Research Article

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The slice spectral sequence for the $C_4$ analog of real $K$-theory

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Abstract: We describe the slice spectral sequence of a 32-periodic $C_4$-spectrum $K_{[2]}$ related to the $C_4$ norm $MU(C_4) = N^C_4MU_R$ of the real cobordism spectrum $MU_R$. We will give it as a spectral sequence of Mackey functors converging to the graded Mackey functor $\pi^* K_{[2]}$, complete with differentials and exotic extensions in the Mackey functor structure. The slice spectral sequence for the 8-periodic real $K$-theory spectrum $K_{[2]}$ was first analyzed by Dugger. The $C_8$ analog of $K_{[2]}$ is 256-periodic and detects the Kervaire invariant classes $\theta_j$. A partial analysis of its slice spectral sequence led to the solution to the Kervaire invariant problem, namely the theorem that $\theta_j$ does not exist for $j \geq 7$.

Keywords: Equivariant stable homotopy theory, Kervaire invariant, Mackey functor, slice spectral sequence

MSC 2010: Primary 55Q10; secondary 55Q91, 55P42, 55R45, 55T99

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1 Introduction

In [6] we derived the main theorem about the Kervaire invariant elements from some properties of a $C_8$-equivariant spectrum we called $\Omega$ constructed as follows. We started with the $C_2$-spectrum $MU_R$, meaning the usual complex cobordism spectrum $MU$ equipped with a $C_2$ action defined in terms of complex conjugation. Then we defined a functor $N^{C_8}_{C_2}$, the norm of [6, Section 2.2.3] which we abbreviate here by $N^8_{2}$, from the category of $C_2$-spectra to that of $C_8$-spectra. Roughly speaking, given a $C_2$-spectrum $X$, $N^8_{2}X$ is underlain by the fourfold smash power $X^{\wedge 4}$ where a generator $y$ of $C_8$ acts by cyclically permuting the four factors, each of which is invariant under the given action of the subgroup $C_2$. In a similar way one can define a functor $N^G_H$ from $H$-spectra to $G$-spectra for any finite groups $H \leq G$.

A $C_8$-spectrum such as $N^8_{2}MU_R$, which is a commutative ring spectrum, has equivariant homotopy groups indexed by $RO(C_8)$, the orthogonal representation ring for the group $C_8$. One element of the latter is $\rho_8$, the regular representation. In [6, Section 9] we defined a certain element $D \in \pi_{19}\rho_8 N^8_{2}MU_R$ and then formed the associated mapping telescope, which we denoted by $\Omega_O$. The symbol $O$ was chosen to suggest a connection with the octonions, but there really is none apart from the fact that the octonions are 8-dimensional like $\rho_8$.

Note that $\Omega_O$ is also a $C_8$-equivariant commutative ring spectrum. We then proved that it is equivariantly equivalent to $\Sigma^{256}\Omega_O$; we call this result the Periodicity Theorem. Then our spectrum $\Omega$ is $\Omega^C_8 \Omega_O$, the fixed point spectrum of $\Omega_O$.

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It is possible to do this with $C_8$ replaced by $C_{2^n}$ for any $n$. The dimension of the periodicity is then $2^{1+n+2^n-1}$. For example it is 32 for the group $C_4$ and $2^{13}$ for $C_{16}$. We chose the group $C_8$ because it is the smallest that suits our purposes, namely it is the smallest one yielding a fixed point spectrum that detects the Kervaire invariant elements $\theta_j$.

We know almost nothing about $\pi_*, \Omega$, only that it is periodic with periodic 256, that $\pi_{-2} = 0$ (the Gap Theorem of [6, Section 8]), and that when $\theta_j$ exists its image in $\pi_*, \Omega$ is nontrivial (the Detection Theorem of [6, Section 11]).

We also know, although we did not say so in [6], that more explicit computations would be much easier if we cut $N^2_2\text{MU}_R$ down to size in the following way. Its underlying homotopy, meaning that of the spectrum $\text{MU}^{\wedge A}$, is known classically to be a polynomial algebra over the integers with four generators (cyclically permuted up to sign by the group action) in every positive even dimension. This can be proved with methods described by Adams in [1]. For the cyclic group $C_{2^n}$ one has $2^{n-1}$ generators in each positive even degree. Specific generators $r_{i,j} \in \pi_{2i}[\text{MU}^{2n-1}]$ for $i > 0$ and $0 \leq j < n-1$ are defined in [6, Section 5.4.2].

There is a way to kill all the generators above dimension $2k$ that was described in [6, Section 2.4]. Roughly speaking, let $A$ be a wedge of suspensions of the sphere spectrum, one for each monomial in the generators one wants to kill. One can define a multiplication and group action on $A$ corresponding to the ones in $\pi_* \text{MU}^{\wedge A}$. Then one has a map $A \to \text{MU}^{\wedge A}$ whose restriction to each summand represents the corresponding monomial, and a map $A \to S^0$ (where the target is the sphere spectrum, not the space $S^0$) sending each positive-dimensional summand to a point. This leads to two maps

$$S^0 \wedge A \wedge \text{MU}^{\wedge A} \Rightarrow S^0 \wedge \text{MU}^{\wedge A}$$

whose coequalizer we denote by $S^0 \wedge_A \text{MU}^{\wedge A}$. Its homotopy is the quotient of $\pi_* \text{MU}^{\wedge A}$ obtained by killing the polynomial generators above dimension $2k$. The construction is equivariant, meaning that $S^0 \wedge_A \text{MU}^{\wedge A}$ underlies a $C_8$-spectrum.

In [6, Section 7] we showed that for $k = 0$ the spectrum we get is the integer Eilenberg–Mac Lane spectrum $HZ$; we called this result the Reduction Theorem. In the nonequivariant case this is obvious. We are in effect attaching cells to $\text{MU}^{\wedge A}$ to kill all of its homotopy groups in positive dimensions, which amounts to constructing the $0$th Postnikov section. In the equivariant case the proof is more delicate.

Now consider the case $k = 1$, meaning that we are killing the polynomial generators above dimension 2. Classically we know that doing this to $\text{MU}$ (without the $C_2$-action) produces the connective complex $K$-theory spectrum, some times denoted by $k$, $bu$ or (2-locally) $BP(1)$. Inverting the Bott element via a mapping telescope gives us $K$ itself, which is of course 2-periodic. In the $C_2$-equivariant case one gets the “real $K$-theory” spectrum $K_R$ first studied by Atiyah in [3]. It turns out to be 8-periodic and its fixed point spectrum is $KO$, which is also referred to in other contexts as real $K$-theory.

The spectrum we get by killing the generators above dimension 2 in the $C_8$-spectrum $N^8_2\text{MU}_R$ will be denoted analogously by $k[3]$. We can invert the image of $D$ by forming a mapping telescope, which we will denote by $K[3]$. More generally we denote by $k[n]$ the spectrum obtained from $N^8_2\text{MU}_R$ by killing all generators above dimension 2. In particular, $k[1] = K_R$. Then we denote the mapping telescope (after defining a suitable $D$) by $K[n]$ and its fixed point set by $KO[n]$.

For $n \geq 3$, $KO[n]$ also has a Periodicity, Gap and Detection Theorem, so it could be used to prove the Kervaire Invariant Theorem.

Thus $K[3]$ is a substitute for $\Omega_0$ with much smaller and therefore more tractable homotopy groups. A detailed study of them might shed some light on the fate of $\theta_6$ in the 126-stem, the one hypothetical Kervaire invariant element whose status is still open. If we could show that $\pi_{128}KO[3] = 0$, that would mean that $\theta_6$ does not exist.

The computation of the equivariant homotopy $\pi_*, K[3]$ at this time is daunting. The purpose of this paper is to do a similar computation for the group $C_4$ as a warmup exercise. In the process of describing it we will develop some techniques that are likely to be needed in the $C_8$ case. We start with $N^8_2\text{MU}_R$, kill its polynomial generators (of which there are two in every positive even dimension) above dimension 2 as described previously, and then invert a certain element in $\pi_{sp^4}$. We denote the resulting spectrum by $K[2]$, see Definition 7.3 below. This spectrum is known to be 32-periodic. In an earlier draft of this paper it was denoted by $K_{H}$. 


The computational tool for finding these homotopy groups is the slice spectral sequence introduced in [6, Section 4]. Indeed we do not know of any other way to do it. For $K_R$ it was first analyzed by Dugger [4] and his work is described below in Section 8. In this paper we will study the slice spectral sequence of Mackey functors associated with $K_{[2]}$. We will rely extensively on the results, methods and terminology of [6].

We warn the reader that the computation for $K_{[2]}$ is more intricate than the one for $K_R$. For example, the slice spectral sequence for $K_R$, which is shown in Figure 7, involves five different Mackey functors for the group $C_2$. We abbreviate them with certain symbols indicated in Table 1. The one for $K_{[2]}$, partly shown in Figure 16, involves over twenty Mackey functors for the group $C_4$, with symbols indicated in Table 2.

Part of this spectral sequence is also illustrated in an unpublished poster produced in late 2008 and shown in Figure 1. It shows the spectral sequence converging to the homotopy of the fixed point spectrum $K^C_{[2]}$. The corresponding spectral sequence of Mackey functors converges to the graded Mackey functor $\pi_\ast K_{[2]}$.

In both illustrations some patterns of $d_3$s and families of elements in low filtration are excluded to avoid clutter. In the poster, representative examples of these are shown in the second and fourth quadrants, the spectral sequence itself being concentrated in the first and third quadrants. In this paper those patterns are spelled out in Section 12 and Section 13.

We now outline the rest of the paper. Briefly, the next five sections introduce various tools we need. Our objects of study, the spectra $k_{[2]}$ and $K_{[2]}$, are formally introduced in Section 7. Dugger's computation for $K_R$ is recalled in Section 8. The final six sections describe the computation for $k_{[2]}$ and $K_{[2]}$.

In more detail, Section 2 collects some notions from equivariant stable homotopy theory with an emphasis on Mackey functors. Definition 2.7 introduces new notation that we will occasionally need.

Section 3 concerns the equivariant analog of the homology of a point namely, the RO$(G)$-graded homotopy of the integer Eilenberg–Mac Lane spectrum $HZ$. In particular, Lemma 3.6 describes some relations among certain elements in it including the “gold relation” between $a_V$ and $u_V$.

Section 4 describes some general properties of spectral sequences of Mackey functors. These include Theorem 4.4 about the relation between differential and exotic extensions in the Mackey functor structure and Theorem 4.7 on the norm of a differential.
Section 5 lists some concise symbols for various specific Mackey functors for the groups $C_2$ and $C_4$ that we will need. Such functors can be spelled out explicitly by means of Lewis diagrams (5.1), which we usually abbreviate by symbols shown in Tables 1 and 2.

In Section 6 we study some chain complexes of Mackey functors that arise as cellular chain complexes for $G$-CW complexes of the form $S^V$.

In Section 7 we formally define (in Definition 7.3) the $C_2$-spectra of interest in this paper, $k_{[2]}$ and $K_{[2]}$.

In Section 8 we shall describe the slice spectral sequence for an easier case, the $C_2$-spectrum for real $K$-theory, $K_R$. This is due to Dugger [4] and serves as a warmup exercise for us. It turns out that everything in the spectral sequence is formally determined by the structure of its $E_2$-term and Bott periodicity.

In Section 9 we introduce various elements in the homotopy groups of $k_{[2]}$ and $K_{[2]}$. They are collected in Table 3, which spans several pages. In Section 10 we determine the $E_2$-term of the slice spectral sequence for $k_{[2]}$ and $K_{[2]}$.

In Section 11 we use the Slice Differentials Theorem of [6] to determine some differentials in our spectral sequence.

In Section 12 we examine the $C_4$-spectrum $k_{[2]}$ as a $C_2$-spectrum. This leads to a calculation only slightly more complicated than Dugger’s. It gives a way to remove a lot of clutter from the $C_4$ calculation.

In Section 13 we determine the $E_4$-term of our spectral sequence. It is far smaller than $E_2$ and the results of Section 12 enable us to ignore most of it. What is left is small enough to be shown legibly in the spectral sequence charts of Figures 14 and 16. They illustrate integrally graded (as opposed to RO$(C_4)$-graded) spectral sequences of Mackey functors, which are discussed in Section 5. In order to read these charts one needs to refer to Table 2 which defines the “hieroglyphic” symbols we use for the specific Mackey functors that we need.

We finish the calculation in Section 14 by dealing with the remaining differentials and exotic Mackey functor extensions. It turns out that they are all formal consequences of $C_2$ differentials of the previous section along with the results of Section 4.

The result is a complete description of the integrally graded portion of $\pi_* k_{[2]}$. It is best seen in the spectral sequence charts of Figures 14 and 16. Unfortunately, we do not have a clean description, much less an effective way to display the full RO$(C_4)$-graded homotopy groups.

For $G = C_2$, the two irreducible orthogonal representations are the trivial one of degree 1, denoted by $\sigma$. Thus RO$(G)$ is additively a free abelian group of rank 2, and the spectral sequence of interest is trigraded. In the RO$(C_2)$-graded homotopy of $K_R$, a certain element of degree $1 + \sigma$ (the degree of the regular representation $\rho_2$) is invertible. This means that each component of $\pi_* K_R$ is canonically isomorphic to a Mackey functor indexed by an ordinary integer. See Theorem 8.6 for a more precise statement. Thus the full (trigraded) RO$(C_2)$-graded slice spectral sequence is determined by bigraded one shown in Figure 7.

For $G = C_4$, the representation ring RO$(G)$ is additively a free abelian group of rank 3, so it leads to a quadigraded spectral sequence. The three irreducible representations are the trivial and sign representations 1 and $\sigma$ (each having degree one) and a degree two representation $\lambda$ given by a rotation of the plane $\mathbb{R}^2$ of order 4. The regular representation $\rho_4$ is isomorphic to $1 + \sigma + \lambda$. As in the case of $K_R$, there is an invertible element $\delta_1$ (see Table 3) in $\pi_* K_{[2]}$ of degree $\rho_4$. This means we can reduce the quadigraded slice spectral sequence to a trigraded one, but finding a full description of it is a problem for the future.

# 2 Recollections about equivariant stable homotopy theory

We first discuss some structure on the equivariant homotopy groups of a $G$-spectrum $X$. We will assume throughout that $G$ is a finite cyclic $p$-group. This means that its subgroups are well ordered by inclusion and each is uniquely determined by its order. The results of this section hold for any prime $p$, but the rest of the paper concerns only the case $p = 2$. We will define several maps indexed by pairs of subgroups of $G$. We will often replace these indices by the orders of the subgroups, sometimes denoting $|H|$ by $h$. 
The homotopy groups can be defined in terms of finite $G$-sets $T$. Let
\[ \pi^G_*(T) = [T, , X]^G \]
be the set of homotopy classes of equivariant maps from $T_*$, the suspension spectrum of the union of $T$ with a disjoint base point, to the spectrum $X$. We will often omit $G$ from the notation when it is clear from the context. For an orthogonal representation $V$ of $G$, we define
\[ \pi^V_*(T) = [S^V \wedge T_*, , X]^G. \]
As an RO($G$)-graded contravariant abelian group valued functor of $T$, this converts disjoint unions to direct sums. This means it is determined by its values on the sets $G/H$ for subgroups $H \subseteq G$.

Since $G$ is abelian, $H$ is normal and $\pi^V_*(G/H)$ is a $\mathbb{Z}[G/H]$-module.

Given subgroups $K \subseteq H \subseteq G$, one has pinch and fold maps between the $H$-spectra $H/H_+$ and $H/K_+$. This leads to a diagram
\[
\begin{array}{ccc}
H/H_+ & \xrightarrow{\text{pinch}} & H/K_+ \\
\downarrow & & \downarrow \\
G/H_+ & \xrightarrow{\text{pinch}} & G/K_+ \\
G/H \rightarrow G_\wedge H/H_+ & \xrightarrow{\text{fold}} & G_\wedge H/K_+ & \xrightarrow{\text{fold}} & G_\wedge K/K_+ & \rightarrow G/K_+.
\end{array}
\]

Note that while the fold map is induced by a map of $H$-sets, the pinch map is not. It only exists in the stable category.

**Definition 2.2** (The Mackey functor structure maps in $\pi^G_*(X)$). The fixed point transfer and restriction maps
\[ \pi^G_*(G/H) \xrightarrow{\text{res}^H} \pi^G_*(G/K) \]
are the ones induced by the composite maps in the bottom row of (2.1).

These satisfy the formal properties needed to make $\pi^G_*(X)$ into a Mackey functor; see [6, Definition 3.1]. They are usually referred to simply as the transfer and restriction maps. We use the words “fixed point” to distinguish them from another similar pair of maps specified below in Definition 2.11.

We remind the reader that a Mackey functor $M$ for a finite group $G$ assigns an abelian group $M(T)$ to every finite $G$-set $T$. It converts disjoint unions to direct sums. It is therefore determined by its values on orbits, meaning $G$-sets for the form $G/H$ for various subgroups $H$ of $G$. For subgroups $K \subseteq H \subseteq G$, one has a map of $G$-sets $G/K \rightarrow G/H$. In categorical language $M$ is actually a pair of functors, one covariant and one contravariant, both behaving the same way on objects. Hence we get maps both ways between $M(G/K)$ and $M(G/H)$. For the Mackey functor $\pi^G_*(X)$, these are the two maps of Definition 2.2.

One can generalize the definition of a Mackey functor by replacing the target category of abelian groups by one’s favorite abelian category, such as that of $R$-modules over graded abelian groups.

**Definition 2.3.** A graded Green functor $R_*$ for a group $G$ is a Mackey functor for $G$ with values in the category of graded abelian groups such that $R_*(G/H)$ is a graded commutative ring for each subgroup $H$ and for each pair of subgroups $K \subseteq H \subseteq G$, the restriction map $\text{res}^H_K$ is a ring homomorphism and the transfer map $\text{tr}^H_K$ satisfies the Frobenius relation
\[ \text{tr}^H_K(\text{res}^H_K(a)b) = a(\text{tr}^H_K(b)) \quad \text{for } a \in R_*(G/H) \text{ and } b \in R_*(G/K). \]
When $X$ is a ring spectrum, we have the fixed point Frobenius relation
\[ \text{tr}^H_K(\text{res}^H_K(a)b) = a(\text{tr}^H_K(b)) \quad \text{for } a \in R_*(X(G/H)) \text{ and } b \in R_*(X(G/K)). \]
In particular, this means that
\[ a(\text{tr}_r^H(b)) = 0 \quad \text{when} \quad \text{res}_r^H(a) = 0. \]  
(2.5)

For a representation \( V \) of \( G \), the group
\[ \pi^G_V(X/GH) = \pi^H_V X = [S^V, X]^H \]
is isomorphic to
\[ [S^0, S^{-V} \wedge X]^H = \pi_0(S^{-V} \wedge X)^H. \]

However fixed points do not respect smash products, so we cannot equate this group with
\[ \pi_0(S^{-V} \wedge X)^H = \pi_0(S^{-V} \wedge X)^H = \pi^G_{S^{-V} \wedge X}(G/H). \]

Conversely a \( G \)-equivariant map \( S^V \to X \) represents an element in
\[ [S^V, X]^G = \pi^G_V X = \pi^G_{S^V}(G/G). \]

The following notion is useful.

**Definition 2.6 (Mackey functor induction and restriction).** For a subgroup \( H \) of \( G \) and an \( H \)-Mackey functor \( M \), the induced \( G \)-Mackey functor \( \uparrow^G_H M \) is given by
\[ \uparrow^G_H M(T) = M(i_H^* T) \]
for each finite \( G \)-set \( T \), where \( i_H^* \) denotes the forgetful functor from \( G \)-sets (or spaces or spectra) to \( H \)-sets.

For a \( G \)-Mackey functor \( N \), the restricted \( H \)-Mackey functor \( \downarrow^G_H N \) is given by
\[ \downarrow^G_H N(S) = N(G \times_H S) \]
for each finite \( H \)-set \( S \).

This notation is due to Thévenaz–Webb [10]. They put the decorated arrow on the right and denote \( G \times_H S \) by \( S \uparrow^G_H \) and \( i_H^* T \) by \( T \downarrow^G_H \).

We also need notation for \( X \) as an \( H \)-spectrum for subgroups \( H \subseteq G \). For this purpose we will enlarge the orthogonal representation ring of \( G \), \( RO(G) \), to the representation ring Mackey functor \( RO(G) \) defined by \( RO(G)(G/H) = RO(H) \). This was the motivating example for the definition of a Mackey functor in the first place.

In it the transfer map on a representation \( V \) of \( H \) is the induced representation of a supergroup \( K \supseteq H \), and its restriction to a subgroup is defined in the obvious way. In particular, the restriction of the transfer of \( V \) is \( |K/H| \cdot V \).

More generally for a finite \( G \)-set \( T \), \( RO(G)(T) \) is the ring (under pointwise direct sum and tensor product) of functors to the category of finite-dimensional orthogonal real vector spaces from \( B_G T \), the split groupoid (see [9, A1.1.22]) whose objects are the elements of \( T \) with morphisms defined by the action of \( G \).

**Definition 2.7 (RO(\( G \))-graded homotopy groups).** For each \( G \)-spectrum \( X \) and each pair \((H, V)\) consisting of a subgroup \( H \subseteq G \) and a virtual orthogonal representation \( V \) of \( H \), let the \( G \)-Mackey functor \( \pi_{H,V}(X) \) be defined by
\[ \pi_{H,V}(X)(T) := [(G \wedge T)^H \wedge T + X]^G = [S^V \wedge T + i_H^* X]^H = \pi^H_{S^V}(i_H^* X)(i_H^* T), \]
for each finite \( G \)-set \( T \). Equivalently, \( \pi_{H,V}(X) = \uparrow^G_H \pi^G_V(i_H^* X) \) (see 2.6) as Mackey functors. We will often denote \( \pi_{G,V} \) by \( \pi^G_V \) or \( \pi_V \).

We will be studying the \( RO(G) \)-graded slice spectral sequence \( (E^r_{s,*}) \) of Mackey functors with \( r, s \in \mathbb{Z} \) and \( * \in RO(G) \). We will use the notation \( E^s_{r,(H,V)} \) for such Mackey functors, abbreviating to \( E^s_{r,V} \) when the subgroup is \( G \). Most of our spectral sequence charts will display the values of \( E^s_{r,t} \) for integral values of \( t \) only.

The following definition should be compared with [2, (2.3)].

**Definition 2.8 (An equivariant homeomorphism).** Let \( X \) be a \( G \)-space and \( Y \) an \( H \)-space for a subgroup \( H \subseteq G \).

We define the equivariant homeomorphism
\[ \tilde{u}^G_{H}(Y, X) : G \times_H (Y \times i_H^* X) \to (G \times_H Y) \times X \]
Lemma 2.12

As abelian groups, we have

\[ \bar{u}_H^G(Y, X) : G \vee_H (Y \wedge i^*_H Y) \to (G \vee_H Y) \wedge X \]

for a $G$-spectrum $X$ and $H$-spectrum $Y$. We will abbreviate

\[ \bar{u}_H^G(S^0, X) : G \vee_H i^*_H X \to G/H \wedge X \]

by $\bar{u}_H^G(X)$.

