Gamma \otimes_A B) \square_\Sigma N) = \mathrm{Cotor}_\Sigma(M \otimes_A B, N).$$

In particular

$$\mathrm{Ext}_\Gamma(A, (\Gamma \otimes_A B) \square_\Sigma N) = \mathrm{Ext}_\Sigma(B, N),$$

Proof. By A1.1.19 and A1.2.8(b) we have

$$\mathrm{Cotor}_\Sigma^s(\Gamma \otimes_A B, N) = 0 \quad \text{for } s > 0.$$

A1.3.11(b) gives

$$\mathrm{Cotor}_\Gamma(M, \mathrm{Cotor}_\Sigma^0(\Gamma \otimes_A B, N)) = \mathrm{Cotor}_\Sigma(M \otimes_A B, N).$$

Since N is flat over B,

$$\mathrm{Cotor}_\Sigma^0(\Gamma \otimes_A B, N) = (\Gamma \otimes_A B) \square_\Sigma N$$

and the result follows. ☐

A1.3.13. Corollary. *Let K be a field and $f\colon (K,\Gamma)\to(K,\Sigma)$ be a surjective map of Hopf algebras. If N is a left Σ-comodule then*

$$\mathrm{Ext}_\Gamma(K,\Gamma\,\square_\Sigma\,N)=\mathrm{Ext}_\Sigma(K,N). \qquad \square$$

Next we will construct a change-of-rings spectral sequence for an extension of Hopf algebroids (A1.1.15) similar to that of Cartan and Eilenberg [**1**, XVI 6.1], which we will refer to as the Cartan–Eilenberg spectral sequence.

A1.3.14. Cartan–Eilenberg Spectral Sequence Theorem. *Let*

$$(D,\Phi)\xrightarrow{i}(A,\Gamma)\xrightarrow{f}(A,\Sigma)$$

be an extension of graded connected Hopf algebroids (A1.1.15). Let M be a right Φ-comodule and N a left Γ-comodule.

(a) $\mathrm{Cotor}_\Sigma(A,N)$ *is a left Φ-comodule. If N is a comodule algebra, then so is this* Cotor .

(b) *There is a natural spectral sequence converging to* $\mathrm{Cotor}_\Gamma(M\otimes_D A,N)$ *with*

$$E_2^{s,t}=\mathrm{Cotor}_\Phi^s(M,\mathrm{Cotor}_\Sigma^t(A,N))$$

and

$$d_r\colon E_r^{s,t}\to E_r^{s+r,t-r+1}.$$

(c) *If M and N are comodule algebras, then the spectral sequence is one of algebras.*

Proof. Applying A1.3.11 to the map i shows that $\mathrm{Cotor}_\Gamma(\Phi\otimes_D A,N)$ is a left Φ-comodule algebra and there is a spectral sequence converging to $\mathrm{Cotor}_\Gamma(M\otimes_D A,N)$ with

$$E_2=\mathrm{Cotor}_\Phi(M,\mathrm{Cotor}_\Gamma(\Phi\otimes_A D,N)).$$

Hence the theorem will follow if we can show that $\mathrm{Cotor}_\Gamma(\Phi\otimes_D A,N)=\mathrm{Cotor}_\Sigma(A,N)$. Now $\Phi\otimes_D A=A\,\square_\Sigma\,\Gamma$ by A1.1.16. We can apply A1.3.12 to f and get $\mathrm{Cotor}_\Gamma(P\,\square_\Sigma\,\Gamma,R)=\mathrm{Cotor}_\Sigma(P,R)$ for a right Σ-comodule P and left Γ-comodule R. Setting $P=A$ and $R=N$ gives the desired isomorphism

$$\mathrm{Cotor}_\Gamma(\Phi\otimes_D A,N)=\mathrm{Cotor}_\Gamma(A\,\square_\Sigma\,\Gamma,N)=\mathrm{Cotor}_\Sigma(A,N). \qquad \square$$

The case $M=D$ gives

A1.3.15. Corollary. *With notation as above, there is a spectral sequence of algebras converging to $\mathrm{Ext}_\Gamma(A,N)$ with $E_2=\mathrm{Ext}_\Phi(D,\mathrm{Ext}_\Sigma(A,N))$.* ☐

Now we will give an alternative formulation of the Cartan–Eilenberg spectral sequence (A1.3.14) suggested by Adams [**12**], 2.3.1 which will be needed to apply the results of the next sections on Massey products and Steenrod operations. A similar formulation was given by Hochschild-Serre [**1**] in the context of group cohomology. Using the notation of A1.2.14, we define a decreasing filtration on $C_\Gamma(M\otimes_D A,N)$ by saying that $m\gamma_1|\dots|\gamma_s n\in F^i$ if i of the γ's are in $\ker f_2$.

A1.3.16. Theorem. *The spectral sequence associated with the above filtration of $C_\Gamma(M\otimes_D A,N)$ coincides with the Cartan–Eilenberg spectral sequence of* A1.3.14.

Proof. The Cartan–Eilenberg spectral sequence is obtained by filtering the double complex $C^*_\Phi(M, C^*_\Gamma(\Phi \otimes_D A, N))$ by the first degree. We define a filtration-preserving map θ from this complex to $C_\Gamma(M \otimes_D A, N)$ by

$$\theta(m \otimes \phi_1 \otimes \cdots \phi_s \otimes \phi \otimes \gamma_{s+1} \otimes \cdots \gamma_{s+t} \otimes n) \\ = m \otimes i_2(\phi_1) \otimes \cdots i_2(\phi_s) i_1 \varepsilon(\phi) \otimes \gamma_{s+1} \otimes \cdots \gamma_{s+t} \otimes n.$$

Let $E_1^{s,t}(M,N) = C^s_\Phi(M, \operatorname{Cotor}^t_\Gamma(\Phi \otimes_D A, N)) = C^s_\Phi(M, \operatorname{Cotor}^t_\Sigma(A, N))$ be the E_1-term of the Cartan–Eilenberg spectral sequence and $\widetilde{E}_1(M,N)$ the E_1-term of the spectral sequence in question. It suffices to show that

$$\theta_*\colon E_1(M,N) \to \widetilde{E}_1(M,N)$$

is an isomorphism.

First consider the case $s = 0$. We have

$$F_0/F^1 = C_\Sigma(M \otimes_D A, N) = M \otimes_D C_\Sigma(A, N)$$

so this is the target of θ for $s = 0$. The source is $M \otimes_D C_\Gamma(\phi \otimes_D A, N)$. The argument in the proof of Theorem A1.3.14 showing that

$$\operatorname{Cotor}_\Gamma(\Phi \otimes_D A, N) = \operatorname{Cotor}_\Sigma(A, N)$$

shows that our two complexes are equivalent so we have the desired isomorphism for $s = 0$.

For $s > 0$ we use the following argument due to E. Ossa.

The differential

$$d_0\colon E_0^{s,t}(M,N) \to E_0^{s,t+1}(M,N)$$

depends only on the Σ-comodule structures of M and N. In fact we may define a complex $\widetilde{D}_\Sigma(N)$ formally by

$$\widetilde{D}^{s,t}_\Sigma(N) = \widetilde{E}_0^{s,t}(\Sigma, N).$$

Then we have

$$\widetilde{E}_0^{s,t}(M,n) = M \,\square_\Sigma\, \widetilde{D}^{s,t}_\Sigma(N).$$

Observe that

$$\widetilde{D}^{0,t}_\Sigma(N) = C^t_\Sigma(\Sigma, N).$$

Now let $G = \ker f$ and

$$C^{s+1} = G^s \,\square_\Sigma\, G = G \,\square_\Sigma\, G \,\square_\Sigma \cdots \square_\Sigma\, G$$

with $s+1$ factors.

Note that

$$G = \Sigma \otimes \overline{\Phi} \quad \text{and hence}$$
$$G^s = \Sigma \otimes \overline{\Phi}^{\otimes s}$$

as left Σ-comodules, where the tensor products are over D.

Define

$$\beta_s\colon G^s \,\square_\Sigma\, \widetilde{D}^{0,t}_\Sigma(N) \to \widetilde{D}^{s,t}_\Sigma(N)$$

by

$$\beta_s((g_1 \otimes \ldots g_s) \otimes \sigma_1 \otimes \cdots \otimes \sigma_t \otimes n) \\ = \Sigma f(g_1') g_1'' \otimes g_2 \cdots \otimes g_s \otimes \sigma_1 \otimes \cdots \otimes \sigma_t \otimes n.$$

Then β_s is a map of differential Σ-comodules and the diagram

$$\begin{tikzcd}
\widetilde{E}_0^{s,t}(M,N) \arrow[r, equal] \arrow[d, "\theta^{s,t}"] & \widetilde{E}_0^{0,t}(M\otimes\Phi^{\otimes s},N) \arrow[d, "\theta^{0,t}"] \\
\widetilde{E}_0^{s,t}(M,N) \arrow[d, equal] & \widetilde{E}_0^{0,t}(M\otimes\Phi^{\otimes s},N) \arrow[d, equal] \\
M \square_\Sigma \widetilde{D}_\Sigma^{s,t}(N) & M \square_\Sigma G^s \square_\Sigma \widetilde{D}_\Sigma^{s,t}(N) \arrow[l, "\beta_s"']
\end{tikzcd}$$

commutes.

We know that $\theta^{0,t}$ is a chain equivalence so it suffices to show that β_s is one by induction on s. To start this induction note that β_0 is the identity map by definition.

Let

$$F^{s,t}(\Gamma,N) = F^s C_{s+t}(\Gamma,N)$$

and

$$\begin{aligned} F^{s,t}(\Gamma,N) &= F^{s,t}(G,N) + F^{s+1,t-1}(\Gamma,N) \\ &= F^{s,t}(\Gamma,N). \end{aligned}$$

Then $F^{s,*}(\Gamma,N)$ is a Σ-comodule subcomplex of $C_\Gamma(\Gamma,N)$ which is invariant under the contraction

$$S(\gamma\otimes\gamma_1\ldots\gamma_s\otimes n) = \varepsilon(\gamma)\otimes\gamma_1\ldots\gamma_s\otimes n.$$

Since $H_0(F^{s,*}(\Gamma,N)) = 0$, the complex $F^{s,*}(\Gamma,N)$ is acyclic.

Now look at the short exact sequence of complexes

$$\begin{tikzcd}
0 \arrow[r] & \dfrac{F^{s+1}(\Gamma,N)}{\tilde{F}^{s+1}(\Gamma,N)} \arrow[r] \arrow[d, "\phi"', "\cong"] & \dfrac{\tilde{F}^{s}(\Gamma,N)}{\tilde{F}^{s+1}(\Gamma,N)} \arrow[r] & \dfrac{\tilde{F}^{s}(\Gamma,N)}{F^{s+1}(\Gamma,N)} \arrow[r] \arrow[d, "\psi"', "\cong"] & 0 \\
& \tilde{D}_\Sigma^{s+1}(N) & & G \square_\Sigma \tilde{D}_\Sigma^{s}(N) &
\end{tikzcd}$$

The connecting homomorphism in cohomology is an isomorphism.

We use this for the inductive step. By the inductive hypothesis, the composite

$$G \square_\Sigma (G^s \square_\Sigma N) \to G \square_\Sigma (G^s \square_\Sigma D_\Sigma^0(N)) \to G \square_\Sigma D_\Sigma^s(N)$$

is an equivalence. If we follow it by $\phi\,\partial\psi$ we get β_{s+1}. This completes the inductive step and the proof. $\square$

A1.3.17. THEOREM. *Let $\Phi\to\Gamma\to\Sigma$ be a cocentral extension* (A1.1.15) *of Hopf algebras over a field K; M a left Φ-comodule and N a trivial left Γ-comodule. Then $\mathrm{Ext}_\Sigma(K,N)$ is trivial as a left Φ-comodule, so the Cartan–Eilenberg spectral sequence* (A1.3.14) *E_2-term is $\mathrm{Ext}_\Phi(M,K)\otimes\mathrm{Ext}_\Sigma(K,K)\otimes N$.*

PROOF. We show first that the coaction of Φ on $\mathrm{Ext}_\Sigma(K,N)$ is essentially unique and then give an alternative description of it which is clearly trivial when the extension is cocentral. The coaction is defined for any (not necessarily trivial) left Γ-comodule N. It is natural and determined by its effect when $N=\Gamma$ since we can use an injective resolution of N to reduce to this case. Hence any natural Φ-coaction on $\mathrm{Ext}_\Sigma(K,N)$ giving the standard coaction on $\mathrm{Ext}_\Sigma(K,\Gamma)=\Phi$ must be identical to the one defined above.

Now we need some results of Singer [**5**]. Our Hopf algebra extension is a special case of the type he studies. In Proposition 2.3 he defines a Φ-coaction on Σ, $\rho_\Sigma\colon \Sigma\to\Phi\otimes\Sigma$ via a sort of coconjugation. Its analog for a group extension $N\to G\to H$ is the action of H on N by conjugation. This action is trivial when the extension is central, as is Singer's coaction in the cocentral case.

The following argument is due to Singer.

Since Σ is a Φ-comodule it is a Γ-comodule so for any N as above $\Sigma\otimes_K N$ is a Γ-comodule. It follows that the cobar resolution $D_\Sigma N$ is a differential Γ-comodule and that $\mathrm{Hom}_\Sigma(K,D_\Sigma N)$ is a differential comodule over $\mathrm{Hom}_\Sigma(K,\Gamma)=\Phi$. Hence we have a natural Φ-coaction on $\mathrm{Ext}_\Sigma(K,N)$ which is clearly trivial when N has the trivial Γ-comodule structure and the extension is cocentral.

It remains only to show that this Φ-coaction is identical to the standard one by evaluating it when $N=\Gamma$. In that case we can replace $D_\Sigma N$ by N, since N is an extended Σ-comodule. Hence we have the standard Γ-coaction on Γ inducing the standard Φ-coaction on $\mathrm{Hom}_\Sigma(K,\Gamma)=\Phi$. $\square$

4. Massey Products

In this section we give an informal account of Massey products, a useful structure in the Ext over a Hopf algebroid which will figure in various computations in the text. A parallel structure in the ASS is discussed in Kochman [**4**] and Kochman [**2**, Section 12]. These products were first introduced by Massey [**3**], but the best account of them is May [**3**]. We will give little more than an introduction to May's paper, referring to it for all the proofs and illustrating the more complicated statements with simple examples.

The setting for defining Massey products is a differential graded algebra (DGA) C over a commutative ring K. The relevant example is the cobar complex $C_\Gamma(L,M)$ of A1.2.11, where L and M are Γ-comodule algebras and Γ is a Hopf algebroid (A1.1.1) over K. The product in this complex is given by A1.2.15.

We use the following notation to keep track of signs. For $x\in C$, let $\bar{x}$ denote $(-1)^{1+\deg x}x$, where $\deg x$ is the total degree of x; i.e., if C is a complex of graded objects, $\deg x$ is the sum of the internal and cohomological degrees of x. Hence we have $d(\bar{x})=-\overline{d(x)}$, $(\overline{xy})=-\bar{x}\bar{y}$, and $d(xy)=d(x)y-\bar{x}d(y)$.

Now let $\alpha_i\in H^*(C)$ be represented by cocycles $a_i\in C$ for $i=1,2,3$. If $\alpha_i\alpha_{i+1}=0$ then there are cochains u_i such that $d(u_i)=\bar{a}_i a_{i+1}$, and $\bar{u}_1a_3+\bar{a}_1u_2$ is a cocycle. The corresponding class in $H^*(C)$ is the *Massey product* $\langle\alpha_1,\alpha_2,\alpha_3\rangle$. If $\alpha_i\in H^{s_i}$ the this $\langle\alpha_1,\alpha_2,\alpha_3\rangle\in H^{s-1}$, where $s=\sum s_i$. Unfortunately, this triple product is not well defined because the choices made in its construction are not unique. The choices of a_i do not matter but the u_i could each be altered by adding a cocycle, which means $\langle\alpha_1,\alpha_2,\alpha_3\rangle$ could be altered by any element of the form $x\alpha_3+\alpha_1y$ with $x\in H^{s_1+s_2-1}$ and $y\in H^{s_2+s_3-1}$. The group $\alpha_1H^{s_3+s_2-1}\oplus\alpha_3H^{s_1+s_2-1}$ is called the *indeterminacy*, denoted by $\mathrm{In}\langle\alpha_1,\alpha_2,\alpha_3\rangle$. It may be trivial, in which case $\langle\alpha_1,\alpha_2,\alpha_3\rangle$ is well defined.

A1.4.1. DEFINITION. *With notation as above,* $\langle\alpha_1,\alpha_2,\alpha_3\rangle \subset H^s(C)$ *is the coset of* $\mathrm{In}\langle\alpha_1,\alpha_2,\alpha_2\rangle$ *represented by* $\bar{a}_1u_2+\bar{u}_1a_3$. *Note that* $\langle\alpha_1,\alpha_2,\alpha_3\rangle$ *is only defined when* $\bar{\alpha}_1\alpha_2=\bar{\alpha}_2\alpha_3=0$.

This construction can be generalized in two ways. First the relations $\alpha_i\bar{\alpha}_{i+1}=0$ can be replaced by

$$\sum_{j=1}^{m}(\bar{\alpha}_1)_j(\alpha_2)_{j,k}=0 \quad \text{for } 1\le k\le n$$

and

$$\sum_{k=1}^{n}(\bar{\alpha}_2)_{j,k}(\alpha_3)_k=0 \quad \text{for } 1\le j\le m.$$

Hence the α_i become matrices with entries in $H^*(C)$. We will denote the set of matrices with entries in a ring R by MR. For $x\in MC$ or $MH^*(C)$, define $\bar{x}$ by $(\bar{x})_{j,k}=\bar{x}_{j,k}$.

As before, let $a_i\in MC$ represent $\alpha_i\in MH^*(C)$ and let $u_1\in MC$ be such that $d(u_i)=\bar{a}_ia_{i+1}$. Then u_1 and u_2 are $(1\times n)$- and $(m\times 1)$-matrices, respectively, and $\bar{a}_1u_2+\bar{u}_1a_3$ is a cocycle (not a matrix thereof) that represents the coset $\langle\alpha_1,\alpha_2,\alpha_3\rangle$.

Note that the matrices α_i need not be homogeneous (i.e., their entries need not all have the same degree) in order to yield a homogeneous triple product. In order to multiply two such matrices we require that, in addition to having compatible sizes, the degrees of their entries be such that the entries of the product are all homogeneous. These conditions are easy to work out and are given in 1.1 of May [**3**]. They hold in all of the applications we will consider and *will be tacitly assumed in subsequent definitions.*

A1.4.2. DEFINITION. *With notation as above, the* matric Massey product $\langle\alpha_1,\alpha_2,\alpha_3\rangle$ *is the coset of* $\mathrm{In}\langle\alpha_1,\alpha_2,\alpha_3\rangle$ *represented the cocycle* $\bar{a}_1u_2+\bar{u}_1a_3$, *where* $\mathrm{In}\langle\alpha_1,\alpha_2,\alpha_3\rangle$ *is the group generated by elements of the form* $x\alpha_3+\alpha_1y$ *where* $x,y\in MH^*(C)$ *have the appropriate form.*

The second generalization is to higher (than triple) order products. The Massey product $\langle\alpha_1,\alpha_2,\ldots,\alpha_n\rangle$ for $\alpha_i\in MH^*(C)$ is defined when all of the lower products $\langle\alpha_i,\alpha_{i+1},\ldots,\alpha_j\rangle$ for $1\le i<j\le n$ and $j-i<n-1$ are defined and contain zero. Here the double product $\langle\alpha_i\alpha_{i+1}\rangle$ is understood to be the ordinary product $\alpha_i\alpha_{i+1}$. Let $a_{i-1,i}$ be a matrix of cocycles representing α_i. Since $\alpha_i\alpha_{i+1}=0$ there are cochains $a_{i-1,i+1}$ with $d(a_{i-1,i+1})=\bar{a}_{i-1,i}a_{i,i+1}$. Then the triple product $\langle\alpha_i,\alpha_{i+1},\alpha_{i+2}\rangle$ is represented by $b_{i-1,i+2}=\bar{a}_{i-1,i+1}a_{i+1,i+2}+\bar{a}_{i-1,i}a_{i,i+2}$. Since this triple product is assumed to contain zero, the above choices can be made so that there is a matrix of cochains $a_{i-1,i+2}$ whose coboundary is $b_{i-1,i+2}$.

Then the fourfold product $\langle\alpha_1,\alpha_2,\alpha_3,\alpha_4\rangle$ is represented by the cocycle $\bar{a}_{0,3}a_{3,4}+\bar{a}_{0,2}a_{2,4}+\bar{a}_{0,1}a_{1,4}$. More generally, we can choose elements $a_{i,j}$ and $b_{i,j}$ by induction on $j-i$ satisfying $b_{i,j}=\sum_{i<k<j}\bar{a}_{i,k}a_{k,j}$ and $d(a_{i,j})=b_{i,j}$ for $i-j<n-1$.

A1.4.3. DEFINITION. *The n-fold Massey product* $\langle\alpha_1,\alpha_2,\ldots,\alpha_n\rangle$ *is defined when all of the lowerproducts* $\langle\alpha_i,\ldots,\alpha_j\rangle$ *contain zero for* $i<j$ *and* $j-i<n-1$. *It is* strictly defined *when these lower products also have trivial indeterminacy, e.g., all triple products are strictly defined. In either case the matrices* $a_{i,j}$ *chosen above*

for $0 < i \le j \le n$ *and* $j - i < n$ *constitute a* defining system *for the product in question, which is, modulo indeterminacy (to be described below), the class represented by the cocycle*

$$\sum_{0<i<n} \bar{a}_{0,i} a_{i,n}.$$

Note that if $\alpha_i \in H^{s_i}(C)$*, then* $\langle \alpha_1, \ldots, \alpha_n \rangle \subset H^{s+2-n}(C)$ *where* $s = \sum s_i$.

In 1.5 of May [3] it is shown that this product is natural with respect to DGA maps f in the sense that $\langle f_*(\alpha_1), \ldots, f_*(\alpha) \rangle$ is defined and contains $f_*(\langle \alpha_1, \ldots, \alpha_n \rangle)$.

The indeterminacy for $n \ge 4$ is problematic in that without additional technical assumptions it need not even be a subgroup. Upper bounds on it are given by the following result, which is part of 2.3, 2.4, and 2.7 of May [3]. It expresses the indeterminacy of n-fold products in terms of $(n-1)$-fold products, which is to be expected since that of a triple product is a certain matric double product.

A1.4.4. Indeterminacy Theorem. *Let* $\langle \alpha_1, \ldots, \alpha_n \rangle$ *be defined. For* $1 \le k \le n-1$ *let the degree of* x_k *be one less than that of* $\alpha_k \alpha_{k+1}$.

(a) *Define matrices* W_k *by*

$$W_1 = (\alpha_1 \ x_1),$$
$$W_k = \begin{pmatrix} \alpha_k & x_k \\ 0 & \alpha_{k+1} \end{pmatrix} \quad \text{for } 2 \le k \le n-2$$

and

$$W_{n-1} = \begin{pmatrix} x_{n-1} \\ \alpha_n \end{pmatrix}.$$

Then $\mathrm{In}\langle \alpha_1, \ldots, \alpha_n \rangle \subset \bigcup \langle W_1, \ldots, W_n \rangle$ *where the union is over all* x_k *for which* $\langle W_1, \ldots, W_n \rangle$ *is defined.*

(b) *Let* $\langle \alpha_1, \ldots, \alpha_n \rangle$ *be strictly defined. Then for* $1 \le k \le n-1$ $\langle \alpha_1, \ldots, \alpha_{k-1}, x_k, \alpha_{k+2}, \ldots, \alpha_n \rangle$ *is strictly defined and*

$$\mathrm{In}\langle \alpha_1, \ldots, \alpha_n \rangle \subset \bigcup \sum_{k=1}^{n-1} \langle \alpha_1, \ldots, \alpha_{k-1}, x_k, \alpha_{k+2}, \alpha_n \rangle$$

where the union is over all possible x_k. *Equality holds when* $n = 4$.

(c) *If* $\alpha_k = \alpha'_k + \alpha''_k$ *and* $\langle \alpha_1, \ldots, \alpha'_k, \ldots, \alpha_n \rangle$ *is strictly defined, then*

$$\langle \alpha_1, \ldots, \alpha_n \rangle \subset \langle \alpha_1, \ldots, \alpha'_k, \ldots, \alpha_n \rangle + \langle \alpha_1, \ldots, \alpha''_k, \ldots, \alpha_n \rangle. \qquad \square$$

There is a more general formula for the sum of two products, which generalizes the equation

$$\alpha_1\beta_1 + \alpha_2\beta_2 = \left\langle (\alpha_1 \alpha_2), \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} \right\rangle$$

and is part of 2.9 of May [3].

A1.4.5. Addition Theorem. *Let* $\langle \alpha_1, \ldots, \alpha_n \rangle$ *and* $\langle \beta_1, \ldots, \beta_n \rangle$ *be defined. Then so is* $\langle \gamma_1, \ldots, \gamma_n \rangle$ *where*

$$\gamma_1 = (\alpha_1, \beta_1), \quad \gamma_k = \begin{pmatrix} \alpha_k & 0 \\ 0 & \beta_k \end{pmatrix} \quad \textit{for } 1 < k < n, \quad \textit{and} \quad \gamma_n = \begin{pmatrix} \alpha_n \\ \beta_n \end{pmatrix}.$$

Moreover $\langle \alpha_1, \ldots, \alpha_n \rangle + \langle \beta_1, \ldots, \beta_n \rangle \subset \langle \gamma_1, \ldots, \gamma_n \rangle$.

In Section 3 of May [**3**] certain associativity formulas are proved, the most useful of which (3.2 and 3.4) relate Massey products and ordinary products and are listed below. The manipulations allowed by this result are commonly known as *juggling*.