For representations $V$ and $V'$ of $G$ both restricting to $W$ on $H$, but having distinct restrictions to all larger subgroups, we define $u_{V \to V'} = \bar{u}_H^G(S^0) \bar{u}_H^G(S^1)^{-1}$, so the following diagram of equivariant homeomorphisms commutes:

\[
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
G \wedge_H S^W \\
\xrightarrow{\bar{u}_H^G(S^0)} \\
\xrightarrow{\bar{u}_H^G(S^1)} \\
\end{array}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
G/H \wedge S^V \\
\xleftarrow{\bar{u}_V^W(S^0)} \\
\xleftarrow{\bar{u}_V^W(S^1)} \\
\end{array}
\end{array}
\end{array}
\xrightarrow{u_{V \to V'}} \\
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
G \wedge_H S^W \\
\xrightarrow{\bar{u}_H^G(S^0)} \\
\xrightarrow{\bar{u}_H^G(S^1)} \\
\end{array}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
G/H \wedge S^V' \\
\xleftarrow{\bar{u}_V^W(S^0)} \\
\xleftarrow{\bar{u}_V^W(S^1)} \\
\end{array}
\end{array}
\end{array}
\end{array}
\]

When $V' = |V|$ (meaning that $H = G_V$ acts trivially on $W$), then we abbreviate $u_{V \to V'}$ by $u_V$.

If $V$ is a representation of $H$ restricting to $W$ on $K$, we can smash the diagram (2.1) with $S^V$ and get

\[ S^V \xrightarrow{\text{pinch}} H/K_+ \wedge S^V \]

(2.10)

where the homeomorphism is induced by that of Definition 2.8.

Definition 2.11 (The group action restriction and transfer maps). For subgroups $K \subseteq H \subseteq G$, let $V \in RO(H)$ be a virtual representation of $H$ restricting to $W \in RO(K)$. The group action transfer and restriction maps

\[ \bar{r}_H^G(i^*_H X) = \mathbb{P}_{H,V} \mathbb{P}_{K,W} \bar{t}_K^H(i^*_K X) \]

(see 2.6) are the ones induced by the composite maps in the bottom row of (2.10). The symbols $t$ and $r$ here are underlined because they are maps of Mackey functors rather than maps within Mackey functors.

We include $V$ as an index for the group action transfer $t_k^H V$ because its target is not determined by its source. Thus we have abelian groups $\mathbb{P}_{H',V}(X)(G/H'')$ for all subgroups $H', H'' \subseteq G$ and representations $V$ of $H'$. Most of them are redundant in view of Theorem 2.13 below. In what follows, we will use the notation $H_0 = H' \cup H''$ and $H_\cap = H' \cap H''$.

Lemma 2.12 (An equivariant module structure). For a $G$-spectrum $X$ and $H'$-spectrum $Y$,

\[ [G \wedge_{H'} Y, X]^{H''} = \mathbb{Z}[G/H_\cup] \otimes [H_0 \wedge_{H'} Y, X]^{H''} \]

as $\mathbb{Z}[G/H'']$-modules.

Proof. As abelian groups,

\[ [G \wedge_{H'} Y, X]^{H''} = [i^*_H \sigma_{H'}^G(G \wedge_{H'} Y), X]^{H''} = \bigvee_{[G/H_\cup]} H_\cup \wedge_{H'} Y, X]^{H''} = \bigoplus_{[G/H_\cup]} [H_\cup \wedge_{H'} Y, X]^{H''} \]

and $G/H''$ permutes the wedge summands of $\bigvee_{[G/H_\cup]} H_\cup \wedge_{H'} Y$ as it permutes the elements of $G/H_\cup$. \qed
**Theorem 2.13** (The module structure for $RO(G)$-graded homotopy groups). For subgroups $H', H'' \subseteq G$ with $H_0 = H' \cup H''$ and $H_\gamma = H' \cap H''$, and a virtual representation $V$ of $H'$ restricting to $W$ on $H_\gamma$,

$$\pi_{H', V}(G/H'') \equiv [G/H_\gamma] \otimes \pi_{H_\gamma, W}X(G/G) \equiv \mathbb{Z}[G/H_\gamma] \otimes \pi_{H_\gamma}^* \pi_{H_\gamma, W}X(H_\gamma/H_\gamma)$$

as $\mathbb{Z}[G/H'']$-modules.

Suppose that $H''$ is a proper subgroup of $H'$ and $\gamma \in H'$ is a generator. Then as an element in $\mathbb{Z}[G/H'']$, $\gamma$ induces multiplication by $-1$ in $\pi_{H', V}(G/H'')$ if and only if $V$ is nonorientable.

**Proof.** We start with the definition and use the homeomorphism of Definition 2.8 and the module structure of Lemma 2.12:

$$\pi_{H', V}(G/H'') = [(G_+ \wedge H^* H'' \wedge X/H', X)^G$$

$$= [G_+ \wedge H'' (G_+ \wedge H^* S^V), X]^G$$

$$= [G_+ \wedge H'' S^V, X]^{H''} = \mathbb{Z}[G/H_\gamma] \otimes [H_\gamma \wedge H^* S^V, X]^{H''},$$

$$[H_\gamma \wedge H^* S^V, X]^{H''} = [S^W, X]^{H_\gamma}$$

$$= [G_+ \wedge H_\gamma S^W, X]^G$$

$$= \pi_{H_\gamma}^* \pi_{H_\gamma, W}(H_\gamma/H_\gamma) = \pi_{H_\gamma, W}X(G/G).$$

For the statement about nonoriented $V$, we have

$$\pi_{H', V}(G/H'') = \mathbb{Z}[G/H'] \otimes \mathbb{Z}[G/H'']^{H''} \otimes \pi_{H_\gamma}^* \pi_{H_\gamma, W}(H_\gamma/H_\gamma) = \mathbb{Z}[G/H'] \otimes [S^W, X]^{H''}.$$ 

Then $\gamma$ induces a map of degree $\pm 1$ on the sphere depending on the orientability of $V$. \qed

Theorem 2.13 means that we need only consider the groups

$$\pi_{H, V}(G/G) \equiv \mathbb{Z}^H \otimes \mathbb{Z}^{H_\gamma} \pi_{H_\gamma, W}(H/H).$$

When $H \subset G$ and $V$ is a virtual representation of $G$, we have

$$\pi_{H, V}(G/H) \equiv \pi_{H, V}(G/G) \equiv \mathbb{Z}^H \otimes \mathbb{Z}^{H_\gamma} \pi_{H_\gamma, W}(H/H). \quad (2.14)$$

This isomorphism makes the following diagram commute for $K \subseteq H$:

$$\begin{array}{ccc}
\pi_V X(G/K) & \cong & \pi_{H', V} X(G/H) \\
\downarrow \text{res}_V & & \downarrow \text{res}_{H/H} \\
\pi_V X(G/H) & \cong & \pi_{H', V}^* \pi_{H_\gamma, W} X(H/H) \\
\end{array}$$

We will use the three groups of (2.14) interchangeably as convenient and use the same notation for elements in each related by this canonical isomorphism. Note that the group on the left is indexed by $RO(G)$ while the two on the right are indexed by $RO(H)$. This means that if $V$ and $V'$ are representations of $G$ each restricting to $W$ on $H$, then $\pi_V X(G/H)$ and $\pi_{V'} X(G/H)$ are canonically isomorphic. The first of these is

$$[G/H_\gamma \wedge S^V, X]^G \equiv [G_+ \wedge H_\gamma S^W, X]^G \equiv [S^W, \pi_{H_\gamma}^* X]^H,$$

where the first isomorphism is induced by the homeomorphism $\bar{\nu}_V^H(X)$ of Definition 2.8 and the second is the fact that $G_+ \wedge H_\gamma (\cdot)$ is the left adjoint of the forgetful functor $\pi_{H_\gamma}^*$. 

**Remark 2.15** (Factorization via restriction). For a ring spectrum $X$, such as the one we are studying in this paper, an indecomposable element in $\pi_V X(G/H)$ may map to a product $xy \in \pi_{H_\gamma} X(G/G)$ of elements in groups indexed by representations of $H$ that are not restrictions of representations of $G$. When this happens we may denote the indecomposable element in $\pi_V X(G/H)$ by $[xy]$. This factorization can make some computations easier.
3 The RO(\(G\))-graded homotopy of \(HZ\)

We describe part of the RO(\(G\))-graded Green functor \(\pi_\ast(HZ)\), where \(HZ\) is the integer Eilenberg–Mac Lane spectrum \(HZ\) in the \(G\)-equivariant category, for some finite cyclic 2-group \(G\). For each actual (as opposed to virtual) \(G\)-representation \(V\) we have an equivariant reduced cellular chain complex \(C^V_\ast\) for the space \(S^V\). It is a complex of \(\mathbb{Z}[G]\)-modules with \(H_\ast(C^V_\ast) = H_\ast(S^V)\).

One can convert such a chain complex \(C^V_\ast\) of \(\mathbb{Z}[G]\)-modules to one of Mackey functors as follows. Given a \(\mathbb{Z}[G]\)-module \(M\), we get a Mackey functor \(M\) defined by

\[
M(G/H) = M^H \quad \text{for each subgroup } H \subseteq G.
\]  

(3.1)

We call this a fixed point Mackey functor. In it each restriction map \(\text{res}_{K}^H\) (for \(K \subseteq H \subseteq G\)) is one-to-one. When \(M\) is a permutation module, meaning the free abelian group on a \(G\)-set \(B\), we call \(M\) a permutation Mackey functor [6, Section 3.2].

In particular, the \(\mathbb{Z}[G]\)-module \(\mathbb{Z}\) with trivial group action (the free abelian group on the \(G\)-set \(G/G\)) leads to a Mackey functor \(\mathbb{Z}\) in which each restriction map is an isomorphism and the transfer map \(\text{tr}_K^H\) is multiplication by \(|H/K|\). For each Mackey functor \(M\) there is an Eilenberg–Mac Lane spectrum \(HM\) (see [5, Section 5]), and \(HZ\) is the same as \(HZ\) with trivial group action.

Given a finite \(G\)-CW spectrum \(X\), meaning one built out of cells of the form \(G_\ast \wedge_H e^n\), we get a reduced cellular chain complex of \(\mathbb{Z}[G]\)-modules \(C_\ast X\), leading to a chain complex of fixed point Mackey functors \(C_\ast X\).

Its homology is a graded Mackey functor \(H_\ast X\) with

\[
H_\ast X(G/H) = \pi_\ast(X \wedge HZ)(G/H) = \pi_\ast(X \wedge HZ)^H.
\]

In particular, \(H_\ast X(G/\{e\}) = H_\ast X\), the underlying homology of \(X\). In general \(H_\ast X(G/H)\) is not the same as \(H_\ast \pi(X^H)\) because fixed points do not commute with smash products.

For a finite cyclic 2-group \(G = C_{2^k}\), the irreducible representations are the 2-dimensional ones \(\lambda(m)\) corresponding to rotation through an angle of \(2\pi m/2^k\) for \(0 < m < 2^{k-1}\), the sign representation \(\sigma\) and the trivial one of degree one, which we denote by \(1\). The 2-local equivariant homotopy type of \(S^{\lambda(m)}\) depends only on the 2-adic valuation of \(m\), so we will only consider \(\lambda(2^j)\) for \(0 \leq j \leq k - 2\) and denote it by \(\lambda_j\). The planar rotation \(\lambda_{k-1}\) through angle \(\pi\) is the same representation as \(2\sigma\). We will denote \(\lambda(1) = \lambda_0\) simply by \(\lambda\).

We will describe the chain complex \(C^V_\ast\) for

\[
V = a + b\sigma + \sum_{2^i \leq j \leq 2^k} c_j \lambda_{k-j}
\]

for nonnegative integers \(a\), \(b\), and \(c_j\). The isotropy group of \(V\) (the largest subgroup fixing all of \(V\)) is

\[
G_\ast V = \begin{cases} 
C_{2^k} = G & \text{for } b = c_2 = \cdots = c_k = 0, \\
C_{2^k-1} =: G' & \text{for } b > 0 \text{ and } c_2 = \cdots = c_k = 0, \\
C_{2^k-2} & \text{for } c_\ell > 0 \text{ and } c_{1+\ell} = \cdots = c_k = 0.
\end{cases}
\]

The sphere \(S^V\) has a \(G\)-CW structure with reduced cellular chain complex \(C^V_\ast\) of the form

\[
C^V_n = \begin{cases} 
\mathbb{Z} & \text{for } n = d_0, \\
\mathbb{Z}[G/G'] & \text{for } d_0 \leq n \leq d_1, \\
\mathbb{Z}[G/C_{2^{k-1}}] & \text{for } d_{j-1} \leq n \leq d_j \text{ and } 2 \leq j \leq \ell, \\
0 & \text{otherwise},
\end{cases}
\]

(3.2)

where

\[
d_j = \begin{cases} 
a & \text{for } j = 0, \\
a + b & \text{for } j = 1, \\
a + b + 2c_2 + \cdots + 2c_j & \text{for } 2 \leq j \leq \ell,
\end{cases}
\]

so \(d_\ell = |V|\).
The boundary map $\partial_n : C_n^V \to C_{n-1}^V$ is determined by the fact that $H_*(C^V) = H_*(\Sigma^V)$. More explicitly, let $y$ be a generator of $G$ and

$$\zeta_j = \sum_{0 \leq t < c} y^t \quad \text{for } 1 \leq j \leq k.$$ 

Then we have

$$\partial_n = \begin{cases} 
V & \text{for } n = 1 + d_0, \\
(1 - y)x_n & \text{for } n = d_0 \text{ even and } 2 + d_0 \leq n \leq d_n, \\
x_n & \text{for } n = d_0 \text{ odd and } 2 + d_0 \leq n \leq d_n, \\
0 & \text{otherwise,}
\end{cases}$$

where $V$ is the fold mapping sending $y \mapsto 1$, and $x_n$ denotes multiplication by an element in $\mathbb{Z}[G]$ to be named below. We will use the same symbol below for the quotient map $\mathbb{Z}[G/H] \to \mathbb{Z}[G/K]$ for $H \subseteq K \subseteq G$. The elements $x_n \in \mathbb{Z}[G]$ for $2 + d_0 \leq n \leq |V|$ are determined recursively by $x_{2+d_0} = 1$ and

$$x_n x_{n-1} = \zeta_j \quad \text{for } 2 + d_{j-1} < n \leq 2 + d_j.$$ 

It follows that $H_{|V|}C^V = \mathbb{Z}$ generated by either $x_{1+|V|}$ or its product with $1 - y$, depending on the parity of $b$. This complex is

$$C^V = \Sigma^{V_0} C^{V_0},$$

where $V_0 = V^G$. This means we can assume without loss of generality that $V_0 = 0$. An element

$$x \in H_n C^V(G/H) = H_n S^V(G/H)$$

corresponds to an element $x \in \pi_{n-V} H\mathbb{Z}(G/H)$.

We will denote the dual complex $\text{Hom}_G(C^V, \mathbb{Z})$ by $C^V$. Its chains lie in dimensions $-n$ for $0 \leq n \leq |V|$. An element $x \in H_n(\Sigma^V)(G/H)$ corresponds to an element $x \in \pi_{n-V} H\mathbb{Z}(G/H)$.

The method we have just described determines only a portion of the $RO(G)$-graded Mackey functor $\pi_{(G,v)} H\mathbb{Z}$, namely the groups in which the index differs by an integer from an actual representation $V$ or its negative. For example, it does not give us $\pi_{n-1} H\mathbb{Z}$ for $|G| \geq 4$.

We leave the proof of the following as an exercise for the reader.

**Proposition 3.3** (The top (bottom) homology groups for $S^V (S^{-V})$). Let $G$ be a finite cyclic 2-group and $V$ a non-trivial representation of $G$ of degree $d$ with $V^G = 0$ and isotropy group $G_V$. Then

$$C_d^V = C_{-d}^V = \mathbb{Z}[G/G_V]$$

and the following hold:

(i) If $V$ is oriented, then $H_d S^V = \mathbb{Z}$, the constant $\mathbb{Z}$-valued Mackey functor in which each restriction map is an isomorphism and each transfer $\text{tr}^K_H$ is multiplication by $|K/H|$.

(ii) $H_{-d} S^{-V} = \mathbb{Z}(G, G_V)$, the constant $\mathbb{Z}$-valued Mackey functor in which

$$\text{res}^K_H = \begin{cases} 
1 & \text{for } K \subseteq G_V, \\
|K/H| & \text{for } G_V \subseteq H,
\end{cases}$$

and

$$\text{tr}^K_H = \begin{cases} 
|K/H| & \text{for } K \subseteq G_V, \\
1 & \text{for } G_V \subseteq H.
\end{cases}$$

(The above completely describes the cases where $|K/H| = 2$, and they determine all other restrictions and transfers.) The functor $\mathbb{Z}(G, e)$ is also known as the dual $\mathbb{Z}^*$. These isomorphisms are induced by the maps

$$\begin{array}{ccc}
H_d S^V & \cong & H_{-d} S^{-V} \\
\mathbb{Z} & \overset{\Lambda}{\longrightarrow} & \mathbb{Z}(G/G_V) & \overset{\nu}{\longrightarrow} & \mathbb{Z}(G, G_V).
\end{array}$$
We also have

\[ H_i \cdot dS^V = Z_\cdot \]

where each restriction map \( \text{res}_H^K \) is an isomorphism and each transfer \( \text{tr}_H^K \) is multiplication by \(|K/H|\) for each proper subgroup \( K \).

(iv) We also have \( H_i \cdot dS^V = Z(G, G_V)_\cdot \), where

\[ Z(G, G_V)_\cdot (G/H) = \begin{cases} 0 & \text{for } H = G, \\ Z_\cdot := Z(G)/(1 + \gamma) & \text{otherwise}, \end{cases} \]

with the same restrictions and transfers as \( Z(G, G_V) \). These isomorphisms are induced by the maps

\[ H_i dS^V \quad \text{and} \quad H_i dS^V \]

The Mackey functor \( Z(G, G_V) \) is one of those defined (with different notation) in [7, Definition 2.1].

**Definition 3.4** (Three elements in \( \pi_*^G(HZ) \)). Let \( V \) be an actual (as opposed to virtual) representation of the finite cyclic 2-group \( G \) with \( V^G = 0 \) and isotropy group \( G_V \).

(i) The equivariant inclusion \( S^0 \rightarrow S^V \) defines an element in \( \pi_*^V S^0(G/G) \) via the isomorphisms

\[ \pi_*^V S^0(G/G) = \pi_0^V S^V(G/G) = \pi_0^V S^0 = \pi_0 S^0 = Z, \]

and we will use the symbol \( a_V \) to denote its image in \( \pi_*^V HZ(G/G) \).

(ii) The underlying equivalence \( S^V \rightarrow S^V \) defines an element in

\[ \pi_*^V S^V(G/G_V) = \pi_*^{V-}[V] S^0(G/G_V) \]

and we will use the symbol \( e_V \) to denote its image in \( \pi_*^{V-}[V] HZ(G/G_V) \).

(iii) If \( W \) is an oriented representation of \( G \) (we do not require that \( W^G = 0 \)), there is a map

\[ \Delta : Z \rightarrow c_1^W_{[W]} = Z[G/G_W] \]

as in Proposition 3.3 giving an element

\[ u_W \in H_{[W]} S^W(G/G) = \pi_{[W]-w} HZ(G/G). \]

For nonoriented \( W \), Proposition 3.3 gives a map

\[ \Delta_\cdot : Z_\cdot \rightarrow c_1^W_{[W]} \]

and an element

\[ u_W \in H_{[W]} S^W(G/G') = \pi_{[W]-w} HZ(G/G'). \]

The element \( u_W \) above is related to the element \( \tilde{u}_V \) of (2.9) as follows.

**Lemma 3.5** (The restriction of \( u_W \) to a unit and permanent cycle). Let \( W \) be a nontrivial representation of \( G \) with \( H = G_W \). Then the homeomorphism

\[ \Sigma \cdot u_W : G/H_+ \wedge S^{[W]-W} \rightarrow G/H_+ \]

of (2.9) induces an isomorphism \( \pi_* HZ(G/H) \rightarrow \pi_{[W]-W} HZ(G/H) \) sending the unit to \( \text{res}_H^K(u_W) \) for \( u_W \) as defined in (iii) above and \( K = G \) or \( G' \) depending on the orientability of \( W \).

The product

\[ \text{res}_H^K(u_W) e_W \in \pi_* HZ(G/H) = Z \]

is a generator, so \( e_W \) and \( \text{res}_H^K(u_W) \) are units in the ring \( \pi_* HZ(G/H) \), and \( \text{res}_H^K(u_W) \) is in the Hurewicz image of \( \pi_* S^0(G/H) \).
Proof. The diagram
\[
G/K \wedge S^{[W]-W} \xleftarrow{\text{fold}} G/H \wedge S^{[W]-W} \xrightarrow{u_W} G/H
\]
induces (via the functor $[-, H\mathbb{Z}^G]$)
\[
\begin{array}{ccc}
\pi_{[W]-W} H\mathbb{Z}(G/K) & \xrightarrow{\text{res}_{G}} & \pi_{[W]-W} H\mathbb{Z}(G/H) \\
\downarrow & & \downarrow \\
H_{[W]} S^W(G/K) & \xrightarrow{=} & H_{[W]} S^W(G/H) \\
\end{array}
\]
The restriction map is an isomorphism by Proposition 3.3 and the group on the left is generated by $u_W$.

The product is the composite of $H$-maps
\[
S^W \xrightarrow{e_W} S^{[W]} \xrightarrow{\text{res}_{G}} \Sigma^W H\mathbb{Z},
\]
which is the standard inclusion. \hfill \Box

Note that $a_V$ and $e_V$ are induced by maps to equivariant spheres while $u_W$ is not. This means that in any spectral sequence based on a filtration where the subquotients are equivariant $H\mathbb{Z}$-modules, elements defined in terms of $a_V$ and $e_V$ will be permanent cycles, while multiples and powers of $u_W$ can support nontrivial differentials. Lemma 3.5 says a certain restriction of $u_W$ is a permanent cycle.

Each nonoriented $V$ has the form $W + \sigma$ where $\sigma$ is the sign representation and $W$ is oriented. It follows that
\[
u_V = a_0 \text{ res}_{G}^\sigma(u_W) \in \pi_{[W]-V} H\mathbb{Z}(G/G').
\]
Note also that $a_0 = e_0 = u_0 = 1$. The trivial representations contribute nothing to $\pi_*(H\mathbb{Z})$. We can limit our attention to representations $V$ with $V^G = 0$. Among such representations of cyclic 2-groups, the oriented ones are precisely the ones of even degree.

Lemma 3.6 (Properties of $a_V, e_V$ and $u_W$). The three elements $a_V \in \pi_{-V} H\mathbb{Z}(G/G)$, $e_V \in \pi_{-V-|V|} H\mathbb{Z}(G/G)$ and $u_W \in \pi_{-W-|V|} H\mathbb{Z}(G/G)$ for $W$ oriented of Definition 3.4 satisfy the following:

1. $a_{V+W} = a_V a_W$ and $u_{V+W} = u_V u_W$.
2. $|G/G_V| a_V = 0$, where $G_V$ is the isotropy group of $V$.
3. For oriented $V$, $\text{tr}_{G_V}^G (e_V)$ and $\text{tr}_{G_V}^G (e_{V+\sigma})$ have infinite order, while $\text{tr}_{G_V}^G (e_{V+\sigma})$ has order 2 if $|V| > 0$ and $\text{tr}_{G_V}^G (e_{\sigma}) = \text{tr}_{G_V}^G (e_0) = 0$.
4. For oriented $V$ and $G_V \subseteq H \subseteq G$,
\[
\text{tr}_{G_V}^G (e_V) u_V = |G/G_V| \in \pi_{0} H\mathbb{Z}(G/G) = \mathbb{Z},
\]
\[
\text{tr}_{G_V}^G (e_{V+\sigma}) u_{V+\sigma} = |G'/G_V| \in \pi_{-|V|} H\mathbb{Z}(G'/G') = \mathbb{Z}
\]
for $|V| > 0$.

5. $a_{V+W} \text{tr}_{G_V}^G (e_{V+\sigma}) = 0$ if $|V| > 0$.
6. For $V$ and $W$ oriented, $u_W \text{tr}_{G_V}^G (e_{V+W}) = |G_V/G_V+W| \text{tr}_{G_V}^G (e_V)$.
7. The gold (or au) relation. For $V$ and $W$ oriented representations of degree 2 with $G_V \subseteq G_W$, $a_W u_V = |G_W/G_V| a_V u_W$.

For nonoriented $W$ similar statements hold in $\pi_{*} H\mathbb{Z}(G/G')$. Moreover, $2W$ is oriented and $u_{2W}$ is defined in $\pi_{2|W|-2W} H\mathbb{Z}(G/G)$ with $\text{res}_{G}^{2W} (u_{2W}) = u_{2W}^2$.

Proof. (1) This follows from the existence of the pairing $C^V \otimes C^W \rightarrow C^{V+W}$. It induces an isomorphism in $H_0$ and (when both $V$ and $W$ are oriented) in $H_{[V+W]}$.

(2) This holds because $H_0(V)$ is killed by $|G/G_V|$.

(3) This follows from Proposition 3.3.

(4) Using the Frobenius relation we have
\[
\text{tr}_{G_V}^G (e_V) u_V = \text{tr}_{G_V}^G (e_V \text{ res}_{G_V}^\sigma (u_V)) = \text{tr}_{G_V}^G (1) \text{ by Lemma 3.5}
\]
\[
= |G/G_V|,
\]
and
\[ \text{tr}_{G/\nu \sigma}^G(e_{\nu \sigma})u_{\nu \sigma} = \text{tr}_{G/\nu \sigma}^G(e_{\nu \sigma} \text{res}_{G/\nu}^G(u_{\nu \sigma})) = \text{tr}_{G/\nu}^G(1) = |G'/G\nu|. \]

(5) We have
\[ a_{\nu \nu} \text{tr}_{G/\nu}^G(e_{\nu U}) : S^{-|V| - |U|} \to S^{U - U}. \]
It is null because the bottom cell of \( S^{U - U} \) is in dimension \(-|U|\).

(6) Since \( V \) is oriented, we are computing in a torsion free group so we can tensor with the rationals. It follows from (4) that
\[ \text{tr}_{G/\nu \nu}^G(e_{\nu \nu} \nu) = \frac{|G/G_{\nu \nu}|}{u_{\nu \nu} u_{\nu \nu}} \quad \text{and} \quad \text{tr}_{G/\nu}^G(e_{\nu}) = \frac{|G/G\nu|}{u_{\nu}} \]
so
\[ u_{\nu} \text{tr}_{G/\nu \nu}^G(e_{\nu \nu} \nu) = \frac{|G/G_{\nu \nu}|}{u_{\nu}} = |G/G_{\nu \nu}| \text{tr}_{G/\nu}^G(e_{\nu}). \]

(7) For \( G = C_{2^n} \), each oriented representation of degree 2 is \( 2 \)-locally equivalent to a \( \lambda_j \) for \( 0 \leq j < n \). The isotropy group is \( G_{\lambda_j} = C_{2^j} \). Hence the assumption that \( G_V \subset G_W \) is can be replaced with \( V = \lambda_j \) and \( W = \lambda_k \) with \( 0 \leq j < k < n \). The statement we wish to prove is
\[ a_{\lambda_k} u_{\lambda_j} = 2^{k-j} a_{\lambda_j} u_{\lambda_k}. \]

One has a map \( S^{\lambda_j} \to S^{\lambda_k} \) which is the suspension of the \( 2^{k-j} \)th power map on the equatorial circle. Hence its underlying degree is \( 2^{k-j} \). We will denote it by \( a_{\lambda_k} / a_{\lambda_j} \) since there is a diagram

\[ \begin{array}{ccc} S^0 & \xrightarrow{a_{\lambda_j}} & S^{\lambda_j} \\ \downarrow{a_{\lambda_k}} & & \downarrow{a_{\lambda_k}/a_{\lambda_j}} \\ S^{\lambda_k} & \xrightarrow{a_{\lambda_j}} & S^{\lambda_k}. \end{array} \]

We claim there is a similar diagram

\[ \begin{array}{ccc} S^2 & \xrightarrow{u_{\lambda_j}} & S^{\lambda_k} \wedge HZ \\ \downarrow{u_{\lambda_k}} & & \downarrow{u_{\lambda_k}/u_{\lambda_j}} \\ S^{\lambda_k} \wedge HZ & \xrightarrow{u_{\lambda_j}} & S^{\lambda_k} \wedge HZ. \end{array} \]

in which the underlying degree of the vertical map is one.

Smashing \( a_{\lambda_k} / a_{\lambda_j} \) with \( HZ \) and composing with \( u_{\lambda_k} / u_{\lambda_j} \) gives a factorization of the degree \( 2^{k-j} \) map on \( S^{\lambda_k} \wedge HZ \). Thus we have
\[ \frac{u_{\lambda_k} a_{\lambda_k}}{u_{\lambda_j}} = 2^{k-j}, \]
\[ u_{\lambda_j} a_{\lambda_k} = 2^{k-j} u_{\lambda_k} a_{\lambda_j}, \]
as desired.

The vertical map in (3.7) would follow from a map
\[ S^{\lambda_k - \lambda_j} \to HZ \]
with underlying degree one. Let \( G = C_{2^n} \) and \( G \supset H = C_{2^j} \). Then \( S^{-\lambda_j} \) has a cellular structure of the form
\[ G/H_+ \wedge S^{-2} \cup G/H_+ \wedge e^{-1} \cup e^0. \]

We need to smash this with \( S^{\lambda_k} \). Since \( \lambda_k \) restricts trivially to \( H \),
\[ G/H_+ \wedge S^{\lambda_k} = G/H_+ \wedge S^2. \]
This means
\[ S^{k_i - \lambda_i} = S^{k_i} \wedge S^{-\lambda_i} = G/H_+ \wedge S^0 \cup G/H_+ \wedge e^1 \cup e^0 \wedge S^{k_i}. \]

Thus its cellular chain complex has the form
\[
\begin{array}{ccc}
2 & & \mathbb{Z}[G/K] \\
| & & \downarrow 1-y \\
1 & & \mathbb{Z}[G/K] \rightarrow \mathbb{Z}[G/H]/\Delta \\
| & & \downarrow 1-y \\
0 & & \mathbb{Z} \rightarrow \mathbb{Z}[G/H]
\end{array}
\]

where \( K = G/C_{p^k} \) and the left column is the chain complex for \( S^{k_i} \).

There is a corresponding chain complex of fixed point Mackey functors. Its value on the \( G \)-set \( G/L \) for an arbitrary subgroup \( L \) is
\[
\begin{array}{ccc}
2 & & \mathbb{Z}[G/\text{max}(K, L)] \\
| & & \downarrow 1-y \\
1 & & \mathbb{Z}[G/\text{max}(K, L)] \rightarrow \mathbb{Z}[G/\text{max}(H, L)]/\Delta \\
| & & \downarrow 1-y \\
0 & & \mathbb{Z} \rightarrow \mathbb{Z}[G/\text{max}(H, L)].
\end{array}
\]

For each \( L \) the map \( \Delta \) is injective and maps the kernel of the first \( 1-y \) isomorphically to the kernel of the second one. This means we can replace the above by a diagram of the form
\[
\begin{array}{ccc}
2 & & \text{coker}(1-y) \\
| & & \downarrow 1-y \\
1 & & \text{coker}(1-y) \rightarrow \mathbb{Z}/\Delta \\
| & & \downarrow 1-y \\
0 & & \mathbb{Z} \rightarrow \text{coker}(1-y),
\end{array}
\]

where each cokernel is isomorphic to \( \mathbb{Z} \) and each map is injective.

This means that \( H_+ S^{k_i - \lambda_i} \) is concentrated in degree 0 where it is the pushout of the diagram above, meaning a Mackey functor whose value on each subgroup is \( \mathbb{Z} \). Any such Mackey functor admits a map to \( \mathbb{Z} \) with underlying degree one. This proves the claim of (3.7). □

The \( \mathbb{Z} \)-valued Mackey functor \( H_0 S^{k_i - \lambda_i} \) is discussed in more detail in [7], where it is denoted by \( \mathbb{Z}(k, j) \).

## 4 Generalities on differentials and Mackey functor extensions

Before proceeding with a discussion about spectral sequences, we need the following.

**Remark 4.1** (Abusive spectral sequence notation). When \( d_r(x) \) is a nontrivial element of order 2, the elements \( 2x \) and \( x^2 \) both survive to \( E_{r+1} \), but in that group they are not the products indicated by these symbols since \( x \) itself is no longer present. More generally if \( d_r(x) = y \) and \( ay = 0 \) for some \( a \), then \( ax \) is present in \( E_{r+1} \).

This abuse of notation is customary because it would be cumbersome to rename these elements when passing from \( E_r \) to \( E_{r+1} \). We will sometimes denote them by \([2x]\), \([x^2]\) and \([ax]\) respectively to emphasize their indecomposability.

Now we make some observations about the relation between exotic transfers and restriction with certain differentials in the slice spectral sequence. By “exotic” we mean in a higher filtration. In a spectral sequence of Mackey functors converging to \( \pi_* X \), it can happen that an element \( x \in \pi_{r'}(G/H) \) has filtration \( s \), but its restriction or transfer has a higher filtration. In the spectral sequence charts in this paper, exotic transfers and restrictions will be indicated by blue and dashed green lines respectively.
Lemma 4.2 (Restriction kills \(a_\sigma\) and \(a_\delta\) kills transfers). Let \(G\) be a finite cyclic 2-group with sign representation \(\sigma\) and index 2 subgroup \(G'\), and let \(X\) be a \(G\)-spectrum. Then in \(\pi_\ast X(G/G)\) the image of \(\text{tr}^G_{G'}\) is the kernel of multiplication by \(a_\sigma\), and the kernel of \(\text{res}^G_{G'}\) is the image of multiplication by \(a_\delta\).

Suppose further that \(a\) divides the order of \(G\) and let \(\lambda\) be the degree 2 representation sending a generator \(y \in G\) to a rotation of order 4. Then restriction kills \(2a_\lambda\) and \(2a_\lambda\) kills transfers.

**Proof.** Consider the cofiber sequence obtained by smashing \(X\) with

\[
\Sigma^{-1} X \xrightarrow{a_\sigma} \Sigma^{-1} X / G' \xrightarrow{a_\delta} \Sigma^0 S^0 \xrightarrow{a_\sigma} S^0.
\]

Since \((G, \wedge G')^G\) is equivalent to \(X^{G'}\), passage to fixed point spectra gives

\[
\Sigma^{-1} X^{G'} \xrightarrow{a_\sigma} (\Sigma^{-1} X)^{G'} \xrightarrow{a_\delta} X^{G'} \xrightarrow{a_\sigma} (\Sigma^0 X)^{G'},
\]

so the exact sequence of homotopy groups is

\[
\pi_{k+1} X(G/G) \xrightarrow{a_\sigma} \pi_{k+1} X(G/G) \xrightarrow{a_\delta} \pi_k (G \wedge G') X(G/G) \xrightarrow{a_\delta} \pi_k X(G/G).
\]

Note that the isomorphism \(a_\sigma\) is invertible. This gives the exactness required by both statements.

For the statements about \(a_\lambda\), note that \(\lambda\) restricts to \(2\sigma_{G'}\), where \(\sigma_{G'}\) is the sign representation for the index 2 subgroup \(G'\). It follows that \(\text{res}^G_{G'}(a_\lambda) = a_\lambda^2\), which has order 2. Using the Frobenius relation, we have for \(x \in \pi_\ast X(G/G)\),

\[
2a_\lambda \text{tr}^G_{G'}(x) = \text{tr}^G_{G'}(\text{res}^G_{G'}(2a_\lambda)x) = \text{tr}^G_{G'}(2a_\lambda^2 x) = 0.
\]

This implies that when \(a_\sigma x\) is killed by a differential but \(x \in \pi_{r-1} X(G/G)\) is not, then \(x\) represents an element that is \(\text{tr}^G_{G'}(y)\) for some \(y\) in lower filtration. Similarly if \(x\) supports a nontrivial differential but \(a_\sigma x\) is a nontrivial permanent cycle, then the latter represents an element with a nontrivial restriction to \(G'\) of higher filtration. In both cases the converse also holds.

**Theorem 4.4 (Exotic transfers and restrictions in the RO\((G)\)-graded slice spectral sequence).** Let \(G\) be a finite cyclic 2-group with index 2 subgroup \(G'\) and sign representation \(\sigma\), and let \(X\) be a \(G\)-equivariant spectrum with \(x \in E^{s+r-1, V+r+\sigma-2}_r X(G/G)\) (for \(V \in \text{RO}(G)\)) in the slice spectral sequence for \(X\). Then:

(i) Suppose there is a permanent cycle \(y' \in E^{s+r-1, V+r+\sigma-2}_r X(G/G)\). Then there is a nontrivial differential

\[
d_r(x) = \text{tr}^G_{G'}(y')
\]

if and only if \([a_\sigma x]\) is a permanent cycle with \(\text{res}^G_{G'}(a_\sigma x) = u_\sigma y'\). In this case \([a_\sigma x]\) represents the Toda bracket \(\langle a_\sigma, \text{tr}^G_{G'}, y' \rangle\).

(ii) Suppose there is a permanent cycle \(y \in E^{s+r-1, V+r+\sigma-2}_r X(G/G)\). Then there is a nontrivial differential

\[
d_r(x) = a_\sigma y
\]

if and only if \(\text{res}^G_{G'}(x)\) is a permanent cycle with \(\text{tr}^G_{G'}(u_\sigma^{-1} \text{res}^G_{G'}(x)) = y\). In this case \(\text{res}^G_{G'}(x)\) represents the Toda bracket \(\langle \text{res}^G_{G'}, a_\sigma, y \rangle\).

In each case a nontrivial \(d_r\) is equivalent to a Mackey functor extension raising filtration by \(r - 1\). In (i) the permanent cycle \(a_\sigma x\) is not divisible in \(\pi_* X\) by \(a_\sigma\) and therefore could have a nontrivial restriction in a higher filtration. Similarly in (ii) the element denoted by \(\text{res}^G_{G'}(x)\) is not a restriction in \(\pi_* X\), so we cannot use the Frobenius relation to equate \(\text{tr}^G_{G'}(u_\sigma^{-1} \text{res}^G_{G'}(x))\) with \(\text{tr}^G_{G'}(u_\sigma^{-1})x\).

We remark that the proof below makes no use of any properties specific to the slice filtration. The result holds for any equivariant filtration with suitable formal properties.

Before giving the proof we need the following.
Lemma 4.5 (A formal observation). Suppose we have a commutative diagram up to sign

\[
\begin{array}{c}
A_{0,0} \xrightarrow{a_{0,0}} A_{0,1} \xrightarrow{a_{0,1}} A_{0,2} \xrightarrow{a_{0,2}} \Sigma A_{0,0} \\
\downarrow b_{0,0} \quad \quad \quad \quad \quad \downarrow b_{0,1} \quad \quad \quad \quad \quad \downarrow b_{0,2} \quad \quad \quad \quad \quad \downarrow b_{0,3} \\
A_{1,0} \xrightarrow{a_{1,0}} A_{1,1} \xrightarrow{a_{1,1}} A_{1,2} \xrightarrow{a_{1,2}} \Sigma A_{1,0} \\
\downarrow b_{1,0} \quad \quad \quad \quad \quad \downarrow b_{1,1} \quad \quad \quad \quad \quad \downarrow b_{1,2} \quad \quad \quad \quad \quad \downarrow b_{1,3} \\
A_{2,0} \xrightarrow{a_{2,0}} A_{2,1} \xrightarrow{a_{2,1}} A_{2,2} \xrightarrow{a_{2,2}} \Sigma A_{2,0} \\
\downarrow b_{2,0} \quad \quad \quad \quad \quad \downarrow b_{2,1} \quad \quad \quad \quad \quad \downarrow b_{2,2} \\
\Sigma A_{0,0} \xrightarrow{a_{0,0}} \Sigma A_{0,1} \xrightarrow{a_{0,1}} \Sigma A_{0,2} \xrightarrow{a_{0,2}} \Sigma^2 A_{0,0}
\end{array}
\]

in which each row and column is a cofiber sequence. Suppose that from some spectrum \( W \) we have a map \( f_3 \) and hypothetical maps \( f_1 \) and \( f_2 \) making the following diagram commute up to sign, where \( c_{i,j} = b_{i+1,j} a_{i,j} a_{i+1,j} b_{i,j} \):

\[
\begin{array}{c}
W \xrightarrow{f_3} \Sigma A_{0,0} \quad \quad \quad \quad \quad \downarrow c_{0,0} \\
\downarrow f_2 \quad \quad \quad \quad \quad \downarrow \quad \quad \quad \quad \quad \downarrow \quad \quad \quad \quad \quad \downarrow \\
A_{1,0} \xrightarrow{a_{1,0}} \xrightarrow{a_{1,1}} A_{1,1} \xrightarrow{a_{1,2}} \Sigma A_{1,0} \xrightarrow{a_{1,0}} \Sigma A_{1,1} \\
\downarrow b_{1,0} \quad \quad \quad \quad \quad \downarrow b_{1,1} \quad \quad \quad \quad \quad \downarrow b_{1,2} \\
A_{2,0} \xrightarrow{a_{2,0}} \xrightarrow{a_{2,1}} A_{2,1} \xrightarrow{a_{2,2}} \Sigma A_{2,0} \xrightarrow{a_{2,0}} \Sigma A_{2,1} \\
\downarrow b_{2,0} \quad \quad \quad \quad \quad \downarrow b_{2,1} \quad \quad \quad \quad \quad \downarrow b_{2,2} \\
\Sigma A_{0,0} \xrightarrow{a_{0,0}} \xrightarrow{a_{0,1}} \xrightarrow{a_{0,2}} \Sigma A_{0,1} \xrightarrow{a_{0,1}} \Sigma A_{0,2} \\
\downarrow c_{0,0} \quad \quad \quad \quad \quad \downarrow b_{0,1} \quad \quad \quad \quad \quad \downarrow b_{0,2} \\
\Sigma A_{1,0} \xrightarrow{a_{1,0}} \xrightarrow{a_{1,1}} \Sigma A_{1,1} \xrightarrow{a_{1,2}} \Sigma^2 A_{1,0}
\end{array}
\]

Then \( f_1 \) exists if and only if \( f_2 \) does. When this happens, \( c_{0,0} f_3 \) is null and we have Toda brackets

\[
\langle a_{1,1}, c_{0,0}, f_3 \rangle \ni f_2 \quad \text{and} \quad \langle b_{1,1}, c_{0,0}, f_3 \rangle \ni f_1.
\]

Proof. Let \( R \) be the pullback of \( a_{2,1} \) and \( b_{1,2} \), so we have a diagram

\[
\begin{array}{c}
A_{0,2} \xrightarrow{a_{0,2}} A_{0,2} \\
\downarrow b_{0,2} \\
A_{2,0} \xrightarrow{a_{2,0}} R \xrightarrow{c_{1,2}} A_{1,2} \xrightarrow{a_{1,2}} \Sigma A_{2,0} \\
\downarrow b_{1,2} \quad \quad \quad \quad \quad \downarrow \quad \quad \quad \quad \quad \downarrow \quad \quad \quad \quad \quad \downarrow \\
A_{2,0} \xrightarrow{a_{2,0}} A_{2,1} \xrightarrow{a_{2,1}} A_{2,2} \xrightarrow{a_{2,2}} \Sigma A_{2,0} \\
\downarrow c_{2,1} \quad \quad \quad \quad \quad \downarrow b_{1,2} \quad \quad \quad \quad \quad \downarrow b_{2,2} \\
\Sigma A_{0,2} \xrightarrow{a_{0,2}} \quad \quad \quad \quad \quad \Sigma A_{0,2}
\end{array}
\]

in which each row and column is a cofiber sequence. Thus we see that \( R \) is the fiber of both \( c_{1,2} \) and \( c_{2,1} \). If \( f_1 \) exists, then

\[
c_{2,1} f_1 = a_{0,1} b_{2,1} f_1 = a_{0,1} a_{0,0} f_3
\]

which is null homotopic, so \( f_1 \) lifts to \( R \), which comes equipped with a map to \( A_{1,2} \), giving us \( f_2 \). Conversely if \( f_2 \) exists, it lifts to \( R \), which comes equipped with a map to \( A_{2,1} \), giving us \( f_1 \).

The statement about Toda brackets follows from the way they are defined. \( \square \)
Proof of Theorem 4.4. For a $G$-spectrum $X$ and integers $a < b < c \leq \infty$ there is a cofiber sequence

$$P_{b+1}^c X \xrightarrow{i} P_b^c X \xrightarrow{j} P_0^b X \xrightarrow{k} \Sigma P_{b+1}^c X.$$ 

When $c = \infty$, we omit it from the notation. We will combine this and the one of (4.3) to get a diagram similar to (4.6) with $W = S^k$ to prove our two statements.

For (i) note that $x \in \pi_{-3}^G S^2 X(G/G)$ is by definition an element in $\pi_{-3}^G S^2 X(G/G)$. We will assume for simplicity that $s = 0$, so $x$ is represented by a map from some $S^V$ to $(P_0^b X)^G$. Its survival to $E_2$ and supporting a nontrivial differential means that it lifts to $(P_0^{b-2} X)^G$ but not to $(P_0^{b-1} X)^G$. The value of $d_i(x)$ is represented by the composite $kx$ in the diagram below, where we can use Lemma 4.5:

The commutativity of the lower left trapezoid is the differential of (i), $d_i(x) = tr_{w}^G(y')$. The existence of the map $w$ making the diagram commute follows from that of $x$ and $y'$. It is the representative of $a_s X$ as a permanent cycle, which represents the indicated Toda bracket. The commutativity of the upper right trapezoid identifies $y'$ as $u_s^{-1} res_{H}^G(x)$ as claimed. For the converse we have the existence of $y'$ and $w$ and hence that of $x$.

The second statement follows by a similar argument based on the diagram

Here $w$ represents $u^{-1}_s res_{H}^G(x)$ as a permanent cycle, so we get a Toda bracket containing $res_{H}^G(x)$ as indicated.