A1.4.6. First Juggling Theorem. (a) *If $\langle\alpha_2,\ldots,\alpha_n\rangle$ is defined, then so is $\langle\bar\alpha_1\alpha_2,\alpha_3,\ldots,\alpha_n\rangle$ and*

$$\alpha_1\langle\alpha_2,\ldots,\alpha_n\rangle\subset-\langle\bar\alpha_1\alpha_2,\alpha_3,\ldots,\alpha_n\rangle.$$

(b) *If $\langle\alpha_1,\ldots,\alpha_{n-1}\rangle$ is defined, then so is $\langle\alpha_1,\ldots,\alpha_{n-2},\alpha_{n-1}\alpha_n\rangle$ and*

$$\langle\alpha_1,\ldots,\alpha_{n-1}\rangle\alpha_n\subset\langle\alpha_1,\ldots,\alpha_{n-2},\alpha_{n-1}\alpha_n\rangle.$$

(c) *If $\langle\alpha_1,\ldots,\alpha_{n-1}\rangle$ and $\langle\alpha_2,\ldots,\alpha_n\rangle$ are strictly defined, then*

$$\alpha_1\langle\alpha_2,\ldots,\alpha_n\rangle=\langle\bar\alpha_1,\ldots,\bar\alpha_{n-1}\rangle\alpha_n.$$

(d) *If $\langle\alpha_1\alpha_2,\alpha_3,\ldots,\alpha_n\rangle$ is defined, then so is $\langle\alpha_1,\bar\alpha_2\alpha_3,\alpha_4,\ldots,\alpha_n\rangle$ and*

$$\langle\alpha_1\alpha_2,\alpha_3,\ldots,\alpha_n\rangle\subset-\langle\alpha_1,\bar\alpha_2\alpha_3,\alpha_4,\ldots,\alpha_n\rangle.$$

(e) *If $\langle\alpha_1,\ldots,\alpha_{n-2},\bar\alpha_{n-1}\alpha_n\rangle$ is defined, then so is $\langle\alpha_1,\ldots,\alpha_{n-3},\alpha_{n-1},\alpha_n\rangle$ and*

$$\langle\alpha_1,\ldots,\alpha_{n-2},\bar\alpha_{n-1}\alpha_n\rangle\subset-\langle\alpha_1,\ldots,\alpha_{n-3},\alpha_{n-2}\alpha_{n-1},\alpha_n\rangle.$$

(f) *If $\langle\alpha_1,\ldots,\alpha_{k-1},\alpha_k\alpha_{k+1},\alpha_{k+2},\ldots,\alpha_n\rangle$ and $\langle\alpha_1,\ldots,\alpha_k,\bar\alpha_{k+1}\alpha_{k+2},\alpha_{k+3},\ldots,\alpha_n\rangle$ are strictly defined, then the intersection of the former with minus the latter is nonempty.* □

Now we come to some commutativity formulas. For these the DGA C must satisfy certain conditions (e.g., the cup product must be commutative) which always hold in the cobar complex. We must assume (if $2\neq0$ in K) that in each matrix α_i the degrees of the entries all have the same parity ε_i; i.e., ε_i is 0 if the degrees are all even and 1 if they are all odd. Then we define

$$s(i,j)=j-i+\sum_{i\le k\le m\le j}(1+\varepsilon_k)(1+\varepsilon_m) \tag{A1.4.7}$$

and

$$t(k)=(1+\varepsilon_1)\sum_{j=2}^{k}(1+\varepsilon_j).$$

The transpose of a matrix α will be denoted by α'. The following result is 3.7 of May [**3**].

A1.4.8. Second Juggling Theorem. *Let $\langle\alpha_1,\ldots,\alpha_n\rangle$ be defined and assume that either $2=0$ in K or the degrees of all of the entries of each α_i have the same parity ε_i. Then $\langle\alpha_n',\ldots,\alpha_1'\rangle$ is also defined and*

$$\langle\alpha_1,\ldots,\alpha_n\rangle'=(-1)^{s(1,n)}\langle\alpha_n',\ldots,\alpha_1'\rangle.$$

(*For the sign see* A1.4.7) □

The next result involves more complicated permutations of the factors. In order to ensure that the permuted products make sense we must assume that we have ordinary, as opposed to matric, Massey products. The following result is 3.8 and 3.9 of May [**3**].

A1.4.9. THIRD JUGGLING THEOREM. *Let $\langle\alpha_1,\ldots,\alpha_n\rangle$ be defined as an ordinary Massey product.*

(a) *If $\langle\alpha_{k+1},\ldots,\alpha_n,\alpha_1,\ldots,\alpha_k\rangle$ is strictly defined for $1\le k<n$, then*

$$(-1)^{s(1,n)}\langle\alpha_1,\ldots,\alpha_n\rangle\subset\sum_{k=1}^{n-1}(-1)^{s(1,k)+s(k+1,n)}\langle\alpha_{k+1},\ldots,\alpha_n,\ldots,\alpha_k\rangle.$$

(b) *If $\langle\alpha_2,\ldots,\alpha_k,\alpha_{k+1},\ldots,\alpha_n\rangle$ is strictly defined for $1\le k\le n$ then*

$$\langle\alpha_1,\ldots,\alpha_n\rangle\subset-\sum_{k=2}^{n}(-1)^{t(k)}\langle\alpha_2,\ldots,\alpha_k,\alpha_1,\alpha_{k+1},\ldots,\alpha_n\rangle.$$

(*For the signs see* A1.4.7)

Now we consider the behavior of Massey products in spectral sequences. In the previous section we considered essentially three types: the one associated with a resolution (A1.3.2), the one associated with a filtration (decreasing or increasing) of the Hopf algebroid Γ (A1.3.9), and the Cartan–Eilenberg spectral sequence associated with an extension (A1.3.14). In each case the spectral sequence arises from a filtration of a suitable complex. In the latter two cases this complex is the cobar complex of A1.2.11 (in the case of the Cartan–Eilenberg spectral sequence this result is A1.3.16), which is known to be a DGA (A1.2.14) that satisfies the additional hypotheses (not specified here) needed for the commutativity formulas A1.4.8 and A1.4.9. Hence all of the machinery of this section is applicable to those two spectral sequences; its applicability to the resolution spectral sequence of A1.3.2 will be discussed as needed in specific cases.

To fix notation, suppose that our DGA C is equipped with a decreasing filtration $\{F^pC\}$ which respects the differential and the product. We do not require $F^0C=C$, but only that $\lim_{p\to\infty}F^pC=C$ and $\lim_{p\to\infty}F^pC=0$. Hence we can have an increasing filtration $\{F_pC\}$ by defining $F_pC=F^{-p}C$. Then we get a spectral sequence with

$$\begin{aligned}E_0^{p,q}&=F^pC^{p+q}/F^{p+1}C^{p+q},\\E_1^{p,q}&=H^{p+q}(F^p/F^{p+1}),\\d_r\colon E_r^{p,q}&\to E_r^{p+r,q-r+1},\end{aligned}$$

and

$$E_\infty^{p,q}=F^pH^{p+q}/F^{p+1}H^{p+q}.$$

We let $E_{r,\infty}^{p,q}\subset E_r^{p,q}$ denote the permanent cycles and $i\colon E_{r,\infty}^{p,q}\to E_\infty^{p,q}$ and $\pi\colon F^pC^{p+q}\to E_0^{p,q}$ the natural surjections. If $x\in E_{r,\infty}^{p,q}$ and $y\in F^pH^{p+q}$ projects to $i(x)\in E_\infty^{p,q}$ we say that x *converges to* y. If the entries of a matrix $B\in MC$ are all known to survive to E_r, we indicate this by writing $\pi(B)\in ME_r$. In the following discussions α_i will denote an element in ME_r represented by $a_i\in MC$. If $\alpha_i\in ME_{r,\infty}$, $\beta_i\in MH^*(C)$ will denote an element to which it converges.

Each E_r is a DGA in whose cohomology, E_{r+1}, Massey products can be defined. Suppose $\langle\alpha_1,\ldots,\alpha_n\rangle$ is defined in E_{r+1} and that the total bidegree of the α_i is (s,t), i.e., that the ordinary product $\alpha_1\alpha_2\ldots\alpha_n$ (which is of course zero if $n\ge3$) lies in $E_{r+1}^{s,t}$. Then the indexing of d_r implies that the Massey product is a subset of $E_{r+1}^{s-r(n-2),t+(r-1)(n-2)}$.

May's first spectral sequence result concerns convergence of Massey products. Suppose that the ordinary triple product $\langle\beta_1,\beta_2,\beta_3\rangle \subset H^*(C)$ is defined and that $\langle\alpha_1,\alpha_2,\alpha_3\rangle$ is defined in E_{r+1}. Then one can ask if an element in the latter product is a permanent cycle converging to an element of the former product. Unfortunately, the answer is not always yes. To see how counterexamples can occur, let $\hat{u}_i \in E_r$ be such that $d_r(\hat{u}_i) = \alpha_i\alpha_{i+1}$. Let (p,q) be the bidegree of one of the $\hat{u}_i$. Since $\langle\beta_1,\beta_2,\beta_3\rangle$ is defined we have as before $u_i \in C$ such that $d(u_i) = a_ia_{i+1}$. The difficulty is that $\bar{a}_ia_{i+1}$ need *not* be a coboundary in F^pC; i.e., it may not be possible to find a $u_i \in F^pC$. Equivalently, the best possible representative $\tilde{u}_i \in F^pC$ of $\hat{u}_i$, may have coboundary $\bar{a}_ia_{i+1} - e_i$ with $0 \neq \pi(e_i) \in E_t^{p+t,q-t+1}$ for some $t > r$. Then we have $d(u_i - \tilde{u}_i) = e_i$. and $\pi(u_i - \tilde{u}_i) = \pi(u_i) \in E_{m+t}^{p-m,q+m}$ for some $m > 0$, so $d_{m+t}(\pi(u_i)) = \pi(e_i)$. In other words, the failure of the Massey product in E_{r+1} to converge as desired is reflected in the presence of a certain higher differential. Thus we can ensure convergence by hypothesizing that all elements in $E_{m+r+1}^{p-m,q+m}$ for $m \geq 0$ are permanent cycles.

The case $m = 0$ is included for the following reason (we had $m > 0$ in the discussion above). We may be able to find a $u_i \in F^pC$ with $d(u_i) = \bar{a}_ia_{i+1}$ but with $\pi(u_i) \neq u_i$, so $d_t(\pi(u - \tilde{u}_i)) = \pi(e_i) \neq 0$. In this case we can find a convergent element in the Massey product in E_{r+1}, but it would not be the one we started with.

The general convergence result, which is 4.1 and 4.2 of May [3], is

A1.4.10. CONVERGENCE THEOREM. (a) *With notation as above let $\langle\alpha_1,\ldots,\alpha_n\rangle$ be defined in E_{r+1}. Assume that $\alpha_i \in ME_{r+1,\infty}$ and α_i converges to β_i, where $\langle\beta_1,\ldots,\beta_n\rangle$ is strictly defined in $H^*(C)$. Assume further that if (p,q) is the bidegree of an entry of some $a_{i,j}$ (for $1 < j-1 < n$) in a defining system for $\langle\alpha_1,\ldots,\alpha_n\rangle$ then each element in $E_{r+m+1}^{p-m,q+m}$ for all $m \geq 0$ is a permanent cycle. Then each element of $\langle\alpha_1,\ldots,\alpha_n\rangle$ is a permanent cycle converging to an element of $\langle\beta_1,\ldots,\beta_n\rangle$.*

(b) *Suppose all of the above conditions are met except that $\langle\alpha_1,\ldots,\alpha_n\rangle$ is not known to be defined in E_{r+1}. If for (p,q) as above every element of $E_{r+m}^{p-m,q+m}$ for $m \geq 1$ is a permanent cycle then $\langle\alpha_1,\ldots,\alpha_n\rangle$ is strictly defined so the conclusion above is valid.* □

The above result does not prevent the product in question from being hit by a higher differential. In this case $\langle\beta_1,\ldots,\beta_n\rangle$ projects to a higher filtration.

May's next result is a generalized Leibnitz formula which computes the differential on a Massey product in terms of differentials on its factors. The statement is complicated so we first describe the simplest nontrivial situation to which it applies. For this discussion we assume that we are in characteristic 2 so we can ignore signs. Suppose $\langle\alpha_1,\alpha_2,\alpha_3\rangle$ is defined in E_{r+1} but that the factors are not necessarily permanent cycles. We wish to compute d_{r+1} of this product. Let α_i have bidegree (p_i,q_i). Then we have $u_i \in F^{p_i+p_{i+1}-2}C$ with $d(u_i) = a_ia_{i+1}$ mod $F^{p_i+p_{i+1}+1}C$. The product is represented by $u_1a_3 + a_1u_2$. Now let $d(a_i) = a_i'$ and $d(u_i) = a_ia_{i+1} + u_i'$. Then we have $d(u_1a_3 + a_1u_2) = u_1'a_3 + u_1a_3' + a_1'u_2 + a_2u_2'$. This expression projects to a permanent cycle which we want to describe as a Massey product in E_{r+1}. Consider

$$\left\langle \begin{pmatrix} d_{r+1}(\alpha_1) & \alpha_1 \end{pmatrix}, \begin{pmatrix} \alpha_2 & 0 \\ d_{r+1}(\alpha_2) & \alpha_2 \end{pmatrix}, \begin{pmatrix} \alpha_3 \\ d_{r+1}(\alpha_3). \end{pmatrix} \right\rangle.$$

Since $d(u_i) = \alpha_i\alpha_{i+1} + u_i'$ is a cycle, we have $d(u_i') = d(a_i a_{i+1}) = a_i' a_{i+1} + a_i a_{i+1}'$, so $d_r(\pi(u_i')) = d_{r+1}(\alpha_i)\alpha_{i+1} + \alpha_i d_{r+1}(\alpha_{i+1})$. It follows that the above product contains $\pi(u_1' a_3 + u_1 a_3' + a_1' u_2 + a_2 u_2') \in E_{r+1}$.

Hence we have shown that

$$d_{r+1}(\langle \alpha_1, \alpha_2, \alpha_3\rangle) \subset \left\langle \begin{pmatrix} d_{r+1}(\alpha_1) & \alpha_1 \end{pmatrix}, \begin{pmatrix} \alpha_2 & 0 \\ d_{r+1}(\alpha_2) & \alpha_2 \end{pmatrix}, \begin{pmatrix} \alpha_3 \\ d_{r+1}(\alpha_3) \end{pmatrix} \right\rangle.$$

We would like to show more generally that for $s > r$ with $d_t(\alpha_i) = 0$ for $r < t < s$, the product is a d_t-cycle and d_s on it is given by a similar formula. As in A1.4.10, there are potential obstacles which must be excluded by appropriate technical hypotheses which are vacuous when $s = r + 1$. Let (p, q) be the bidegree of some u_i. By assumption $u_i' \in F^{p+r+1}C$ and $d(u_i') = a_i' a_{i+1} + a_i a_{i+1}'$. Hence $\pi(a_i a_{i+1}' + a_i' a_{i+1}) \in E_*^{p+r+s, q-r-s+2}$ is killed by a d_{r+s-t} for $r < t \le s$. If the new product is to be defined this class must in fact be hit by a d_r and we can ensure this by requiring $E_{r+s-t}^{p+t, q-t+1} = 0$ for $r < t < s$. We also need to know that the original product is a d_t-cycle for $r < t < s$. This may not be the case if $\pi(u_i') \neq 0 \in E_t^{p+t, q-t+1}$ for $r < t < s$, because then we could not get rid of $\pi(u_i')$ by adding to u_i an element in $F^{p+1}C$ with coboundary in $F^{p+r+1}C$ (such a modification of u_i would not alter the original Massey product) and the expression for the Massey product's coboundary could have lower filtration than needed. Hence we also require $E_t^{p+t, q-t+1} = 0$ for $r < t < s$.

We are now ready to state the general result, which is 4.3 and 4.4 of May [**3**].

A1.4.11. THEOREM (LEIBNITZ FORMULA). (a) *With notation as above let* $\langle \alpha_1, \alpha_2, \dots, \alpha_n\rangle$ *be defined in* E_{r+1} *and let* $s > r$ *be given with* $d_t(\alpha_i) = 0$ *for all* $t < s$ *and* $1 \le i \le n$. *Assume further that for* (p, q) *as in* A1.4.10 *and for each* t *with* $r < t < s$,

$$E_t^{p+t, q-t+1} = 0 \quad \text{and} \quad E_{r+s-t}^{p+t, q-t+1} = 0$$

(*for each* t *one of these implies the other*). *Then each element* α *of the product is a* d_t*-cycle for* $r < t < s$ *and there are permanent cycles* $\alpha_i' \in ME_{r+1,\infty}$ *which survive to* $d_s(\alpha_i)$ *such that* $\langle \gamma_1, \dots, \gamma_n\rangle$ *is defined in* E_{r+1} *and contains an element* γ *which survives to* $-d_s(\alpha)$, *where*

$$\gamma_1 = (\alpha_1' \bar{\alpha}_1), \quad \gamma_i = \begin{pmatrix} \alpha_i & 0 \\ \alpha_i & \bar{\alpha}_i \end{pmatrix} \quad \text{for } 1 < i < n,$$

and

$$\gamma_n = \begin{pmatrix} \alpha_n \\ \alpha_n' \end{pmatrix}.$$

(b) *Suppose further that each* α_i' *is unique, that each* $\langle \bar{\alpha}_1, \dots, \bar{\alpha}_{i-1}, \alpha_i, \alpha_{i+1}, \dots, \alpha_n\rangle$ *is strictly defined, and that all products in sight have zero indeterminacy. Then*

$$d_s(\langle \alpha_1, \dots, \alpha_s\rangle) = -\sum_{i=1}^{n} \langle \bar{\alpha}_1, \dots, \bar{\alpha}_{i-1}, \alpha_i', \alpha_{i+1}, \dots, \alpha_n\rangle. \qquad \square$$

The last result of May [**3**] concerns the case when $\langle \alpha_1, \dots, \alpha_n\rangle$ is defined in E_{r+1}, the α_i are all permanent cycles, but the corresponding product in $H^*(C)$ is not defined, so the product in E_{r+1} supports some nontrivial higher differential. One could ask for a more general result; one could assume $d_t(\alpha_i) = 0$ for $t < s$ and, without the vanishing hypotheses of the previous theorem, show that the product supports a nontrivial d_t. In many specific cases it may be possible to derive such a result from the one below by passing from the DGA C to a suitable quotient in which the α_i are permanent cycles.

As usual we begin by discussing the situation for ordinary triple products, ignoring signs, and using the notation of the previous discussion. If $\langle\alpha_1,\alpha_2,\alpha_3\rangle$ is defined in E_{r+1} and the a_i are cocycles in C but the corresponding product in $H^*(C)$ is not defined, it is because the a_ia_{i+1} are not both coboundaries; i.e., at least one of the $u_i' = d(u_i)+a_ia_{i+1}$ is nonzero. Suppose $\pi(u_i')$ is nontrivial in $E_{r+1}^{p+r+1,q-r}$. As before, the product is represented by $u_1a_3+a_1u_2$ and its coboundary is $u_1'a_3+a_1u_2'$, so $d_{r+1}(\langle\alpha_1,\alpha_2,\alpha_3\rangle) = \pi(u_1'a_3+a_1u_2')$. Here u_i' represents the product $\beta_i\beta_{i+1} \in H^*(C)$, where $\beta_i \in H^*(C)$ is the class represented by a_i. The product $\beta_i\beta_{i+1}$ has filtration greater than the sum of those of β_i and β_{i+1}, and the target of the differential represents the associator $(\beta_1\beta_2)\beta_3+\beta_1(\beta_2\beta_3)$.

Next we generalize by replacing $r+1$ by some $s>r$; i.e., we assume that the filtration of $\beta_i\beta_{i+1}$ exceeds the sum of those of β_i, and β_{i+1} by $s-r$. As in the previous result we need to assume

$$E_t^{p+t,q-t+1} = 0 \quad \text{for } r<t<s;$$

this condition ensures that the triple product is a d_t-cycle.

The general theorem has some hypotheses which are vacuous for triple products, so in order to illustrate them we must discuss quadruple products, again ignoring signs. Recall the notation used in definition A1.4.3. The elements in the defining system for the product in E_{r+1} have cochain representatives corresponding to the defining system the product would have if it were defined in $H^*(C)$. As above, we denote $a_{i-1,i}$ by $a_i, a_{i-1,i+1}$ by u_i, and also $a_{i-1,i+2}$ by v_i. Hence we have $d(a_i) = 0$, $d(u_i) = a_ia_{i+1}+u_i'$, $d(v_i) = a_iu_{i+1}+u_ia_{i+2}+v_i'$, and the product contains an element α represented by $m = a_1v_2+u_1u_3+v_1a_4$, so $d(m) = a_1v_2' + u_1'u_3+u_1u_3'+v_1'a_4$. We also have $d(u_i') = 0$ and $d(v_i') = u_i'a_{i+2}+a_iu_{i+1}'$.

We are assuming that $\langle\beta_1,\beta_2,\beta_3,\beta_4\rangle$ is not defined. There are two possible reasons for this. First, the double products $\beta_i\beta_{i+1}$ may not all vanish. Second, the double products all vanish, in which case $u_i'=0$, but the two triple products $\langle\beta_i,\beta_{i+1},\beta_{i+2}\rangle$ must not both contain zero, so $v_i'\neq 0$. More generally there are $n-2$ reasons why an n-fold product may fail to be defined. The theorem will express the differential of the n-fold product in E_{r+1} in terms of the highest order subproducts which are defined in $H^*(C)$. We will treat these two cases separately.

Let (p_i,q_i) be the bidegree of α_i. Then the filtrations of u_i, v_i, and m are, respectively, $p_i+p_{i+1}-r$, $p_i+p_{i+1}+p_{i+2}-2r$, and $p_1+p_2+p_3+p_4-2r$.

Suppose the double products do not all vanish. Let $s>r$ be the largest integer such that each u_i' has filtration $\geq s-r+p_i+p_{i+1}$. We want to give conditions which will ensure that $\langle\alpha_1,\alpha_2,\alpha_3,\alpha_4\rangle$ is a d_t-cycle for $r<t<s$ and that the triple product

$$\left\langle (\pi(u_1')\,\alpha_1), \begin{pmatrix}\alpha_3 & 0\\ \pi(u_2') & \alpha_2\end{pmatrix}, \begin{pmatrix}\alpha_4\\ \pi(u_3')\end{pmatrix}\right\rangle$$

is defined in E_{r+1} and contains an element which survives to $d_s(\alpha)$; note that if all goes well this triple product contains an element represented by $d(m)$. These conditions will be similar to those of the Leibnitz formula A1.4.11. Let (p,q) be the bidegree of some v_i. As before, we ensure that $d_t(\alpha)=0$ by requiring $E_t^{p+t,q-t+1}$, and that the triple product is defined in E_{r+1} by requiring $E_{r+s-t}^{p+t,q-t+1} = 0$. The former condition is the same one we made above while discussing the theorem for triple products, but the latter condition is new.

Now we treat the case when the double products vanish but the triple products do not. First consider what would happen if the above discussion were applied here. We would have $s = \infty$ and α would be a permanent cycle provided that $E_t^{p+t,q-t+1} = 0$ for all $t > r$. However, this condition implies that v can be chosen so that $v' = 0$, i.e., that the triple products vanish. Hence the above discussion is not relevant here.

Since $u_i' = 0$, the coboundary of the Massey product m is $a_1 v_2' + v_1' a_4$. Since $d(v_i) = a_i u_{i+1} + u_i a_{i+2} + v_i'$, v_i' is a cocycle representing an element of $\langle \beta_i, \beta_{i+1}, \beta_{i+2}\rangle$. Hence if all goes well we will have $d_s(\alpha) = \alpha_1 \pi(v_2') + \pi(v_1')\alpha_4$, where $s > r$ is the largest integer such that each v_i' has filtration at least $p_i + p_{i+1} + p_{i+2} + s - 2r$. To ensure that $d_t(\alpha) = 0$ for $t < s$, we require $E_t^{p+t,q-t+1} = 0$ for $r < t < s$ as before, where (p,q) is the degree of v_i. We also need to know that $\langle \alpha_i, \alpha_{i+1}, \alpha_{i+2}\rangle$ converges to $\langle \beta_i, \beta_{i+1}, \beta_{i+2}\rangle$; since the former contains zero, this means that the latter has filtration greater than $p_i + p_{i+1} + p_{i+2} - r$. We get this convergence from A1.4.10, so we must require that if (p,q) is the bidegree of $\pi(u_i)$, then each element of $E_{r+m+1}^{p-m,q+m+1}$ for all $m \geq 0$ is a permanent cycle.

Now we state the general result, which is 4.5 and 4.6 of May [**3**].