Next we study the way differentials interact with the norm. Suppose we have a subgroup $H \subset G$ and an $H$-equivariant ring spectrum $X$ with $Y = N_H^G X$. Suppose we have spectral sequences converging to $\pi_* X$ and $\pi_* Y$ based on towers

$$\ldots \rightarrow P^H_{m-1} X \rightarrow P^H_m X \rightarrow \ldots \quad \text{and} \quad \ldots \rightarrow P^G_{m-1} Y \rightarrow P^G_m Y \rightarrow \ldots$$

for functors $P^H_m$ and $P^G_m$ equipped with suitable maps

$$P^H_m X \wedge P^H_{m+n} X, \quad P^G_m Y \wedge P^G_{m+n} Y \quad \text{and} \quad N_{H}^G P^H_{m+n} X \rightarrow P^G_{m[G/H]} Y.$$ 

Our slice spectral sequence for each of the spectra studied in this paper fits this description.
Theorem 4.7 (The norm of a differential). Suppose we have spectral sequences as described above and a differential \(d_r(x) = y\) for \(x \in E^r_{s,t}(X/H/H)\). Let \(\rho = \text{Ind}_H^G 1\) and suppose that \(a_{\rho}\) has filtration \(|G/H| - 1\). Then in the spectral sequence for \(Y = N^G_H X\),

\[
d_{[G/H](r-1)+1}(a_{\rho}N^G_H x) = N^G_H y \in E^{[G/H](r-1)+1}_{s,t} X(G/H).
\]

Proof. The differential can be represented by a diagram

\[
\begin{array}{cccccccc}
S^V & \longrightarrow & S(1 + V) & \longrightarrow & D(1 + V) & \longrightarrow & S^{1+V} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\rho^H_s, X & \longrightarrow & \rho^H_s X & \longrightarrow & \rho^H_s X / \rho^H_{s+r}, X
\end{array}
\]

for some orthogonal representation \(V\) of \(H\), where each row is a cofiber sequence. We want to apply the norm functor \(N^G_H\) to it. Let \(W = \text{Ind}_H^G V\). Then we get

\[
\begin{array}{cccccccc}
S^W & \longrightarrow & N^G_H S(1 + V) & \longrightarrow & D(\rho + W) & \longrightarrow & S^{\rho+W} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\rho^H_s, X & \longrightarrow & \rho^H_s X & \longrightarrow & \rho^H_s X / \rho^H_{s+r}, X
\end{array}
\]

Neither row of this diagram is a cofiber sequence, but we can enlarge it to one where the top and bottom rows are, namely

\[
\begin{array}{cccccccc}
S^W & \longrightarrow & D(1 + W) & \longrightarrow & S^{1+W} \\
\downarrow & & \downarrow a_{\rho} & & \downarrow a_{\rho} \\
S^W & \longrightarrow & D(\rho + W) & \longrightarrow & S^{\rho+W} \\
\downarrow & & \downarrow N^G_H & & \downarrow N^G_H \\
N^G_H \rho^H_s, X & \longrightarrow & N^G_H \rho^H_s X & \longrightarrow & N^G_H (\rho^H_s X / \rho^H_{s+r}, X) \\
\downarrow & & \downarrow p^G_{(s+r)G/H} & & \downarrow p^G_{(s+r)G/H} \\
p^G_{s(G/H)} Y & \longrightarrow & p^G_{s(G/H)} Y & \longrightarrow & p^G_{s(G/H)} Y
\end{array}
\]

Here the first two bottom vertical maps are part of the multiplicative structure the spectral sequence is assumed to have. Composing the maps in the three columns gives us the diagram for the desired differential. \(\square\)

Given a \(G\)-equivariant ring spectrum \(X\), let \(X' = i_{\rho}X\) denote its restriction as an \(H\)-spectrum. Then we have \(N^G_{H} X' = X^{([G/H])}\) and the multiplication on \(X\) gives us a map from this smash product to \(X\). This gives us a map \(\pi_* X' \to \pi_* X\) called the internal norm, which we denote abusively by \(N^G_H\). The argument above yields the following.

Corollary 4.8 (The internal norm of a differential). With notation as above, suppose we have a differential \(d_r(x) = y\) for \(x \in E^r_{s,t}(X'(H/H))\). Then

\[
d_{[G/H](r-1)+1}(a_{\rho}N^G_{H} x) = N^G_{H} y \in E^{[G/H](r-1)+1}_{s,t} X(G/H).
\]

The following is useful in making such calculations. It is very similar to [6, Lemma 3.13].

Lemma 4.9 (The norm of a \(V\) and \(W\)). With notation as above, let \(V\) be a representation of \(H\) with \(V^H = 0\) and let \(W = \text{Ind}_H^G V\). Then \(N^G_{H}(a_{\rho}) = a_{W}\). If \(V\) is oriented (and hence even-dimensional, making \(|V|\rho\) oriented), then

\[
u_{|V|\rho} N(a_{\rho}) = u_W.
\]
Proof. The element $a_V$ is represented by the map $S^0 \to S^V$, the inclusion of the fixed point set. Applying the norm functor to this map gives

$$S^0 = N_H^G S^0 \to N_H^G S^V = S^W,$$

which is $a_W$.

When $V$ is oriented, $u_V$ is represented by a map $S^{V|\rho} \to S^V \wedge HZ$. Applying the norm functor and using the multiplication in $HZ$ leads to a map

$$S^{V|\rho} = N_H^G S^{V|\rho} \xrightarrow{N_H^G u_V} S^W \wedge HZ.$$

Now smash both sides with $HZ$, precompose with $u_{V|\rho}$ and follow with the multiplication on $HZ$, giving

$$S^{V|\rho} \xrightarrow{u_{V|\rho}} S^{V|\rho} \wedge HZ \xrightarrow{N_H u_V \wedge HZ} S^W \wedge HZ \xrightarrow{\rho} S^W \wedge HZ,$$

which is $u_W$ since $|W| = |V|/\rho$.

\[\square\]

5 Some Mackey functors for $C_4$ and $C_2$

We need some notation for Mackey functors to be used in spectral sequence charts. In this paper, when a cyclic group or subgroup appears as an index, we will often replace it by its order. We can specify Mackey functors $M$ for the group $C_2$ and $N$ for $C_4$ by means of Lewis diagrams (first introduced in [8]),

$$M(C_2/C_2) \quad \text{and} \quad N(C_4/C_4)$$

\[5.1\]

\[
\begin{array}{c}
\text{res}^2_1 \left( \begin{array}{c}
C_2/C_2 \\
C_2/e
\end{array} \right) \\
\text{M}(C_2/e) \\
\text{res}^4_1 \left( \begin{array}{c}
C_4/C_4 \\
C_4/e
\end{array} \right) \\
\text{N}(C_4/e)
\end{array}
\]

We omit Lewis’ looped arrow indicating the Weyl group action on $M(G/H)$ for proper subgroups $H$. This notation is prohibitively cumbersome in spectral sequence charts, so we will abbreviate specific examples by more concise symbols. These are shown in Tables 1 and 2. Admittedly some of these symbols are arbitrary and take some getting used to, but we have to start somewhere. Lewis denotes the fixed point Mackey functor for a $ZG$-module $M$ by $R(M)$. He abbreviates $R(Z)$ and $R(Z_\ast)$ by $R$ and $R\ast$. He also defines (with similar abbreviations) the orbit group Mackey functor $L(M)$ by

$$L(M)(G/H) = M/H.$$  

In this case each transfer map is the surjection of the orbit space for a smaller subgroup onto that of a larger one. The functors $R$ and $L$ are the left and right adjoints of the forgetful functor $M \mapsto M(G/e)$ from Mackey functors to $ZG$-modules.

Over $C_2$ we have short exact sequences

$$0 \longrightarrow \Box \longrightarrow \blacksquare \longrightarrow \ast \longrightarrow 0,$$

$$0 \longrightarrow \ast \longrightarrow \Box \longrightarrow \blacksquare \longrightarrow 0,$$

$$0 \longrightarrow \Blacksquare \longrightarrow \Box \longrightarrow \blacksquare \longrightarrow 0.$$

We can apply the induction functor to each of them to get a short exact sequence of Mackey functors over $C_4$.

Five of the Mackey functors in Table 2 are fixed point Mackey functors (3.1), meaning they are fixed points of an underlying $Z[G]$-module $M$, such as $Z[G]$ or

$$Z = Z[G]/(y - 1), \quad Z[G/G'] = Z[G]/(y^2 - 1),$$

Table 1. Some $C_2$-Mackey functors.

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Lewis diagram</th>
<th>Lewis symbol</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\square = \mathbb{Z}$</td>
<td>$\begin{array}{c} \mathbb{Z} \downarrow \downarrow \ \mathbb{Z} \downarrow \downarrow \ \mathbb{Z} \end{array}$</td>
<td>$R$</td>
</tr>
<tr>
<td>$\hat{\square} = \mathbb{Z}[G/G']$</td>
<td>$\begin{array}{c} \mathbb{Z} \downarrow \downarrow \ \mathbb{Z}[G/G'] \downarrow \downarrow \ \mathbb{Z}[G/G'] \end{array}$</td>
<td>$\langle \mathbb{Z}/2 \rangle$</td>
</tr>
<tr>
<td>$\hat{\hat{\square}} = \mathbb{Z}[G]$</td>
<td>$\begin{array}{c} \mathbb{Z} \downarrow \downarrow \ \mathbb{Z}[G/G'] \downarrow \downarrow \ \mathbb{Z}[G/G'] \end{array}$</td>
<td>$L$</td>
</tr>
<tr>
<td>$\hat{\hat{\hat{\square}}}$</td>
<td>$\begin{array}{c} \mathbb{Z} \downarrow \downarrow \ \mathbb{Z}[G/G'] \downarrow \downarrow \ \mathbb{Z}[G/G'] \end{array}$</td>
<td>$R(\mathbb{Z})$</td>
</tr>
</tbody>
</table>

Table 2. Some $C_4$-Mackey functors, where $G = C_4$ and $G'$ is its index 2 subgroup. The notation $\mathbb{Z}(G, H)$ is defined in Proposition 3.3 (i).

\[
\begin{array}{c|c|c}
\mathbb{Z}/2 & \mathbb{Z} & \mathbb{Z}/2 \\
\hline
\mathbb{Z}/2 & \mathbb{Z}[G/G'] & \mathbb{Z}/4 \\
\hline
\mathbb{Z}/2 & \mathbb{Z}[G/G'] & \mathbb{Z}/4 \\
\hline
\mathbb{Z}/2 & \mathbb{Z}[G/G'] & \mathbb{Z}/4 \\
\hline
\mathbb{Z}/2 & \mathbb{Z}[G/G'] & \mathbb{Z}/4 \\
\hline
\mathbb{Z}/2 & \mathbb{Z}[G/G'] & \mathbb{Z}/4 \\
\end{array}
\]

\[
\begin{array}{c|c|c}
\mathbb{Z}/2 & \mathbb{Z} & \mathbb{Z}/2 \\
\hline
\mathbb{Z}/2 & \mathbb{Z}[G/G'] & \mathbb{Z}/4 \\
\hline
\mathbb{Z}/2 & \mathbb{Z}[G/G'] & \mathbb{Z}/4 \\
\hline
\mathbb{Z}/2 & \mathbb{Z}[G/G'] & \mathbb{Z}/4 \\
\hline
\mathbb{Z}/2 & \mathbb{Z}[G/G'] & \mathbb{Z}/4 \\
\hline
\mathbb{Z}/2 & \mathbb{Z}[G/G'] & \mathbb{Z}/4 \\
\end{array}
\]

\[
\begin{array}{c|c|c}
\mathbb{Z}/2 & \mathbb{Z} & \mathbb{Z}/2 \\
\hline
\mathbb{Z}/2 & \mathbb{Z}[G/G'] & \mathbb{Z}/4 \\
\hline
\mathbb{Z}/2 & \mathbb{Z}[G/G'] & \mathbb{Z}/4 \\
\hline
\mathbb{Z}/2 & \mathbb{Z}[G/G'] & \mathbb{Z}/4 \\
\hline
\mathbb{Z}/2 & \mathbb{Z}[G/G'] & \mathbb{Z}/4 \\
\hline
\mathbb{Z}/2 & \mathbb{Z}[G/G'] & \mathbb{Z}/4 \\
\end{array}
\]

\[
\begin{array}{c|c|c}
\mathbb{Z}/2 & \mathbb{Z} & \mathbb{Z}/2 \\
\hline
\mathbb{Z}/2 & \mathbb{Z}[G/G'] & \mathbb{Z}/4 \\
\hline
\mathbb{Z}/2 & \mathbb{Z}[G/G'] & \mathbb{Z}/4 \\
\hline
\mathbb{Z}/2 & \mathbb{Z}[G/G'] & \mathbb{Z}/4 \\
\hline
\mathbb{Z}/2 & \mathbb{Z}[G/G'] & \mathbb{Z}/4 \\
\hline
\mathbb{Z}/2 & \mathbb{Z}[G/G'] & \mathbb{Z}/4 \\
\end{array}
\]

\[
\begin{array}{c|c|c}
\mathbb{Z}/2 & \mathbb{Z} & \mathbb{Z}/2 \\
\hline
\mathbb{Z}/2 & \mathbb{Z}[G/G'] & \mathbb{Z}/4 \\
\hline
\mathbb{Z}/2 & \mathbb{Z}[G/G'] & \mathbb{Z}/4 \\
\hline
\mathbb{Z}/2 & \mathbb{Z}[G/G'] & \mathbb{Z}/4 \\
\hline
\mathbb{Z}/2 & \mathbb{Z}[G/G'] & \mathbb{Z}/4 \\
\hline
\mathbb{Z}/2 & \mathbb{Z}[G/G'] & \mathbb{Z}/4 \\
\end{array}
\]

\[
\begin{array}{c|c|c}
\mathbb{Z}/2 & \mathbb{Z} & \mathbb{Z}/2 \\
\hline
\mathbb{Z}/2 & \mathbb{Z}[G/G'] & \mathbb{Z}/4 \\
\hline
\mathbb{Z}/2 & \mathbb{Z}[G/G'] & \mathbb{Z}/4 \\
\hline
\mathbb{Z}/2 & \mathbb{Z}[G/G'] & \mathbb{Z}/4 \\
\hline
\mathbb{Z}/2 & \mathbb{Z}[G/G'] & \mathbb{Z}/4 \\
\hline
\mathbb{Z}/2 & \mathbb{Z}[G/G'] & \mathbb{Z}/4 \\
\end{array}
\]

\[
\begin{array}{c|c|c}
\mathbb{Z}/2 & \mathbb{Z} & \mathbb{Z}/2 \\
\hline
\mathbb{Z}/2 & \mathbb{Z}[G/G'] & \mathbb{Z}/4 \\
\hline
\mathbb{Z}/2 & \mathbb{Z}[G/G'] & \mathbb{Z}/4 \\
\hline
\mathbb{Z}/2 & \mathbb{Z}[G/G'] & \mathbb{Z}/4 \\
\hline
\mathbb{Z}/2 & \mathbb{Z}[G/G'] & \mathbb{Z}/4 \\
\hline
\mathbb{Z}/2 & \mathbb{Z}[G/G'] & \mathbb{Z}/4 \\
\end{array}
\]

\[
\begin{array}{c|c|c}
\mathbb{Z}/2 & \mathbb{Z} & \mathbb{Z}/2 \\
\hline
\mathbb{Z}/2 & \mathbb{Z}[G/G'] & \mathbb{Z}/4 \\
\hline
\mathbb{Z}/2 & \mathbb{Z}[G/G'] & \mathbb{Z}/4 \\
\hline
\mathbb{Z}/2 & \mathbb{Z}[G/G'] & \mathbb{Z}/4 \\
\hline
\mathbb{Z}/2 & \mathbb{Z}[G/G'] & \mathbb{Z}/4 \\
\hline
\mathbb{Z}/2 & \mathbb{Z}[G/G'] & \mathbb{Z}/4 \\
\end{array}
\]

\[
\begin{array}{c|c|c}
\mathbb{Z}/2 & \mathbb{Z} & \mathbb{Z}/2 \\
\hline
\mathbb{Z}/2 & \mathbb{Z}[G/G'] & \mathbb{Z}/4 \\
\hline
\mathbb{Z}/2 & \mathbb{Z}[G/G'] & \mathbb{Z}/4 \\
\hline
\mathbb{Z}/2 & \mathbb{Z}[G/G'] & \mathbb{Z}/4 \\
\hline
\mathbb{Z}/2 & \mathbb{Z}[G/G'] & \mathbb{Z}/4 \\
\hline
\mathbb{Z}/2 & \mathbb{Z}[G/G'] & \mathbb{Z}/4 \\
\end{array}
\]

\[
\begin{array}{c|c|c}
\mathbb{Z}/2 & \mathbb{Z} & \mathbb{Z}/2 \\
\hline
\mathbb{Z}/2 & \mathbb{Z}[G/G'] & \mathbb{Z}/4 \\
\hline
\mathbb{Z}/2 & \mathbb{Z}[G/G'] & \mathbb{Z}/4 \\
\hline
\mathbb{Z}/2 & \mathbb{Z}[G/G'] & \mathbb{Z}/4 \\
\hline
\mathbb{Z}/2 & \mathbb{Z}[G/G'] & \mathbb{Z}/4 \\
\hline
\mathbb{Z}/2 & \mathbb{Z}[G/G'] & \mathbb{Z}/4 \\
\end{array}
\]}
We will use the following notational conventions for $C_4$-Mackey functors.

(i) Given a $C_2$-Mackey functor $M$ with Lewis diagram

\[
\begin{array}{ccc}
A & \xrightarrow{\alpha} & B \\
\beta & & \\
\end{array}
\]

with $A$ and $B$ cyclic, we will use the symbols $M, \overline{M}$ and $\underline{M}$ for the $C_4$-Mackey functors with Lewis diagrams

\[
\begin{array}{ccc}
A & \xrightarrow{\alpha} & 0 & \xrightarrow{\tau} Z/2 \\
\beta & & & \\
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
1 & \xrightarrow{2} & B & \xrightarrow{\beta} A \\
& & & \\
& & & \\
\end{array}
\]

where a generator $\gamma \in C_4$ acts via multiplication by $-1$ on $A$ and $B$ in the second two, and the transfer $\tau$ is nontrivial.

(ii) For a $C_2$-Mackey functor $M$ we will denote $\uparrow^2_1 M$ (see Definition 2.6) by $\overline{M}$. For a Mackey functor $M$ defined over the trivial group, we will denote $\uparrow^1_1 M$ and $\uparrow^2_1 M$ by $\overline{M}$ and $\underline{M}$. Over $C_4$, in addition to the short exact sequences induced up from $C_2$, we have

\[
\begin{array}{c}
0 \longrightarrow \bullet \longrightarrow \diamondsuit \longrightarrow \square \longrightarrow 0, \\
0 \longrightarrow \nabla \longrightarrow \circ \longrightarrow \bullet \longrightarrow 0, \\
0 \longrightarrow \nabla \longrightarrow \Box \longrightarrow \square \longrightarrow 0, \\
0 \longrightarrow \bullet \longrightarrow \circ \longrightarrow \triangle \longrightarrow 0, \\
0 \longrightarrow \Box \longrightarrow \square \longrightarrow \circ \longrightarrow 0, \\
0 \longrightarrow \Box \longrightarrow \square \longrightarrow \bullet \longrightarrow 0.
\end{array}
\]

\[\text{(5.2)}\]

**Definition 5.3 (A $C_4$-enriched $C_2$-Mackey functor).** For a $C_2$-Mackey functor $M$ as above, $\overline{M}$ will denote the $C_2$-Mackey functor enriched over $Z[C_4]$ defined by

\[
\overline{M}(S) = Z[C_4] \otimes_{Z[C_2]} M(S)
\]

for a finite $C_2$-set $S$. Equivalently, in the notation of Definition 2.6, $\overline{M} = \uparrow^2_1 \uparrow^1_1 M$.

## 6 Some chain complexes of Mackey functors

As noted above, a $G$-CW complex $X$, meaning one built out of cells of the form $G, \wedge_H e^n$, has a reduced cellular chain complex of $Z[G]$-modules $C_* X$, leading to a chain complex of fixed point Mackey functors (see (3.1)) $C_* X$. When $X = S^V$ for a representation $V$, we will denote this complex by $C_*^V$; see (3.2). Its homology is the graded Mackey functor $H_* X$. Here we will apply the methods of Section 3 to three examples.

**Example (i).** Let $G = C_2$ with generator $\gamma$, and $X = S_0^n$ for $n > 0$, where $\rho$ denotes the regular representation. We have seen before [6, Example 3.7] that it has a reduced cellular chain complex $C$ with

\[
\begin{cases}
C_i^{\rho} = Z[G]/(\gamma - 1) & \text{for } i = n, \\
Z[G] & \text{for } n < i \leq 2n, \\
0 & \text{otherwise}.
\end{cases}
\]

\[\text{(6.1)}\]
Let $c_{i}^{(n)}$ denote a generator of $C_{i}^{np_{2}}$. The boundary operator $d$ is given by

$$d(c_{i+1}^{(n)}) = \begin{cases} 
    c_{i}^{(n)} & \text{for } i = n, \\
    y_{i+1-n}(c_{i}^{(n)}) & \text{for } n < i \leq 2n, \\
    0 & \text{otherwise},
\end{cases} \quad (6.2)$$

where $y_{i} = 1 - (-1)^{i}$. For future reference, let

$$e_{i} = 1 - (-1)^{i} = \begin{cases} 
    0 & \text{for } i \text{ even}, \\
    2 & \text{for } i \text{ odd}.
\end{cases}$$

This chain complex has the form

\[
\begin{array}{ccccccc}
  n & n+1 & n+2 & n+3 & 2n \\
  \square & \downarrow & \square & \downarrow & \square & \downarrow & \square & \cdots & \downarrow & \square & \cdots \\
  Z & \xleftarrow{0} & Z & \xleftarrow{2} & Z & \xleftarrow{0} & Z & \xleftarrow{2} & Z & \xleftarrow{0} & Z & \xleftarrow{2} & Z & \cdots & \xleftarrow{e_{n}} & Z \\
  1 \xleftarrow{\Delta} & \xleftarrow{\Delta} & \xleftarrow{\Delta} & \xleftarrow{\Delta} & \xleftarrow{\Delta} & \xleftarrow{\Delta} & \xleftarrow{\Delta} & \cdots & \xleftarrow{\Delta} & \xleftarrow{\Delta} & \xleftarrow{\Delta} & \xleftarrow{\Delta} & \xleftarrow{\Delta} & \cdots & \xleftarrow{\Delta} & \xleftarrow{\Delta} \\
  Z/2 & \xleftarrow{0} & Z/2 & \xleftarrow{0} & \cdots & H_{2n}(G/G) & \xleftarrow{0} & \cdots & \xleftarrow{0} & \xleftarrow{0} & \cdots & \xleftarrow{0} & \xleftarrow{0} & \cdots & \xleftarrow{0} & \xleftarrow{0} \\
  0 & \xleftarrow{0} & 0 & \xleftarrow{0} & \cdots & H_{2n}(G/G) & \xleftarrow{0} & \cdots & \xleftarrow{0} & \xleftarrow{0} & \cdots & \xleftarrow{0} & \xleftarrow{0} & \cdots & \xleftarrow{0} & \xleftarrow{0} \\
\end{array}
\]

Passing to homology we get

\[
\begin{array}{ccccccc}
  n & n+1 & n+2 & n+3 & 2n \\
  \bullet & 0 & \bullet & 0 & \cdots & H_{2n} \\
  Z/2 & 0 & Z/2 & 0 & \cdots & H_{2n}(G/G) & \xleftarrow{0} & \cdots & \xleftarrow{0} & \xleftarrow{0} & \cdots & \xleftarrow{0} & \xleftarrow{0} & \cdots & \xleftarrow{0} & \xleftarrow{0} \\
  0 & 0 & 0 & 0 & \cdots & Z(G)/(y_{n+1}) \\
\end{array}
\]

where

$$H_{2n}(G/G) = \begin{cases} 
    \square & \text{for } n \text{ even}, \\
    0 & \text{for } n \text{ odd},
\end{cases} \quad \text{and} \quad H_{2n} = \begin{cases} 
    \square & \text{for } n \text{ even}, \\
    \uparrow & \text{for } n \text{ odd}.
\end{cases}$$

Here $\square$ and $\uparrow$ are fixed point Mackey functors but $\bullet$ is not.

Similar calculations can be made for $S^{np_{2}}$ for $n < 0$. The results are indicated in Figure 2. This is originally due to unpublished work of Stong and is reported in [8, Theorem 2.1 and Table 2.2]. This information will be used in Section 8.