A1.4.12. Differential and Extension Theorem. (a) *With notation as above, let $\langle \alpha_1, \dots, \alpha_n\rangle$ be defined in E_{r+1} where each α_i is a permanent cycle converging to $\beta_i \in H^*(C)$. Let k with $1 \leq k \leq n-2$ be such that each $\langle \beta_i, \dots, \beta_{i+k}\rangle$ is strictly defined in $H^*(C)$ and such that if (p,q) is the bidegree of an entry of some $a_{i,j}$ for $1 < j - i \leq k$ in a defining system for $\langle \alpha_1, \dots, \alpha_n\rangle$ then each element of $E_{r+m+1}^{p-m,q+m}$ for all $m \geq 0$ is a permanent cycle. Furthermore, let $s > r$ be such that for each (p,q) as above with $k < j - i < n$ and each t with $r < t < s$, $E_t^{p+t,q-t+1} = 0$, and, if $j - i > k+1$, $E_{r+s-t}^{p+t,q-t+1} = 0$.*

Then for each $\alpha \in \langle \alpha_1, \dots, \alpha_n\rangle$, $d_t(\alpha) = 0$ for $r < t < s$, and there are permanent cycles $\delta_i \in ME_{r+1,\infty}$ for $1 \leq i \leq n-k$ which converge to elements of $\langle \beta_i, \dots, \beta_{i+k}\rangle \subset H^*(C)$ such that $\langle \gamma_1, \dots, \gamma_{n-k}\rangle$ is defined in E_{r+1} and contains an element γ which survives to $-d_s(\alpha)$, where

$$\gamma_1 = (\delta_1\, \bar{\alpha}_1), \quad \gamma_i = \begin{pmatrix} \alpha_{i+k} & 0 \\ \delta_i & \bar{\alpha}_i \end{pmatrix} \quad \textit{for } i < n-k,$$

and

$$\gamma_{n-k} = \begin{pmatrix} \alpha_n \\ \delta_{n-k} \end{pmatrix}.$$

(b) *Suppose in addition to the above that each δ_i is unique, that each $\langle \bar{\alpha}_1, \dots, \bar{\alpha}_{i-1}, \delta_i, \alpha_{i+k+1}, \dots, \alpha_n\rangle$ is strictly defined in E_{r+1} and that all Massey products in sight (except possibly $\langle \beta_i, \dots, \beta_{i+k}\rangle$) have zero indeterminacy. Then*

$$d_s(\langle \alpha_1, \dots, \alpha_n\rangle) = \sum_{i=1}^{n-k} \langle \bar{\alpha}_1, \dots, \bar{\alpha}_{i-1}, \delta_i, \alpha_{i+k+1}, \dots, \alpha_n\rangle. \qquad \square$$

Note that in (b) the uniqueness of δ_i does not make $\langle \beta_i, \dots, \beta_{i+k}\rangle$ have zero indeterminacy, but merely indeterminacy in a higher filtration. The theorem does not prevent δ_i from being killed by a higher differential. The requirement that $E_{r+m+1}^{p-m,q+m} \subset E_{r+m+1,\infty}$ is vacuous for $k = 1$, e.g., if $n = 3$. The condition

$E_{r+s-t}^{p+t,q-t+1} = 0$ is vacuous when $k = n-2$; both it and $E_t^{p+t,q-t+1} = 0$ are vacuous when $s = r+1$.

A1.4.13. Remark. The above result relates differentials to nontrivial extensions in the multiplicative structure (where this is understood to include Massey product structure) since δ_i represents $\langle\beta_i, \ldots, \beta_{i+k}\rangle$ but has filtration greater than that of $\langle\alpha_i, \ldots, \alpha_{i+k}\rangle$. The theorem can be used not only to compute differentials given knowledge of multiplicative extensions, but also vice versa. If $d_s(\alpha)$ is known, the hypotheses are met, and there are unique δ_i which fit into the expression for γ, then these δ_i necessarily converge to $\langle\beta_i, \ldots, \beta_{i+k}\rangle$.

5. Algebraic Steenrod Operations

In this section we describe operations defined in $\mathrm{Cotor}_\Gamma(M,N)$, where Γ is a Hopf algebroid over $\mathbf{Z}/(p)$ for p prime and M and N are right and left comodule algebras (A1.1.2) over Γ. These operations were first introduced by Liulevicius [2], although some of the ideas were implicit in Adams [12]. The most thorough account is in May [5], to which we will refer for most of the proofs. Much of the material presented here will also be found in Bruner *et al.* [1]; we are grateful to its authors for sending us the relevant portion of their manuscript. The construction of these operations is a generalization of Steenrod's original construction (see Steenrod [1]) of his operations in the mod (p) cohomology of a topological space X. We recall his method briefly. Let $G = \mathbf{Z}/(p)$ and let E be a contractible space on which G acts freely with orbit space B. X^p denotes the p-fold Cartesian product of X and $X^p \times_G E$ denotes the orbit space of $X^p \times E$ where G acts canonically on E and on X^p by cyclic permutation of coordinates. Choosing a base point in E gives maps $X \to X \times B$ and $X^p \to X^p \times_G E$. Let $\Delta\colon X \to X^p$ be the diagonal embedding. Then there is a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\Delta} & X^p \\ \downarrow & & \downarrow \\ X \times B & \longrightarrow & X^p \times_G E \end{array}$$

Given $x \in H^*(X)$ [all H^* groups are understood to have coefficients in $\mathbf{Z}/(p)$ it can be shown that $x \otimes x \otimes \cdots x \in H^*(X^p)$ pulls back canonically to a class $Px \in H^*(X^p \times_G E)$. We have $H^i(B) = \mathbf{Z}/(p)$ generated by e_i for each $i \geq 0$. Hence the image of Px in $H^*(X \times B)$ has the form $\sum_{i\geq 0} x_i \otimes e_i$ with $x_i \in H^*(X)$ and $x_0 = x^p$. These x_i are certain scalar multiples of various Steenrod operations on x.

If C is a suitable DGA whose cohomology is $H^*(X)$ and W is a free R-resolution (where $R = \mathbf{Z}/(p)[G]$) of $\mathbf{Z}/(p)$, then we get a diagram

$$\begin{array}{ccc} C & \longleftarrow & C_p \\ \downarrow & & \downarrow \\ C \otimes_R W & \longleftarrow & C_p \otimes_R W \end{array}$$

where C_p is the p-fold tensor power of C, R acts trivially on C and by cyclic permutation on C_p, and the top map is the iterated product in C. It is this diagram (with suitable properties) that is essential to defining the operations. The fact that C is associated with a space X is not essential. Any DGA C which admits such a diagram has Steenrod operations in its cohomology. The existence of such a diagram is a strong condition on C; it requires the product to be homotopy commutative in a very strong sense. If the product is strictly commutative the diagram exists but gives trivial operations.

In 11.3 of May [**5**] it is shown that the cobar complex (A1.2.11) $C_\Gamma(M, N)$, for M, N as above and Γ a Hopf algebra, has the requisite properties. The generalization to Hopf algebroids is not obvious so we give a partial proof of it here, referring to Bruner *et al.* [**1**] for certain details.

We need some notation to state the result. Let $C = C_\Gamma(M, N)$ for Γ a Hopf algebroid over K (which need not have characteristic p) and M, N comodule algebras. Let C_r denote the r-fold tensor product of C over K. Let π be a subgroup of the r-fold symmetric group Σ_r and let W be a negatively graded $K[\pi]$-free resolution of K. Let π act on C_r by permuting the factors. We will define a map of complexes

$$\theta\colon W \otimes_{K[\pi]} C_r \to C$$

with certain properties.

We define θ by reducing to the case $M = \Gamma$, which is easier to handle because the complex $d = C_\Gamma(\Gamma, N)$ is a Γ-comodule with a contracting homotopy. We have $C = M \mathbin{\square}_\Gamma D$ and an obvious map

$$j\colon W \otimes_{K[\pi]} C_r \to M_r \mathbin{\square}_\Gamma (W \otimes_{K[\pi]} D_r),$$

where the comodule structure on $W \otimes_{K[\pi]} D_r$ is defined by

$$\psi(w \otimes d_1 \cdots \otimes d_r) = d_1' d_2' \dots d_r' \otimes w \otimes d_1'' \otimes \cdots d_r''$$

for $w \in W$, $d_i \in D$, and $C(d_i) = d_i' \otimes d_i''$, and the comodule structure on M_r is defined similarly. Given a suitable map

$$\tilde{\theta}\colon W \otimes_{K[\pi]} D_r \to D,$$

we define θ to be the composite $(\mu \mathbin{\square} \tilde{\theta})j$, where $\mu\colon M_r \to M$ is the product.

A1.5.1. Theorem. *With notation as above assume $W_0 = K[\pi]$ with generator e_0. Then there are maps $\theta, \tilde{\theta}$ as above with the following properties.*

(i) *The restriction of θ to $e_0 \otimes C_r$ is the iterated product* (A1.2.15) *$C_r \to C$.*

(ii) *θ is natural in M, N, and Γ up to chain homotopy.*

(iii) *The analogs of* (i) *and* (ii) *for $\tilde{\theta}$ characterize it up to chain homotopy.*

(iv) *Let $\Delta\colon W \to W \otimes W$ be a coassociative differential coproduct on W which is a $K[\pi]$-map (where $K[\pi]$ acts diagonally on $W \otimes W$, i.e., given $\alpha \in \pi$, and $w_1, w_2 \in W$, $\alpha \in (w_1 \otimes w_2) = \alpha(w_1) \otimes \alpha(w_2)$); such coproducts are known to exist. Let $\mu\colon C \otimes C \to C$ be the product of* A1.2.15*. Then the following diagram commutes* *up to natural chain homotopy.*

$$\begin{tikzcd}
W \otimes_{K[\pi]} (C \otimes C)_r \arrow[r, "W\otimes\mu_r"] \arrow[d, "\Delta\otimes(C\otimes C)_r"'] & W \otimes_{K[\pi]} C_r \arrow[ddd, "\theta"] \\
W \otimes W \otimes_{K[\pi]} (C \otimes C)_r \arrow[d, "W\otimes T"'] & \\
W \otimes_{K[\pi]} C_r \otimes W \otimes_{K[\pi]} C_r \arrow[d, "\theta\otimes\theta"'] & \\
C \otimes C \arrow[r, "\mu"] & C
\end{tikzcd}$$

where T is the evident shuffle map.

(v) *Let $\pi = \nu = \mathbf{Z}/(p)$, $\sigma = \Sigma_{p^2}$ and let τ be the split extensions of ν^p by π in which π permutes the factors of ν^p. Let W, V, and Y be resolutions of K over $K[\pi]$, $K[\nu]$, and $K[\sigma]$, respectively. Let $j\colon \tau \to \sigma$ (τ is a p-Sylow subgroup of K) induce a map $j\colon W \otimes V_p \to Y$ ($W \otimes V_p$ is a free $K[\tau]$ resolution of K). Then there is a map $\omega\colon Y \otimes_{K[\sigma]} C_{p^2} \to C$ such that the following diagram commutes up to natural homotopy*

$$\begin{tikzcd}
(W \otimes V_p) \otimes_{K[\tau]} C_{p^2} \arrow[r, "j\otimes C_p^2"] \arrow[dd, "U"] & Y \otimes_{K[\sigma]} C_{p^2} \arrow[dr, "\omega"] & \\
& & C \\
W \otimes_{K[\pi]} (V \otimes_{K[v]} C_p)_p \arrow[r, "W\otimes\theta_p"] & W \otimes_{K[\pi]} C_p \arrow[ur, "\theta"] &
\end{tikzcd}$$

where U is the evident shuffle.

Proof. The map $\tilde{\theta}$ satisfying (i), (ii), and (iii) is constructed in Lemma 2.3 of Bruner's chapter in Bruner *et al.* [1]. In his notation let $M = N$ and $K = L = C(A, N)$, which is our D. Thus his map Φ is our $\tilde{\theta}$. Since $\tilde{\theta}$ extends the product on N it satisfies (i). For (ii), naturality in M is obvious since cotensor products are natural and everything in sight is natural in Γ. For naturality in N consider the (not necessarily commutative) diagram

$$\begin{tikzcd}
W \otimes_{k[\pi]} C_\Gamma(\Gamma, N)_r \arrow[r] \arrow[d, "\tilde{\theta}"] & W \otimes_{K[\pi]} C_\Gamma(\Gamma, N')_r \arrow[d, "\tilde{\theta}'"] \\
C_\Gamma(\Gamma, N) \arrow[r] & C_\Gamma(\Gamma, N').
\end{tikzcd}$$

Bruner's result gives a map

$$W \otimes_{K[\pi]} C_\Gamma(\Gamma, N)_r \to C_\Gamma(\Gamma, N')$$

extending the map $N_r \to N'$. Both the composites in the diagram have the appropriate properties so they are chain homotopic and $\tilde{\theta}$ is natural in $\widetilde{N}$ up to the chain homotopy.

For (iv) note that π acts on $(C\otimes C)_r = C_{2r}$ by permutation, so π is a subgroup of Σ_{2r}. The two composites in the diagram satisfy (i) and (ii) as maps from $W\otimes_{K[\pi]} C_{2r}$ to C, so they are naturally homotopic by (iii).

To prove (v), construct ω for the group σ in the same way we constructed θ for the group π. Then the compositions $\omega(j\otimes C_{p^2})$ and $(W\otimes\theta_p)U$ both satisfy (i) and (ii) for the group τ, so they are naturally homotopic by (iii). $\square$

With the above result in hand the machinery of May [5] applies to $C_\Gamma(M,N)$ and we get Steenrod operations in $\mathrm{Cotor}_\Gamma(M,N)$ when $K=\mathbf{Z}/(p)$. Parts (i), (ii), and (iii) guarantee the existence, naturality, and uniqueness of the operations, while (iv) and (v) give the Cartan formula and Adem relations. These operations have properties similar to those of the topological Steenrod operations with the following three exceptions. First, there is in general no Bockstein operation β. There are operations βP^i, but they need not be decomposable. Recall that in the classical case β was the connecting homomorphism for the short exact sequence

$$0\to C\to \widetilde{C}\otimes\mathbf{Z}/(p^2)\to C\to 0,$$

where $\widetilde{C}$ is a DGA which is free over $\mathbf{Z}$, whose cohomology is the integral cohomology of X and which is such that $\widetilde{C}\otimes\mathbf{Z}/(p) = C$. If C is a cobar complex as above then such a $\widetilde{C}$ may not exist. For example, it does not exist if $C = C_{A_*}(\mathbf{Z}/(p),\mathbf{Z}/(p))$ where A_* is the dual Steenrod algebra, but if $C = C_{BP_*(BP)/(p)}(BP_*/(p))$ we have $\widetilde{C} = C_{BP_*(BP)}(BP_*)$.

Second, when dealing with bigraded complexes there are at least two possible ways to index the operations; these two coincide in the classical singly graded case. In May [5] one has $P^i\colon \mathrm{Cotor}^{s,t}\to\mathrm{Cotor}^{s+(2i-t)(p-1),pt}$, which means that $P^i=0$ if either $2i<t$ or $2i>s+t$. (Classically one would always have $t=0$.) We prefer to index our P^i so that they raise cohomological degree by $2i(p-1)$ and are trivial if $i<0$ or $2i>s$ (in May [5] such operations are denoted by $\widetilde{P}^i$). This means that we must allow i to be a half-integer with P^i nontrivial only if $2i\equiv t \mod (2)$. (This is not a serious inconvenience because in most of our applications for $p>2$ the complex C^{**} will be trivial for odd t.) The Cartan formula and Adem relations below must be read with this in mind.

Finally, $P^0\colon \mathrm{Cotor}^{s,2t}\to\mathrm{Cotor}^{s,2pt}$ is not the identity as in the classical case. The following is a reindexed form of 11.8 of May [5].

A1.5.2. Steenrod Operations Theorem. *Let Γ be a Hopf algebroid over $\mathbf{Z}/(p)$ and M and N right and left Γ-comodule algebras. Denote $\mathrm{Cotor}_\Gamma^{s,t}(M,N)$ by $H^{s,t}$. Then there exist natural homomorphisms*

$$Sq^i\colon H^{s,t}\to H^{s+i,2t}\quad\text{for } p=2,$$
$$P^{i/2}\colon H^{s,t}\to H^{qi/2+s,pt}$$

and

$$\beta P^{i/2}\colon H^{s,t}\to H^{qi/2+s+1,pt}\quad\text{for } p>2 \text{ and } q=2p-2,$$

all with $i\ge 0$, having the following properties.

(a) *For $p=2$, $Sq^i(x)=0$ if $i>s$. For $p>2$, $P^{i/2}(x)=0$ and $\beta P^{i/2}(x)=0$ if $i>s$ or $2i\not\equiv t \mod (2)$.*

(b) *For $p=2$, $Sq^i(x)=x^2$ if $i=s$. For $p>2$ and $s+t$ even, $P^i(x)=x^p$ if $2i=s$.*

(c) *If there exists a Hopf algebroid* $\widetilde{\Gamma}$ *and* $\widetilde{\Gamma}$*-comodule algebras* $\widetilde{M}$ *and* $\widetilde{N}$ *all flat over* $\mathbf{Z}_{(p)}$ *with* $\Gamma = \widetilde{\Gamma} \otimes \mathbf{Z}/(p)$, $M = \widetilde{M} \otimes \mathbf{Z}/(p)$, *and* $N = \widetilde{N} \otimes \mathbf{Z}/(p)$, *then* $\beta Sq^i = (i+1)Sq^{i+1}$ *for* $p = 2$ *and for* $p > 2$ βP^i *is the composition of* β *and* P^i, *where* $\beta\colon H^{i,t} \to H^{s+1,t}$ *is the connecting homomorphism for the short exact sequence* $0 \to N \to \widetilde{N} \otimes \mathbf{Z}/(p^2) \to N \to 0$.

(d)

$$Sq^i(xy) = \sum_{0 \le j \le i} Sq^j(x) Sq^{i-j}(y) \quad \text{for } p = 2.$$

For $p > 2$

$$P^{i/2}(xy) = \sum_{0 \le j \le i} P^{j/2}(x) P^{(i-j)/2}(y)$$

and

$$\beta P^{i/2}(xy) = \sum_{0 \le j \le i} \beta P^{j/2}(x) P^{(i-j)/2}(y) + P^{i/2}(x) \beta P^{(i-j)/2}(y).$$

Similar external Cartan formulas hold.

(e) *The following Adem relations hold. For* $p = 2$ *and* $a < 2b$,

$$Sq^a Sq^b = \sum_{i \ge 0} \binom{b-i-1}{a-2i} Sq^{a+b-i} Sq^i.$$

For $p > 2$, $a < pb$, *and* $\varepsilon = 0$ *or* 1 (*and, by abuse of notation,* $\beta^0 P^i = P^i$ *and* $\beta^1 P^i = \beta P^i$),

$$\beta^\varepsilon P^{a/2} P^{b/2} = \sum_{i \ge 0} (-1)^{(a+i)/2} \binom{(p-1)(b-i)/2 - 1}{(a-pi)/2} \beta^\varepsilon P^{(a+b-i)/2} P^{i/2}$$

and

$$\begin{aligned}\beta^\varepsilon P^{a/2} \beta P^{b/2} = {} & (1-\varepsilon) \sum_{i \ge 0} (-1)^{(a+i)/2} \binom{(p-1)(b-i)/2 - 1}{(a-pi)/2} \beta P^{(a+b-i)/2} P^{i/2} \\ & - \sum_{i \ge 0} (-1)^{(a+i)/2} \binom{(p-1)(b-i)/2 - 1}{(a-pi)/2 - 1} \beta^\varepsilon P^{(a+b-i)/2} P^{i/2},\end{aligned}$$

where, in view of (a), *one only considers terms in which* a, b, *and* i *all have the same parity* (*so the signs and binomial coefficients all make sense*). □

To compute Sq^0 or P^0 we have the following, which is 11.10 of May [5].

A1.5.3. Proposition. *With notation as above, let* $x \in H^{s,t}$, *where* t *is even if* $p > 2$, *be represented by a cochain which is a sum of elements of the form* $m\gamma_1|\cdots|\gamma_s n$. *Then* $Sq^0(x)$ *or* $P^0(x)$ *is represented by a similar sum of elements of the form* $m^p \gamma_1^p|\cdots|\gamma_s^p n^p$. □

The operations also satisfy a certain suspension axiom. Consider the category $\mathbf{C}$ of triples (M, Γ, N) with M, Γ, N as above. A morphism in $\mathbf{C}$ consists of maps $M \to M'$, $\Gamma \to \Gamma'$, and $N \to N'$ which respect all the structure in sight. Let C_i, $i = 1, 2, 3$, be the cobar complexes for three objects in $\mathbf{C}$ and suppose there are morphisms which induce maps

$$C_1 \xrightarrow{f} C_2 \xrightarrow{g} C_3$$

such that the composite gf is trivial in positive cohomological degree. Let H^{**}, $i = 1, 2, 3$, denote the corresponding Cotor groups. Define a homomorphism σ (the *suspension*) from $\ker f^* \subset H_1^{s+t,t}$ to $H_3^{s,t}/\operatorname{im} g^*$ as follows. Given $x \in \ker f^*$, choose a cocycle $a \in C_1$ representing x and a cochain $b \in C_2$ such that $d(b) = f(a)$. Then $g(b)$ is a cocycle representing $\sigma(x)$. It is routine to verify that $\sigma(x)$ is well defined.

A1.5.4. Suspension Lemma. *Let σ be as above. Then for $p > 2$, $\sigma(P^i(x)) = P^i(\sigma(x))$ and $\sigma(\beta P^i(x) = \beta P^i(\sigma(x))$ and similarly for $p = 2$.*

Proof. We show how this statement can be derived from ones proved in May [**5**]. Let $\overline{C}_1 \subset C_1$ be the subcomplex of elements of positive cohomological degree. It has the structure necessary for defining Steenrod operations in its cohomology since C_1 does. Then May's theorem 3.3 applies to

$$\overline{C}_1 \xrightarrow{f} C_2 \xrightarrow{g} C_3$$

and shows that suspension commutes with the operations in $\ker f^* \subset H^*(C_1)$. We have $H^s(\overline{C}_1) = H^s(C_1)$ for $s > 1$ and a four-term exact sequence

$$0 \to M_1 \square_{\Gamma_1} N_1 \to M_1 \otimes_{A_1} M_1 \to H^1(\overline{C}_1) \to H^1(C_1) \to 0$$

so the result follows. □

A1.5.5. Corollary. *Let δ be the connecting homomorphism associated with an short exact sequence of commutative associative Γ-comodule algebras. Then $P^i\delta = \delta P^i$ and $\beta P^i \delta = -\delta\beta P^i$ for $p > 2$ and similarly for $p = 2$. (In this situation the subcomodule algebra must fail to have a unit.)*

Proof. Let $0 \to M_1 \to M_2 \to M_3 \to 0$ be such a short exact sequence. Then set $N_i = N$ and $\Gamma_i = \Gamma$ in the previous lemma. Then δ is the inverse of σ so the result follows. □

We need a transgression theorem.

A1.5.6. Corollary. *Let $(D, \Phi) \xrightarrow{i} (A, \Gamma) \xrightarrow{f} (A, \Sigma)$ be an extension of Hopf algebroids over $\mathbf{Z}/(p)$ (A1.1.15); let M be a right Φ-comodule algebra and N a left Γ-comodule algebra, both commutative and associative. Then there is a suspension map σ from $\ker i^* \subset \operatorname{Cotor}_\Phi^{s+1,t}(M, A \square_\Sigma N)$ to $\operatorname{Cotor}_\Sigma^{s,t}(M \otimes_D A, N)/\operatorname{im} f^*$ which commutes with Steenrod operations as in A1.5.4.*

Proof. $A \square_\Sigma N$ is a left Φ-comodule algebra by A1.3.14(a). We claim the composite $\overline{\Phi} \xrightarrow{i} \overline{\Gamma} \xrightarrow{f} \overline{\Sigma}$ is zero; since $\Phi = A \square_\Sigma \Gamma \square_\Sigma A$, $fi(\Phi) = A \square_\Sigma \Sigma \square_\Sigma A = A \square_\Sigma A = D$, so $fi(\overline{\Phi}) = 0$. Hence $C_\Phi(M, A \square_\Sigma N) \to C_\Gamma(M \otimes_D A, N) \to C_\Sigma(M \otimes_D A, N)$ is zero in positive cohomological degree. Hence the result follows from A1.5.4. □

The following is a reformulation of theorem 3.4 of May[**5**].

A1.5.7. Kudo Transgression Theorem. *Let $\Phi \to \Gamma \to \Sigma$ be a cocentral extension (A1.1.15) of Hopf algebras over a field K of characteristic p. In the Cartan–Eilenberg spectral sequence (A1.3.14) for $\operatorname{Ext}_\Gamma(K, K)$ we have $E_2^{s,t} = \operatorname{Ext}_\Phi^s(K, K) \otimes \operatorname{Ext}_\Sigma^t(K, K)$ with $d_r \colon E_r^{s,t} \to E_r^{s+r,t-r+1}$. Then the transgression $d_r \colon E_r^{0,r-1} \to E_r^{r,0}$ commutes with Steenrod operations up to sign as in A1.5.4; e.g., if $d_r(x) = y$ then $d_{r+2s(p+1)}(P^s(x)) = P^s(y)$. Moreover for $p > 2$ and $r - 1$ even we have $d_{(p-1)(r-1)+1}(x^{p-1}y) = -\beta P^{(r-1)/2}(y)$.* □

APPENDIX A2

Formal Group Laws

In this appendix we will give a self-contained account of the relevant aspects of the theory of commutative one-dimensional formal group laws. This theory was developed by various algebraists for reasons having nothing to do with algebraic topology. The bridge between the two subjects is the famous result of Quillen [**2**] (4.1.6) which asserts that the Lazard ring L (A2.1.8) over which the universal formal group law is defined is naturally isomorphic to the complex cobordism ring. A most thorough and helpful treatment of this subject is given in Hazewinkel [**1**]. An account of the Lazard ring is also given in Adams [**5**], while the classification in characteristic p can also be found in Fröhlich [**1**].

We now outline the main results of Section 1. We define formal group laws (A2.1.1) and homomorphisms between them (A2.1.5) and show that over a field of characteristic 0 every formal group law is isomorphic to the additive one (A2.1.6). The universal formal group law is constructed (A2.1.8) and the structure of the ring L over which it is defined is determined (A2.1.10). This result is originally due to Lazard [**1**]. Its proof depends on a difficult lemma (A2.1.12) whose proof is postponed to the end of the section.

Then we define p-typical formal group laws (A2.1.17 and A2.1.22) and determine the structure of the p-typical analog of the Lazard ring, V (A2.1.24). This result is due to Carrier [**1**]; Quillen [**2**] showed that V is naturally isomorphic to $\pi^*(BP)$ (4.1.12). Using a point of view due to Landweber [**1**], we determine the structure of algebraic objects LB (A2.1.16) and VT (A2.1.26), which turn out to be isomorphic to $MU_*(MU)$ (4.1.11) and $BP_*(BP)$ (4.1.19), respectively.

All of the results of this section can be found in Adams [**5**], although our treatment of it differs from his.

In Section 2 we give the explicit generators of V [i.e., of $\pi_*(BP)$] given by Hazewinkel [**2**] (A2.2.1) and Araki [**1**] (A2.2.2) and determine the behavior of the right unit η_R on Araki's generators (A2.2.6).

For the Morava theory of Chapter 6 we will need the classification of formal group laws over separably closed fields of characteristic $p > 0$ (A2.2.12) originally due to Lazard [**2**], and a description of the relevant endomorphism rings (A2.2.18 and A2.2.19) originally due to Dieudonné [**1**] and Lubin [**1**].

For a scheme theoretic approach to this subject, see Strickland [**1**].

1. Universal Formal Group Laws and Strict Isomorphisms

A2.1.1. Definition. *Let R be a commutative ring with unit. A* formal group law *over R is a power series $F(x, y) \in R[[x, y]]$ satisfying*

(i) $F(x, 0) = F(0, x) = x$,

(ii) $F(x, y) = F(y, x)$*, and*

(iii) $F(x, F(y, z)) = F(F(x, y)z)$.

Strictly speaking, such an object should be called a commutative one-dimensional formal group law; we omit the first two adjectives as this is the only type of formal group law we will consider. It is known (Lazard [**3**]) that (ii) is redundant if R has no nilpotent elements.

The reason for this terminology is as follows. Suppose G is a one-dimensional commutative Lie group and $g\colon \mathbf{R} \to U \subset G$ is a homomorphism to a neighborhood U of the identity which sends 0 to the identity. Then the group operation $G \times G \to G$ can be described locally by a real-valued function of two real variables. If the group is analytic then this function has a power series expansion about the origin that satisfies (i)–(iii). These three conditions correspond, respectively, to the identity, commutativity, and associativity axioms of the group. In terms of the power series, the existence of an inverse is automatic, i.e.,

A2.1.2. Proposition. *If F is a formal group law over R then there is a power series $i(x) \in R[[x]]$ (called the* formal inverse*) such that $F(x, i(x)) = 0$.*

In the Lie group case this power series must of course converge, but in the formal theory convergence does not concern us. Formal group laws arise in more algebraic situations; e.g., one can extract a formal group law from an elliptic curve defined over R; see Chapter 7 of Silverman [**1**]. One can also reverse the procedure and get a group out of a formal group law; if R is a complete local ring then $F(x, y)$ will converge whenever x and y are in the maximal ideal, so a group structure is defined on the latter which may differ from the usual additive one.

Before proceeding further note that A2.1.1(i) implies

A2.1.3. Proposition. *If F is a formal group law then*

$$F(x, y) \equiv x + y \mod (x, y)^2. \qquad \square$$

A2.1.4. Examples of Formal Group Lows. (a) $F_a(x, y) = x + y$, the additive formal group law.

(b) $F(x, y) = x + y + uxy$ (where u is a unit in R), the multiplicative formal group law, so named because $1 + uF = (1 + ux)(1 + uy)$.

(c) $F(x, y) = (x + y)/(1 + xy)$.

(d) $F(x, y) = (x\sqrt{1 - y^4} + y\sqrt{1 - x^4})/(1 + x^2y^2)$, a formal group law over $\mathbf{Z}[1/2]$.

The last example is due to Euler and is the addition formula for the elliptic integral

$$\int_0^x \frac{dt}{\sqrt{1 - t^4}}$$

(see Siegel [**1**, pp. 1-9]). These examples will be studied further below (A2.2.10).

The astute reader will recognize (c) as the addition formula for the hyperbolic tangent function; i.e., if $x = \tanh(u)$ and $y = \tanh(v)$ then $F(x, y) = \tanh(u + v)$. Hence we have

$$\tanh^{-1}(F(x, y)) = \tanh^{-1}(x) + \tanh^{-1}(y)$$

or

$$F(x, y) = \tanh(\tanh^{-1}(x) + \tanh^{-1}(y)),$$

where $\tanh^{-1}(x) = \sum_{i \geq 0} x^{2i+1}/(2i + 1) \in R \otimes \mathbf{Q}[[x]]$.

We have a similar situation in (b), i.e.,

$$\log(1 + uF) = \log(1 + ux) + \log(1 + uy),$$

where $\log(l+ux)=\sum_{i>0}(-1)^{i+1}(ux)^i/i\in R\otimes\mathbf{Q}[[x]]$.

This means that the formal group laws of (b) and (c) are isomorphic over $\mathbf{Q}$ to the additive formal group law (a) in the following sense.

A2.1.5. DEFINITION. *Let F and G be formal group laws. A* homomorphism *from F to G is a power series $f(x)\in R[[x]]$ with constant term 0 such that $f(F(x,y))=G(f(x),f(y))$. It is an* isomorphism *if it is invertible, i.e., if $f'(0)$ (the coefficient of x) is a unit in R, and a* strict isomorphism *if $f'(0)=1$. A strict isomorphism from F to the addition formal group law $x+y$ is a logarithm for F, denoted by $\log_F(x)$.*

Hence the logarithms for A2.1.4(b) and (c) are

$$\sum_{i>0}\frac{(-u)^{i-1}x^i}{i}\qquad\text{and}\qquad\tanh^{-1}(x)$$

respectively.

On the other hand, these formal group laws are not isomorphic to the additive one over $\mathbf{Z}$. To see this for (b), set $u=1$. Then $F(x,x)=2x+x^2\equiv x^2\mod 2$, while $F_a(x,x)=2x\equiv 0\mod 2$, so the two formal group laws are not isomorphic over $\mathbf{Z}/(2)$. The formal group law of (c) is isomorphic to F_a over $\mathbf{Z}_{(2)}$, since its logarithm $\tanh^{-1}x$ has coefficients in $\mathbf{Z}_{(2)}$, but we have $F(F(x,x),x)=(3x+x^3)/(1+3x^2)\equiv x^3\mod(3)$ while $F_a(F_a(x,x),x)=3x\equiv 0\mod 3$. Similarly, it can be shown that F and F_a are distinct at every odd prime (see A2.2.10).

A2.1.6. THEOREM. *Let F be a formal group law and let $f(x)\in R\otimes\mathbf{Q}[[x]]$ be given by*

$$f(x)=\int_0^x\frac{dt}{F_2(t,0)}$$

where $F_2(x,y)=\partial F/\partial y$. Then f is a logarithm for F, i.e., $F(x,y)=f^{-1}(f(x)+f(y))$, and F is isomorphic over $R\otimes\mathbf{Q}$ to the additive formal group law.

PROOF. Let $w=f(F(x,y))-f(x)-f(y)$. We wish to show $w=0$. We have $F(F(x,y),z)=F(x,F(y,z))$. Differentiating with respect to z and setting $z=0$ we get

$$F_2(F(x,y),0)=F_2(x,y)F_2(y,0).\tag{A2.1.7}$$

On the other hand, we have $\partial w/\partial y=f'(F(x,y))F_2(y,0)-f'(y)$, which by the definition of f becomes

$$\frac{\partial w}{\partial y}=\frac{F_2(x,y)}{F_2(F(x,y),0)}-\frac{1}{F_2(y,0)}=0\quad\text{by A2.1.7.}$$

By symmetry we also have $\partial w/\partial x=0$, so w is a constant. But f and F both have trivial constant terms, so $w=0$. □

Now we wish to consider the universal formal group law. Its construction is easy.

A2.1.8. THEOREM. *There is a ring L (called the Lazard ring) and a formal group law*

$$F(x,y)=\sum a_{i,j}x^iy^j$$

defined over it such that for any formal group law G over any commutative ring with unit R there is a unique ring homomorphism $\theta\colon L \to R$ such that $G(x,y) = \sum \theta(a_{i,j})x^i y^j$.

PROOF. Simply set $L = \mathbf{Z}[a_{i,j}]/I$, where I is the ideal generated by the relations among the $a_{i,j}$ required by the definition A2.1.1, i.e., by $a_{1,0}-1$, $a_{0,1}-1$, $a_{i,0}$, and $a_{0,i}$ for (i), $a_{i,j}-a_{ji}$ for (ii), and b_{ijk} for (iii), where

$$F(F(x,y),z) - F(x,F(y,z)) = \sum b_{ijk}x^i y^j z^k.$$

Then θ can be defined by the equation it is supposed to satisfy. $\square$

Determining the structure of L explicitly is more difficult. At this point it is convenient to introduce a grading on L by setting $|a_{i,j}| = 2(i+j-1)$. Note that if we have $|x| = |y| = -2$ then $F(x,y)$ is a homogeneous expression of degree -2.

A2.1.9. LEMMA. (a) *$L\otimes \mathbf{Q} = \mathbf{Q}[m_1, m_2, \dots]$ with $|m_i| = 2i$ and $F(x,y) = f^{-1}(f(x)+f(y))$ where $f(x) = x + \sum_{i>0} m_i x^{i+1}$.*

(b) *Let $M \subset L\otimes\mathbf{Q}$ be $\mathbf{Z}[m_1, m_2, \dots]$. Then $\operatorname{im} L \subset M$.*

PROOF. (a) By A2.1.6 every formal group law G over a $\mathbf{Q}$-algebra R has a logarithm $g(x)$ so there is a unique $\phi\colon \mathbf{Q}[m_1 m_2, \dots] \to R$ such that $\phi(f(x)) = g(x)$. In particular we have $\phi\colon \mathbf{Q}[m_1, m_2, \dots] \to L\otimes\mathbf{Q}$ as well as $\theta\colon L\otimes\mathbf{Q} \to \mathbf{Q}[m_1, m_2, \dots]$ with $\theta\phi$ and $\phi\theta$ being identity maps, so θ and ϕ are isomorphisms.

(b) $F(x,y)$ is a power series with coefficients in M, so the map from L to $L\otimes\mathbf{Q}$ factors through M. $\square$

Now recall that if R is a graded connected ring (e.g., $L\otimes\mathbf{Q}$) the group of indecomposables QR is I/I^2 where $I \subset R$ is the ideal of elements of positive degree.

A2.1.10. THEOREM (Lazard [**1**]). (a) *$L = \mathbf{Z}[x_1, x_2, \dots]$ with $|x_i| = 2i$ for $i>0$.*

(b) *x_i can be chosen so that its image in $QL\otimes\mathbf{Q}$ is*

$$\begin{cases} pm_i & \text{if } i = p^k-1 \text{ for some prime } p \\ m_i & \text{otherwise.} \end{cases} \qquad \square$$

(c) *L is a subring of M* [A2.1.9(b)].

The proof of this is not easy and we will postpone the hardest part of it (A2.1.12) to the end of this section. The difficulty is in effect showing that L is torsion-free. Without proving A2.1.12 we can determine L/torsion with relative ease. We will not give F in terms of the x_i, nor will the latter be given explicitly. Such formulas can be found, however, in Hazewinkel [**3**] and in Section 5 of Hazewinkel [**1**].

Before stating the hard lemma we need the following exercise in binomial coefficients.

A2.1.11. PROPOSITION. *Let u_n be the greatest common divisor of the numbers $\binom{n}{i}$ for $0<i<n$. Then*

$$u_n = \begin{cases} p & \text{if } n = p^k \text{ for some prime } p \\ 1 & \text{otherwise.} \end{cases} \qquad \square$$

Now we are ready for the hard lemma. Define homogeneous symmetric polynomials $B_n(x,y)$ and $C_n(x,y)$ of degree n for all $n > 0$ by

$$B_n(x,y) = (x+y)^n - x^n - y^n$$
$$C_n(x,y) = \begin{cases} B_n/p & \text{if } n = p^k \text{ for some prime } p \\ B_n & \text{otherwise.} \end{cases}$$

It follows from A2.1.11 that $C_n(x,y)$ is integral and that it is not divisible by any integer greater than one.

A2.1.12. Comparison Lemma (Lazard [**1**]). *Let F and G be two formal group laws over R such that $F \equiv G \mod (x,y)^n$. Then $F \equiv G + aC_n \mod (x,y)^{n+1}$ for some $a \in R$.* □

The proof for general R will be given at the end of this section. For now we give a proof for torsion-free R.

In this case we lose no information by passing to $R \otimes \mathbf{Q}$, where we know (A2.1.6) that both formal group laws have logarithms, say $f(x)$ and $g(x)$, respectively. Computing mod $(x,y)^{n+1}$ we have

$$f(x) \equiv g(x) + bx^n \quad \text{for some } b \in R \otimes \mathbf{Q} \quad \text{so} \quad f^{-1}(x) = g^{-1}(x) - bx^n$$

and

$$\begin{aligned} F - G &= f^{-1}(f(x) + f(y)) - g^{-1}(g(x) + g(y)) \\ &\equiv g^{-1}(g(x) + g(y) + b(x^n + y^n)) - b(x+y)^n - g^{-1}(g(x) + g(y)) \\ &\equiv g^{-1}(g(x) + g(y)) + b(x^n + y^n) - b(x+y)^n - g^{-1}(g(x) + g(y)) \\ &\equiv -bB_n(x,y). \end{aligned}$$

Since this must lie in R it must have the form $aC_n(x,y)$, completing the proof for torsion-free R.

A2.1.13. Lemma. (a) *In $QL \otimes \mathbf{Q}$, $a_{i,j} = -\binom{i+j}{j} m_{i+j-1}$.*
(b) *QL is torsion-free.*

Proof. (a) Over $L \otimes \mathbf{Q}$ we have $\sum m_{n-1}(\sum a_{i,j}x^iy^j)^n = \sum m_{n-1}(x^n + y^n)$. Using A2.1.3 to pass to $QL \otimes \mathbf{Q}$ we get

$$\sum a_{i,j}x^iy^j + \sum_{n>1} m_{n-1}(x+y)^n = \sum_{n>0} m_{n-1}(x^n + y^n),$$

which gives the desired formula.

(b) Let $Q_{2n}L$ denote the component of QL in degree $2n$, and let R be the graded ring $\mathbf{Z} \oplus Q_{2n}L$. Let F be the formal group law over R induced by the obvious map $\theta\colon L \to R$, and let G be the additive formal group law over R. Then by A2.1.12, $F(x,y) \equiv x + y + aC_{n+1}(x,y)$ for $a \in Q_{2n}L$. It follows that $Q_{2n}L$ is a cyclic group generated by a. By (a) $Q_{2n}L \otimes \mathbf{Q} = \mathbf{Q}$, so $\mathbf{Q}_{2n}L = \mathbf{Z}$ and QL is torsion-free. □

It follows from the above that L is generated by elements x_i whose images in $QL \otimes \mathbf{Q}$ are u_im_i, where u_i is as in A2.1.11, i.e., that L is a quotient of $\mathbf{Z}[x_i]$. By A2.1.9 it is the quotient by the trivial ideal, so A2.1.10 is proved.

Note that having A2.1.12 for torsion-free R implies that L/torsion is as claimed.

The reader familiar with Quillen's theorem (4.1.6) will recognize L as $\pi_*(MU) = MU_*$. We will now define an object which is canonically isomorphic to $\pi_*(MU \wedge MU) = MU_*(MU)$. This description of the latter is due to Landweber [**1**].

A2.1.14. DEFINITION. *Let R be a commutative ring with unit. Then $FGL(R)$ is the set of formal group laws over R (A2.1.1) and $SI(R)$ is the set of triples (F, f, G) where $F, G \in FGL(R)$ and $f\colon F \to G$ is a strict isomorphism (A2.1.5), i.e., $f(x) \in R[[x]]$ with $f(0) = 0$, $f'(0) = 1$, and $f(F(x, y)) = G(f(x), f(y))$. We call such a triple a* matched pair

A2.1.15. PROPOSITION. *$FGL(-)$ and $SI(-)$ are covariant functors on the category of commutative rings with unit. $FGL(-)$ is represented by the Lazard ring L and $SI(-)$ is represented by the ring $LB = L \otimes \mathbf{Z}[b_1, b_2, \dots]$. In the grading introduced above, $|b_i| = 2i$.*

PROOF. All but the last statement are obvious. Note that a matched pair (F, f, G) is determined by F and f and that f can be any power series of the form $f(x) = x + \sum_{i>0} f_i x^{i+1}$. Hence such objects are in 1-1 correspondence with ring homomorphisms $\theta\colon LB \to R$ with $\theta(b_i) = f_i$. □

Now LB has some additional structure which we wish to describe. Note that $FGL(R)$ and $SI(R)$ are the sets of objects and morphisms, respectively, of a groupoid, i.e., a small category in which every morphism is an equivalence. Hence these functors come equipped with certain natural transformations reflecting this structure. The most complicated is the one corresponding to composition of morphisms, which gives a natural (in R) map from a certain subset of $SI(R) \times SI(R)$ to $SI(R)$. This structure also endows (L, LB) with the structure of a Hopf algebroid (A1.1.1). Indeed that term was invented by Haynes Miller with this example in mind. We now describe this structure.

A2.1.16. THEOREM. *In the Hopf algebroid (L, LB) defined above $\varepsilon\colon LB \to L$ is defined by $\varepsilon(b_i) = 0$; $\eta_L\colon L \to LB$ is the standard inclusion while $\eta_R\colon L \otimes \mathbf{Q} \to LB \otimes \mathbf{Q}$ is given by*

$$\sum_{i \geq 0} \eta_R(m_i) = \sum_{i \geq 0} m_i \left(\sum_{j \geq 0} c(b_j) \right)^{i+1},$$

where $m_0 = b_0 = 1$; $\sum_{i \geq 0} \Delta(b_i) = \sum_{j \geq 0} (\sum_{i \geq 0} b_i)^{j+1} \otimes b_j$; and $c\colon LB \to LB$ is determined by $c(m_i) = \eta_R(m_i)$ and $\sum_{i \geq 0} c(b_i) \left(\sum_{j \geq 0} b_j \right)^{i+1} = 1$.

These are the structure formulas for $MU_*(MU)$ (4.1.11).

PROOF. ε and η_L are obvious. For c, if $f(x) = \sum b_i x^{i+1}$ then $f^{-1}(x) = \sum c(b_i) x^{i+1}$. Expanding $f^{-1}(f(1)) = 1$ gives the formula for $c(b_i)$. For η_R, let $\log x = \sum m_i x^{i+1}$ and $\operatorname{mog} x = \sum \eta_R(m_i) x^{i+1}$ be the logarithms for F and G, respectively. Then we have

$$f^{-1}(G(x, y)) = F(f^{-1}(x), f^{-1}(y))$$

so

$$\log(f^{-1}(G(x, y))) = \log(f^{-1}(x)) + \log(f^{-1}(y)).$$

We also have

$$\operatorname{mog}(G(x, y)) = \operatorname{mog}(x) + \operatorname{mog}(y)$$

for which we deduce

$$\mathrm{mog}(x) = \log f^{-1}(x).$$

Setting $x = 1$ gives the formula for η_R. For Δ let $f_1(x) = b_i' x^{i+1}$, $f_2(x) = \sum b_i'' x^{i+1}$, and $f(x) = f_2(f_1(x))$. Then expanding and setting $x = 1$ gives $\sum b_i = \sum b_i''(\sum b_j')^{i+1}$. Since f_2 follows f_1 this gives the formula for Δ. $\square$

Note that (L, LB) is split (Al.1.22) since Δ defines a Hopf algebra structure on $B = \mathbf{Z}[b_i]$.

Next we will show how the theory simplifies when we localize at a prime p, and this will lead us to BP_* and $BP_*(BP)$.

A2.1.17. DEFINITION. *A formal group law over a torsion-free $\mathbf{Z}_{(p)}$-algebra is* p-typical *if its logarithm has the form $\sum_{i\geq 0} \ell_i x^{p^i}$ with $\ell_0 = 1$.*

Later (A2.1.22) we will give a form of this definition which works even when the $\mathbf{Z}_{(p)}$-algebra R has torsion. Assuming this can be done, we have

A2.1.18. THEOREM (Cartier [**1**]). *Every formal group law over a $\mathbf{Z}_{(p)}$-algebra is canonically strictly isomorphic to a p-typical one.*

Actually A2.1.17 is adequate for proving the theorem because it suffices to show that the universal formal group law is isomorphic over $L \otimes \mathbf{Z}_{(p)}$ to a p-typical one.

The following notation will be used repeatedly.

A2.1.19. DEFINITION. *Let F be a formal group law over R. If x and y are elements in an R-algebra A which also contains the power series $F(x, y)$, let*

$$x +_F y = F(x, y).$$

This notation may be iterated, e.g., $x +_F y +_F z = F(F(x, y), z)$. Similarly, $x -_F y = F(x, i(y))$ (A2.1.2). *For nonnegative integers n, $[n]_F(x) = F(x, [n-1]_F(x))$ with $[0]_F(x) = 0$. (The subscript F will be omitted whenever possible.) $\sum^F(\ )$ will denote the formal sum of the indicated elements.*

A2.1.20. PROPOSITION. *If the formal group law F above is defined over a K-algebra R where K is a subring of $\mathbf{Q}$, then for each $r \in K$ there is a unique power series $[r]_F(x)$ such that*

(a) *if r is a nonnegative integer, $[r]_F(x)$ is the power series defined above,*
(b) *$[r_1 + r_2]_F(x) = F([r_1]_F(x), [r_2]_F(x))$,*
(c) *$[r_1 r_2]_F(x) = [r_1]_F([r_2]_F(x))$.*

PROOF. Let $[-1]_F(x) = i(x)$ (A2.1.2), so $[r]_F(x)$ is defined by (b) for all $r \in \mathbf{Z}$. We have $[r]_F(x) \equiv rx \mod (x^2)$, so if $d \in \mathbf{Z}$ is invertible in K, the power series $[d]_F(x)$ is invertible and we can define $[d^{-1}]_F(x) = [d]_F^{-1}(x)$. $\square$

Now we suppose q is a natural number which is invertible in R. Let

$$f_q(x) = [1/q]\left(\sum_{i=1}^{q}{}^{F} \zeta^i x\right) \tag{A2.1.21}$$

where ζ is a primitive qth root of unity. A priori this is a power series over $R[\zeta]$, but since it is symmetric in the ζ^i it is actually defined over R.

If R is torsion-free and $\log(x) = \sum_{i\geq 0} m_i x^{i+1}$, we have

$$\begin{aligned}\log(f_q(x)) &= \frac{1}{q}\sum_{i=1}^{q}\log(\zeta^i x)\\ &= \frac{1}{q}\sum_{i=1}^{q}\sum_{j\geq 0} m_j x^{j+1}\zeta^{i(j+1)}\\ &= \frac{1}{q}\sum_{j\geq 0} m_j x^{j+1}\sum_{i=1}^{q}\zeta^{i(j+1)}.\end{aligned}$$

The expression $\sum_{i=1}^{q}\zeta^{i(j+1)}$ vanishes unless $(j+1)$ is divisible by q, in which case its value is q. Hence, we have

$$\log(f_q(x)) = \sum_{j>0} m_{qj-1}x^{qj}.$$

If F is p-typical for $p \neq q$, this expression vanishes, so we make

A2.1.22. Definition. *A formal group law F over a $\mathbf{Z}_{(p)}$-algebra is* p-typical *if $f_q(x) = 0$ for all primes $q \neq p$.*

Clearly this is equivalent to our earlier definition A2.1.17 for torsion-free R.

To prove Cartier's theorem (A2.1.18) we claim that it suffices to construct a strict isomorphism $f(x) = \sum f_i x^i \in L\otimes\mathbf{Z}_{(p)}[[x]]$ from the image of F over $L\otimes\mathbf{Z}_{(p)}$ to a p-typical formal group law F'. Then if G is a formal group law over a $\mathbf{Z}_{(p)}$-algebra R induced by a homomorphism $\theta\colon L\otimes\mathbf{Z}_{(p)} \to R$, $g(x) = \sum\theta(f_i)x^i \in R[[x]]$ is a strict isomorphism from G to a p-typical formal group law G'.

Recall that if $\operatorname{mog}(x)$ is the logarithm for F' then

$$\operatorname{mog}(x) = \log(f^{-1}(x)).$$

We want to use the $f_q(x)$ for various primes $q \neq p$ to concoct an $f^{-1}(x)$ such that

$$\log(f^{-1}(x)) = \sum_{i\geq 0} m_{p^i-1}x^{p^i}.$$

It would not do to set

$$f^{-1}(x) = x -_F \sum_{q\neq p}^{F} f_q(x)$$

because if n is a product of two or more primes $\neq p$ then a negative multiple of $M_{n-1}x^n$ would appear in $\log f^{-1}(x)$. What we need is the Möbius function $\mu(n)$ defined on natural numbers n by

$$\mu(n) = \begin{cases} 0 & \text{if } n \text{ is divisible by a square}\\ (-1)^r & \text{if } n \text{ is the product of } r \text{ distinct primes.}\end{cases}$$

Note that $\mu(1) = 1$ and $\mu(q) = -1$ if q is prime. Then we define $f(x)$ by

$$f^{-1}(x) = \sum_{p\nmid q}^{F}[\mu(q)]_F(f_q(x)). \tag{A2.1.23}$$

[Note also that $f_1(x) = x$.] The sum is over all natural numbers q not divisible by p. This infinite formal sum is well defined because $f_q(x) \equiv 0 \mod (x^q)$.

Now

$$\log(f^{-1}(x)) = \sum_{p\nmid q} \mu(q) \sum_{j>0} m_{qj-1} x^{qj} = \sum_{n>0} \left(\sum_{\substack{p\nmid q \\ q|n}} \mu(q) \right) m_{n-1} x^n.$$

It is elementary to verify that

$$\sum_{\substack{p\nmid q \\ q|n}} \mu(q) = \begin{cases} 1 & \text{if } n = p^k \\ 0 & \text{otherwise.} \end{cases}$$

It follows that F' has logarithm

$$\text{(A2.1.24)} \qquad \operatorname{mog}(x) = \sum_{i\geq 0} m_{p^i-1} x^{p^i},$$

so F' is p-typical. This completes the proof of A2.1.18.

Now we will construct the universal p-typical formal group law.

A2.1.25. THEOREM. *Let $V = \mathbf{Z}_{(p)}[v_1, v_2, \ldots]$ with $|v_n| = 2(p^n - 1)$. Then there is a universal p-typical formal group law F defined over V; i.e., for any p-typical formal group law G over a commutative $\mathbf{Z}_{(p)}$-algebra R, there is a unique ring homomorphism $\theta\colon V \to R$ such that $G(x, y) = \theta(F(x, y))$. Moreover the homomorphism from $L \otimes \mathbf{Z}_{(p)}$ to V corresponding* (A2.1.8) *to this formal group law is surjective, i.e., V is isomorphic to a direct summand $L \otimes \mathbf{Z}_{(p)}$.* □

We will give an explicit formula for the v_n's in terms of the log coefficients m_{p^n-1} below (A2.2.2). In 4.1.12 it is shown that V is canonically isomorphic to $\pi_*(BP)$.