In other words the RO(G)-graded Mackey functor valued homotopy of $HZ$ is as follows. For $n \geq -1$ we have

\[
\pi_{i}^{\Sigma^{np_{2}}} Hz = \pi_{i-np_{2}} Hz = \begin{cases} 
    \square & \text{for } n \text{ even and } i = 2n, \\
    \square & \text{for } n \text{ odd and } i = 2n, \\
    \bullet & \text{for } n \leq i < 2n \text{ and } i + n \text{ even}, \\
    0 & \text{otherwise}.
\end{cases}
\]

For $n \leq -2$ we have

\[
\pi_{i}^{\Sigma^{np_{2}}} Hz = \pi_{i-np_{2}} Hz = \begin{cases} 
    \square & \text{for } n \text{ even and } i = 2n, \\
    \square & \text{for } n \text{ odd and } i = 2n, \\
    \bullet & \text{for } 2n < i \leq n - 3 \text{ and } i + n \text{ odd}, \\
    0 & \text{otherwise}.
\end{cases}
\]
We can use Definition 3.4 to name some elements of these groups.

Note that $HZ$ is a commutative ring spectrum, so there is a commutative multiplication in $\pi_* HZ$, making it a commutative RO($G$)-graded Green functor. For such a functor $M$ on a general group $G$, the restriction maps are ring homomorphisms while the transfer maps satisfy the Frobenius relations (2.4).

Then the generators of various groups in $\pi_* HZ$ are

- $(4m - 2)$-slices for $m > 0$:
  
  \[ a^{2m-1-2i}u^i = a_{(2m-1-2i)\sigma} u_{2i\sigma} \in \pi_{2m-1+2i} HZ(G/G) = \pi_{2i-(2m-1)\sigma} HZ(G/G) \]
  \[ x^{2m-1} = u_{(2m-1)\sigma} \in \pi_{4m-2} \Sigma^{2m-1}\rho_1 HZ(G/\{e\}) = \pi_{2i-(1-\sigma)(2m-1)} HZ(G/\{e\}) \]
  
  with $y(x) = -x$.

- for $4m$-slices for $m > 0$:
  
  \[ a^{2m-2i}u^i = a_{(2m-2i)\sigma} u_{2i\sigma} \in \pi_{2m-1+2i} \Sigma^{2m-1}\rho_1 HZ(G/G) = \pi_{2i-(2m-1)\sigma} HZ(G/G) \]  
  \[ z_n = e_{2n\rho_1} \in \pi_{-4n} \Sigma^{-2n}\rho_1 HZ(G/\{e\}) = \pi_{3m(\sigma-1)} HZ(G/\{e\}) \]
  
  for $n > 0$,  
  
  \[ a^{-i} \text{tr}(x^{-2n-1}) \in \pi_{-4n-2i} \Sigma^{-2n+1}\rho_1 HZ(G/G) = \pi_{(2n+1)(\sigma-1)+i\sigma} HZ(G/G) \]
  
  for $n > 0$ and $i \geq 0$.

We have relations

- $2a = 0$, $\text{res}(a) = 0$,
- $z_n = x^{-2n}$, $\text{tr}(x^n) = \begin{cases} 2u^{n/2} & \text{for } n \text{ even and } n \geq 0, \\ \text{tr}(z_{-n/2}) & \text{for } n \text{ even and } n < 0, \\ 0 & \text{for } n \text{ odd and } n > -3. \end{cases}$

**Example (ii).** Let $G = C_4$ with generator $y$, $G' = C_2 \leq G$, the subgroup generated by $y^2$, and

\[ \tilde{S}(n, G') = G_\ast \wedge_{G'} S^{op}. \]

Thus we have

\[ C_* (\tilde{S}(n, G')) = \mathcal{Z}[G] \otimes_{\mathcal{Z}[G']} C_*^{op}. \]

with $C_*^{op}$ as in (6.1). The calculations of the previous example carry over verbatim by the exactness of Mackey functor induction of Definition 2.6.
Figure 3. The Mackey functor slice spectral sequence for $\Omega_{\text{res}}^* \mathbb{Z}^n \mathbb{H}\mathbb{Z}$. The symbols are defined in Table 2. The Mackey functor at position $(4n - s, s)$ is $\pi_{(4n - s, -s)}^* \mathbb{H}\mathbb{Z} = H^*_{\text{res}} S^n \mathbb{Z}^\rho$.

Example (iii). Let $G = C_4$ and $X = S^{n\rho}$. Then the reduced cellular chain complex (3.2) has the form

$$c_i^{n\rho} = \begin{cases} \mathbb{Z} & \text{for } i = n, \\ \mathbb{Z}[G/G'] & \text{for } n < i \leq 2n, \\ \mathbb{Z}[G] & \text{for } 2n < i \leq 4n, \\ 0 & \text{otherwise,} \end{cases}$$

in which generators $c_i^{(n)} \in c_i^{n\rho}$ satisfy

$$d(c_i^{(n)}) = \begin{cases} c_i^{(n)} & \text{for } i = n, \\ y_{i+1-n} c_i^{(n)} & \text{for } n < i \leq 2n, \\ \mu_{i+1-n} c_i^{(n)} & \text{for } 2n < i < 4n \text{ and } i \text{ even}, \\ y_{i+1-n} c_i^{(n)} & \text{for } 2n < i < 4n \text{ and } i \text{ odd,} \\ 0 & \text{otherwise,} \end{cases}$$

where

$$\mu_i = y_i(1 + y^2) = (1 - (-1)^i y)(1 + y^2).$$

The values of $H_* S^{n\rho}$ are illustrated in Figure 3. The Mackey functors in filtration 0 (the horizontal axis) are the ones described in Proposition 3.3.

As in (i), we name some of these elements. Let $G = C_4$ and $G' = C_2 \subseteq G$. Recall that the regular representation $\rho_A$ is $1 + \sigma + \lambda$ where $\sigma$ is the sign representation and $\lambda$ is the 2-dimensional representation given by a rotation of order 4.

Note that while Figure 2 shows all of $\pi_* H\mathbb{Z}$ for $G = C_2$, Figure 3 shows only a bigraded portion of this trigraded Mackey functor for $G = C_4$, namely the groups for which the index differs by an integer from a multiple of $\rho_A$. We will need to refer to some elements not shown in the latter chart, namely

$$\begin{align*}
a_0 & \in H_0 S^0(G/G), & a_1 & \in H_0 S^0(G/G), & \overline{a}_\lambda &= \text{res}_1^2(a_1), \\
u_2 a & \in H_2 S^2(G/G), & u_0 & \in H_1 S^0(G/G'), & \overline{u}_\lambda &= \text{res}_1^2(u_0), \\
u_1 & \in H_2 S^1(G/G), & \overline{u}_\lambda &= \text{res}_1^2(u_1),
\end{align*}$$

(6.3)
subject to the relations

\[
\begin{align*}
2a_\sigma &= 0, & \text{res}_2^x(a_\sigma) &= 0, \\
4a_\lambda &= 0, & 2\overline{a}_\lambda &= 0, & \text{res}_1^x(a_\lambda) &= 0,
\end{align*}
\]

(6.4)

\[
\text{res}_2^x(\overline{u}_2\sigma) = u_2^2, & \quad a_\sigma^2 u_\lambda = 2a_\lambda u_{2\sigma} \quad \text{(gold relation)};
\]

see Definition 3.4 and Lemma 3.6.

We will denote the generator of \(E_{-k}^{x,y}(G/H)\) (when it is nontrivial) by \(x_{-r, s}, y_{-r, s}\) and \(z_{l-r, s}\) for \(H = G, G'\) and \(\{ e \}\) respectively. Then the generators for the groups in the 4-slice are

\[
y_{4,0} = u_\rho = u_\sigma \text{ res}_1^x(u_\lambda) \in \gamma_4 \Sigma^{2\rho}_H(Z(G/G) = \gamma_{2-2i-3} Z(G/G) \quad \text{with } y_{4,0} = -x_{4,0},
\]

\[
x_{3,1} = a_\sigma u_\lambda \in \gamma_4 \Sigma^{2\rho}_H(Z(G/G) = \gamma_{2-2i+1} Z(G/G),
\]

\[
y_{2,2} = \text{res}_2^x(a_\lambda) u_\sigma \in \gamma_4 \Sigma^{2\rho}_H(Z(G/G) = \gamma_{2-2i-1} Z(G/G),
\]

\[
x_{1,3} = d_{\rho \sigma} = a_\sigma a_\lambda \in \gamma_4 \Sigma^{2\rho}_H(Z(G/G) = \gamma_{2-2i-3} Z(G/G)
\]

and the ones for the 8-slice are

\[
x_{8,0} = u_{3\lambda + 2\sigma} = u_{2\rho \sigma} \in \gamma_4 \Sigma^{2\rho}_H(Z(G/G) = \gamma_{2-2i-3} Z(G/G) \quad \text{with } y_{4,0}^2 = y_{8,0}^2 = \text{res}_2^x(x_{8,0}),
\]

\[
x_{6,2} = a_\lambda u_{3\lambda + 2\sigma} \in \gamma_4 \Sigma^{2\rho}_H(Z(G/G) = \gamma_{2-2i-3} Z(G/G) \quad \text{with } x_{6,2}^2 = 2x_{6,2}, y_{4,0} y_{2,2} = y_{6,2} = \text{res}_2^x(x_{6,2}),
\]

\[
x_{4,4} = a_\lambda u_{2\rho \sigma} \in \gamma_4 \Sigma^{2\rho}_H(Z(G/G) = \gamma_{2-2i-3} Z(G/G) \quad \text{with } y_{4,4}^2 = y_{4,4} \text{ res}_2^x(x_{4,4}), x_{1,3} x_{3,1} = 2x_{4,4},
\]

\[
x_{2,6} = x_{1,3}^2 \in \gamma_4 \Sigma^{2\rho}_H(Z(G/G) = \gamma_{2-2i-3} Z(G/G).
\]

These elements and their restrictions generate \(\gamma_4 \Sigma^{m\rho}_H Z\) for \(m = 1\) and \(2\). For \(m > 2\) the groups are generated by products of these elements.

The element

\[
z_{4,0} = \text{res}_2^x(y_{4,0}) = \text{res}_2^x(u_\lambda) \in \gamma_4 \Sigma^{2\rho}_H(Z(G/G))
\]

is invertible with \(y(y_{4,0}) = -y_{4,0}, z_{4,0}^2 = z_{8,0} = \text{res}_2^x(x_{8,0})\) and

\[
z_{-4m,0} := z_{4,0}^m = e_{m\rho \sigma} \in \gamma_{-4m} \Sigma^{-m\rho}_H Z(G/\{ e \}) \quad \text{for } m > 0,
\]

where \(e_{m\rho \sigma}\) is as in Definition 3.4. These elements and their transfers generate the groups in

\[
\gamma_{-4m} \Sigma^{-m\rho}_H Z
\]

for \(m > 0\).

**Theorem 6.5** (Divisibilities in the negative regular slices for \(C_4\)). There are the following infinite divisibilities in the third quadrant of the spectral sequence in Figure 3.

(i) \(x_{-4,0} = 4^{x_{4,0}}(x_{-4,0})\) is divisible by any monomial in \(x_{1,3}\) and \(x_{4,4}\), meaning that

\[
x_{1,3}^{i} x_{4,4}^{j} x_{-4,0}^{i - j - 3k} = x_{-4,0} \quad \text{for } i, j \geq 0.
\]

Moreover, no other basis element killed by \(x_{3,1}\) and \(x_{4,4}\) has this property.

(ii) \(x_{-4,0}, y_{-7,1}\) are divisible by any monomial in \(x_{4,4}, x_{6,2}\) and \(x_{8,0}\), subject to the relation \(x_{6,2}^2 = x_{8,0} x_{4,4}\). Note here that \(x_{6,2}^2 = 2x_{6,2}\). Moreover, no other basis element killed by \(x_{4,4}, x_{6,2}\) and \(x_{8,0}\) has this property.

(iii) \(y_{-7,1} = \text{res}_2^x(x_{-7,1})\) is divisible by any monomial in \(y_{2,2}\) and \(y_{4,0}\), meaning that

\[
y_{2,2}^j y_{4,0}^k y_{-7,2} y_{-7,1} = y_{-7,1} \quad \text{for } j, k \geq 0.
\]

Moreover, no other basis element killed by \(y_{2,2}\) and \(y_{4,0}\) has this property.

We will prove Theorem 6.5 as a corollary of a more general statement (Lemma 6.11 and Corollary 6.13) in which we consider all representations of the form \(m\Lambda + n\sigma\) for \(m, n \geq 0\). Let

\[
R = \bigoplus_{m,n \geq 0} \{ H_S^{m\lambda + n\sigma} \}
\]

It is generated by the elements of (6.3) subject to the relations of (6.4).

In the larger ring

\[
\hat{R} = \bigoplus_{m,n \in \mathbb{Z}} \{ H_S^{m\lambda + n\sigma} \},
\]
the elements $u_\alpha$, $\overline{u}_\alpha$ and $\overline{u}_\lambda$ are invertible with
\[ e_\alpha = u_\alpha^{-1} \in H_{-1}S^{-\alpha}(G/G'), \quad e_\lambda = \overline{u}_\lambda^{-1} \in H_{-2}S^{-\lambda}(G/e). \]

Define spectra $L_m$ and $K_n$ to be the cofibers of $a_{m\lambda}$ and $a_{n\sigma}$. Thus we have cofiber sequences
\[
\Sigma^{-1}L_m \xrightarrow{c_{m\lambda}} S^0 \xrightarrow{a_{m\lambda}} S^{m\lambda} \xrightarrow{b_{m\lambda}} L_m, \quad \Sigma^{-1}K_n \xrightarrow{c_{n\sigma}} S^0 \xrightarrow{a_{n\sigma}} S^{n\sigma} \xrightarrow{b_{n\sigma}} K_n.
\]
Dualizing gives
\[
DL_m \xrightarrow{D_{b_{m\lambda}}} S^{-m\lambda} \xrightarrow{D_{a_{m\lambda}}} S^0 \xrightarrow{D_{c_{m\lambda}}} \Sigma DL_m, \quad DK_n \xrightarrow{D_{b_{n\sigma}}} S^{-n\sigma} \xrightarrow{D_{a_{n\sigma}}} S^0 \xrightarrow{D_{c_{n\sigma}}} \Sigma DK_n.
\]
The maps $Da_{m\lambda}$ and $Da_{n\sigma}$ are the same as desuspensions of $a_{m\lambda}$ and $a_{n\sigma}$, which implies that
\[
DL_m = \Sigma^{-1-m\lambda}K_m \quad \text{and} \quad DK_n = \Sigma^{-1-n\sigma}K_n.
\]
Inspection of the cellular chain complexes for $L_m$ and $K_n$ and certain of their suspensions reveals that
\[
\Sigma^{2-\lambda}L_m \wedge HZ = L_m \wedge HZ = \Sigma^{2-2g}L_m \wedge HZ
\]
and
\[
\Sigma^{2-2g}K_n \wedge HZ = K_n \wedge HZ,
\]
while $\Sigma^{1-\sigma}$ alters both $L_m \wedge HZ$ and $K_n \wedge HZ$. We will denote $\Sigma^{k(1-\sigma)}L_m \wedge HZ$ by $L_m^{(1)} \wedge HZ$ and similarly for $K_n$.

The homology groups of $L_m^k$ and $K_n^k$ for $m, n > 0$ are indicated in Figures 4 and 5, and those for $S^{m\lambda}$ and $S^{n\sigma}$ are shown in Figure 6.

**Figure 4.** Charts for $H_1L_m^k$. The horizontal coordinate is $i$ and the vertical one is $m$; $L_m$ is on the left and $L_n$ is on the right.

**Figure 5.** Charts for $H_1K_n^k$. The horizontal coordinate is $i$ and the vertical one is $n$; $K_n$ is on the left and $K^r_n$ is on the right.
In the following diagrams we will use the same notation for a map and its smash product with any identity map. Let $V = m\lambda + n\sigma$ with $m, n > 0$, and let $R_V$ denote the fiber of $a_V$. Since $a_V$ is self-dual up to suspension, we have $DR_V = \Sigma^{-1-V}R_V$. In the following each row and column is a cofiber sequence:

\[
\begin{align*}
\Sigma^{n-1}L_m & \xrightarrow{c_{m\lambda}} \Sigma^{n-1}L_m \\
\Sigma^{-1}K_n & \xrightarrow{c_{n\sigma}} S^0 \xrightarrow{a_{m\lambda}} S^{n\sigma} \xrightarrow{b_{n\sigma}} K_n \\
\Sigma^{-1}R_V & \xrightarrow{c_V} S^0 \xrightarrow{a_V} S^V \xrightarrow{b_V} R_V \\
\Sigma^{n\sigma}L_m & \xrightarrow{c_{n\sigma}} \Sigma^{n\sigma}L_m.
\end{align*}
\]

The homology sequence for the third column is the easiest way to compute $H_* S^V$. That column is

\[
\begin{align*}
\Sigma^{n-1}L_m & \xrightarrow{c_{m\lambda}} \Sigma^{n-1}L_m \\
\Sigma^{-1}DL_m & \xrightarrow{c_{n\sigma}} S^{n\sigma} \xrightarrow{a_{n\sigma}} S^{-V} \xrightarrow{b_{n\sigma}} \Sigma^{-1}DL_m \\
\Sigma^{-1}V L_m & \xrightarrow{c_{n\sigma}} S^{-V} \xrightarrow{a_{n\sigma}} S^{-n\sigma} \xrightarrow{b_{n\sigma}} \Sigma^{-1}V L_m.
\end{align*}
\]

For (6.7) the long exact sequence in homology includes

\[
H_{i+1-n} L_m^{(-1)^n} \xrightarrow{c_{m\lambda}} H_i S^{n\sigma} \xrightarrow{a_{n\sigma}} H_i S^V \xrightarrow{b_{n\sigma}} H_{i-n} L_m^{(-1)^n} \xrightarrow{c_{m\lambda}} H_{i-1} S^{n\sigma}.
\]

**Divisibility by $a_{\lambda}$.** Multiplication by $a_{\lambda}$ leads to

\[
\begin{align*}
H_{i+1-n} L_m^{(-1)^n} & \xrightarrow{c_{m\lambda}} H_i S^{n\sigma} \xrightarrow{a_{m\lambda}} H_i S^V \xrightarrow{b_{m\lambda}} H_{i-n} L_m^{(-1)^n} \xrightarrow{c_{m\lambda}} H_{i-1} S^{n\sigma} \\
H_{i+1-n} L_{m'}^{(-1)^n} & \xrightarrow{c_{m'\lambda}} H_i S^{n\sigma} \xrightarrow{a_{m'\lambda}} H_i S^{V+\lambda} \xrightarrow{b_{m'\lambda}} H_{i-n} L_{m'}^{(-1)^n} \xrightarrow{c_{m'\lambda}} H_{i-1} S^{n\sigma},
\end{align*}
\]

where $m' = m + 1$ and $a'_{\lambda}$ is induced by the inclusion $L_m \to L_{m'}$. 

---

*Figure 6. Charts for $H_* S^{m\lambda}$ and $H_* S^{n\sigma}$. The horizontal coordinates are $i$ and the vertical ones are $m$ and $n$; $S^{m\lambda}$ is on the left and $S^{n\sigma}$ is on the right.*
In the dual case we get
\[
\begin{align*}
H_{i+1}S^{-\nu} &\quad b \quad H_{i+1+i}[V]^{-1}c \quad H_iS^{-\nu} \quad a \quad H_iS^{-\nu} \quad b \quad H_i[V]^{-1}c \\
\end{align*}
\]
(6.9)

Here the subscripts on the horizontal maps \((m\lambda\lambda)\) in the top row and \((m'\lambda\lambda)\) in the bottom row have been omitted to save space. The five lemma implies that the middle vertical map is onto when the left hand \(Da'_\lambda\) is onto and the right hand one is one-to-one. The left version of \(Da'_\lambda\) is onto in every case except \(i = -V\) and the right version of it is one-to-one in all cases except \(i = -V\) and \(i = -V\). This is illustrated for small \(m\) in the following diagram in which trivial Mackey functors are indicated by blank spaces.

\[
\begin{array}{cccccccc}
 j & H_jL_1 & H_jL_2 & H_jL_3 & H_jL_4 & H_jL_5 & H_jL_6 & H_jL_7 & H_jL_8 \\
 -1 & & & & & & & & \\
 0 & & & & & & & & \\
 1 & & & & & & & & \\
 2 & & & & & & & & \\
 3 & & & & & & & & \\
 4 & & & & & & & & \\
 5 & & & & & & & & \\
 6 & & & & & & & & \\
 7 & & & & & & & & \\
 8 & & & & & & & & \\
\end{array}
\]

It follows that the map \(a_\lambda\) in (6.9) is onto for all \(i\) except \(-V\). This is a divisibility result. Note that \(a_\lambda\) is trivial on \(H_i X(G/e)\) for any \(X\) since \(\text{res}_1^0(a_\lambda) = 0\).

**Divisibility by \(u_\lambda\).** For \(u_\lambda\) multiplication we use the diagram
\[
\begin{align*}
H_{i+1}S^{-\nu} &\quad b \quad H_{i+1+i}[V]^{-1}c \quad H_iS^{-\nu} \quad a \quad H_iS^{-\nu} \quad b \quad H_i[V]^{-1}c \\
\end{align*}
\]
(6.10)

The rightmost \(u_\lambda\) is onto in all cases except \(i = -V\) and \(n\) even. This is illustrated for \(n = 6\) and \(7\) in the following diagram.

\[
\begin{array}{cccccccc}
 j & -1 & -2 & -3 & -4 & -5 & -6 & -7 & -8 & -9 \\
 H_iS^{-6\nu} & & & & & & & & & \\
 H_iS^{-6\nu} & & & & & & & & & \\
 H_iS^{-7\nu} & & & & & & & & & \\
 H_iS^{-7\nu} & & & & & & & & & \\
\end{array}
\]

Thus the central \(u_\lambda\) in (6.10) fails to be onto only in when \(i = -V\) and \(n\) is even.

**Divisibility by \(a_\sigma\).** The corresponding diagram is
\[
\begin{align*}
H_{i+1}S^{-\nu} &\quad b \quad H_{i+1+i}[V]^{-1}c \quad H_iS^{-\nu} \quad a \quad H_iS^{-\nu} \quad b \quad H_i[V]^{-1}c \\
\end{align*}
\]
Here we have abbreviated \( n + 1 \) by \( n' \). Since \( \text{res}^i_2(a_{\sigma}) = 0 \), the map \( a_{\sigma} \) must vanish on \( H_*X(G/G') \) and \( H_*X(G/e) \). It can be nontrivial only on \( G/G \).

By Lemma 4.2, the image of \( a_{\sigma} \) is the kernel of the restriction map \( u_{\sigma}^{-1} \text{res}^i_2 \) and the kernel of \( a_{\sigma} \) is the image of the transfer \( t^H_{23} \). From Figure 6 we see that \( \text{res}^i_2 \) kills \( H_*S^{-n\sigma}(G/G) \) except the case \( i = -n \) for even \( n \). From Figure 4 we see that it kills \( H_*L_m(G/G) \) for all \( j \) and \( H_*L_m(G/G) \) for odd \( j > 1 \), but not the generators for \( j = 1 \) nor the ones for even values of \( j \) from 2 to 2m. The transfer has nontrivial image in \( H_*L_m \) only for \( j = 1 \) and in \( H_*L_m \) only for \( j = 1 \) and for even \( j \) from 2 to 2m.

It follows that for odd \( n \), each element of \( H_*S^{-V}(G/G) \) is divisible by \( a_{\sigma} \) except when \( i = -|V| = -2m - n \). For even \( n \) it is onto except when \( i = -n, i = -n - 2m, \) and \( i \) odd from \( 1 - n - 2m \) to \(-1 - n \).