PROOF. Recall that the canonical isomorphism f above corresponds to an endomorphism ϕ of $L \otimes \mathbf{Z}_{(p)}$ given by

$$\phi(m_i) = \begin{cases} m_i & \text{if } i = p^k - 1 \\ 0 & \text{otherwise.} \end{cases}$$

This ϕ is idempotent, i.e., $\phi^2 = \phi$ and its image is a subring $V \subset L \otimes \mathbf{Z}_{(p)}$ over which the universal p-typical formal group law is defined. An argument similar to the proof of Lazard's theorem A2.1.9 shows that V has the indicated structure. □

Now we will construct a ring VT canonically isomorphic to $BP_*(BP)$ and representing the set of p-typical matched pairs (F, f, G) (A2.1.14), i.e., matched pairs with F and G p-typical. The power series f must be chosen carefully to ensure that G is p-typical, and this choice depends on F. There is no such thing as a "p-typical power series," i.e., one that sends any p-typical F to a p-typical G. To characterize the appropriate f we have

A2.1.26. LEMMA. *Let F be a p-typical formal group law over a $\mathbf{Z}_{(p)}$-algebra R. Let $f(x)$ be an isomorphism* (A2.1.5) *from F to a formal group law G. Then G is p-typical if*

$$f^{-1}(x) = \sum_{i\geq 0}^{F} t_i x^{p^i}$$

for $t_i \in R$ with t_0 a unit in R.

PROOF. For a prime number $\neq p$ let

$$h_q(x) = [q^{-1}]_G\left(\sum_{i=1}^{q}{}^{G} \zeta^i x\right)$$

where ζ is a primitive pth root of unity. By A2.1.22 we need to show that $h_q(x) = 0$ for all $q \neq p$ iff f is as specified. From the relation

$$G(x, y) = f(F(f^{-1}(x), f^{-1}(y)))$$

we deduce

$$f^{-1}(h_q(x)) = [q^{-1}]_F\left(\sum_{j=1}^{q}{}^{F} f^{-1}(\zeta^j x)\right).$$

Now for isomorphism $f(x)$ there are unique $c_i \in R$ such that

$$f^{-1}(x) = \sum_{i>0}{}^{F} c_i x^i$$

with c_1 a unit in R. Hence we have

$$\begin{aligned} f^{-1}(h_q(x)) &= [q^{-1}]_F\left(\sum_{i,j}{}^{F} c_i\zeta^{ij}x^i\right) \\ &= [q^{-1}]_F\left(\sum_{q\nmid i}{}^{F}\sum_{j}{}^{F} \zeta^j c_i x^i\right) +_F [q^{-1}] +_F \left(\sum_{i}{}^{F} [q]_F(c_{qi}x^{qi})\right) \\ &= \sum_{q\nmid i}{}^{F} f_q(c_i x^i) +_F \sum_{i}{}^{F} c_{qi}x^{qi} = \sum_{i>0} c_{qi}x^{qi}. \end{aligned}$$

This expression vanishes for all $q \neq p$ iff $c_{qi} = 0$ for all $i > 0$ and $q \neq p$, i.e., iff f is as specified. $\square$

It follows immediately that $VT = V \otimes \mathbf{Z}_{(p)}[t_1, t_2, \dots]$ as a ring since for a strict isomorphism $t_0 = 1$. The rings V and VT represent the sets of objects and morphisms in the groupoid of strict isomorphisms of p-typical formal group laws over a $\mathbf{Z}_{(p)}$-algebra. Hence (V, VT), like (L, LB), is a Hopf algebroid (Al.1.1) and it is isomorphic to $(BP_*, BP_*(BP))$. Its structure is as follows.

A2.1.27. THEOREM. *In the Hopf algebroid* (V, VT) (*see* A1.1.1)

(a) $V = \mathbf{Z}_{(p)}[v_1, v_2, \dots]$ *with* $|v_n| = 2(p^n - 1)$,

(b) $VT = V \otimes \mathbf{Z}_{(p)}[t_1, t_2, \dots]$ *with* $|t_n| = 2(p^n - 1)$, *and*

(c) $\eta_L\colon V \to VT$ *is the standard inclusion and* $\varepsilon\colon VT \to V$ *is defined by* $\varepsilon(t_i) = 0$, $\varepsilon(v_i) = v_i$.

Let $\ell_i \in V \otimes \mathbf{Q}$ *denote the image of* $m_{p^i-1} \in L \otimes \mathbf{Q}$ (*see* A2.1.9). *Then*

(d) $\eta_R\colon V \to VT$ *is determined by* $\eta_R(\ell_n) = \sum_{0\leq i\leq n} \ell_i t_{n-i}^{p^i}$ *where* $\ell_0 = t_0 = 1$,

(e) Δ *is determined by* $\sum_{i,j\geq 0} \ell_i \Delta(t_j)^{p^i} = \sum_{i,k,j\geq 0} \ell_i t_j^{p^i} \otimes t_k^{p^{i+j}}$, *and*

(f) c *is determined by* $\sum_{i,j,k\geq 0} \ell_i t_j^{p^i} c(t_k)^{p^{i+j}} = \sum_{i\geq 0} \ell_i$.

(g) *The forgetful functor from p-typical formal group laws to formal group laws induces a surjection of Hopf algebroids* (Al.1.19) $(L \otimes \mathbf{Z}_{(p)}, LB \otimes \mathbf{Z}_{(p)}) \to (V, VT)$.

Note that (e) and (f) are equivalent to

$$\sum_{i\geq 0}^{F} \Delta(t_i) = \sum_{i,j\geq 0}^{F} t_i \otimes t_j^{p^i} \quad \text{and} \quad \sum_{i,j\geq 0}^{F} t_i c(t_j)^{p^i} = 1,$$

respectively.

It can be shown that unlike (L, LB) (A2.1.16), (V, VT) is not split (A1.1.22).

Proof. Part (a) was proved in A2.1.23, (b) follows from A2.1.23, and (c) is obvious, as is (g).

For (d) let f be a strict isomorphism between p-typical formal group law F and G with logarithms $\log(x)$ and $\operatorname{mog}(x)$, respectively. If $f(x)$ satisfies

$$f^{-1}(x) = \sum_{i\geq 0}^{F} t_i x^{p^i}$$

and

$$\log(x) = \sum_{i\geq 0} \ell_i x^{p^i}$$

then by definition of η_R

$$\operatorname{mog}(x) = \sum_{i\geq 0} \eta_R(\ell_i) x^{p^i}.$$

We have (see the proof of A2.1.16)

$$\begin{aligned}\operatorname{mog}(x) = \log(f^{-1}(x)) &= \log\left(\sum_{i\geq 0}^{F} t_i x^{p^i}\right)\\ &= \sum_{i\geq 0} \log(t_i x^{p^i}) = \sum_{i,j\geq 0} \ell_i t_j^{p^i} x^{p^{i+j}}\end{aligned}$$

and (d) follows.

For (e) let $F \xrightarrow{f_1} G \xrightarrow{f_2} H$ be strict isomorphisms of p-typical formal group laws with

$$f_1^{-1}(x) = \sum_{i\geq 0}^{F} t_i' x^{p^i} \quad \text{and} \quad f_2^{-1}(x) = \sum_{j\geq 0}^{G} t_j'' x^{p^j}.$$

If we set $f = f_2 \circ f_1$, with

$$f^{-1}(x) = \sum_{i\geq 0}^{F} t_i x^{p^i}$$

then a formula for t_i in terms of t_i' and t_i'' will translate to a formula for $\Delta(t_i)$.

We have

$$\begin{aligned}f^{-1}(x) = f_1^{-1}(f_2^{-1}(x)) &= f_1^{-1}\left(\sum_{j\geq 0}^{G} t_j'' x^{p^j}\right)\\ &= \sum_{j}^{F} f_1^{-1}(t_j'' x^{p^j}) = \sum_{i,j}^{F} t_i'(t_j'' x^{p^j})^{p^i}.\end{aligned}$$

This gives

$$\sum_{i}^{F} \Delta(t_i) = \sum_{i,j}^{F} t_i \otimes t_j^{p^i}$$

as claimed.

For (f) let $f\colon F \to G$ be as above. Then

$$f(x) = \sum\nolimits^G c(t_j)x^{p^j}$$

so

$$\begin{aligned} x = f^{-}1(f(x)) &= f^{-1}\Bigl(\sum_j{}^G c(t_j)x^{p^j}\Bigr) \\ &= \sum_j{}^F f^{-1}(c(t_j)x^{p^j}) = \sum_{i,j}{}^F t_i(c(t_j)x^{p^j})^{p^i} \end{aligned}$$

setting $x = 1$ gives (f). □

Our only remaining task is to prove Lazard's comparison lemma A2.1.12. The proof below is due to Fröhlich [**1**]. The lemma states that if F and G are formal group laws with $F \equiv G \mod (x,y)^n$ then

$$F \equiv G + aC_n(x,y) \mod (x,y)^{n+1},$$

where

$$C_n(x,y) = \begin{cases} \dfrac{(x+y)^n - x^n - y^n}{p} & \text{if } n = p^k \text{ for some prime } p \\ (x+y)^n - x^n - y^n & \text{otherwise.} \end{cases}$$

Let $\Gamma(x,y)$ be the degree n component of $F - G$.

A2.1.28. Lemma. *$\Gamma(x,y)$ above is a homogeneous polynomial satisfying*
(i) $\Gamma(x,y) = \Gamma(y,x)$,
(ii) $\Gamma(x,0) = \Gamma(0,x) = 0$,
(iii) $\Gamma(x,y) + \Gamma(x+y,z) = \Gamma(x,y+z) + \Gamma(y,z)$.

Proof. Parts (i) and (ii) follow immediately A2.1.1(ii) and (i), respectively. For (iii) let $G(x,y) = x + y + G'(x,y)$. Then mod $(x,y,z)^{n+1}$ we have

$$\begin{aligned} F(F(x,y),z) &\equiv G(F(x,y),z) + \Gamma(F(x,y),z) \\ &\equiv F(x,y) + z + G'(F(x,y),z) + \Gamma(x+y,z) \\ &\equiv G(x,y) + \Gamma(x,y) + z + G'(G(x,y),z) + \Gamma(x+y,z) \\ &\equiv G(G(x,y),z) + \Gamma(x,y) + \Gamma(x+y,z). \end{aligned}$$

Similarly,

$$F(x,F(y,z)) = G(x,G(y,z)) + \Gamma(x,y+z) + \Gamma(y,z)$$

from which (iii) follows. □

It suffices to show that any such Γ must be a multiple of C_n.

A2.1.29. Lemma. *Let R be a field of characteristic $p > 0$. Then any $\Gamma(x,y)$ over R as above is a multiple of $C_n(x,y)$.*

Proof. It is easy to verify that C_n satisfies the conditions of A2.1.28, so it suffices to show that the set of all such Γ is one-dimensional vector space. Let $\Gamma(x,y) = \sum a_i x^i y^{n-i}$. Then from A2.1.28 we have

$$a_0 = a_n = 0, \quad a_i = a_{n-i},$$

and

$$a_i\binom{n-i}{j} = a_{i+j}\binom{i+j}{j} \quad \text{for } 0 < i, i+j < n. \tag{A2.1.30}$$

The case $n = 1$ is trivial so we write $n = sp^k$ with either $s = p$ or $s > 1$ and $s \not\equiv 0 \mod p$. We will prove the lemma by showing $a_i = 0$ if $i \not\equiv 0 \mod (p^k)$ and that a_{cp^k} is a fixed multiple of a_{p^k}.

If $i \not\equiv 0 \mod (p^k)$ we can assume by symmetry that $i < (s-1)p^k$ and write $i = cp^k - j$ with $0 < c < s$ and $0 < j < p^k$. Then A2.1.30 gives

$$a_i \binom{(s-c)p^k + j}{j} = a_{cp^k} \binom{cp^k}{j},$$

i.e., $a_i = 0$.

To show a_{cp^k} is determined by a_{p^k} for $c < s$ let $i = p^k$ and $j = (c-1)p^k$. Then A2.1.30 gives

$$a_{p^k} \binom{(s-1)p^k}{(c-1)p^k} = a_{cp^k} \binom{cp^k}{(c-1)p^k},$$

i.e.,

$$a_{p^k} \binom{s-1}{c-1} = a_{cp^k} c.$$

This determines a_{cp^k} provided $c \not\equiv 0 \mod (p)$. Since $c < s$ we are done for the case $s = p$. Otherwise $a_{cp^k} = a_{(s-c)p^k}$ by symmetry and since $s \not\equiv 0 \mod (p)$ either c or $s - c$ is $\not\equiv 0 \mod (p)$. $\square$

Note that A2.1.29 is also true for fields of characteristic 0; this can be deduced immediately from A2.1.30. Alternatively, we have already proved A2.1.12, which is equivalent to A2.1.29, for torsion-free rings.

The proof of A2.1.29 is the last hard computation we have to do. Now we will prove the analogous statement for $R = \mathbf{Z}/(p^m)$ by induction on m. We have

$$\Gamma(x, y) = aC_n(x, y) + p^{m-1}\Gamma'(x, y),$$

where Γ' satisfies A2.1.28 mod p. Hence by A2.1.29 $\Gamma'(x, y) = bC_n(x, y)$ so

$$\Gamma(x, y) = (a + bp^{m-1})C_n(x, y)$$

as claimed.

To prove A2.1.29 (and hence A2.1.12) for general R note that the key ingredient A2.1.30 involves only the additive structure of R; i.e., we only have to compute in a finitely generated abelian group A containing the coefficient of Γ. We have to show that symmetry and A2.1.30 imply that the coefficients a_i are fixed in relation to each other as are the coefficients of C_n. We have shown that this is true for $A = Z$ (from the case $R = \mathbf{Q}$) and $A = \mathbf{Z}/(p^m)$. It is clear that if it is true for groups A_1 and A_2 then it is true for $A_1 \oplus A_2$, so it is true for all finitely generated abelian groups A. This completes the proof of A2.1.12.

2. Classification and Endomorphism Rings

In order to proceed further we need an explicit choice of the generators v_n. The first such choice was given by Hazewinkel [**2**], which was circulating in preprint form six years before it was published. The same generators for $p = 2$ were defined earlier still by Liulevicius [**3**]. A second choice, which we will use, was given by Araki [**1**].

Hazewinkel's generators are defined by

(A2.2.1) $$p\ell_n = \sum_{0 \le i < n} \ell_i v_{n-i}^{p^i}$$

which gives, for example,

$$\ell_1 = \frac{v_1}{p}, \qquad \ell_2 = \frac{v_2}{p} + \frac{v_1^{1+p}}{p^2},$$
$$\ell_3 = \frac{v_3}{p} + \frac{v_1 v_2^p + v_2 v_1^{p^2}}{p^2} + \frac{v_1^{1+p+p^2}}{p^3}.$$

Of course, it is nontrivial to prove that these v_n are contained in and generate V.

Araki's formula is nearly identical,

$$p\ell_n = \sum_{0\le i\le n} \ell_i v_{n-i}^{p^i} \tag{A2.2.2}$$

where $v_0 = p$. These v_n can be shown to agree with Hazewinkel's mod (p). They give messier formulas for ℓ_n, e.g.,

$$\ell_1 = \frac{v_1}{p-p^p}, \quad (p-p^{p^2})\ell_2 = v_2 + \frac{v_1^{1+p}}{p-p^p},$$
$$(p-p^{p^3})\ell_3 = v_3 + \frac{v_1 v_2^p}{p-p^p} + \frac{v_2 v_1^{p^2}}{p-p^{p^2}} + \frac{v_1^{1+p+p^2}}{(p-p^p)(p-p^{p^2})},$$

but a nicer formula (A2.2.6) for η_R.

A2.2.3. THEOREM (Hazewinkel [**2**], Araki [**1**]). *The sets of elements defined by* A2.2.1 *and* A2.2.2 *are contained in and generate V as a ring, and they are congruent* mod (p).

PROOF. We first show that Araki's elements generate V. Equation A2.2.2 yields

$$\sum_{i\ge 0} p\ell_i x^{p^i} = \sum_{i,j\ge 0} \ell_i v_j^{p^i} x^{p^{i+j}}.$$

Applying exp (the inverse of log) to both sides gives

$$[p]_F(x) = \sum_{i\ge 0}{}^F v_i x^{p^i}, \tag{A2.2.4}$$

which proves the integrality of the v_n, i.e., that $v_n \in V$. To show that they generate V it suffices by A2.1.10 to show $v_n = pu_n\ell_n$ in $QV \otimes \mathbf{Q}$, where u_n is a unit in $\mathbf{Z}_{(p)}$. Reducing A2.2.2 modulo decomposables gives

$$p\ell_n = v_n + \ell_n p^{p^n}$$

so the result follows.

We now denote Hazewinkel's generators of (A2.2.1) by w_i. Then A2.2.1 gives

$$p\log x - px = \sum_{i>0} \log w_i x^{p^i}$$

or

$$px = p\log x - \sum_{i>0} \log w_i x^{p^i}.$$

Exponentiating both sides gives

$$\begin{aligned}\exp px &= [p](x) -_F \sum_{i>0}^F w_i x^{p^i} \\ &= px +_F \sum_{i>0}^F v_i x^{p^i} -_F \sum_{i>0}^F w_i x^{p^i} \text{ by A2.2.4.}\end{aligned}$$

If we can show that $(\exp px)/p$ is integral then the above equation will give

$$\sum_{i>0}^F v_i x^{p^i} \equiv \sum_{i>0}^F w_i x^{p^i} \mod (p)$$

and hence $v_i \equiv w_i \mod (p)$ as desired.

To show that $(\exp px)/p$ is integral simply note that its formal inverse is $(\log px)/p = \sum \ell_i p^{p^i-1} x^{p^i}$, which is integral since $p^i \ell_i$ is. □

A2.2.5. LEMMA. *Let F by a p-typical formal group law over a torsion-free $\mathbf{Z}_{(p)}$-algebra. Then for $p > 2$, $[-1]_F(x) = -x$, and for $p = 2$ it is*

$$[-1]_F(x) = -x -_F \sum_{i>0}^F w_i x^{2^i} = x -_F \sum_{i\geq 0}^F v_i x^{2^i}$$

where w_i is the Hazewinkel generator of (A2.2.1) and v_i is the Araki generator of (A2.2.2).

PROOF. The series $[-1]_F(x)$ is characterized by

$$\log([-1]_F(x)) = -\log x.$$

For $p > 2$, the p-typical logarithm is an odd function, so $\log(-x) - \log(x)$. For $p = 2$ we need to compute the logarithm of the indicated series, first the one given in terms of the Hazewinkel generators. We have

$$\begin{aligned}\log\left(-x -_F \sum_{i>0}^F w_i x^{2^i}\right) &= \log(-x) - \sum_{j>0} \log(w_j x^{2^j}) \\ &= \sum_{i\geq 0} \ell_i (-x)^{2^i} - \sum_{\substack{i\geq 0\\ j>0}} \ell_i (w_j x^{2^j})^{2^i} \\ &= -x + \sum_{i>0} \ell_i x^{2^i} - \sum_{\substack{i\geq 0\\ j>0}} \ell_i w_j^{2^i} x^{2^{i+j}} \\ &= -x + \sum_{i>0} \ell_i x^{2^i} - \sum_{i>0} \left(\sum_{0\leq j<i} \ell_j w_{i-j}^{2^j}\right) x^{2^i} \\ &= -x + \sum_{i>0} \ell_i x^{2^i} - \sum_{i>0} 2\ell_i x^{2^i} \\ &= -\log(x).\end{aligned}$$

The result follows.

For the formula in terms of the Araki generators, note that by (A2.2.4), the expression is the same as $[1]_F x -_F [2]_F(x) = [-1]_F(x)$. □

From now on v_n will denote the Araki generator defined by A2.2.2 or equivalently by A2.2.4. The following formula for $\eta_R(v_n)$ first appeared in Ravenel [**1**], where it was stated mod (p) in terms of the Hazewinkel generators; see also Moreira [**2**].

A2.2.6. THEOREM. *The behavior of η_R on v_n is defined by*

$$\sum_{i,j\geq 0}^F t_i\eta_R(v_j)^{p^i} = \sum_{i,j\geq 0}^F v_i t_j^{p^i}.$$

PROOF. Applying η_R to A2.2.2 and reindexing we get by A2.1.27(d)

$$\sum p\ell_i t_j^{p^i} = \sum \ell_i t_j^{p^i}\eta_R(v_k)^{p^{i+j}}.$$

Substituting A2.2.2 on the left-hand side and reindexing gives

$$\sum \ell_i v_j^{p^i} t_k^{p^{i+j}} = \sum \ell_i t_j^{p^i}\eta_R(v_k)^{p^{i+j}}.$$

Applying the inverse of log to this gives the desired formula. $\square$

This formula will be used to prove the classification theorem A2.2.12 below. Computational corollaries of it are given in Section 4.3.

We now turn to the classification in characteristic p. We will see that formal group laws over a field are characterized up to isomorphism over the separable algebraic closure by an invariant called the height (A2.2.8). In order to define it we need

A2.2.7. LEMMA. *Let F be a formal group law over a commutative $\mathbf{F}_p$-algebra R and let $f(x)$ be a nontrivial endomorphism of F (A2.1.5). Then for some n, $f(x) = g(x^{p^n})$ with $g'(0)\neq 0$. In particular f has leading term ax^{p^n}.*

When R is a perfect field K, we can replace $g(x^{p^n})$ by $h(x)^{p^n}$ with $h'(0)\neq 0$.

For our immediate purpose we only need the statement about the leading term, which is easier to prove. The additional strength of the lemma will be needed below (A2.2.19). The argument we use can be adapted to prove a similar statement about a homomorphism to another formal group law G.

PROOF. Suppose inductively we have shown that $f(x) = f_i(x^{p^i})$, this being trivial for $i = 0$, and suppose $f_i'(0) = 0$, as otherwise we are done. Define $F^{(i)}(x,y)$

$$F(x,y)^{p^i} = F^{(i)}(x^{p^i}, y^{p^i}).$$

It is straightforward to show that $F^{(i)}$ is also a formal group law. Then we have

$$\begin{aligned} f_i(F^{(i)}(x^{p^i}, y^{p^i})) &= f_i(F(x,y)^{p^i}) = f(F(x,y)) \\ &= F(f(x), f(y)) = F(f_i(x^{p^i}), f_i(y^{p^i})) \end{aligned}$$

so

$$f_i(F^{(i)}(x,y)) = F(f_i(x), f_i(y)).$$

Differentiating with respect to y and setting $y = 0$ we get

$$f_i'(F^{(i)}(x,0))F_2^{(i)}(x,0) = F_2(f_i(x), f_i(0))f_i'(0).$$

Since $f_i'(0) = 0$, $F_2^{(i)}(x,0)\neq 0$, and $F^{(i)}(x,0) = x$, this gives us

$$f_i'(x) = 0 \quad \text{so} \quad f_i(x) = f_{i+1}(x^p).$$

We repeat this process until we get an $f_n(x)$ with $f_n'(0)\neq 0$ and set $g = f_n$.

The statement about the perfect field case follwos from the fact taht each coeffcient of g has a p^nth root. $\square$

A2.2.8. Definition. *A formal group law F over a commutative $\mathbf{F}_p$-algebra R has* height n *if $[p]_F(x)$ has leading term ax^{p^n}. If $[p]_F(x) = 0$ then F has* height ∞.

A2.2.9. Lemma. *The height of a formal group law is an isomorphism invariant.*

Proof. Let f be an isomorphism from F to G. Then

$$f([p]_F(x)) = [p]_G(f(x));$$

since $f(x)$ has leading term ux for u a unit in R and the result follows. $\square$

A2.2.10. Examples. Just for fun we will compute the heights of the mod (p) reductions of the formal group laws in A2.1.4.

(a) $[p]_F(x) = 0$ for all p so F has height ∞.

(b) $[p]_F(x) = u_{p-1}x^p$ so F has height 1.

(c) As remarked earlier, F is isomorphic over $\mathbf{Z}_{(2)}$ to the additive formal group law, so its height at $p = 2$ is ∞. Its logarithm is

$$\sum_{i\geq 0} \frac{x^{2i+1}}{2i+1}$$

so for each odd prime p we have $\ell_1 = m_{p-1} = 1/p$, so $v_1 \neq 0 \mod p$ by A2.2.2, so the height is 1 by A2.2.4 and A2.2.8.

(d) Since F is not defined over $\mathbf{Z}_{(2)}$ (as can be seen by expanding it through degree 5) it does not have a mod 2 reduction. To compute its logarithm we have

$$F_2(x, 0) = \sqrt{1 - x^4}$$

so by A2.1.6

$$\begin{aligned}
\log(x) &= \int_0 \frac{dt}{\sqrt{1 - t^4}} \\
&= \sum_{i\geq 0} \binom{-1/2}{i} \frac{(-1)^i x^{4i+1}}{4i+1} \\
&= \sum_{i\geq 0} \binom{(2i-1)/2}{i} \frac{x^{4i+1}}{4i+1} \\
&= \sum_{i\geq 0} \frac{1\cdot 3\cdot 5\cdot \cdots \cdot (2i-1)x^{4i+1}}{2^i i!(4i+1)} \\
&= \sum_{i\geq 0} \frac{(2i)!\, x^{4i+1}}{2^{2i}(i!)^2(4i+1)}.
\end{aligned}$$

Now if $p \equiv 1 \mod (4)$, we find that $\ell_1 = m_{p-1}$ is a unit (in $\mathbf{Z}_{(p)}$) multiple of $1/p$, so as in (c) the height is 1. However, if $p \equiv -1 \mod (4)$, $v_1 = \ell_1 = 0$ so so the height is at least 2. We have

$$\ell_2 = m_{p^2-1} = \frac{(2i)!}{4(i!)^2p^2} \quad \text{where} \quad i = \frac{p^2-1}{4}.$$

Since

$$\frac{p^2-1}{2} = \frac{p(p-1)}{2} + \frac{p-1}{2},$$

$(2i)!$ is a unit multiple of $p^{(p-1)/2}$; since

$$\frac{p^2-1}{4} = p\left(\frac{p-3}{4}\right) + \frac{3p-1}{4}$$

$(i!)$ is a unit multiple of $p^{(p-3)/4}$. It follows that ℓ_2 is a unit multiple of $1/p$, so $v_2 \not\equiv 0 \mod p$ and the height is 2.

It is known that the formal group law attached to a nonsingular elliptic curve always has height 1 or 2. (See Corollary 7.5 of Silverman [**1**]).

Now we will specify a formal group law of height n for each n.

A2.2.11. DEFINITION. $F_\infty(x,y) = x+y$. *For a natural number h let F_n be the p-typical formal group law (of height n) induced by the homomorphism $\theta\colon V \to R$* (A2.1.25) *defined by $\theta(v_n)=1$ and $\theta(v_i)=0$ for $i \neq n$.*

A2.2.12. THEOREM (Lazard [**2**]). *Let K be a separably closed field of characteristic $p>0$. A formal group law G over K of height n is isomorphic to F_n.*

PROOF. By Cartier's theorem (A2.1.18) we can assume G is p-typical (A2.1.22) and hence induced by a homomorphism $\theta\colon V \to K$ (A2.1.24). If $n=\infty$ then by A2.2.4 $\theta(v_n)=0$ for all n and $G=F_\infty$. For n finite we have $\theta(v_i)=0$ for $i<n$ and $\theta(v_n)\neq 0$. Let $F=F_n$. We want to construct an isomorphism $f\colon F\to G$ with $f^{-1}(x) = \sum_{i\geq 0}^F t_i x^{p^i}$. It follows from A2.2.6 that these t_i must satisfy

$$\text{(A2.2.13)} \qquad \sum_{i,j}{}^{F} t_i\theta(v_j)^{p^i} x^{p^{i+j}} = \sum_{j}{}^{F} t_j^{p^n} x^{p^{n+j}}$$

since the homomorphism from V inducing F is given in A2.2.11, and the $\eta_R(v_j)$ in A2.2.6 correspond to $\theta(v_j)$. Here we are not assuming $t_0=1$; the proof of A2.2.6 is still valid if $t_0\neq 1$.

Equating the coefficient of x^{p^n} in A2.2.13, we get $t_0\theta(v_n) = t_0^{p^n}$, which we can solve for t_0 since K is separably closed. Now assume inductively that we have solved A2.2.13 for $t_0, t_1, \ldots, t_{i-1}$. Then equating coefficients of $x^{p^{i+n}}$ gives

$$t_i\theta(v_n)^{p^i} + c = t_i^{p^n}$$

for some $c\in K$. This can also be solved for t_i, completing the proof. $\square$

Our last objective in this section is to describe the endomorphism rings of the formal group laws F_n of A2.2.11.

A2.2.14. LEMMA. *Let F be a formal group law over a field K of characteristic $p>0$ and let E be the set of endomorphisms of F.*

(a) *E is a ring under composition and formal sum, i.e., the sum of two endomorphisms $f(x)$ and $g(x)$ is $f(x)+_F g(x)$.*

(b) *E is a domain.*

(c) *E is a $\mathbf{Z}_p$-algebra (where $\mathbf{Z}_p$ denotes the p-adic integers) which is a free $\mathbf{Z}_p$-module if F has finite height, and an $\mathbf{F}_p$-vector space if F has infinite height.*

PROOF.

(a) We need to verify the distributive law for these two operations. Let $f(x)$, $g(x)$, and $h(x)$ be endomorphisms. Then

$$f(g(x)+_F f(x)) = f(g(x)) +_F f(h(x))$$

so

$$f(g+h) = (fg) + (fh) \quad \text{in } E.$$

Similarly,

$$(g +_F h)(f(x)) = g(f(x)) +_F h(f(x))$$

so

$$(g+h)f = (gf) + (hf) \quad \text{in } E.$$

(b) Suppose $f(x)$ and $g(x)$ having leading terms ax^{p^n} and bx^{p^n}, respectively, with $a, b \neq 0$ (A2.2.6). Then $f(g(x))$ has leading term $ab^{p^m} x^{p^{m+n}}$, so $fg \neq 0$ in E.

(c) We need to show that $[a]_F(x)$ is defined for $a \in \mathbf{Z}_p$. We can write $a = \sum a_i p^i$ with $a_i \in \mathbf{Z}$. Then we can define

$$[a]_F(x) = \sum\nolimits^F [a_i]_F([p^i]_F(x))$$

because the infinite formal sum on the right is in $K[[x]]$ since $[p^i]_F(x) \equiv 0$ modulo x^{p^i}. If $h < \infty$ then $[a]_F(x) \neq 0$ for all $0 \neq a \in \mathbf{Z}_p$, so E is torsion-free by (b). If $h = \infty$ then $[p]_F(x) = 0$ so E is an $\mathbf{F}_p$-vector space. □

Before describing our endomorphism rings we need to recall some algebra.

A2.2.15. LEMMA. *Let p be a prime and $q = p^i$ for some $i > 0$.*

(a) *There is a unique field $\mathbf{F}_p$ with q elements.*

(b) *Each $x \in \mathbf{F}_q$ satisfies $x^q - x = 0$.*

(c) *$\mathbf{F}_{p^m}$ is a subfield of $\mathbf{F}_{p^n}$ iff $m \mid n$. The extension is Galois with Galois group $Z/(m/n)$ generated by the Frobenius automorphism $x \mapsto x^{p^m}$.*

(d) *$\overline{\mathbf{F}}_p$ the algebraic closure of $\mathbf{F}_p$ and of each $\mathbf{F}_q$, is the union of all the $\mathbf{F}_q$. Its Galois group is $\widehat{\mathbf{Z}} = \varprojlim \mathbf{Z}/(m)$, the profinite integers, generated topologically by the Frobenius automorphism $x \mapsto x^p$. The subgroup $m\mathbf{Z}$ of index m is generated topologically by $x \mapsto x^{p^m}$ and fixes the field $\mathbf{F}_{p^m}$.* □

A proof can be found, for example, in Lang [1, Section VII.5]

Now we need to consider the Witt rings $W(\mathbf{F}_q)$, which can be obtained as follows. Over $\mathbf{F}_p$ the polynomial $x^q - x$ is the product of irreducible factors of degrees at most n (where $q = p^n$) since it splits over $\mathbf{F}_q$, which is a degree n extension of $\mathbf{F}_p$. Let $h(x) \in \mathbf{Z}_p[x]$ be a lifting of an irreducible factor of degree n of $x^q - x$. Then let $W(\mathbf{F}_q) = \mathbf{Z}_p[x]/(h(x))$. It is known to be independent of the choices made and to have the following properties.

A2.2.16. LEMMA. (a) *$W(\mathbf{F}_q)$ is a $\mathbf{Z}_p$-algebra and a free $\mathbf{Z}_p$-module of rank n, where $q = p^n$ [e.g., $W(\mathbf{F}_p) = \mathbf{Z}_p$].*

(b) *$W(\mathbf{F}_q)$ is a complete local ring with maximal ideal (p) and residue field $\mathbf{F}_q$.*

(c) *Each $w \in W(\mathbf{F}_q)$ can be written uniquely as $w = \sum_{i \geq 0} w_i p^i$ with $w_i^q - w_i = 0$ for each i.*

(d) *The Frobenius automorphism of $\mathbf{F}_q$ lifts to an automorphism σ of $W(\mathbf{F}_q)$ defined by*

$$w^\sigma = \sum_{i \geq 0} w_i^p p^i.$$

σ generates the Galois group $\mathbf{Z}/(n)$ of $W(\mathbf{F}_q)$ over $\mathbf{Z}_p$.

(e) $W(\mathbf{F}_q) = \varprojlim W(\mathbf{F}_q)/(p^i)$, *so it is a compact topological ring.*

(f) *The group of units* $W(\mathbf{F}_q)^\times$ *is isomorphic to* $W(\mathbf{F}_q) \oplus \mathbf{F}_q^\times$, *where* $\mathbf{F}_q^\times \cong \mathbf{Z}/(q-1)$, *for* $p > 2$, *and to* $W(\mathbf{F}_q) \oplus \mathbf{F}_q^\times \oplus \mathbf{Z}/(2)$ *for* $p = 2$, *the extra summand being generated by* -1.

(g) $W(\mathbf{F}_q) \otimes \mathbf{Q} = \mathbf{Q}_p[x]/(h(x))$, *the unramified degree* n *extension of* $\mathbf{Q}_p$, *the field of* p*-adic numbers.*

A proof can be found in Mumford [**1**, Lecture 26] and in Serre [**1**, Section 11.5.6]. We will sketch the proof of (f). For $p > 2$ there is a short exact sequence

$$1 \to W(\mathbf{F}_q) \xrightarrow{i} W(\mathbf{F}_q)^\times \xrightarrow{j} \mathbf{F}_q^\times \to 1$$

where j is mod (p) reduction and $i(w) = \exp pw = \sum_{i\geq 0}(pw)^i/i!$ [this power series converges in $W(\mathbf{F}_q)$]. To get a splitting $\mathbf{F}_q^\times \to W(\mathbf{F}_q)^\times$ we need to produce $(q-1)$th roots of unity in $W(\mathbf{F}_q)$, i.e., roots of the equation $x^q - x = 0$. [This construction is also relevant to (c).]

These roots can be produced by a device known as the Teichmüller construction. Choose a lifting u of a given element in $\mathbf{F}_q$, and consider the sequence $\{u, u^q, u^{q^2}, \dots\}$. It can be shown that it converges to a root of $x^q - x = 0$ which is independent of the choice of u.

For $p = 2$ the power series $\exp 2w$ need not converge, so we consider instead the short exact sequence

$$1 \to W(\mathbf{F}_q) \xrightarrow{i} W(\mathbf{F}_q)^\times \xrightarrow{j} W(\mathbf{F}_q)/(4)^\times \to 1,$$

where j is reduction mod (4) and $i(w) = \exp 4w$, which always converges. This sequence does not split. We have $W(\mathbf{F}_q)/(4)^\times \cong \mathbf{F}_q \oplus \mathbf{F}_q^\times$. Since $W(\mathbf{F}_q) \otimes \mathbf{Q}$ is a field, $W(\mathbf{F}_q)^\times$ can have no elements of order 2 other than ± 1, so the other elements of order 2 in $W(\mathbf{F}_q)/(4)^\times$ lift to elements in $W(\mathbf{F}_q)^\times$ with nontrivial squares.

Next we describe the noncommutative $\mathbf{Z}_p$-algebra E_n, which we will show to be isomorphic to the endomorphism ring of F_n, for finite n.

A2.2.17. LEMMA. *Let* E_n *be the algebra obtained from* $W(\mathbf{F}_q)$ *by adjoining an indeterminate* S *and setting* $S^n = p$ *and* $Sw = w^\sigma S$ *for* $w \in W(\mathbf{F}_q)$. *Then*

(a) E_n *is a free* $\mathbf{Z}_p$ *module of rank* n^2.

(b) *Each element* $e \in E_n$ *can be expressed uniquely as* $\sum_{i\geq 0} e_i S^i$ *with* $e_i^q - e_i = 0$.

(c) E_n *is generated as a* $\mathbf{Z}_p$*-algebra by* S *and a primitive* $(q-1)$*th root of unity* ω *with relations* $S^n - p = 0$, $S\omega = \omega^p S = 0$, *and* $h(w) = 0$, *where* $h(x)$ *is an irreducible degree* n *factor of* $x^q - x$ *over* $\mathbf{Z}_p$.

(d) E_n *is the maximal order in* $D_n = E_n \otimes \mathbf{Q}$ *which is a division algebra with center* $\mathbf{Q}_p$ *and invariant* $1/n$.

The proofs of (a), (b), and (c) are elementary. To see that D_n is a division algebra, note that any element in D_n can be multiplied by some power of S to give an element in E_n which is nonzero mod (S). It is elementary to show that such an element is invertible.

The invariant referred to in (d) is an element in $\mathbf{Q}/\mathbf{Z}$ which classifies division algebras over $\mathbf{Q}_p$. Accounts of this theory are given in Serre [**1**, Chapters XII and XIII] Cassels and Fröhlich [**1**, pp. 137–139], Hazewinkel [**1**, Sections 20.2.16 and 23.1.4]. We remark that for $0 < i < n$ and i prime to n a division algebra with invariant i/n has a description similar to that of D_n except that S^n is p^i instead of p.

Our main results on endomorphism rings are as follows.

A2.2.18. THEOREM (Dieudonné [**1**] and Lubin [**1**]). *Let K be a field of characteristic p containing $\mathbf{F}_q$, with $q = p^n$. Then the endomorphism ring of the formal group law F_n* (A2.2.11) *over K is isomorphic to E_n. The generators ω and S* [A2.2.17(c)] *correspond to endomorphisms $\overline{\omega}x$ and x^p, respectively.*

A2.2.19. THEOREM. *Let R be a commutative $\mathbf{F}_p$-algebra. Then the endomorphism ring of the additive formal group law F_∞ over R is the noncommutative power series ring $R\langle\langle S\rangle\rangle$ in which $Sa = a^pS$ for $a \in R$. The elements a and S correspond to the endomorphisms ax and x^p, respectively.*

PROOF OF A2.2.19. An endomorphism $f(x)$ of F_∞ must satisfy $f(x+y) = f(x) + f(y)$. This is equivalent to $f(x) = \sum_{i\geq 0} a_i x^{p^i}$ for $a_i \in R$. The relation $Sa = a^pS$ corresponds to $(ax)^p = a^px^p$. $\square$

There is an amusing connection between this endomorphism ring and the Steenrod algebra. Theorem A2.2.19 implies that the functor which assigns to each commutative $\mathbf{F}_p$-algebra R the strict automorphism group of the additive formal group law is represented by the ring

$$P = \mathbf{F}_p[a_1, a_1, \dots]$$

since $a_0 = 1$ in this case. The group operation is represented by a coproduct $\Delta\colon P \to P \otimes P$. To compute Δa_n let $f_1(x) = \sum a'_j x^{p^k}$, $f_2(x) = \sum a''_k x^{p^k}$, and $f(x) = f_2(f_1(x)) = \sum a_i x^{p^i}$ with $a'_0 = a''_0 = a_0 = 1$. Then we have

$$f(x) = \sum a''_k \left(\sum a'_j x^{p^j}\right)^{p^k} = \sum a''_k (a'_j)^{p^j} x^{p^{j+k}}.$$

It follows that

$$\Delta a_n = \sum_{0\leq i\leq n} a_{n-i}^{p^i} \otimes a_i \quad \text{with } a_0 = 1,$$

i.e., P is isomorphic to the dual of the algebra of Steenrod reduced powers.

Before proving A2.2.18 we need an improvement of A2.2.7. I am grateful to Gerd Laures for finding an error in an earlier version of the following.

A2.2.20. LEMMA. *Let F be a p-typical formal group law over a perfect field K of characteristic $p > 0$, and let $f(x)$ be an endomorphism of F. Then*

$$f(x) = \sum_{i\geq 0}^{F} a_i x^{p^i}$$

for some $a_i \in K$.

PROOF. As in the proof of A2.2.7, we define for each integer n a power series $F^{(n)} \in K[[x, y]]$ by replacing each coefficient by its p^nth power. It is easily seen that it is a p-typical formal group law when F is.

Then we have

$$\begin{aligned} f(F(x,y)) &= h(F(x,y))^{p^n} \\ F(f(x), f(y)) &= F(h(x)^{p^n}, h(y)^{p^n}) = \left(F^{(-n)}(h(x), h(y))\right)^{p^n}. \end{aligned}$$

These two expressions are equal because f is an endomorphism, so

$$h(F(x,y)) = F^{(-n)}(h(x), h(y))$$

and h (and hence h^{-1}) is an isomorphism between two p-typical formal group laws. The result then follows from A2.1.26. $\square$

A2.2.18 will follow easily from the following.

A2.2.21. LEMMA. *Let $E(F_n)$ be the endomorphism ring of F_n (A2.2.11) over a perfect field K containing $\mathbf{F}_q$ where $q = p^n$. Then*

(a) *if $f(x) = \sum^{F_n} a_i x^{p^i}$ is in $E(F_n)$, then each $a_i \in \mathbf{F}_q$;*
(b) *for $a \in \mathbf{F}_q$, $ax \in E(F_n)$;*
(c) *$x^p \in E(F_n)$; and*
(d) *$E(F_n)/(p) = E_n/(p) = \mathbf{F}_q\langle S\rangle/(S^n)$ with $Sa = a^p S$.*

PROOF. (a) By the definition of F_n (A2.2.11) and A2.2.15 we have

$$[p](x) = x^{p^n}. \tag{A2.2.22}$$

Any endomorphism f commutes with $[p]$ so by A2.2.20 we have

$$[p](f(x)) = [p]\left(\sum\nolimits^{F_n} a_i x^{p^i}\right) = \sum\nolimits^{F_n} [p](a_i x^{p^i}) = \sum\nolimits^{F_n} a_i^{p^n} x^{p^{i+n}}.$$

This must equal

$$f([p](x)) = \sum\nolimits^{F_n} a_i([p](x))^{p^i} = \sum\nolimits^{F_n} a_i x^{p^{i+n}}.$$

Hence $a_i^{p^n} = a_i$ for all i and $a_i \in \mathbf{F}_q$.

(b) It suffices to prove this for $K = \mathbf{F}_q$. $\mathbf{F}_n$ can be lifted to a formal group law $\widetilde{F}_n$ over $W(\mathbf{F}_q)$ (A2.2.16) by the obvious lifting of $\theta\colon V \to \mathbf{F}_q$, to $W(\mathbf{F}_q)$. It suffices to show that ωx is an endomorphism of $\widetilde{F}_n$ if $\omega^q - \omega = 0$. By A2.2.2 $\widetilde{F}_n$ has a logarithm of the form

$$\log(x) = \sum a_i x^{q^i}$$

so $\log(\omega x) = \omega \log(x)$ and ωx is an endomorphism.

(c) This follows from the fact that F_n is defined over $\mathbf{F}_p$, so $F_n(x^p, y^p) = F_n(x,y)^p$.

(d) By A2.2.22, (b) and (c), $f(x) \in pE(F_n)$ iff $a_i = 0$ for $i < n$. It follows that for $f(x), g(x) \in E(F_n)$, $f \equiv g \mod (p)E(F_n)$ iff $f(x) \equiv g(x) \mod (x^q)$. Now our lifting $\widetilde{F}_n$ of F_n above has $\log x \equiv x \mod (x^q)$, so $F_n(x,y) \equiv x + y \mod (x,y)^q$. It follows that $E(F_n)/(p)$ is isomorphic to the corresponding quotient of $E(F_\infty)$ over $\mathbf{F}_q$, which is as claimed by A2.2.18. $\square$

PROOF OF A2.2.18. . By A2.2.17(c) E_n is generated by ω and S. The corresponding elements are in $E(F_n)$ by A2.2.21(b) and (c). The relation $S\omega = \omega^p S$ corresponds as before to the fact that $(\overline{\omega}x)^p = \overline{\omega}^p x^p$, where $\overline{\omega}$ is mod (p) reduction of ω. Hence we have a homomorphism $\lambda\colon E_n \to E(F_n)$ which is onto by A2.2.20. We know [A2.2.14(c)] that $E(F_n)$ is a free $\mathbf{Z}_p$-module. It has rank n^2 by A2.2.21(d), so λ is 1-1 by A2.2.17(a). $\square$

APPENDIX A3

Tables of Homotopy Groups of Spheres

In this appendix we collect most of the known values of the stable homotopy groups of spheres for the primes 2, 3, and 5. Online graphic displays of these are given by Hatcher [**1**]. The results of Toda [**6**] on unstable homotopy groups are shown in Table A3.6. A table of unstable 3-primary homotopy groups up to dimension 80 can be found in Toda [**8**].

Extensive online charts of various Ext groups over the Steenrod algebra have been provided by Nassau [**1**] and Bruner [**3**].

In Figs. A3.1a–c we display the classical Adams E_2-term for $p=2$,

$$\mathrm{Ext}_A^{s,t}(\mathbf{Z}/(2), \mathbf{Z}/(2))$$

for $t-s \leq 61$, along the differentials and group extensions. The main reference for the calculation of Ext is Tangora [**1**], which includes a table showing the answer for $t-s \leq 70$. We use his notation for the many generators shown in Ext. His table is preceded by a dictionary (not included here) relating this notation to that of the May spectral sequence, which is his main computational tool.

In our table each basis element is indicated by a small circle. Multiplication by the elements h_0, h_1, and h_2 is indicated, respectively, by vertical lines and lines with slopes 1 and $\frac{1}{3}$. Most multiplicative generators are labeled, but there are a few unlabeled generators due to limitations of space. In each case the unlabeled generator is in the image of the periodicity operator P (denoted by Π in Section 3.4), which sends an element $x \in \mathrm{Ext}^{s,t}$ to the Massey product (Section A1.4)

$$\langle x, h_0^4, h_3 \rangle \in \mathrm{Ext}^{s+4,t+12}.$$

Differentials are indicated by lines with negative slope. For $t-s \leq 20$ these can be derived by combining the calculation of Ext in this range due to May [**1**] with the calculation of the corresponding homotopy groups by Toda [**6**]. For $21 \leq t-s \leq 45$ the results can be found in various papers by Barratt, Mahowald, Milgram, and Tangora and most recently in Bruner [**2**], where precise references to the earlier work can be found.

Differentials in the range $46 \leq t-s \leq 61$ have been computed (tentatively in some cases) by Mahowald (unpublished) and are included here with his kind permission.

Exotic group extensions and some exotic multiplications by h_1 and h_2 are indicated by broken lines with nonnegative slope.

In Fig. A3.2 we display the Adams–Novikov E_2-term for $p=2$ in the range $t-s \leq 39$. The method used is that of Section 4.4, where the calculation is described in detail through dimension 25. The small circles in the chart indicate summands of order 2. Larger cyclic summands are indicated by squares. All such summands in

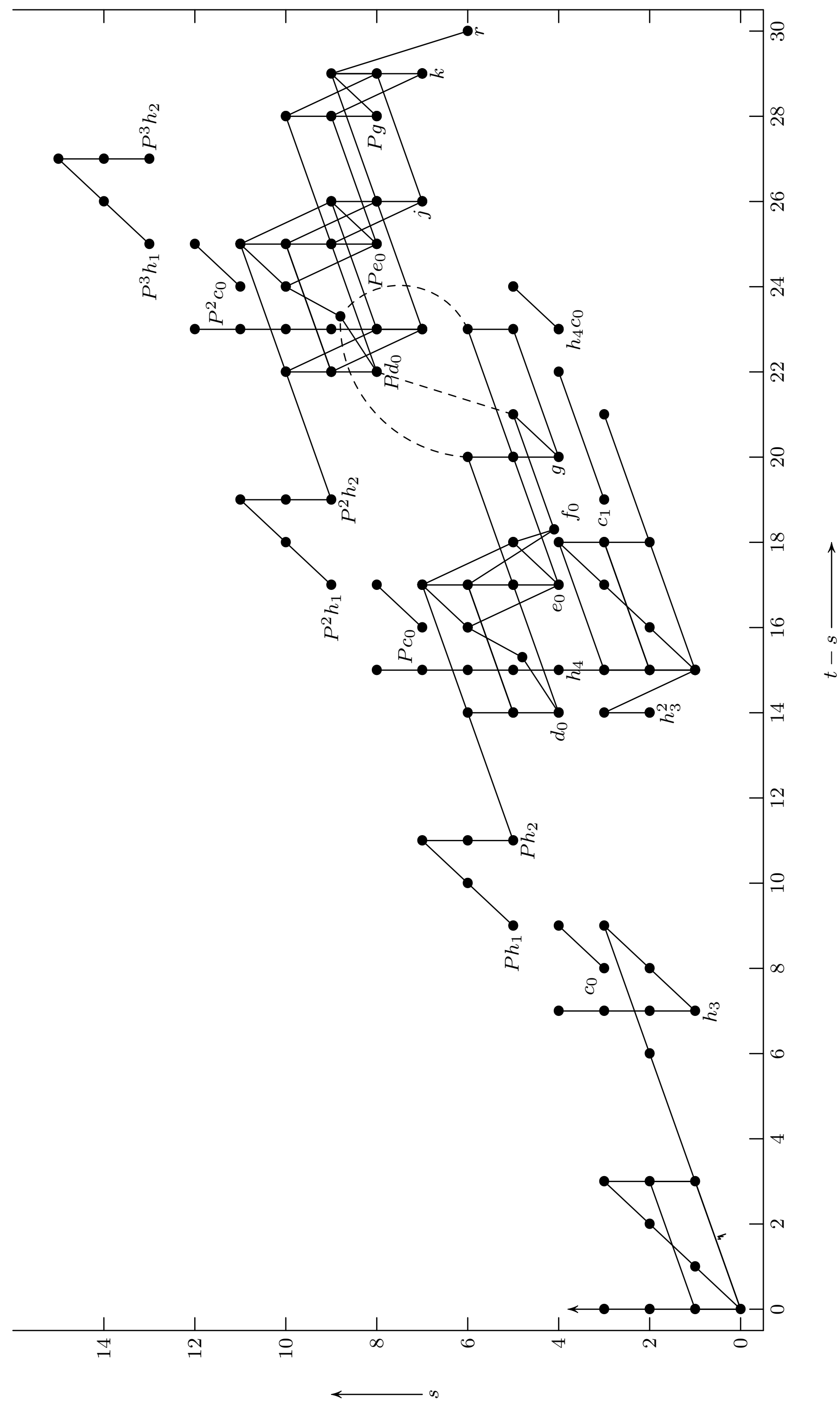
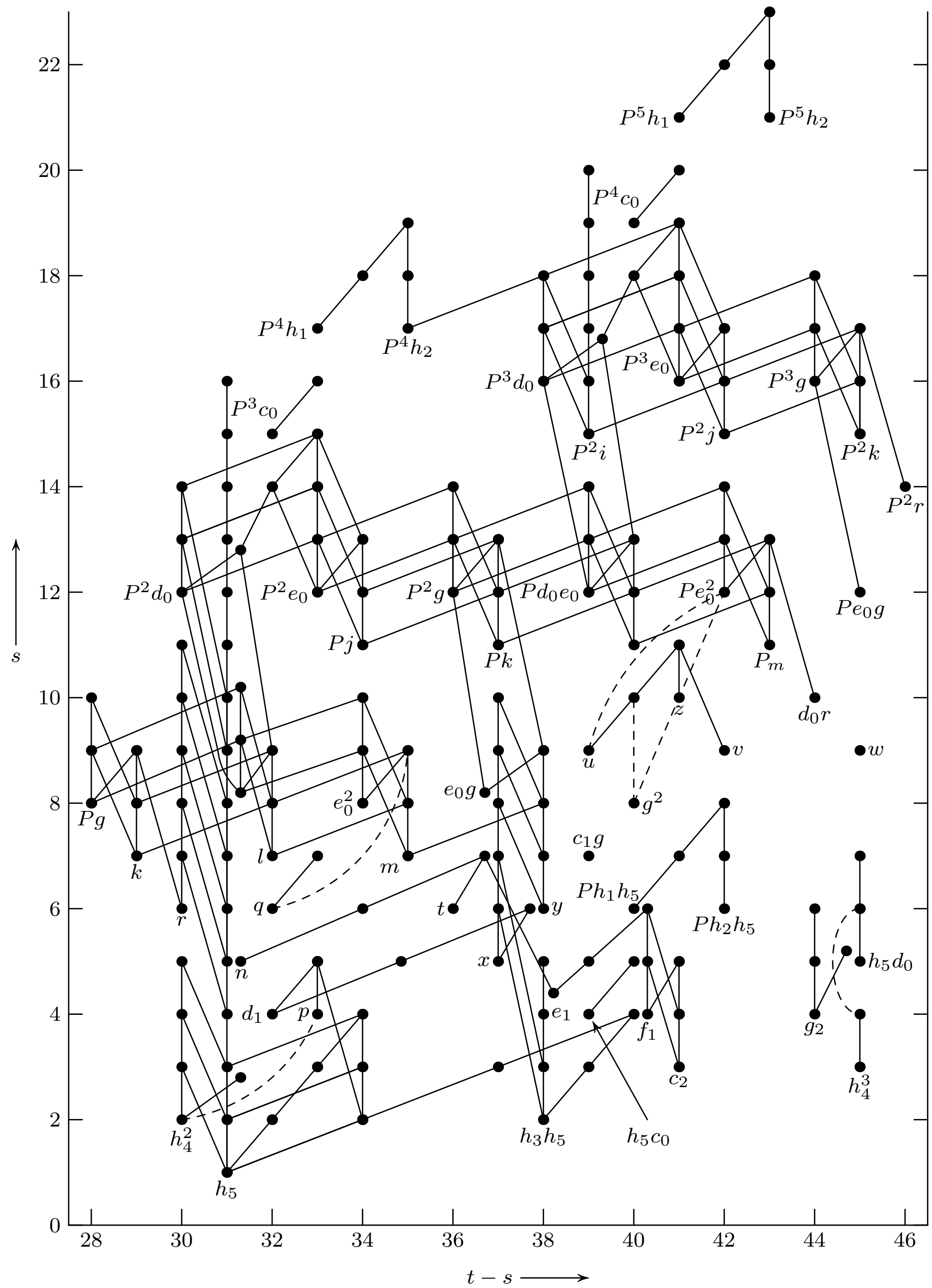


FIGURE A3.1a. The Adams spectral sequence for $p = 2$, $t - s \leq 29$.