### Divisibility by \( u_{2\sigma} \)

For \( u_{2\sigma} \) multiplication, the diagram is

\[
\begin{array}{ccc}
H_{j+1}S^{-n\sigma} & \overset{b}{\to} & H_{j+1}L_m^{-(1)^{n}} \overset{c}{\to} H_jS^{-V} \overset{a}{\to} H_jS^{-n\sigma} \overset{b}{\to} H_jL_m^{-(1)^{n}} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
H_{j-1}S^{-(n+2)^{\sigma}} & \overset{b}{\to} & H_{j-1}L_m^{-(1)^{n}} \overset{c}{\to} H_{j-1}S^{-V-2\sigma} \overset{a}{\to} H_{j-1}S^{-(n+2)^{\sigma}} \overset{b}{\to} H_{j-1}L_m^{-(1)^{n}}.
\end{array}
\]

The rightmost \( u_{2\sigma} \) is onto in all cases, so every element in \( H_*S^{-V} \) is divisible by \( u_{2\sigma} \).

The arguments above prove the following.

**Lemma 6.11 (RO\((G)\)-graded divisibility).** Let \( G = C_4 \) and \( V = m\lambda + n\sigma \) for \( m, n \geq 0 \).

(i) Each element in \( H_*S^{-V}(G/G) \) or \( H_*S^{-V}(G/G') \) is divisible by \( a_{\lambda} \) or \( \bar{a}_{\lambda} \) except when \( i = -|V| \).

(ii) Each element in \( H_*S^{-V}(G/H) \) is divisible by a suitable restriction of \( u_{\lambda} \) except when \( i = -n \) for even \( n \).

(iii) Each element in \( H_*S^{-V}(G/G) \) for odd \( n \) is divisible by \( a_{\sigma} \) except when \( i = -|V| \). For even \( n \) it is divisible by \( a_{\sigma} \) except when \( i = -n, i = -|V| \) and \( i \) odd from \( 1 - |V| \) to \(-1 - n \).

(iv) Each element in \( H_*S^{-V}(G/H) \) is divisible by \( u_{2\sigma}, u_{\lambda} \) or \( \bar{u}_{\lambda} \).

In Theorem 6.5 we are looking for divisibility by

\[
\begin{align*}
x_{1,3} &= a_{\sigma}a_{\lambda} \in H_0S^{\sigma\lambda}(G/G) = H_1S^0(G/G), \\
x_{4,4} &= a_{\sigma}^2u_{2\sigma} \in H_2S^{2\lambda+2\sigma}(G/G) = H_2S^2(G/G), \\
y_{2,2} &= \bar{a}_{\lambda}u_{\sigma} \in H_1S^{1\lambda+1\sigma}(G/G') = H_2S^0(G/G'), \\
x_{6,2} &= a_{\lambda}u_{2\sigma}u_{\lambda} \in H_4S^{2\lambda+2\sigma}(G/G) = H_3S^2(G/G), \\
x_{8,0} &= u_{2\sigma}u_{\lambda}^2 \in H_6S^{2\lambda+2\sigma}(G/G) = H_5S^2(G/G), \\
y_{4,0} &= u_{\sigma}\bar{u}_{\lambda} \in H_4S^{2\lambda+2\sigma}(G/G') = H_3S^0(G/G').
\end{align*}
\]

In view of Lemma 6.11 (iv), we can ignore the factors \( u_{2\sigma} \) and \( u_{\lambda} \) when analyzing such divisibility.

**Corollary 6.13 (Infinite divisibility by the divisors of (6.12)).** Let

\[
V = m\lambda + n\sigma \quad \text{for} \quad m, n \geq 0.
\]

Then the following hold:

- Each element of \( H_*S^{-V}(G/G) \) is infinitely divisible by \( x_{1,3} = a_{\sigma}a_{\lambda} \) for \( i > -n \) when \( n \) is even and for \( i \geq -n \) when \( n \) is odd.

- Each element of \( H_*S^{-V}(G/G) \) is infinitely divisible by \( x_{4,4} = a_{\sigma}^2u_{2\sigma} \) for \( i > -|V| \).

- Each element of \( H_*S^{-V}(G/G') \) is infinitely divisible by \( y_{2,2} = \bar{a}_{\lambda}u_{\sigma} \) for \( i > -|V| \).

- Each element of \( H_*S^{-V}(G/G) \) is infinitely divisible by \( x_{6,2} = a_{\lambda}u_{2\sigma}u_{\lambda} \) for \( i > -|V| \) when \( n \) is odd and for \( -|V| < i < -n \) when \( n \) is even.

- Each element of \( H_*S^{-V}(G/G) \) is infinitely divisible by \( x_{8,0} = u_{2\sigma}u_{\lambda}^2 \) for \( i < -n \) when \( n \) is even and for all \( i \) when \( n \) is odd.

- Each element of \( H_*S^{-V}(G/G') \) is infinitely divisible by \( y_{4,0} = u_{\sigma}\bar{u}_{\lambda} \) for \( i < -n \) when \( n \) is even and for all \( i \) when \( n \) is odd.

This implies Theorem 6.5.
7 The spectra $k_R$ and $k_{[2]}$

Before defining our spectrum we need to recall some definitions and formulas from [6]. Let $H \subset G$ be finite groups. In [6, Section 2.2.3] we define a norm functor $N^G_H$ from the category of $H$-spectra to that of $G$-spectra. Roughly speaking, for an $H$-spectrum $X$, $N^G_H X$ is the $G$-spectrum underlain by the smash power $X^{(G/H)}$ with $G$ permuting the factors and $H$ leaving each one invariant. When $G$ is cyclic, we will denote the orders of $G$ and $H$ by $g$ and $h$, and the norm functor by $N^g_h$.

There is a $C_2$-spectrum $\text{MU}_R$ underlain by the complex cobordism spectrum $\text{MU}$ with group action given by complex conjugation. Its construction is spelled out in [6, Section B.12]. For a finite cyclic 2-group $G$ we define

$$\text{MU}^{(G)} = N^g_2 \text{MU}_R.$$ Choose a generator $y$ of $G$. In [6, (5.47)] we defined generators

$$\overline{r}_k = r^G_k \in \pi^{C_2}_{k-p_2} i^*_G \text{MU}^{(G)}(C_2/C_2) \cong \pi_{C_2,kp_2}^{C_2} \text{MU}^{(G)}(G/G)$$ (note that this group is a module over $G/C_2$) and

$$r_k = r^G_k(\overline{r}_k) \in \pi^{\underline{e}}_{(e),2k} \text{MU}^{(G)}(G/G) \cong \pi_{2k}^{\underline{e}} \text{MU}^{(G)}(\underline{e}/\{e\}) = \pi_{2k}^{\underline{e}} \text{MU}^{(G)}.$$ The Hurewicz images of the $\overline{r}_k$ (for which we use the same notation) are defined in terms of the coefficients (see Definition 2.7)

$$\overline{m}_k \in \pi_{2k}^{C_2} HZ_{(2)} \wedge \text{MU}^{(G)}(C_2/C_2) = \pi_{C_2,kp_2}^{C_2} HZ_{(2)} \wedge \text{MU}^{(G)}(G/G)$$

of the logarithm of the formal group law $\overline{F}$ associated with the left unit map from $\text{MU}$ to $\text{MU}^{(G)}$. The formula is

$$\sum_{k \geq 0} \overline{r}_k x^{k+1} = \left(x + \sum_{\ell > 0} y(\overline{m}_{2^\ell-1}) x^{2^\ell}\right)^{-1} \circ \log_F(x),$$

where

$$\log_F(x) = x + \sum_{k \geq 0} \overline{m}_k x^{k+1}.$$ For small $k$ we have

$$\overline{r}_1 = (1 - y)(\overline{m}_1),$$

$$\overline{r}_2 = \overline{m}_2 - 2y(\overline{m}_1)(1 - y)(\overline{m}_1),$$

$$\overline{r}_3 = (1 - y)(\overline{m}_3) - y(\overline{m}_1)(5y(\overline{m}_1)^2 - 6y(\overline{m}_1)\overline{m}_1 + \overline{m}_1^2 + 2\overline{m}_2).$$

Now let $G = C_2$ or $C_4$ and, in the latter case $G' = C_2 \subseteq G$. The generators $r^G_k$ are the $\overline{r}_k$ defined above. We also have elements $r^{G'}_k$ defined by similar formulas with $y$ replaced by $y^2$; recall that $y^2(\overline{m}_k) = (-1)^k \overline{m}_k$. They are the images of similar generators of

$$\pi_{C_2,kp_2}^{C_2} \text{MU}^{(G')}(C_2/C_2) \cong \pi_{C_2,kp_2}^{C_2} \text{MU}^{(G')}(G'/G')$$ under the left unit map

$$\text{MU}^{(G')} \to \text{MU}^{(G')} \wedge \text{MU}^{(G')} \cong i^*_g \text{MU}^{(G)}.$$ Thus we have

$$\overline{r}^{G'}_1 = 2\overline{m}_1,$$

$$\overline{r}^{G'}_2 = \overline{m}_2 + 4\overline{m}_1^2,$$

$$\overline{r}^{G'}_3 = 2\overline{m}_3 + 2\overline{m}_1\overline{m}_2 + 12\overline{m}_1^3.$$
If we set $\bar{r}_2 = 0$ and $\bar{r}_3 = 0$, we get
\[
\begin{align*}
\bar{r}_1'^0 &= \bar{r}_{1,0} + \bar{r}_{1,1}, \\
\bar{r}_2'^0 &= 3\bar{r}_{1,0}\bar{r}_{1,1} + \bar{r}_{1,1}, \\
\bar{r}_3'^0 &= 5\bar{r}_{1,0}^2\bar{r}_{1,1} + 5\bar{r}_{1,0}\bar{r}_{1,1}^2 + \bar{r}_{1,1} = \bar{r}_{1,1}(5\bar{r}_{1,0}^2 + 5\bar{r}_{1,0}\bar{r}_{1,1} + \bar{r}_{1,1}), \\
y(\bar{r}_3'^0) &= -\bar{r}_{1,0}(5\bar{r}_{1,0}^2 - 5\bar{r}_{1,0}\bar{r}_{1,1} + \bar{r}_{1,1}), \\
-\bar{r}_3'^0 &= \bar{y}(\bar{r}_1'^0)/\bar{r}_{1,0}\bar{r}_{1,1} = (5\bar{r}_{1,0}^2 - 5\bar{r}_{1,0}\bar{r}_{1,1} + \bar{r}_{1,1})(\bar{r}_{1,1} + 5\bar{r}_{1,0}\bar{r}_{1,1} + 5\bar{r}_{1,0}^2) \\
&= (5\bar{r}_{1,0}^2 - 20\bar{r}_{1,0}\bar{r}_{1,1} + \bar{r}_{1,1}^2 + 20\bar{r}_{1,0}\bar{r}_{1,1}^2 + 5\bar{r}_{1,1}^2) \\
&= (5\bar{r}_{1,0}^2 - \bar{r}_{1,1}^2 - 20\bar{r}_{1,0}\bar{r}_{1,1} + \bar{r}_{1,1} + 11(\bar{r}_{1,0}\bar{r}_{1,1})^2),
\end{align*}
\]
where $\bar{r}_{1,0} = \bar{r}_1$ and $\bar{r}_{1,1} = \bar{y}(\bar{r}_1)$.

**Definition 7.3** ($k_R, K_R, k_{[2]}$ and $K_{[2]}$). The $C_2$-spectrum $k_R$ (connective real $K$-theory), is the spectrum obtained from $\text{MU}_R$ by killing the $r_n$s for $n \geq 2$. Its periodic counterpart $K_R$ is the telescope obtained from $k_R$ by inverting $\bar{r}_1 \in \pi_{\text{top}} k_R(C_2/C_2)$.

The $C_4$-spectrum $k_{[2]}$ is obtained from $\text{MU}^{(C_4)}$ by killing the $r_n$s and their conjugates for $n \geq 2$. Its periodic counterpart $K_{[2]}$ is the telescope obtained from $k_{[2]}$ by inverting a certain element $D \in \pi_{\text{top}} k_{[2]}(C_4/C_4)$ defined below in (9.3) and Table 3.

The image of $D$ in $\pi_{\text{top}} k_{[2]}(C_2/C_2) \equiv \pi_{\text{top}} k_{[2]}(C_4/C_4)$ is
\[
\bar{r}_2^G(D) = \bar{r}_{1,0}\bar{r}_{1,1}\bar{r}_3^G y(\bar{r}_3'^0).
\]

It is fixed by the action of $G/G'$, while its factors $\bar{r}_{1,0}\bar{r}_{1,1}$ and $\bar{r}_3^G y(\bar{r}_3'^0)$ are each negated by the action of the generator $y$.

We remark that while $\text{MU}^{(C_4)}$ is $\text{MU}_R \wedge \text{MU}_R$ as a $C_2$-spectrum, $k_{[2]}$ is not $k_R \wedge k_R$ as a $C_2$-spectrum. The former has torsion free underlying homotopy but the latter does not.

## 8 The slice spectral sequence for $K_R$

In this section we describe the slice spectral sequence for $K_R$. These results are originally due to Dugger [4], to which we refer for many of the proofs. This case is far simpler than that of $K_{[2]}$, but it is very instructive.

**Theorem 8.1** (The slice $E_2$-terms for $K_R$ and $k_R$). The slices of $K_R$ are

\[
P_t^0 K_R = \begin{cases} \Sigma^{(t/2)p_0} H\mathbb{Z} & \text{for } t \text{ even}, \\ \ast & \text{otherwise}. \end{cases}
\]

For $k_R$ they are the same in nonnegative dimensions, and contractible below dimension 0.

Hence we know the integrally graded homotopy groups of these slices by the results of Section 6, and they are shown in Figure 2. It shows the $E_2$-term for the wedge of all of the slices of $K_R$, and $K_R$ itself has the same $E_2$-term. It turns out that the differentials and Mackey functor extensions are determined by the fact that $\pi_* K_R$ is $8$-periodic, while the $E_2$-term is far from it. This explanation is admittedly circular in that the proof of the Periodicity Theorem itself of [6, Section 9] relies on the existence of certain differentials described below in (11.2).

**Theorem 8.2** (The slice spectral sequence for $K_R$). The differentials and extensions in the spectral sequence are as indicated in Figure 7.
Figure 7. The slice spectral sequence for $K_3$. Compare with Figure 2. Exotic transfers and restrictions are indicated respectively by solid blue and dashed green lines. Differentials are in red.

Proof. There are four phenomena we need to establish:

(i) The differentials in the first quadrant, which are indicated by red lines.
(ii) The differentials in the third quadrant.
(iii) The exotic transfers in the first quadrant, which are indicated by blue lines.
(iv) The exotic restrictions in the third quadrant, which are indicated by dashed green lines.

For (i), note that there is a nontrivial element in $E^{3,6}_2(G/G)$, which is part of the $3$-stem, but nothing in the $(-5)$-stem. This means the former element must be killed by a differential, and the only possibility is the one indicated. The other differentials in the first quadrant follow from this one and the multiplicative structure.

For (ii), we know that $\pi_6 K_R = 0$, so the same must be true of $\pi_{-9}$. Hence the element in $E^{3,-12}_2$ cannot survive, leading to the indicated third quadrant differentials.

For (iii), note that $\pi_2$ and $\pi_{-6}$ must be the same as Mackey functors. This forces the indicated exotic transfers. For each $m \geq 0$ one has a nonsplit short exact sequence of $C_2$ Mackey functors

$$0 \rightarrow E^{2,8m+4}_2 \rightarrow \pi_{8m+2} K_R \rightarrow E^{0,8m+2}_2 \rightarrow 0.$$  

For (iv), note that $\pi_{-8}$ and $\pi_0$ must also agree. This forces the indicated exotic restrictions. For each $m < 0$ one has a nonsplit short exact sequence

$$0 \rightarrow E^{0,8m}_2 \rightarrow \pi_{8m} K_R \rightarrow E^{2,8m-2}_2 \rightarrow 0$$

as desired.

In order to describe $\pi_*, K_R$ as a graded Green functor, meaning a graded Mackey functor with multiplication, we recall some notation from Section 6 (i) and Definition 3.4. For $G = C_2$ we have elements

$$
\begin{align*}
  a &= a_0 \in \pi_{-2} HZ(G/G), \\
  u &= u_2 \sigma \in \pi_{2-2} HZ(G/G), \\
  x &= u_0 \in \pi_{1} HZ(G/e), \\
  z_n &= e_{2n+1} \sigma \in \pi_{2n+1} HZ(G/e) \\
  a^{-1} \text{tr}(x^{2n-1}) &= a_{2n+1} \sigma HZ(G/G)
\end{align*}
$$

with $x^2 = \text{res}(u)$, for $n > 0$.

(8.3)
We will use the same symbols for the representatives of these elements in the slice $E_2$-term. The filtrations of $u$, $x$ and $z_n$ are zero while that of $a$ is one. It follows that $a^{-1} \text{tr}(x^{-2n-1})$ has filtration $-1$. The element $x$ is invertible.

In $E_2^{i,j}$ we have relations in

\[
\begin{align*}
2a &= 0, \quad \text{res}(a) = 0, \\
\eta &= a \bar{\eta} \\
\end{align*}
\]

\[
\begin{align*}
z_n = x^{-2n}, \quad \text{tr}(x^n) &= \begin{cases} 
2u^{n/2} & \text{for } n \text{ even and } n \geq 0, \\
\text{tr}(x^{-n/2}) & \text{for } n \text{ even and } n < 0, \\
0 & \text{for } n \text{ odd and } n > -3, \\
\ne 0 & \text{for } n \text{ odd and } n \leq -3.
\end{cases}
\]

(8.4)

We also have the element $\bar{\eta} \in \pi_{1+\sigma}K \{G/G\}$, the image of the element of the same name in $\bar{\pi} \in \pi_{1+\sigma}MU \{G/G\}$ of (7.1). We use the same symbol for its representative $E_2^{1,1+\sigma}(G/G)$. Then we have integrally graded elements

\[
\begin{align*}
\eta &= a \bar{\eta} \\
v_1 &= x \cdot \text{res}(\bar{\eta}) \in E_2^{0,2} \{G/e\} \quad \text{with } \gamma(v_1) = -v_1, \\
u \bar{r}_1^2 &= E_2^{0,4} \{G/G\}, \\
w &= 2u \bar{r}_1^2 \in E_2^{0,4} \{G/G\}, \\
b &= u \bar{r}_1^2 \in E_2^{0,4} \{G/G\} \quad \text{with } w^2 = 4b,
\end{align*}
\]

where $\eta$ and $v_1$ are the images of the elements of the same name in $\pi_1S^0$ and $\pi_2k$, and $w$ and $b$ are permanent cycles. The elements $x$, $v_1$ and $b$ are invertible. Note that for $n < 0$,

\[
E_2^{0,2n}(G/G) = \begin{cases}
\mathbb{Z} \text{ generated by } \text{tr}(v_1^n) & \text{for } n \text{ even}, \\
\mathbb{Z}/2 \text{ generated by } \text{tr}(v_1^n) & \text{for } n \text{ odd and } n < -1,
\end{cases}
\]

so each group is killed by $\eta = a \bar{\eta}$ by Lemma 4.2.

Then we have

\[
\begin{align*}
d_3(u) &= a^3 \bar{\eta} \quad \text{by (11.3) below}, \\
d_3(u) \bar{r}_1^2 &= a^3 \bar{r}_1^2 = \eta^3,
\end{align*}
\]

so

\[
\begin{align*}
\text{tr}(x) &= a^2 \bar{\eta} \quad \text{by (11.4), raising filtration by 2}, \\
\text{tr}(v_1) &= \eta^2.
\end{align*}
\]

Thus we get:

**Theorem 8.5** (The homotopy of $K \{G\}$ as an integrally graded Green functor). With notation as above,

\[
\pi_\ast K \{G\} = \mathbb{Z}[v_1^{\pm 1}], \quad \pi_\ast K \{G/G\} = \mathbb{Z}[b^{\pm 1}, w, \eta]/(2\eta, \eta^3, w\eta, w^2 - 4b)
\]

with

\[
\text{tr}(v_1^n) = \begin{cases}
2b^i & \text{for } i = 4j, \\
\eta^2b^i & \text{for } i = 4j + 1, \\
w^ib^i & \text{for } i = 4j + 2, \\
0 & \text{for } i = 4j + 3.
\end{cases}
\]

For each $j < 0$, $b^i$ has filtration $-2$ and supports an exotic restriction in the slice spectral sequence as indicated in Figure 7. Both $v_1$ res$(b^i)$ and $\eta^2b^i$ have filtration zero, so the transfer relating them is does not raise filtration.

Now we will describe the RO$(G)$-graded slice spectral sequence and homotopy of $K \{G\}$. The former is trigraded since RO$(G)$ itself is bigraded, being isomorphic as an abelian group to $\mathbb{Z} \oplus \mathbb{Z}$. For each integer $k$, one can imagine a chart similar to Figure 7 converging to the graded Mackey functor $\pi_{k\sigma + 1}K \{G\}$. Figure 7 itself is the
one for \( k = 0 \). The product of the elements in the \( k \)th and \( \ell \)th charts lies in the \( (k + \ell) \)th chart. We have elements as in (8.3):

\[
\begin{align*}
\alpha &= \alpha_0 \in E_2^{1,1-\alpha}(G/G), \\
\beta &= \beta_0 \in E_2^{0,2-2\alpha}(G/G), \\
\gamma &= \gamma_0 \in E_2^{0,1-\alpha}(G/e)) \\
\delta &= \delta_0 \in E_2^{0,0-2\alpha}(G/e) \\
\end{align*}
\]

with \( \gamma(x) = -1 \) and \( x^2 = \text{res}(\beta) \),

\[
\begin{align*}
&x = x_0 \in E_2^{x,2-2n}(G/e) \\
&\text{for } n > 0, \\
&z_n = z_n \in E_2^{x,2-2n+2\alpha}(G/e) \\
&\text{for } n > 0,
\end{align*}
\]

\[
\begin{align*}
&z_n = \alpha^{-2n} \in E_2^{x,2-2n+2\alpha}(G/G) \\
&\text{for } n > 0 \\
&\bar{v}_1 \in E_2^{0,1-\alpha}(G/G),
\end{align*}
\]

where \( a, x, z_n \) and \( \bar{v}_1 \) are permanent cycles, both \( x \) and \( \bar{v}_1 \) are invertible, and there are relations as in (8.4).

We also know that

\[
d_3(u) = a^3 \bar{v}_1 \quad \text{by (11.3) below,} \quad \text{tr}^3_1(x) = a^3 \bar{v}_1 \quad \text{by (11.4).}
\]

**Theorem 8.6.** The RO(G)-graded slice spectral sequence for \( K_\mathbb{R} \) can be obtained by tensoring that of Figure 7 with \( \mathbb{Z}[\mathbb{F}_1^{-1}] \), that is for any integer \( k \),

\[
E_2^{s,t+k\alpha}(G/G) \cong \mathbb{F}_1^{s,k\alpha}E_2^{s,t-k}(G/G) \quad \text{and} \quad E_2^{s,t+k\alpha}(G/e) \cong \mathbb{F}_1^{s,k\alpha}E_2^{s,t-k}(G/e)
\]

and \( \mathbb{F}_1^{s,t+k\alpha}K_\mathbb{R} \) has a similar description.

**Proof.** The element \( \bar{v}_1 \) and its restriction are invertible permanent cycles, so multiplication by either induces an isomorphism in the spectral sequence.