$d_4(e_0d_0 + h_0^7h_5) = P^2d_0$ $\quad h_4g = h_2d_1$ $\quad x = \sigma\theta_4$ $\quad h_1x = h_3u = c_1^2$ $\quad h_3x = h_0^2g_2$

$\rho\theta_4 = h_0^2h_5d_0$

$\pi_{45}^S = \mathbf{Z}/(16) \oplus (\mathbf{Z}/(2))^3$

FIGURE A3.1b. The Adams spectral sequence for $p = 2$, $28 \le t - s \le 45$

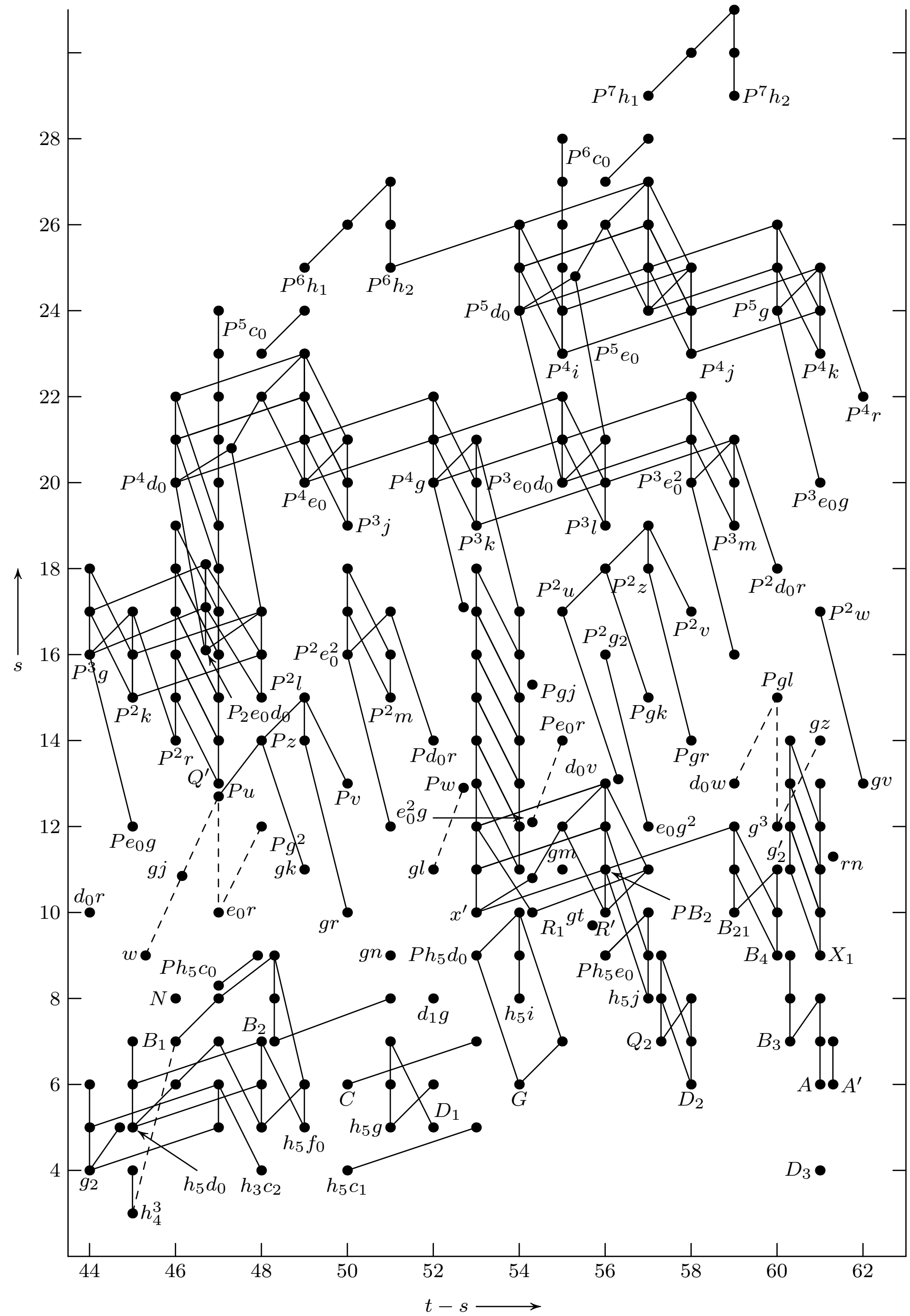


FIGURE A3.1c. The Adams spectral sequence for $p = 2$, $44 \le t - s \le 61$. Differentials are tentative, see Isaksen [2] for a better chart.

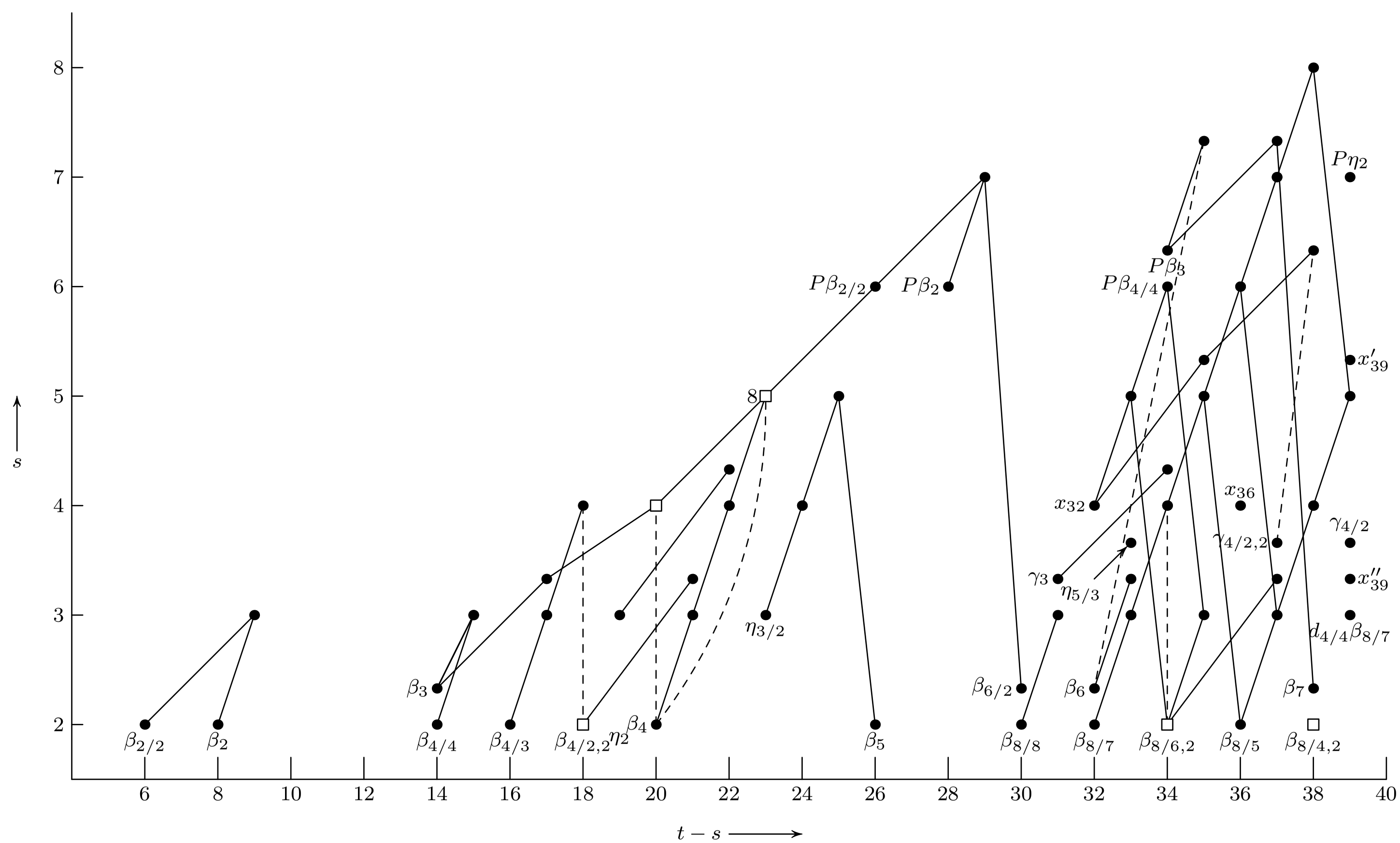


FIGURE A3.2. The Adams–Novikov spectral sequence for $p = 2$, $t - s \leq 39$. v_1-periodic elements omited. Computations for $t - s \geq 30$ are tentative. As of February, 2016, this chart is known to contain some errors. A much better chart through a higher range (59) can be found in Isaksen [1].

this range have order 4 except the one in $\mathrm{Ext}^{5,28}$, which has order 8. The solid and broken lines in this figure means the same thing as in Figs. A3.1a–c as described above. This figure does *not* include the v_1-periodic elements described in 5.3.7, i.e., the elements in the image of the J-homomorphism and the elements constructed in Adams [**1**].

In TABLE A3.3 we list the values of the 2-component of the stable stems π_k^s for $k \leq 45$, showing the name of each element given by Toda [**7**] (where applicable), by Tangora [**1**] in the Adams spectral sequence, and by us in the Adams–Novikov spectral sequence. Again we omit the v_1-periodic elements described in 5.3.7. These omitted summands are as follows.

$$\begin{array}{ll}
\mathbf{Z} & \text{for } k = 0, \\
\mathbf{Z}/(2) & \text{for } k = 1 \text{ or } 2, \\
\mathbf{Z}/(4) & \text{for } k = 3, \\
\mathbf{Z}/(2^{m+4}) & \text{for } k = 8t-1, \text{ where } t \text{ is an odd multiple of } 2^m, \\
\mathbf{Z}/(2) & \text{for } k \equiv 0 \text{ or } 2 \mod (8) \text{ and } k > 7, \\
(\mathbf{Z}/(2))^2 & \text{for } k \equiv 1 \mod (8) \text{ and } k > 7, \text{ and}, \\
\mathbf{Z}/(8) & \text{for } k \equiv 3 \mod (8) \text{ and } k > 7.
\end{array}$$

In TABLES A3.4 and A3.5 we do the same for the primes 3 and 5, recapitulating the results obtained in Sections 7.4 and 7.5, respectively. Again we omit the v_1-periodic elements described in 5.3.7, which in these cases are (in positive dimensions) precisely im J, i.e.,

$$\begin{array}{ll}
\mathbf{Z} & \text{for } k = 0 \text{ and} \\
\mathbf{Z}/(p^{m+1}) & \text{for } k = (2p-2)t - 1,
\end{array}$$

where $t = sp^m$ and s is prime to p.

In FIG. A3.6 we reproduce the table of unstable homotopy groups of spheres through the 19-stem, given in Toda [**6**].

TABLE A3.3. π_*^S at $p = 2^a$

Stem	Toda's name	Tangora's name	Adams–Novikov name
6	ν^2	h_2^2	$\beta_{2/2}$
8	$\varepsilon = \langle \nu^2, 2, \eta \rangle$	c_0	β_2
9	ν^3	$h_1^2 h_3$	$\alpha_1 \beta_2$
14	σ^2	h_3^2	$\beta_{4/4}$
	κ	d_0	β_3
15	$\eta\kappa$	$h_1 d_0$	$\alpha_1 \beta_3 = \alpha_1 \beta_{4/4}$
16	$\eta^* \langle \sigma, 2\sigma, \eta \rangle$	$h_1 h_4$	$\beta_{4/3}$

TABLE A3.3 (continued)

Stem	Group	Tangora's name	Adams–Novikov name
17	$\eta\eta^*$	$h_1^2h_4$	$\alpha_1\beta_{4/3}$
	$\nu\kappa$	h_2d_0	$\alpha_{2/2}\beta_3 = \alpha_{2/2}\beta_{4/4}$
18	$\nu^*\langle\sigma, 2\sigma, \nu\rangle$	$h_2h_4, h_0h_2h_4$	$\beta_{4/2,2}$
		$h_0^2h_2h_4 = h_1^3h_4$	$4\beta_{4/2,2} = \alpha_1^2\beta_{4/3}$
19	$\bar{\sigma} = \langle\sigma^2+\kappa, \eta, \nu\rangle$	c_1	η_2
20	$\bar{\kappa}$	g	β_4
	$2\bar{\kappa}$	h_0g	$2\beta_4 = x_{20} = \langle 2, \alpha_1^3, \beta_{4/3}\rangle$
	$4\bar{\kappa}$	h_0^2g	$2x_{20} = \alpha_{2/2}^2\beta_3$
21	σ^3	$h_2^2h_4$	$\alpha_{2/2}\beta_{4/2,2}$
	$\eta\bar{\kappa}$	h_1g	$\alpha_1\beta_4$
22	$\nu\bar{\sigma}$	h_2c_1	$\alpha_{2/2}\eta_2$
	$\eta^2\bar{\kappa}$	Pd_0	$\alpha_1^2\beta_4$
23	$\mathbf{Z}/2 \otimes \mathbf{Z}/8$	h_4c_0	$\eta_{3/2}$
		h_2g	$x_{23} = \langle\alpha_{2/2}, \alpha_1^3, \beta_{4/3}\rangle$
		h_0h_2g	$2x_{23}$
		Ph_1d_0	$4x_{23}$
24	$\mathbf{Z}/2$	$h_1h_4c_0$	$\alpha_1\eta_{3/2}$
26	$\mathbf{Z}/2$	h_2^2g	$\alpha_{2/2}x_{23}$
28	$\mathbf{Z}/2$	$Pg = d_0^2$	$x_{28} = \langle\beta_2, \alpha_1^3, \beta_{4/3}\rangle$
30	$\mathbf{Z}/2$	h_4^2	$\beta_{8/8}$
31	$(\mathbf{Z}/2)^2$	$h_1^2h_4$	$\alpha_1\beta_{8/8}$
		n	γ_3
32	$(\mathbf{Z}/2)^3$	h_1h_5	$\beta_{8/7}$
		d_1	$x_{32} = \langle\alpha_1, \beta_{4/4}+\beta_3,$ $\alpha_1, \beta_{4/4}+\beta_3\rangle$
		q	β_6
33	$(\mathbf{Z}/2)^3$	$h_1^2h_5$	$\alpha_1\beta_{8/7}$
		p	$\eta_{5/6}$
		h_1q	$\alpha_1\beta_3$
34	$\mathbf{Z}/4 \otimes (\mathbf{Z}/2)^2$	$h_0h_2h_5$	$\beta_{8/6,2}$
		$h_0^2h_2h_5 = h_1^3h_5$	$\alpha_1^2\beta_{8/7}$
		e_0^2	$\alpha_{2/2}\gamma_3$
		C_0^2	$x_{34} = \langle\beta_3, \alpha_1^3, \beta_{4/3}\rangle = P\beta_3$
35	$(\mathbf{Z}/2)^2$	h_2d_1	$\alpha_{2/2}x_{32}$
		$h_1e_0^2$	α_1x_{34}
36	$\mathbf{Z}/2$	t	$x_{36} = ?$
37	$(\mathbf{Z}/2)^2$	$h_2^2h_5$	$\alpha_{2/2}\beta_{8/6,2}$
		x	$\gamma_{4/2,2}$
38	$(\mathbf{Z}/4) \oplus \mathbf{Z}/2$	$h_0^2h_3h_5, h_0^3h_3h_5$	$\beta_{8/4,2}$
		h_1x	$\alpha_1\gamma_{4/2,2}$

TABLE A3.3 (continued)

Stem	Group	Tangora's name	Adams–Novikov name
39	$(\mathbf{Z}/2)^5$	$h_1h_3h_5$	$\alpha_{4/4}\beta_{8/7}$
		h_5c_0	$\gamma_{4/2}$
		h_1c_1	$x_{39} = \langle\alpha_1, \beta_{2/2}, \gamma_3\rangle$
		c_1g	$x'_{39} = \langle\eta_2, \alpha_1^3, \beta_{4/3}\rangle = P\eta_2$
		u	$x''_{39} = \langle\beta_{8/6}, a_1, \alpha_{2/2}\rangle$
40	$\mathbf{Z}/4 + (\mathbf{Z}/2)^4$	$h_1^2h_3h_5$	$\alpha_1\alpha_{4/4}\beta_{8/7}$
		f_1	x_{40}?
		$h_1h_5c_0$	$\beta_2\beta_{8/7} = \gamma_{4/2}\alpha_1$
		Ph_1h_5	$\beta_{8/3}$
		g^2	?
		h_1n	$\alpha_1x''_{39}$
41	$(\mathbf{Z}/2)^3$	h_1f_1	α_1x_{40}
		$Ph_1^2h_5$	α_1x_{40}
		z	?
42	$\mathbf{Z}/8 \oplus \mathbf{Z}/2$	$Ph_2h_5, Ph_0h_2h_5$	$\beta_{\alpha 2,2}$
		$Ph_0^2h_2h_5 = Ph_1^3h_5$	$4\beta_{8/2,2} = \alpha_1^2\beta_{8/3}$
		Pe_0^2	?
44	$\mathbf{Z}/8$	g_2	β_8?
		h_0g_2	
		$h_0^2g_2$	
45	$(\mathbf{Z}/16) \oplus (\mathbf{Z}/2)^3$	h_4^3	γ_4?
		$h_0h_4^3$	
		hg_2	$\alpha_1\beta_8$?
		h_5d_0	
		$h_0h_5d_0$	
		$h_0^2h_5d_0$	
		w	

[a] All element have order 2 unless otherwise indicated. (im J and μ_{8k+1}, μ_{8k+2} omitted.)

TABLE A3.4. 3-Primary Stable Homotopy Excluding im J^a

Stem	Element	Stem	Element
10	β_1	81	γ_2
13	$\alpha_1\beta_1$		$x_{81} = \langle\alpha_1, \alpha_1, \beta_5\rangle$
20	β_1^2	82	$\beta_{6/3}$
23	$\alpha_1\beta_1^2$	84	$\alpha_1\gamma_2$
26	β_2		$\beta_1\beta_5 = \alpha_1 x_{81}$
29	$\alpha_1\beta_2$	85	$\langle\alpha_1, \alpha_1, \beta_2^3\rangle = \beta_1\mu$
30	$\beta_1^3 = \langle\beta_2, 3, \alpha_1\rangle$		$\alpha_1\beta_{6/3}$
36	$\beta_1\beta_2$	86	$\beta_{6/2}$
37	$\langle\alpha_1, \alpha_1, \beta_1^3\rangle = \langle\beta_1, 3, \beta_2\rangle$	90	β_6
38	$\beta_{3/2} = \langle\alpha_1, \beta_1^3, 3, \alpha_1\rangle$	91	$\beta_1\gamma_2$
39	$\alpha_1\beta_1\beta_2$		$\beta_1 x_{81}$
40	β_1^4	92	$\beta_1\beta_{6/3}$
42	β_3		$x_{92} = \langle\beta_1, 3, \gamma_2\rangle$
45	$x_{45} = \langle\alpha_1, \alpha_1, \beta_{3/2}\rangle$ with	93	$x_{93} = \langle\alpha_1, \alpha_1, \beta_{6/2\rangle}$ with
	$3x_{45} = \alpha_1\beta_3$		$3x_{93} = \alpha_1\beta_6$
46	$\beta_1^2\beta_2$	94	$\alpha_1\beta_1\gamma_2$
47	$\langle\alpha_1, \alpha_1, \beta_1^4\rangle$		$\beta_1^2\beta_5$
49	$\alpha_1\beta_1^2\beta_2$	95	$\alpha_1\beta_1\beta_{6/3}$
50	β_1^5	99	$\langle\alpha_1, \alpha_1, x_{92}\rangle$
52	$\beta_2^2 = \langle\alpha_1, \alpha_1, x_{45}\rangle$	100	$\beta_2\beta_5$
55	$\alpha_1\beta_2^2$	101	$\beta_1^2\gamma_2$
62	$\beta_1\beta_2^2$		$\beta_1^2 x_{81}$
65	$\alpha_1\beta_1\beta_2^2$	102	$\beta_1^2\beta_{6/3}$
68	$x_{68} = \langle\alpha_1, \beta_{3/2}, \beta_2\rangle$		$\beta_1 x_{92}$
72	$\beta_1^2\beta_2^2 = \langle\alpha_1, 3, x_{68}\rangle$	104	$\alpha_1\beta_1^2\gamma_2$
74	β_5	106	$x_{106} = \beta_7 \pm \beta_{9/9}$
75	$x_{75} = \langle\alpha_1, \alpha_1, x_{68}\rangle = \langle\beta_1, \beta_{3/2,\beta_2\rangle}$	107	$\gamma_2\beta_2$
	with $3x_{75} = \alpha_1\beta_1^2\beta_2^2$		$\beta_2 x_{81}$
78	$\beta_2^3 = \beta_1 x_{68}$	108	$\beta_2\beta_{6/3} = \langle\alpha_1, \alpha_1, \beta_1^2 x_{81}\rangle$

[a] (See 7.5.3 and subsequent discussion.) All elements have order 3 unless otherwise indicated.