**Remark 8.7.** In the RO(G)-graded slice spectral sequence for \( K_\mathbb{R} \) one has \( d_3(u) = \bar{v}_1 a^3 \), but \( a^3 \) itself, and indeed all higher powers of \( a \), survive to \( E_2 = E_\infty \). Hence the \( E_\infty \)-term of this spectral sequence does not have the horizontal vanishing line that we see in \( E_\infty \)-term of Figure 7. However when we pass from \( K_\mathbb{R} \) to \( K_\mathbb{R} \), \( \bar{v}_1 \) becomes invertible and we have

\[
d_3(\bar{v}_1^{-1}u) = a^3.
\]

We can keep track of the groups in this trigraded spectral sequence with the help of four variable Poincaré series \( g(E_2(G/g)) \in \mathbb{Z}[x, y, z, t] \) in which the rank of \( E_2^{s,t+k\alpha}(G/G) \) is the coefficient in \( \mathbb{Z}[t] \) of \( x^{-s}y^t z^s \). The variable \( t \) keeps track of powers of two. Thus a copy of the integers is represented by \( 1/(1 - t) \) or (when it is the kernel of a differential of the form \( Z \to Z/(2) \)) \( t/(1 - t) \). Let

\[
\tilde{a} = y^{-1}z, \quad \tilde{u} = x^2y^{-1} \quad \text{and} \quad \bar{v} = xy.
\]

Since \( E_2(G/G) = \mathbb{Z}[a, u, \bar{v}_1]/(2a) \), we have

\[
\begin{align*}
g(E_2(G/G)) &= \left( \frac{1}{1 - t} + \frac{\tilde{a}}{1 - \tilde{a}} \right) \frac{1}{(1 - \tilde{u})(1 - \bar{v})}, \\
g(E_2(G/G)) &= \left( \frac{1}{1 - t} + \frac{\tilde{a}}{1 - \tilde{a}} \right) \frac{1}{(1 - \tilde{u})(1 - \bar{v})}. \\
\end{align*}
\]

We subtract the indicated expression from \( g(E_2(G/G)) \) because we have differentials

\[
d_3(a^{i+j+k} \bar{v}_1^{-1}) = a^{i+j+k+1} \bar{v}_1 u^{2k} \quad \text{for all } i, j, k \geq 0.
\]

Pursuing this further we get

\[
\begin{align*}
g(E_2(G/G)) &= \left( \frac{1}{1 - t} + \frac{\tilde{a}}{1 - \tilde{a}} \right) \frac{1}{(1 - \tilde{u})(1 - \bar{v})} - \frac{\tilde{u}^2}{(1 - \tilde{u})(1 - \bar{v})} - \frac{a \tilde{u} + \bar{v}}{(1 - \tilde{u})(1 - \bar{v})} \\
&= \left( \frac{1}{1 - t} - \frac{a \tilde{u} + \bar{v}}{(1 - \tilde{u})(1 - \bar{v})} \right) \frac{1}{(1 - \tilde{u})(1 - \bar{v})} + \frac{\tilde{u}}{(1 - \tilde{u})(1 - \bar{v})} \\
&= \left( \frac{1}{1 - t} - \frac{a \tilde{u} + \bar{v}}{(1 - \tilde{u})(1 - \bar{v})} \right) \frac{1}{(1 - \tilde{u})(1 - \bar{v})} + \frac{\tilde{u}}{(1 - \tilde{u})(1 - \bar{v})} \\
&= \left( \frac{1}{1 - t} - \frac{a \tilde{u} + \bar{v}}{(1 - \tilde{u})(1 - \bar{v})} \right) \frac{1}{(1 - \tilde{u})(1 - \bar{v})} + \frac{\tilde{u}}{(1 - \tilde{u})(1 - \bar{v})}.
\end{align*}
\]
The third term of this expression represents the elements of filtration above two (referred to in Remark 8.7) which disappear when we pass to $K_R$. The first term represents the elements of filtration zero, which include

$$1, \ [2u] \in \langle 2, a, a^2r_1 \rangle \quad \text{and} \quad [u^2] \in \langle a, a^2r_1, a^2r_1 \rangle. \quad (8.9)$$

Here we use the notation $[2u]$ and $[u^2]$ to indicate the images in $E_2$ of the elements $2u$ and $u^2$ in $E_1$; see Remark 4.1 below. The former not divisible by 2 and the latter is not a square since $u$ itself is not present in $E_0$, where the Massey products are defined. For an introduction to Massey products, we refer the reader to [9, A1.4].

We now make a similar computation where we enlarge $E_1(G/G)$ by adjoining $r_1^{-1}u$ and denote the resulting spectral sequence terms by $E'_2$ and $E'_3$.

Let

$$\tilde{w} = r^{-1}a = xy^{-3}.$$  

Then since

$$E'_2(G/G) = \mathbb{Z}[a, r_1^{-1}u, r_1]/(2a),$$

we have

$$g(E'_2(G/G)) = \left(\frac{1}{1-t} + \frac{\tilde{a}}{1-\tilde{a}}\right) \frac{1}{(1-\tilde{w})(1-\tilde{r})},$$

$$g(E'_3(G/G)) = g(E'_2(G/G)) - \frac{\tilde{w}}{(1-\tilde{a})(1-\tilde{w}^2)(1-\tilde{r})} - a\tilde{w} + \tilde{a}^2 = \frac{1 + \tilde{w} - \tilde{w}(1-t)}{(1-t)(1-\tilde{w}^2)(1-\tilde{r})} + \frac{\tilde{a}(1+\tilde{w}) - \tilde{a}(\tilde{w} + \tilde{a}^2)}{(1-t)(1-\tilde{w}^2)(1-\tilde{r})} - \frac{\tilde{a} + \tilde{a}^2}{(1-\tilde{a})(1-\tilde{w}^2)(1-\tilde{r})},$$

and there is nothing in $E'_3$ with filtration above two. As far as we know there is no modification of the spectrum $k_R$ corresponding to this modification of $E_*$. However the map $E_* k_R \to E_* K_R$ clearly factors through $E'_*$. 

## 9 Some elements in the homotopy groups of $k_{[2]}$ and $K_{[2]}$

For $G = C_4$ we will often use a (second) subscript $e$ on elements such as $r_n$ to indicate the action of a generator $g$ of $G = C_4$, so $\gamma(x_e) = x_{1+e}$ and $x_{2+e} = \pm x_e$. Then we have

$$\pi^e_* k_{[2]} = \pi_* k_{[2]}(G/e) = \mathbb{Z}[r_1, r_1/e] = \mathbb{Z}[r_1, \gamma(r_1)] = \mathbb{Z}[r_1, r_1, r_1], \quad (9.1)$$

where $\gamma^2(r_1,e) = -r_1,e$. Here we use $r_1,e$ and $r_1,e'$ to denote the images of elements of the same name in the homotopy of $MU^{((G))}$.

<table>
<thead>
<tr>
<th>2</th>
<th>$a_\lambda$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$a_r^2$</td>
</tr>
<tr>
<td>0</td>
<td>$u_{2r}$</td>
</tr>
<tr>
<td>$-2$</td>
<td>$-1$</td>
</tr>
</tbody>
</table>

Here the vertical coordinate is $s$ and the horizontal coordinate is $|t| - s$. More information about these elements can be found in Table 3 below.
Similarly now discuss.

The Toda bracket wedge summand of \(r\) where \(\bar{t}_2\).

This is spelled out in Theorem 10.2 below.

\[ G/\bar{t}_2. \]

The group action (by \(G\) on \(\bar{t}_1,e\), \(a_\sigma\), and \(u_{\sigma_2}\), and by \(G\) on all the others) fixes each generator but \(u_\sigma\) and \(u_{\sigma_2}\). For them the action is given by

\[ u_\sigma \xrightarrow{\gamma} -u_\sigma \quad \text{and} \quad u_{\sigma_2} \xrightarrow{\gamma} -u_{\sigma_2}. \]

by Theorem 2.13. This is compatible with the following \(G\)-action:

\[ \begin{array}{c|c|c}
  r_{1,0} & y & r_{1,1} \\
  \downarrow & \downarrow & \downarrow \\
  r_{1,1} & y & -r_{1,0}, \\
\end{array} \]

where \(r_{1,e} = \tau^2_1(\bar{t}_1,e) \in \pi_{[2]}(G/G)\).

We will see below (Theorem 11.13) that \(d_2(u_{\sigma_2}) = a_\sigma^2a_\sigma\bar{d}_1\) and \([u_{\sigma_2}]\) is a permanent cycle. Since all transfers are killed by \(a_\sigma\) multiplication (Lemma 4.2), this implies that \([u_{\sigma_2}]\) is a permanent cycle representing the Toda bracket

\[ [u_{\sigma_2}] = [u_{\sigma_2}\tau^2_1(y)] = \langle x, a_\sigma, a_\sigma^2a_\sigma\bar{d}_1 \rangle. \]

This element is \(x''\) since in \(E_2\) we have (using the Frobenius relation (2.4))

\[ x'' = \tau^0_2(u_{\sigma_2}y) = \tau^0_2(\text{res}^2_2(u_{\sigma_2})y) = u_{\sigma_2}\tau^0_2(y) = u_{\sigma_2}x. \]

Similarly \(x''' = u_{\sigma_2}x\). For \(k \geq 4\), \(x^{(k)} = u_{2\sigma}x^{(k-4)}\) in \(\pi_{[2]}\) as well as \(E_2\).

The Periodicity Theorem [6, Theorem 9.19] states that inverting a class in \(\pi_{[2]}k_{[2]}(G/G)\) whose image under \(\tau^0_2\text{res}^2_2\) is divisible by \(\bar{t}_1^2\) \((\text{see (7.2)})\) and \(\bar{t}_1\bar{t}_1 = \bar{t}_1\bar{t}_1\) makes \(u_{\sigma_2}\) a permanent cycle. One such class is

\[ D = N^2_2(\bar{t}_1^3)\bar{t}_2 = u_{2\sigma_2}\tau^0_2(\bar{t}_1^3)\bar{t}_2 = \bar{t}_1^3(-\bar{t}_1^2 + 2\bar{t}_2\bar{d}_1 + 9\bar{d}_1^2) \in \pi_{[2]}k_{[2]}(G/G), \]

where \(\bar{t}_2 = \tau^0_2(u_{\sigma_2}^{(1)}(\bar{t}_1,1))\) and \(\bar{d}_1\) is as in (9.5) below, and \(k_{[2]} = D^{-1}k_{[2]}\). Then we know that \(\Sigma^{32}k_{[2]}\) is equivalent to \(k_{[2]}\).

The Slice and Reduction Theorems [6, Theorems 6.1 and 6.5] imply that the 2kth slice of \(k_{[2]}\) is the 2kth wedge summand of \(H\mathbb{Z} \wedge N^2_2(\bigvee_{i \geq 0}S^{2i})\).

It follows that over \(G\), the 2kth slice is a wedge of \(k + 1\) copies of \(H\mathbb{Z} \wedge S^{2i} \wedge S^{2i+2}\). Over \(G\) we get the wedge of the appropriate number of copies of \(G_+ \wedge G\mathbb{Z} \wedge S^{2i}\), wedged with a single copy of \(H\mathbb{Z} \wedge S^{(k-2)i}\) for even \(k\). This is spelled out in Theorem 10.2 below.

The group \(\pi^G_{[2]}k_{[2]}(G'/(e))\) is not in the image of the group action restriction \(\tau^2_1\) because \(\rho_2\) is not the restriction of a representation of \(G\). However, \(\pi^G_{[2]}k_{[2]}\) is refined (in the sense of [6, Definition 5.28]) by a map from

\[ S_{\rho_2} := G_+ \wedge G\mathbb{Z} \wedge S^{2i} \wedge S^{2i+2} \rightarrow k_{[2]}. \]

The Reduction Theorem implies that the 2-slice \(P^2_{[2]}k_{[2]}\) is \(S_{\rho_2} \wedge H\mathbb{Z}\). We know that

\[ \pi^2(S_{\rho_2} \wedge H\mathbb{Z}) = 0. \]

We use the symbols \(r_1\) and \(y(r_1)\) to denote the generators of the underlying abelian group of \(\hat{\mathbb{C}}(G/(e)) = \mathbb{Z}^2\). These elements have trivial fixed point transfers and

\[ \pi^2(S_{\rho_2} \wedge H\mathbb{Z})(G/G') = 0. \]

Table 3 describes some elements in the slice spectral sequence for \(k_{[2]}\) in low dimensions, which we now discuss.
Given an element in $\pi_* \text{MU}^{(G)}$, we will often use the same symbol to denote its image in $\pi_* k_{[2]}$. For example, in [6, Section 9.1]

\[
\tilde{\delta}_n \in \pi^{G}_{(2^{n-1})\rho_4} \text{MU}^{(G)} = \pi^{G}_{(2^{n-1})\rho_4} \text{MU}^{(G)}(G/G) \tag{9.5}
\]

was defined to be the composite

\[
S^{(2^{n-1})\rho_4} = N^1_2 S^{(2^{n-1})\rho_2} \xrightarrow{N^1_2 r_{2n-1}} N^1_2 \text{MU}^{(G)} \longrightarrow \text{MU}^{(G)}.
\]

We will use the same symbol to denote its image in the group $\pi^{G}_{(2^{n-1})\rho_4} k_{[2]}(G/G)$.

The element $\eta \in \pi_1 S^0$ (coming from the Hopf map $S^3 \to S^2$) has image $a_0 \tilde{\tau}_1 \in \pi^{G}_{2} k_{R}(G' / G')$. There are two corresponding elements

\[
\eta_\epsilon \in \pi^{G}_{2} k_{[2]}(G' / G') \quad \text{for } \epsilon = 0, 1.
\]

We use the same symbol for their preimages under $r_\epsilon$ in $\pi^{G}_{2} k_{[2]}(G' / G')$, and there we have

\[
\eta_\epsilon = a_0 \tilde{r}_1, \epsilon.
\]

We denote by $\eta$ again the image of either under the transfer $\text{tr}_2^\epsilon$, so

\[
\text{res}^\epsilon_2(\eta) = \eta_0 + \eta_1.
\]

Its cube is killed by a $d_3$ in the slice spectral sequence, as is the sum of any two monomials of degree 3 in the $\eta_\epsilon$. It follows that in $E_4$ each such monomial is equal to $\eta_0^3$. It has a non-trivial transfer, which we denote by $x_3$.

In [6, Definition 5.51] we defined

\[
f_k = a_0^k N_0^k (\tilde{r}_k) \in \pi_1 \text{MU}^{(G)}(G/G) \tag{9.6}
\]

for a finite cyclic 2-group $G$. In particular, $f_{2n-1} = a_0^{2n-1} \tilde{\delta}_n$ for $\tilde{\delta}_n$ as in (9.5). The slice filtration of $f_k$ is $k(g - 1)$ and we will see below (Lemma 4.2 and, for $G = C_4$, Theorem 11.13) that

\[
\text{tr}^G_{2k}(u_{\sigma}) = a_{\sigma} f_{1,k}. \tag{9.7}
\]

Note that $u_{\sigma} \in E_2^{0,1-\sigma}(G / G')$ since the maximal subgroup for which the sign representation $\sigma$ is oriented is $G'$, on which it restricts to the trivial representation of degree 1. This group depends only on the restriction of the RO($G$)-grading to $G'$, and the isomorphism extends to differentials as well. This means that $u_{\sigma}$ is a place holder corresponding to the permanent cycle $1 \in E_2^{0,0}(G / G')$.

For $G = C_4$, equation (9.7) implies

\[
\text{tr}^G_{2}(u_{\sigma}) = a_{\sigma} f_{1} = a_{\sigma}^2 a_1 \tilde{\delta}_1.
\]

For example,

\[
\text{tr}^G_{2}(\eta_0 \eta_1) = \text{tr}^G_{2}(a_0^2 \tilde{r}_1, 0 \tilde{r}_1, 1) = \text{tr}^G_{2}(u_{\sigma} \text{res}^G_{2}(a_1 \tilde{\delta}_1)) = \text{tr}^G_{2}(u_{\sigma} a_1 \tilde{\delta}_1) = a_{\sigma} f_{1} a_1 \tilde{\delta}_1 = f_1^2.
\]

The Hopf element $\nu \in \pi_1 S^0$ has image

\[
a_{\sigma} a_1 \tilde{\delta}_1 \in \pi_2 k_{[2]}(G/G),
\]

so we also denote the latter by $\nu$. (We will see below in (11.7) that $u_{\lambda}$ is not a permanent cycle, but $\nu := a_{\sigma} u_{\lambda}$ is (11.8).) It has an exotic restriction $\eta_0^3$ (filtration jump two), which implies that

\[
2 \nu = \text{tr}^G_{2}(\text{res}^G_{2}(\nu)) = \text{tr}^G_{2}(\eta_0^3) = x_3.
\]

One way to see this is to use the Periodicity Theorem to equate $\pi_2 k_{[2]}$ with $\pi_{-29} k_{[2]}$, which can be shown to be the Mackey functor $\cdot$ in slice filtration $-32$. Another argument not relying on periodicity is given below in Theorem 11.13.

The exotic restriction on $\nu$ implies

\[
\text{res}^G_{2}(\nu^2) = \eta_0^6,
\]

with filtration jump 4.
Theorem 9.8 (The Hurewicz image). The elements \( v \in \pi_* k(2)(G/G) \), \( e \in \pi_* k(2)(G/G) \), \( \kappa \in \pi_* k(2)(G/G) \), and \( \bar{\kappa} \in \pi_* k(2)(G/G) \) are the images of elements of the same names in \( \pi_* S^0 \). The image of the Hopf map \( \eta \in \pi_1 S^0 \) is either \( \eta = \text{tr}_2^s(\kappa_e) \) or its sum with \( f_1 \).

We refer the reader to [9, Table A3.3] for more information about these elements.

Proof. Suppose we know this for \( v \) and \( \bar{v} \). Then \( A^{14}_1 v \) is represented by an element of filtration \(-3\) whose product with \( v^2 \) is nontrivial. This implies that \( v^3 \) has nontrivial image in \( \pi_* k(2)(G/G) \). This is a nontrivial multiplicative extension in the first quadrant, but not in the third. The spectral sequence representative of \( v^3 \) has filtration \( 11 \) instead of \( 3 \). We will see later that \( v^3 = 2n \) where \( n \) has filtration \( 1 \), and \( v^3 \) is the transfer of an element in filtration \( 1 \).

Since \( v^3 = \eta e \) in \( \pi_* S^0 \), this implies that \( \eta \) and \( e \) are both detected and have the images stated in Table 3. It follows that \( v \bar{v} \) has nontrivial image here. Since \( k^2 = e \bar{v} \) in \( \pi_* S^0 \), \( \bar{v} \) must also be detected. Its only possible image is the one indicated.

Both \( v \) and \( \bar{v} \) have images of order \( 8 \) in \( \pi_* \) TMF and its \( K(2) \) localization. The latter is the homotopy fixed point set of an action of the binary tetrahedral group \( G_{24} \), acting on \( E_2 \). This in turn is a retract of the homotopy fixed point set of the quaternion group \( Q_8 \). A restriction and transfer argument shows that both elements have order at least \( 4 \) in the homotopy fixed point set of \( C_4 \subset Q_8 \).

There is an orientation map \( MU \to E_2 \), which extends to a \( C_2 \)-equivariant map \( MU_{\mathbb{R}} \to E_2 \). Norming up and multiplying on the right gives us a \( C_4 \)-equivariant map \( N^4_{\mathbb{R}} \to E_2 \). This \( C_4 \)-action on the target is compatible with the \( G_{24} \)-action leading to \( L(2) \) TMF.

The image of \( \eta \in \pi_1 S^0 \) must restrict to \( \eta_0 + \eta_1 \), so modulo the kernel of \( \text{res}_2 \) it is the element \( \text{tr}_2^s(\eta_e) \), which we are calling \( \eta \). The kernel of \( \text{res}_2 \) is generated by \( f_1 \). \( \square \)

We now discuss the norm \( N^4_{\mathbb{R}} \), which is a functor from the category of \( C_2 \)-spectra to that of \( C_4 \) spectra. As explained above in connection with Corollary 4.8, for a \( C_4 \)-ring spectrum \( X \) we have an internal norm

\[
\pi^G_\Sigma V X(G'/G') = \pi^G_\Sigma V X(G/G) \to \pi^G_{\text{ind}}_\Sigma V X(G/G)
\]

and a similar functor on the slice spectral sequence for \( X \). It preserves multiplication but not addition. Its source is a module over \( G/G' \), which acts trivially on its target. Consider the diagram

\[
\begin{array}{ccc}
\pi^G_{\Sigma V} X(G/G) & \xrightarrow{=} & \pi^G_{\Sigma V} X(G'/G') \\
\downarrow & & \downarrow \text{res}_2 \\
\pi^G_{\Sigma V} X(G'/G') & \xrightarrow{=} & \pi^G_{\text{ind}}_\Sigma V X(G/G')
\end{array}
\]

For \( x \in \pi^G_{\Sigma V} X(G'/G') \) we have \( xy(x) \in \pi^G_{\Sigma V} X(G'/G') \) and 2V is the restriction of some \( W \in RO(G) \). The group \( \pi^G_\Sigma X(G'/G') \) depends only on the restriction of \( W \) to \( RO(G') \). If \( W' \in RO(G) \) is another virtual representation restricting to 2V, then \( W - W' = k(1 - \sigma) \) for some integer \( k \). The canonical isomorphism between \( \pi^G_\Sigma X(G'/G') \) and \( \pi^G_\Sigma X(G/G') \) is given by multiplication by \( u^G_{\Sigma} \).

Definition 9.9 (A second use of square bracket notation). For \( 0 \leq i \leq 2d \), let \( f(\bar{r}_{1,0}, \bar{r}_{1,1}) \) be a homogeneous polynomial of degree \( 2d - i \), so

\[
ad^i_{\bar{r}, f}(\bar{r}_{1,0}, \bar{r}_{1,1}) \in \pi^G_{\Sigma (2d-i)+(2d-2i)\Sigma}[k][2](G'/G').
\]

We will denote by \( [ad^i_{\bar{r}, f}(\bar{r}_{1,0}, \bar{r}_{1,1})] \) its preimage in \( \pi^G_{\Sigma (2d-i)+(2d-2i)\Sigma}[k][2](G'/G') \) under the isomorphism of (2.14).

The first use of square bracket notation is that of Remark 4.1. Note that \( \bar{r}_{1,e} \in \pi^G_{\Sigma^k} \Sigma[2] \) is not the target of such an isomorphism since \( \rho_2 \in RO(G') \) is not the restriction of any element in \( RO(G) \), hence the requirement that \( f \) has even degree.