TABLE A3.5. 5-Primary Stable Homotopy Excluding im J

Stem	Element	Stem	Element
38	β_1	255	$\alpha_1\beta_1^3\beta_3$
45	$\alpha_1\beta_1$	258	$\beta_1^2\beta_4$
76	β_1^2	265	$\alpha_1\beta_1^2\beta_4$
83	$\alpha_1\beta_1^2$	266	β_1^7
86	β_2	268	$\beta_2\beta_4$ with $\beta_1\beta_5 = 0$
93	$\alpha_1\beta_2$	275	$\alpha_1\beta_2\beta_4$
114	β_1^3	278	β_6
121	$\alpha_1\beta_1^3$	281	$\underline{2}\beta_1^7$
124	$\beta_1\beta_2$	285	$\alpha_1\beta_6$
131	$\alpha_1\beta_1\beta_2$	286	$\beta_1^4\beta_3$
134	β_3	293	$\alpha_1\beta_1^4\beta_3$
141	$\alpha_1\beta_3$	296	$\beta_1^3\beta_4$
152	β_1^4	303	$\alpha_1\beta_1^3\beta_4$
159	$\alpha_1\beta_1^4$	304	β_1^8
162	$\beta_1^2\beta_2$	306	$\beta_1\beta_2\beta_4$
169	$\alpha_1\beta_1^2\beta_2$	313	$\alpha_1\beta_1\beta_2\beta_4$
172	$\beta_1\beta_3$	316	$\beta_1\beta_6$
179	$\alpha_1\beta_1\beta_3$	319	$\underline{2}\beta_1^8$
182	β_4	326	β_7
189	$\alpha_1\beta_4 = \gamma_1$	331	$\underline{2}\beta_1\beta_6$
190	β_1^5	333	$\alpha_1\beta_7$
200	$\beta_1^3\beta_2$	334	$\beta_1^4\beta_4$
205	$\underline{2}\beta_1^5 = \langle\alpha_1, \alpha_1, \beta_1^5\rangle$	341	$\alpha_1\beta_1^4\beta_4$
206	$\beta_{5/4} = \langle\alpha_1, \beta_1^5, 5, \alpha_1\rangle$	342	β_1^9
207	$\alpha_1\beta_1^3\beta_2$	344	$\beta_1^2\beta_2\beta_4$
210	$\beta_1^2\beta_3$	351	$\alpha_1\beta_1^2\beta_2\beta_4$
213	$\alpha_1\beta_{5/4}$	354	$\beta_1^2\beta_6$
214	$\beta_{5/3}$	357	$\underline{2}\beta_1^9$
217	$\alpha_1\beta_1^2\beta_3$	364	$\beta_1\beta_7$
220	$\beta_1\beta_4$	369	$\underline{2}\beta_1^2\beta_6$
221	$\alpha_1\beta_{5/3}$	374	β_8
222	$\beta_{5/2}$	379	$\underline{\beta_1}\beta_7$
227	$\alpha_1\beta_1\beta_4$	380	β_1^{10}
228	β_1^6	381	$\alpha_1\beta_8$
230	β_5	382	$\beta_1^3\beta_2\beta_4$
237	$\underline{2}\beta_{5/2}$ with $5(\underline{2}\beta_{5/2}) = \alpha_1\beta_5$	389	$\alpha_1\beta_1^3\beta_2\beta_4$
238	$\beta_1^4\beta_2$	392	$\beta_1^3\beta_6$
243	$\underline{2}\beta_1^6$	402	$\beta_1^2\beta_7$
245	$\alpha_1\beta_1^4\beta_2$	403	$\underline{3}\beta_1^{10}$
248	$\beta_1^3\beta_3$	404	$x_{404} = \langle\alpha_1\beta_1^4, \beta_1, \beta_{5/4}\rangle$

Table A3.5 (continued)

Stem	Element
407	$\underline{2}\beta_1^3\beta_6$
411	$\alpha_1 x_{404} = \beta_{5/4}\underline{2}\beta_1^5$
412	$\beta_1\beta_8$
	$x_{412} = \beta_1\beta_8 + \beta_{5/4}^2$
417	$\underline{2}\beta_1^2\beta_7$
418	β_1^{11}
419	$\alpha_1\beta_1\beta_8$ with $\alpha_1 x_{412} = 0$
420	$\beta_1^4\beta_2\beta_4 = \langle\alpha_1, 5, x_{412}\rangle$
422	β_9
427	$\underline{2}x_{412}$ with $5(\underline{2}x_{412}) = \alpha_1\beta_1^4\beta_2\beta_4$
430	$\beta_1^4\beta_6$
437	$\underline{2}\beta_9$
	γ_2
438	$\beta_{10/5}$
440	$\beta_1^3\beta_7$
441	$\underline{3}\beta_1^{11}$
444	$\alpha_1\gamma_2$
445	$\alpha_1\beta_{10/5}$
	$\underline{2}\beta_1^4\beta_6$
446	$\beta_{10/4}$
450	$\beta_1^2\beta_8$ with $\beta_1\beta_{5/4}^2 = 0$
453	$\alpha_1\beta_{10/4}$
454	$\beta_{10/3}$
455	$\underline{2}\beta_1^3\beta_7$
456	β_1^{12}
460	$\beta_1\beta_9$
461	$\alpha_1\beta_{10/3}$
462	$\beta_{10/2}$
465	$\underline{2}\beta_1^2\beta_8$
470	β_{10}
475	$\beta_1\gamma_2$
476	$\beta_1\beta_{10/5} = \langle\alpha_1, \beta_1\beta_6, \beta_1^4\rangle$
	$\underline{5}\gamma_2 = \langle\beta_1, 5, \gamma_2\rangle$
477	$\underline{2}\beta_{10/2}$ with $5(\underline{2}\beta_{10/2}) = \alpha_1\beta_{10}$
478	$\beta_1^4\beta_7$
479	$\underline{3}\beta_1^{12}$
482	$\alpha_1\beta_1\gamma_2$
483	$\alpha_1\beta_1\beta_{10/5}$
488	$\beta_1^3\beta_8$

Stem	Element
491	$\underline{2}\,\underline{5}\gamma_2 = \langle\beta_1\alpha_2, \gamma_2\rangle = \langle\alpha_1\beta_1, \alpha_1, 5, \gamma_2\rangle$
493	$\underline{2}\beta_1^4\beta_7$
494	β_1^{13}
498	$\beta_1^2\beta_9$
503	$\underline{2}\beta_1^3\beta_8$
508	$\beta_2\beta_9$ with $\beta_1\beta_{10} = 0$
513	$\beta_1^2\gamma_2$
	$\underline{2}\beta_1^2\beta_9$
514	$\beta_1^2\beta_{10/5}$
517	$\underline{3}\beta_1^{13}$
518	β_{11}
520	$\alpha_1\beta_1^2\gamma_2$
523	$\underline{2}\beta_2\beta_9$
	$\beta_1\gamma_2$
524	$\beta_2\beta_{10/5}$
525	$\alpha_1\beta_{11}$
526	$\beta_1^4\beta_8$
529	$\underline{2}\,\underline{5}\beta_1\gamma_2$
530	$\alpha_1\beta_2\gamma_2$
531	$\beta_7\underline{2}\beta_1^5 = \alpha_1\beta_2\beta_{10/5}$
532	β_1^{14}
536	$\beta_1^3\beta_9$
541	$\underline{2}\beta_1^4\beta_8$
546	$\beta_1\beta_2\beta_9$
551	$\beta_1^3\gamma_2$
	$\underline{2}\beta_1^3\beta_9$
552	$\beta_1^2\underline{5}\gamma_2$
555	$\underline{3}\beta_1^{14}$
556	$\beta_1\beta_{11}$
558	$\alpha_1\beta_1^3\gamma_2$
561	$\beta_1\underline{2}\beta_2\beta_9$ with $\beta_1\beta_2\gamma_2 =? \beta_1\underline{2}\beta_2\beta_9$
566	β_{12}
567	$\beta_1^2\underline{2}\,\underline{5}\gamma_2$
570	β_1^{15}
571	$\underline{2}\beta_1\beta_{11}$
572	$\beta_3\beta_{10/5}$
573	$\alpha_1\beta_{12}$
574	$\beta_1^4\beta_9$
579	$\beta_9\underline{2}\beta_1^5 = \alpha_1\beta_3\beta_{10/5}$

TABLE A3.5 (continued)

Stem	Element	Stem	Element
583	$\beta_1^2\beta_2\beta_9$	659	$\alpha_1\beta_1\beta_{13}$
589	$\beta_1^4\gamma_2$		with $\alpha_1 x_{652} = 0$
	$\underline{2}\beta_1^4\beta_9$	660	$\beta_1^4\beta_2\beta_9$
590	$\beta_1^3\underline{5}\gamma_2$	662	β_{14}
594	$\beta_1^2\beta_{11}$	665	$\underline{3}\beta_1^2\beta_{12}$
596	$\alpha_1\beta_1^4\gamma_2$		$\beta_1^6\gamma_2$
599	$\beta_1^2\underline{2}\beta_2\beta_9$	666	$\beta_1^5\underline{5}\gamma_2$
601	$\underline{4}\beta_1^{15}$	667	$\underline{2}x_{652}$
602	$x_{602} = \langle \underline{2}\beta_1^9, \beta_1, \beta_{5/4}\rangle$	670	$\beta_1^4\beta_{11}$
604	$\beta_1\beta_{12}$	675	$\underline{2}\beta_1^4\beta_2\beta_9$
605	$\beta_1^3\underline{2}\,\underline{5}\gamma_2$	677	$\underline{2}\beta_{14}$
608	β_1^{16}		$\underline{4}\beta_1^{17}$
609	$\alpha_1 x_{602}$ with $\underline{2}\beta_1^2\beta_{11} =? \alpha_1 x_{602}$	678	$\beta_{15/5}$
610	$\beta_1\beta_3\beta_{10/5} = \langle \alpha_1, 5, x_{602}\rangle$	680	$\beta_1^3\beta_{12}$
614	β_{13}		$\underline{2}\beta_1^6 5\gamma_2$
617	$x_{617} = \langle \alpha_1, (\alpha_1\underline{2}\beta_2\beta_6), \binom{x_{602}}{\beta_6}\rangle$	685	$\alpha_1\beta_{15/5}$
	with	686	$\beta_{15/4}$
	$5x_{617} = \alpha_1\beta_1\beta_3\beta_{10/5}$	689	$\underline{3}\beta_1^5\underline{5}\gamma_2$
620	$\beta_4\beta_{10/5}$	690	$\beta_1^2\beta_{13}$
621	$\alpha_1\beta_{13}$	692	$x_{692} = \langle \alpha_1, \beta_1^5, \beta_1^{13}\rangle$
622	$\beta_1^3\beta_2\beta_9$	693	$\alpha_1\beta_{15/4}$
627	$\underline{3}\beta_1\beta_{12}$		$\underline{3}\beta_1^4\beta_{11}$
	$\beta_1^5\gamma_2$	694	$\beta_{15/3}$
628	$\beta_1^4\underline{5}\gamma_2$	700	$\beta_1\beta_{14}$
632	$\beta_1^3\beta_{11}$	701	$\alpha_1\beta_{15/3}$
635	$\underline{2}\beta_4\beta_{10/5}$	702	$\beta_{15/2}$
636	$x_{636} = \langle \beta_1^3, \alpha_1\beta_1^2, \beta_{10/5}\rangle$	703	$\underline{3}\beta_1^3\beta_{12}$
	$= \langle \beta_{5/4}, \beta_9, \alpha_1\rangle$		$\beta_1^7\gamma_2$
637	$\underline{2}\beta_1^3\beta_2\beta_9$	704	$\beta_1^6\underline{5}\gamma_2$
639	$\underline{4}\beta_1^{16}$	710	β_{15}
642	$\beta_1^2\beta_{12}$	713	$\underline{\beta}_1^2\beta_{13}$
	$\underline{2}\beta_1^5\gamma_2$	714	$x_{714} = \langle \beta_1^4, \beta_1\beta_2, \gamma_2 + \underline{2}\beta_9\rangle$
643	$\underline{2}\,\underline{5}\beta_1^4\gamma_2 = \beta_{5/4}\gamma_2$	715	$\underline{2}\beta_1\beta_{14}$
	$\alpha_1 x_{636}$		$\underline{3}x_{692}$
644	$\beta_{5/4}\beta_{10/5}$	716	$\beta_1\beta_{15/5}$
646	β_1^{17}	717	$\underline{2}\beta_{15/2}$ with
651	$\alpha_1\beta_{5/4}\beta_{10/5}$		$5(\underline{2}\beta_{15/2}) = \alpha_1\beta_{15}$
652	$\beta_1\beta_{13}$	718	$\beta_1^4\beta_{12}$
	$\beta_{5/3}\beta_{10/5} + \beta_1\beta_{13} = x_{652}$		$\underline{2}\beta_1^7\gamma_2$
655	$\underline{3}\beta_1^3\beta_{11}$	721	$\alpha_1 x_{714}$

TABLE A3.5 (continued)

Stem	Element	Stem	Element
723	$\alpha_1\beta_1\beta_{15/5}$	786	$\beta_1\beta_2\beta_{14}$
724	$x_{724} = \langle \beta_1, 5, \beta_1, \beta_1^{17} \rangle$	789	$\underline{3}\beta_1^4\beta_{13}$
	$x'_{724} = \langle \beta_1^2, \beta_1^2\beta_{11}, \alpha_1\alpha_1 \rangle$	794	$\underline{2}\beta_1^9\gamma_2$
	with $\beta_1\beta_{15/4} = 0$	796	$\beta_1\beta_{16}$
727	$\underline{3}\beta_1^6\underline{5}\gamma_2$	799	$\underline{3}\beta_1^3\beta_{14}$
728	$\beta_1^3\beta_{13}$		$\beta_1 x_{761}$ with
730	$\beta_1 x_{692}$		$\underline{4}\beta_1^2 x_{692} = ?$
731	$\alpha_1 x'_{724}$ with	800	$\beta_1^2 x_{724}$
	$\alpha_1 x_{724} = 0$		$\beta_1^2 x'_{724}$
738	$\beta_1^2\beta_{14}$	803	$\underline{3}\beta_1^8\underline{5}\gamma_2$
739	$\underline{2}x_{724}$	806	β_{17}
741	$\underline{3}\beta_1^4\beta_{12}$		$\alpha_1\beta_1 x_{761} = \beta_1^3 x_{692}$
	$\beta_1^8\gamma_2$	807	$\alpha_1\beta_1^2 x'_{724}$
742	$\beta_1^7\underline{5}\gamma_2$	809	$\beta_1 x_{771}$
748	$\beta_2\beta_{14}$ with	810	$\beta_2 x'_{724}$
	$\beta_1\beta_{15} = 0$	811	$\underline{2}\beta_1\beta_{16}$
751	$\underline{3}\beta_1^3\beta_{13}$	812	$\beta_3\beta_{15/5}$
753	$\underline{3}\beta_1 x_{692}$	813	$\alpha_1\beta_{17}$
756	$\underline{2}\beta_1^8\gamma_2$	814	$\beta_1^4\beta_{14}$
	with $\beta_1^5\beta_{12} = 0$	815	$\underline{2}\beta_1^2 x_{724}$
758	β_{16}	816	$\alpha_1\beta_1 x_{771}$
761	$x_{761} = \langle \beta_3, \gamma_1, \gamma_2 \rangle$	817	$\beta_1^{10}\gamma_2$
	$\underline{3}\beta_1\beta_{14}$		$\underline{4}\beta_1\beta_2\beta_{14}$ with
762	$\beta_1 x_{724}$		$\underline{2}\beta_2\beta_{15/5} = 0$
	$\beta_1 x'_{724}$	818	$\beta_1^{10}\beta_{10/5}$
763	$\underline{2}\beta_2\beta_{14}$	824	$\beta_1^2\beta_1\beta_{14}$
764	$\beta_2\beta_{15/5}$	825	$\underline{2}\beta_2 x'_{724}$
765	$\alpha_1\beta_{16}$	826	$x_{826} = \langle \alpha_1, \beta_1^5, \alpha_1\beta_4, \beta_{10/5} \rangle$
	$\underline{3}\beta_1^7\underline{5}\gamma_2$	827	$\underline{2}\beta_3\beta_{15/5}$
766	$\beta_1^4\beta_{13}$	833	$\alpha_1 x_{826}$
768	$\alpha_1 x_{761} = \gamma_2\underline{2}\beta_1\beta_6$	834	$\beta_1^2\beta_{16}$
	$\beta_1^2 x_{692}$		$x_{834} = \langle \beta_1^4, \underline{2}\beta_1^6, \beta_{10/5} \rangle$
769	$\alpha_1\beta_1 x'_{724}$	837	$\underline{3}\beta_1^4\beta_{14}$
771	$x_{771} = \langle \beta_2, \beta_1^5, \beta_1^{13} \rangle$		$\beta_1^2 x_{761}$
776	$\beta_1^3\beta_{14}$	838	$\beta_1^3 x_{724}$ with
777	$\underline{2}\beta_1 x_{724}$		$\beta_1^3 x'_{724} = 0$
778	$\alpha_1 x_{771}$	840	$\underline{3}\beta_1^{10}\gamma_2$
779	$\beta_1^9\gamma_2$	841	$\alpha_1 x_{834}$
	$\underline{2}\beta_2\beta_{15/5}$		$\underline{3}\beta_1^{10}\beta_{10/5}$
780	$\beta_1^8\underline{5}\gamma_2$	842	$x_{842} = \langle \underline{2}\beta_1^9, \beta_1, \beta_{10/4} \rangle$

Table A3.5 (continued)

Stem	Element
844	$\alpha_1\beta_1^2x_{761}$
	$\beta_1\beta_{17}$
847	$\beta_1^2x_{771}$
849	$\alpha_1x_{842}=\underline{2}\beta_1^2\beta_{16}$
850	$\beta_1\beta_3\beta_{15/5}$
853	$\underline{2}\beta_1^3x_{724}$
854	β_{18}
	$\alpha_1\beta_1^2x_{771}$
855	$\beta_1^{11}\gamma_2$
856	$\beta_1^{11}\beta_{10/5}$ with
	$\underline{4}\,\underline{2}x_{810}=0$
	$\underline{4}\beta_1^2\beta_1\beta_{14}$
857	$x_{857}=\langle\alpha_1(\alpha_1\underline{2}\beta_1\beta_6),\binom{x_{842}}{\beta_{11}}\rangle$
860	$\beta_4\beta_{15/5}$
861	$\alpha_1\beta_{18}$
862	$\beta_1^3\beta_2\beta_{14}$
865	$\underline{2}\beta_1\beta_3\beta_{15/5}$
867	$\underline{3}\beta_1\beta_{17}$
868	x_{868} (see 7.6.5)
872	$\beta_1^3\beta_{16}$ with $\beta_1x_{834}=0$
875	α_1x_{868}
	$\underline{2}\beta_4\beta_{15/5}$
	$\beta_1^3x_{761}=\beta_{10/5}\gamma_2$
876	$\beta_{10/5}^2$
	$\beta_1^4x_{724}$
878	$\underline{3}\beta_1^{11}\gamma_2$
882	$\beta_1^2\beta_{17}$ with
	$\alpha_1\beta_1^3x_{761}=0$
883	$\alpha_1\beta_{10/5}^2$
884	$\beta_{5/4}\beta_{15/5}$
885	$\beta_1^3x_{771}$
887	$\underline{4}\beta_1^{11}\beta_{10/5}$
890	$\underline{2}\beta_1^3x_{761}$
891	$\underline{2}\beta_1^4x_{724}$
	$\alpha_1\beta_{5/4}\beta_{15/3}$
892	$\beta_1\beta_{18}$
	$\alpha_1\beta_1^3x_{771}$
	$x_{892}=\beta_{10/4}^2+\beta_1\beta_{18}$
893	$\beta_1^{12}\gamma_2$
	$\underline{4}\beta_1^3\beta_2\beta_{14}$
894	$\beta_1^{12}\beta_{10/9}$
899	$\alpha_1\beta_1\beta_{18}$ with
	$\alpha_1x_{892}=0$
900	$\beta_1^4\beta_2\beta_{14}$
902	β_{19}
903	$\underline{4}\beta_1^3\beta_{16}$
905	$\underline{3}\beta_1^2\beta_{17}$
906	β_1x_{868}
907	$\underline{2}x_{892}$
910	$\beta_1^4\beta_{16}$
913	$\alpha_1\beta_1x_{868}$
	$\beta_1^4x_{761}$ with
	$\underline{2}\beta_1\beta_4\beta_{15/5}=0$
914	$\beta_1^5x_{724}$ with
	$\beta_1\beta_{10/5}^2=0$
916	$\underline{\beta}_1^{12}\gamma_2$
917	$\underline{2}\beta_{19}$
918	$\beta_{20/5}$
920	$\beta_1^3\beta_{17}$
923	$\beta_1^4x_{771}$
925	$\alpha_1\beta_{20/5}$
	$\underline{4}\beta_1^{12}\beta_{10/5}$
926	$\beta_{20/4}$
928	$\underline{2}\beta_1^4x_{761}$
930	$\alpha_1\beta_1^4x_{771}$
	$\beta_1^2\beta_{18}$ with
	$\beta_1x_{892}=0$
931	$\beta_1^{13}\gamma_2$
	$\underline{4}\beta_1^4\beta_2\beta_{14}$
932	$\beta_1^{13}\beta_{10/5}$?
933	γ_4?
	$\alpha_1\beta_{20/4}$
934	$\beta_{20/3}$
937	$\underline{3}\beta_1^5x_{724}$ with
	$\alpha_1\beta_1\beta_{10/4}^2=0$
940	$\alpha_1\gamma_4$
	$\beta_1\beta_{19}$
941	$\alpha_1\beta_{20/3}$
	$\underline{4}\beta_1^4\beta_{16}$
942	$\beta_{20/2}$

TABLE A3.5 (continued)

Stem	Element	Stem	Element
950	β_{20}	969	$\beta_1^{14}\gamma_2$
951	$\beta_1^5 x_{761}$		$\underline{2}x_{954}$
	$\underline{4}\beta_1^3\beta_{17}$ with	970	$\beta_1^{14}\beta_{10/5}$?
	$\alpha_1\beta_1^2 x_{868} = 0$	971	$\beta_1\gamma_4$? with
952	$\beta_1^6 x_{724}$		$\alpha_1\beta_1\beta_{20/4} = 0$
	$x_{952} = \langle \beta_1, \beta_1\beta_{10/5} + \underline{5}\gamma_2, \gamma_2 \rangle$	972	$\underline{5}\gamma_4$ with
953	$\underline{3}\beta_1^2\beta_{18}$		$\beta_1\beta_{20/3} = 0$
954	$\underline{3}\beta_1^{13}\gamma_2$	975	$\underline{3}\beta_1^6 x_{724}$
	$x_{954} = \langle \alpha_1, \alpha_1, \beta_1\beta_2\beta_{14}, \beta_1^4 \rangle$	978	$\alpha_1\beta_1\gamma_4$
955	$\underline{2}\beta_1\beta_{19}$		$\beta_1^2\beta_{19}$
956	$\beta_1\beta_{20/5}$	979	$\underline{2}x_{964}$ with
957	$\underline{2}\beta_{20/2}$ with		$\alpha_1\underline{5}\gamma_4 = 0$ and
	$5(\underline{2}\beta_{20/2}) = \alpha_1\beta_{20}$		$\alpha_1/\beta_1\beta_{20/3} = 0$
958	$\beta_1^4\beta_{17}$	987	$\underline{2}\,\underline{5}\gamma_4$
959	$\alpha_1 x_{952}$	988	$\beta_2\beta_{19}$ with $\beta_1\beta_{20} = 0$
963	$\alpha_1\beta_1\beta_{20/5}$	989	$\beta_1^6 x_{761}$
	$\underline{4}\beta_1^{13}\beta_{10/5}$	990	$\beta_1^7 x_{724}$
964	$x_{964} = \underline{3}\,\underline{4}\beta_1^4\beta_{16}$		$\beta_1 x_{952}$
	with $\beta_1\beta_{20/4} = 0$	992	$\underline{3}\beta_1^{14}\gamma_2$ with $\beta_1 x_{954} = 0$
966	$\underline{2}\beta_1^5 x_{761}$	998	β_{21}
968	$\beta_1^3\beta_{18}$	999	$x_{999} = \langle \beta_1\beta_2, \gamma_2, \gamma_2 \rangle$
	with $\alpha_1\beta_1^5 x_{771} = 0$		with $\underline{4}\beta_1^3\beta_{18} = 0$

n \ k	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
1	∞	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
2	∞	∞	2	2	12	2	2	3	15	2	2^2	12.2	84.2^2	2^2	6	30	30	6.2	12.2^2	6.2
3		2	2	12	2	2	3	15	2	2^2	12.2	84.2^2	2^2	6	30	30	6.2	12.2^2	12.2^2	132.2
4			2	$\infty.12$	2^2	2^2	24.3	15	2	2^3	120.12.2	84.2^5	2^6	24.6.2	2520.6.2	30	$6^2.2$	$24.12.4.2^2$	$120.12.2^5$	132.2^5
5				24	2	2	2	30	2	2^3	72.2	504.2^2	2^3	6.2	6.2	30.2	2^2	4.2^2	24.2^2	264.2
6					0	∞	2	60	24.2	2^3	72.2	504.4	240	6	12.2	60.6	504.2^2	2^4	24.6.2	1056.8
7						0	2	120	2^3	2^4	24.2	504.2	0	6	24.4	120.2^3	2^4	2^4	24.2	264.2
8							2	$\infty.120$	2^4	2^5	$24^2.2$	504.2	0	6.2	240.24.4	120.2^5	2^7	6.2^4	540.24.2	264.2
9								240	2^3	2^4	24.2	504.2	0	6	16.4	240.2^3	2^4	2^4	24.2	264.2
10									2^2	$\infty.2^3$	12.2	504	12	6	16.2	240.2^2	240.2	2^3	24.2^2	264.6
11										2^3	6.2	504	2	6.2	16.2	240.2	2	2^3	8.4.2	264.2^3
12											6	$\infty.504$	2^2	6.2	48.4.2	240.2	2	2^4	$480.4^2.2$	264.2^5
13												504	2	6	16.2	480.2	2	2^4	$8^2.2$	264.2^3
14													0	$\infty.3$	8.2	480.2	24.2	2^4	$8^2.2$	264.4.2
15														3	4.2	480.2	2^3	2^5	$8^2.2$	264.2^2
16															2^2	$\infty.480.2$	2^4	2^6	$24.8^2.2$	264.2^2
17																480.2	2^3	2^5	$8^2.2$	264.2^2
18																	2^2	$\infty.2^4$	8.4.2	264.2
19																		2^4	8.2^2	264.2
20																			8.2	$\infty.264.2$
21																				264.2

TABLE A3.6. Toda's calculation of unstable homotopy groups $\pi_{n+k}(S^n)$ for $n \leq k+2$ and $k \leq 19$.

For $n > k+1$ the group is isomorphic to the one for $n = k+1$. The notation $a.b.c\ldots$ denotes the direct sum of cyclic groups of order a, b, c, etc. The notation a^j denotes the direct sum of j cyclic groups, each having order a. (After Toda [6].)

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