We will define \( u^G_{\bar{r}, \Sigma}[k^2]_{\bar{r}, e} \in \pi^G_{\Sigma^k} [k][2](G'/G') \) by \( \Sigma_{\bar{r}, e} \). Then we have \( \gamma(\Sigma_{\bar{r}, e}) = -\Sigma_{\bar{r}, e, 1} \) and \( \gamma(\Sigma_{\bar{r}, e}) = -\Sigma_{\bar{r}, e, 0} \). We define \( \Sigma_{\bar{r}, e} = (-1)^e \Sigma_{\bar{r}, e} \).
which is independent of $e$, and we have

$$\text{res}_2^5(\mathfrak{t}_2) = S_{2,0} - S_{2,1}.$$ 

Then we have

$$\text{res}_2^5(N_2^5(\mathfrak{t}_{1,0})) = \text{res}_2^5(\hat{a}_1) = u_0^{-1}[\mathfrak{t}_{1,0} \mathfrak{t}_{1,1}] \in \pi_{2,0}k_{2}[G/G'].$$

More generally, for integers $m$ and $n$,

$$\text{res}_2^5(N_2^5(m\mathfrak{t}_{1,0} + n\mathfrak{t}_{1,1})) = u_0^{-1}[(m\mathfrak{t}_{1,0} + n\mathfrak{t}_{1,1})(m\mathfrak{t}_{1,1} - n\mathfrak{t}_{1,0})]$$

$$= u_0^{-1}((m^2 - n^2)[\mathfrak{t}_{1,0} \mathfrak{t}_{1,1}] + mn[\mathfrak{t}_{1,0}^{-1}]^{-1} - [\mathfrak{t}_{1,0}^{-1}])$$

so

$$N_2^5(m\mathfrak{t}_{1,0} + n\mathfrak{t}_{1,1}) = (m^2 - n^2)\delta_1 - mn\text{res}_2^5(\mathfrak{t}_2),$$

(9.10)

Similarly, for integers $a$, $b$ and $c$,

$$u_0^2 \text{res}_2^5(N_2^5(a\mathfrak{t}_{1,0}^2 + b\mathfrak{t}_{1,0}\mathfrak{t}_{1,1} + c\mathfrak{t}_{1,1}^2)) = [(a\mathfrak{t}_{1,0}^2 + b\mathfrak{t}_{1,0}\mathfrak{t}_{1,1} + c\mathfrak{t}_{1,1}^2)(a\mathfrak{t}_{1,0}^2 - b\mathfrak{t}_{1,0}\mathfrak{t}_{1,1} + c\mathfrak{t}_{1,1}^2)]$$

$$= [ac(\mathfrak{t}_{1,0}^2 + \mathfrak{t}_{1,1}^{-1} + b(c-a)\mathfrak{t}_{1,0}\mathfrak{t}_{1,1}(\mathfrak{t}_{1,0}^2 - \mathfrak{t}_{1,1}^{-2})] + (a^2 - b^2)\mathfrak{t}_{1,0}^{-2}\mathfrak{t}_{1,1}^{-2},$$

so

$$N_2^5(a\mathfrak{t}_{1,0}^2 + b\mathfrak{t}_{1,0}\mathfrak{t}_{1,1} + c\mathfrak{t}_{1,1}^2) = ac\mathfrak{t}_{1,0}^2 + b(c-a)\mathfrak{t}_{1,0}\mathfrak{t}_{1,1} + ((a+c)^2 - b^2)\mathfrak{t}_{1,1}^{-2}$$

(9.11)

For future reference we need

$$N_2^5(5\mathfrak{t}_{1,0}^2\mathfrak{t}_{1,1} + 5\mathfrak{t}_{1,0}\mathfrak{t}_{1,1} + \mathfrak{t}_{1,1}^2) = N_2^5(\mathfrak{t}_{1,1}^2)N_2^5(5\mathfrak{t}_{1,0}^2 + 5\mathfrak{t}_{1,0}\mathfrak{t}_{1,1} + \mathfrak{t}_{1,1}^2) = -\delta_1(5\mathfrak{t}_{1,0}^2 - 20\mathfrak{t}_{1,0}\mathfrak{t}_{1,1} + 11\mathfrak{t}_{1,1}^2).$$

Compare with (7.2). We also denote by

$$\eta_e = [a_{\partial_1}\mathfrak{t}_{1,e}] \in \pi_{1}k_{2}[G/G']$$

the preimage of $a_{\partial_1}\mathfrak{t}_{1,e} \in \pi_{1}G'/k_{2}(G'/G')$ and by $[a_{\partial_1}^2] \in \pi_{-1}k_{2}(G'/G')$ the preimage of $a_{\partial_1}^2$. The latter is $\text{res}_2^5(a_{\partial_1})$. The values of $N_2^5(a_{\partial_1})$ and $N_2^5(u_{\partial_2})$ are given by Lemma 4.9, namely

$$N_2^5(a_{\partial_1}) = a_{\lambda} \quad \text{and} \quad N_2^5(u_{\partial_2}) = u_{2\lambda}/u_{2\alpha}.$$
<table>
<thead>
<tr>
<th>Element</th>
<th>Description</th>
</tr>
</thead>
</table>
| \( u_1 \in \pi_{2,1}k_2^G(G/G) \) with \( [2u_1] \in \pi_{2,1}K_2^G(G/G) \) | Element corresponding to \( u_1 \in \pi_{2,1}HZ(G/G) \)
| \( \alpha_1^2 u_1 = 0 \) | \( 2, \eta, a_{\lambda} \) |
| \( d_3(u_1) = \eta a_\lambda = tr_4^3([a_{\lambda}^2, \tau_1, 0]) \) | \( \eta \alpha \) follows from the gold relation, Lemma 3.6 (vii) |
| \( d_3([u_1]) = \gamma_\alpha^2 \delta_1 \) | \( \eta \alpha \) follows from Theorem 11.13 |
| \( d_3([2u_1]) = \eta_\alpha^2 \delta_1 \) | \( \eta \alpha \) follows from Theorem 11.13 |
| \( [4u_1^2] \in \pi_{4,2}K_2^G(G/G) \) | \( \langle a_\alpha, \eta, a_\lambda \delta_1 \rangle \) |
| \( [2a_\alpha^2 u_1^2] \in \pi_{3,1}K_2^G(G/G) \) | \( \langle a_\alpha, \eta, a_\lambda \delta_1 \rangle \) |
| \( d_3 ([u_1]) = \langle \eta, \alpha, a_\lambda \delta_1 \rangle \) | \( \langle a_\alpha, \eta, a_\lambda \delta_1 \rangle \) |
| \( [2u_1^2] \in \pi_{8,4}K_2^G(G/G) \) | \( \langle a_\alpha, \eta, a_\lambda \delta_1 \rangle \) |
| \( \varpi \in \pi_{2,1}K_2^G(G/G') \) with \( d_3 (\varpi) = \langle a_\alpha^2, \tau_1, 0 + \tau_1, 1 \rangle \) | \( \langle a_\alpha, \eta, a_\lambda \delta_1 \rangle \) |
| \( \sigma \in \pi_{3,2}K_2^G(G/G') \) with \( d_3 (\sigma) = \langle a_\alpha, \eta, a_\lambda \delta_1 \rangle \) | \( \langle a_\alpha, \eta, a_\lambda \delta_1 \rangle \) |
| \( \tau \in \pi_{4,3}K_2^G(G/G') \) with \( d_3 (\tau) = \langle a_\alpha, \eta, a_\lambda \delta_1 \rangle \) | \( \langle a_\alpha, \eta, a_\lambda \delta_1 \rangle \) |
| \( \varphi \in \pi_{5,4}K_2^G(G/G') \) with \( d_3 (\varphi) = \langle a_\alpha, \eta, a_\lambda \delta_1 \rangle \) | \( \langle a_\alpha, \eta, a_\lambda \delta_1 \rangle \) |
| \( \psi \in \pi_{6,5}K_2^G(G/G') \) with \( d_3 (\psi) = \langle a_\alpha, \eta, a_\lambda \delta_1 \rangle \) | \( \langle a_\alpha, \eta, a_\lambda \delta_1 \rangle \) |

**Filtration 1**

**Continued on next page**
In this section we will identify the slices for $k_2$ and $K_2$ and the generators of their integrally graded homotopy groups. For the latter we will use the notation of Table 3. Let

$$X_{m,n} = \begin{cases} \sum_{p \geq 0} HZ & \text{for } m = n \\ G_+ \wedge G' \Sigma^{m-n+p} HZ & \text{for } m < n. \end{cases} \quad (10.1)$$

The slices of $k_2$ are certain finite wedges of these, and those of $K_2$ are a certain infinite wedges. Fortunately we can analyze these slices by considering just one value of $m$ at a time, this index being preserved by the
first differential $d_3$. These are illustrated below in Figures 9–12. They show both $E_2$ and $E_4$ in four cases depending on the sign and parity of $m$.

**Theorem 10.2** (The slice $E_2$-term for $k_{[2]}$). The slices of $k_{[2]}$ are

$$p^t_{[2]}(k_{[2]}) = \begin{cases} \bigvee_{0 \leq m \leq t/4} X_m,t/2 - m & \text{for } t \text{ even and } t \geq 0, \\ \ast & \text{otherwise,} \end{cases}$$

where $X_{m,n}$ is as in (1.1).

The structure of $\pi_* k_{[2]}$ as an $\mathbb{Z} [G]$-module (see (9.1)) leads to four types of orbits and slice summands:

1. $[(r_1,0,0,1)]^{2\ell - 1}$ leading to $X_{2\ell - 1,2\ell}$ for $\ell \geq 0$; see the leftmost diagonal in Figure 9. On the 0-line we have a copy of $\Delta$ (defined in Table 2) generated under restrictions by

$$\Delta_1^t = u_0^{2\ell t} \delta_1^{2\ell} = u_0^{2\ell} u_2^{2\ell} \delta_1^{2\ell} \in E_2^{0,2\ell t} (G/G).$$

In positive filtrations we have

- $\circ \subseteq E_2^{2\ell t,2\ell t}$ generated by $a_0^{2\ell} u_2^{2\ell - j} \delta_1^{2\ell} \in E_2^{2\ell t,2\ell t} (G/G)$ for $0 < j \leq 2\ell$,
- $\bullet \subseteq E_2^{2\ell + k,2\ell t}$ generated by $a_0^{2\ell} a_2^{2\ell - j} u_2^{2\ell - k} \delta_1^{2\ell} \in E_2^{2\ell + k,2\ell t} (G/G)$ for $0 < k \leq \ell$.

2. $[(r_1,0,1,1)]^{2\ell - 1}$ leading to $X_{2\ell + 1,2\ell + 1}$ for $\ell \geq 0$; see the leftmost diagonal in Figure 10. On the 0-line we have a copy of $\tilde{\Delta}$ generated under restrictions by

$$\tilde{\Delta}_{1}^{2\ell} = u_0^{2\ell} \tilde{\delta}_1^{2\ell} \in E_2^{0,2\ell t} (G/G).$$

In positive filtrations we have

- $\circ \subseteq E_2^{2\ell t,2\ell t}$ generated by $a_0^{2\ell} \tilde{\delta}_2^{2\ell - j} \tilde{\delta}_1^{2\ell} \in E_2^{2\ell t,2\ell t} (G/G)$ for $0 < j \leq 2\ell$,
- $\bullet \subseteq E_2^{2\ell + k,2\ell t}$ generated by $a_0^{2\ell} a_2^{2\ell - j} \tilde{\delta}_2^{2\ell - k} \tilde{\delta}_1^{2\ell} \in E_2^{2\ell + k,2\ell t} (G/G)$ for $0 < k \leq \ell$.

3. $[(r_1,0,1,1), r_1, 0, 1, 1]$ leading to $X_{0,2\ell - i}$ for $0 \leq i < \ell$; see other diagonals in Figure 9. On the 0-line we have a copy of $\tilde{\Delta}$ generated (under $\tilde{\Delta}_2$ and the group action) by

$$u_0^{2\ell} \tilde{\delta}_2^{-i} \tilde{\delta}_1^{2\ell} \in E_2^{0,2\ell t} (G/G).$$

In positive filtrations we have

- $\circ \subseteq E_2^{2\ell t,2\ell t}$ generated by $u_0^{2\ell} \tilde{\delta}_2^{-i} \tilde{\delta}_1^{2\ell} \in E_2^{2\ell t,2\ell t} (G/G)$ for $0 < j \leq \ell$,
  $$= \eta_\varepsilon^{2\ell} u_0^{2\ell} \tilde{\delta}_2^{-i} \tilde{\delta}_1^{2\ell} \in E_2^{2\ell t,2\ell t} (G/G)$$

4. $[(r_1,0,1,1), r_1, 0, 1, 1]$ leading to $X_{0,2\ell - i}$ for $0 \leq i < \ell$; see other diagonals in Figure 10. On the 0-line we have a copy of $\tilde{\Delta}$ generated (under transfers and the group action) by

$$r_1, 0 \tilde{\delta}_2^{-i} \tilde{\delta}_1^{2\ell} \in E_2^{0,2\ell t} (G/G).$$

In positive filtrations we have

- $\circ \subseteq E_2^{2\ell t,2\ell t}$ generated by $\eta_\varepsilon^{2\ell} u_0^{2\ell} \tilde{\delta}_2^{-i} \tilde{\delta}_1^{2\ell} \in E_2^{2\ell t,2\ell t} (G/G)$ for $0 < j \leq \ell$,
  $$= \eta_\varepsilon^{2\ell} u_0^{2\ell} \tilde{\delta}_2^{-i} \tilde{\delta}_1^{2\ell} \in E_2^{2\ell t,2\ell t} (G/G)$$

**Corollary 10.3** (A subring of the slice $E_2$-term). The ring $E_2 k_{[2]} (G/G')$ contains

$$\mathbb{Z} [\delta_1, \Sigma_{2,1}, \eta_\varepsilon : \varepsilon = 0, 1]/(2 \eta_\varepsilon, \delta_1^2 - \Sigma_{2,0} \Sigma_{2,1}, \eta_\varepsilon \Sigma_{2,0} \Sigma_{2,1} + \eta_1 + \varepsilon \delta_1);$$

see Table 3 for the definitions of its generators. In particular, the elements $\eta_0$ and $\eta_1$ are algebraically independent modulo 2 with

$$y^m (\eta_0^m \eta_1^n) \in \pi_{m+n} (X_{m,n} (G/G'))$$

for $m \leq n$. 

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The element \((\eta_0\eta_1)^2\) is the fixed point restriction of
\[
 u_2a^1b^1c^2 \in E_2^{0,8} k_2^2(G/G),
\]
which has order 4, and the transfer of the former is twice the latter. The element \(\eta_0\eta_1\) is not in the image of \(\text{res}_2^4\) and has trivial transfer in \(E_2\).

**Proof.** We detect this subring with the monomorphism
\[
 E_2 k_2^2(G/G) \xrightarrow{\rho_k} E_2 k_2^2(G'/G'), \quad \eta_t e \mapsto a_0 \bar{r}_{1,e}, \quad \Sigma_2 e \mapsto u_2 a^2 \bar{r}_{1,e}, \quad \delta_1 \mapsto u_2 a^0 \bar{r}_{1,0} \bar{r}_{1,1},
\]
in which all the relations are transparent. \(\square\)

**Corollary 10.4 (Slices for \(K_2\)).** The slices of \(K_2\) are
\[
P^t K_2 = \begin{cases} 
X_{m,t/2-m} & \text{for } t \text{ even}, \\
* & \text{otherwise},
\end{cases}
\]
where \(X_{m,n}\) is as in Theorem 10.2. Here \(m\) can be any integer, and we still require that \(m \leq n\).

**Proof.** Recall that \(K_2\) is obtained from \(k_2\) by inverting a certain element
\[
 D \in \Sigma_{4p_k} k_2^2(G/G)
\]
described in Table 3. Thus \(K_2\) is the homotopy colimit of the diagram
\[
k_2^2 \xrightarrow{D} \Sigma^{-4p_k} k_2^2 \xrightarrow{D} \Sigma^{-8p_k} k_2^2 \xrightarrow{D} \ldots.
\]
Desuspending by \(4p_k\) converts slices to slices, so for even \(t\) we have
\[
P^t K_2 = \lim_{k \to \infty} \Sigma^{-4k_2} P^{t+16k_2} k_2^2 = \lim_{k \to \infty} \Sigma^{-4k_2} \bigvee_{0 \leq m \leq t/4+4k} X_{m,t/2+8k-m} = \lim_{k \to \infty} \bigvee_{0 \leq m \leq t/4+4k} X_{m-4k,t/2+4k-m} = \lim_{k \to \infty} \bigvee_{m \leq t/4} X_{m,t/2-m} = \bigvee_{m \leq t/4} X_{m,t/2-m}.
\]
\(\square\)

**Corollary 10.5 (A filtration of \(k_2\)).** Consider the diagram
\[
k_2^2 \xleftarrow{\delta_1} \Sigma^{p_0} k_2^2 \xrightarrow{\delta_1} \Sigma^{2p_0} k_2^2 \xleftarrow{\delta_1} \ldots
\]
\[
y_0 \quad y_1 = \Sigma^{p_0} y_0 \quad y_2 = \Sigma^{2p_0} y_0,
\]
where \(y_0\) is the cofiber of the map induced by \(\delta_1\). Then the slices of \(y_m\) are
\[
P^t y_m = \begin{cases} 
X_{m,t/2-m} & \text{for } t \text{ even and } t \geq 4m, \\
* & \text{otherwise}.
\end{cases}
\]

**Corollary 10.6 (A filtration of \(K_2\)).** Let \(R = \mathbb{Z}_2[x]/(11x^2 - 20x + 5)\). After tensoring with \(R\) (by smashing with a suitable Moore spectrum \(M\)) there is a diagram
\[
\ldots \xrightarrow{f_1} \Sigma^{2p_0} k_2^2 \xrightarrow{f_1} \Sigma^{p_0} k_2^2 \xrightarrow{f_0} \Sigma^{-p_0} k_2^2 \xrightarrow{f_{-1}} \ldots
\]
\[
Y_2 \quad Y_1 \quad Y_0 \quad Y_{-1},
\]
where the homotopy colimit of the top row is \(K_2\) and each \(Y_m\) has slices similar to those of \(y_m\) as in Corollary 10.5.
Proof. The periodicity element $D = -\delta_i^2 (5\delta_i^2 - 20\delta_i \delta_0 + 11\delta_i^2)$ can be factored as

$$D = D_0 D_1 D_2 D_3,$$

where $D_i = a_i \delta_1 + b_i \delta_2$ with $a_i \in \mathbb{Z}_{(2)}$ and $b_i \in \mathbb{R}$. Then let $f_{an+i}$ be multiplication by $D_i$. It follows that the composite of any four successive $f_{mn}$ is $D$, making the colimit $K_{(2)}$ as desired. The fact that $a_i$ is a unit means that the $Y$'s here have the same slices as the $y$'s in Corollary 10.5.

Remark 10.7. The 2-adic completion of $R$ is the Witt ring $W(F_4)$ used in Morava $E_2$-theory. This follows from the fact that the roots of the quadratic polynomial involve $\sqrt{5}$, which is in $W(F_4)$ but is not a 2-adic integer.

Moreover, if we assume that $D_0 D_1 = 5\delta_i^2 - 20\delta_i \delta_0 + 11\delta_i^2$, then the composite maps $f_{an} f_{a_{n+1}}$, as well as $f_{a_{n+2}}$ and $f_{a_{n+3}}$, can be constructed without adjoining $\sqrt{5}$.

It turns out that $y_m \wedge M$ and $Y_m$ for $m \geq 0$ not only have the same slices, but the same slice spectral sequence, which is shown in Figures 9–12. See Remark 13.2 below. We do not know if they have the same homotopy type.

11 Some differentials in the slice spectral sequence for $k_{(2)}$

Now we turn to differentials. The only generators in (9.2) that are not permanent cycles are the $u$'s. We will see that it is easy to account for the elements in $E_{2}^{s,t} = V(G/H)$ for proper subgroups $H = G_{4}$, from (9.2) we see that

$$E_{2}^{s,t} = 0 \quad \text{for } |t| \text{ odd.}$$

This sparseness condition implies that $d_r$ can be nontrivial only for odd values of $r$.

Our starting point is the Slice Differentials Theorem of [6, Theorem 9.9], which is derived from the fact that the geometric fixed point spectrum of $MU_{(G)}$ is $MO$. It says that in the slice spectral sequence for $MU_{(G)}$ for an arbitrary finite cyclic 2-group $G$ of order $g$, the first nontrivial differential on various powers of $u_{2g}$ is

$$d_r(u_{2g}^{2k-1}) = a_{2g}^{2k} \delta_2^{2k-1} \prod_{i=0}^{k-1} (2g)^{i} \in E_{r}^{s,r+2k(1-g)-1} MU_{(G)}(G/G),$$

where $r = 1 + (2k-1)g$ and $\delta$ is the reduced regular representation of $G$.

In particular,

$$d_5(u_{2g}) = a_{2g}^{3} \prod_{i=0}^{2g} MU_{(G)}(G/G) \quad \text{for } G = G_{4},$$

$$d_3(u_{2g}) = \prod_{i=0}^{2g} \prod_{i=0}^{2g} MU_{(G)}(G/G) \quad \text{for } G = G_{2},$$

The first of these leads directly to a similar differential in the slice spectral sequence for $k_{(2)}$. The target of the second one has trivial image in $k_{(2)}$ and we shall see that $[u_{2g}^{k}]$ turns out to be a permanent cycle.

There are two ways to leverage the third and fourth differentials of (11.3) into information about $k_{(2)}$.

(i) They both lead to differentials in the slice spectral sequence for the $C_2$ spectrum $i_{0}^{*} k_{(2)}$. They are spelled out in (11.6) and will be studied in detail below in Section 12. They completely determine the slice spectral sequence $E_{2}^{s,t} (G/G')$ for both $k_{(2)}$ and $K_{(2)}$. Since $u_{2g}$ restricts to $u_{1}$, which is isomorphic to $u_{2g}$, we get some information about differentials on powers of $u_{1}$. The $d_5$ on $u_{2g}$ forces a $d_3(u_{1}) = \eta a_{1}$. The target of $d_7(u_{2g}^{3})$ turns out to be the exotic restriction of an element in filtration 5, leading to $d_7(u_{1}^{2}) = v a_{1}^{2}$. We will also see that even though $[u_{2g}^{2}]$ is a permanent cycle, $[u_{1}^{2}]$'s image under the restriction map $res_{2}$ is not.

(ii) One can norm up the differentials on $u_{2g}$ and its square using Corollary 4.8, converting the $d_7$ and $d_5$ to a $d_5$ and a $d_5$. The source of the latter is $[a_{g} u_{1}^{2}]$, which implies that $[u_{1}^{2}]$ is not a permanent cycle.
The differentials of (11.3) lead to Massey products which are permanent cycles,
\[ \langle 2, a_{\sigma}^2, f_1 \rangle = [2u_{2\alpha}] = \text{tr}_{G}^{G}(u_{\alpha}^2) \in \begin{cases} E_{0}^{G, 2-2\alpha}MU^{(G)}(G/G) & \text{for } G = C_{4}, \\ E_{4}^{G, 2-2\alpha}MU_{R}(G/G) & \text{for } G = C_{2}, \end{cases} \]
\[ \langle 2, a_{\sigma}^2, f_3 \rangle = [2u_{2\alpha}^2] = \text{tr}_{G}^{G}(u_{\alpha}^4) \in \begin{cases} E_{14}^{G, 4-4\alpha}MU^{(G)}(G/G) & \text{for } G = C_{4}, \\ E_{8}^{G, 4-4\alpha}MU_{R}(G/G) & \text{for } G = C_{4}, \end{cases} \]
and (by Theorem 4.4) to exotic transfers
\[ a_{\sigma}^2 f_1 = \begin{cases} \text{tr}_{G}^{G}(u_{\alpha}) \in E_{\infty}^{G, 5-\alpha}MU^{(G)}(G/G) & \text{for } G = C_{4} \text{ (filtration jump 4),} \\ \text{tr}_{G}^{G}(u_{\alpha}) \in E_{\infty}^{G, 3-\alpha}MU_{R}(G/G) & \text{for } G = C_{2} \text{ (filtration jump 2),} \end{cases} \]
\[ a_{\sigma}^2 f_3 = \begin{cases} \text{tr}_{G}^{G}(u_{\alpha}^2) \in E_{\infty}^{G, 15-3\alpha}MU_{R}(G/G) & \text{for } G = C_{4} \text{ (filtration jump 12),} \\ \text{tr}_{G}^{G}(u_{\alpha}^2) \in E_{\infty}^{G, 9-3\alpha}MU_{R}(G/G) & \text{for } G = C_{2} \text{ (filtration jump 6).} \end{cases} \]
(11.4)

Since \( a_{\sigma} \) and \( 2a_{\lambda} \) kill transfers by Lemma 4.2, we have Massey products,
\[ [u_{2\alpha}, \text{tr}_{2}^{G}(x)] = [\text{tr}_{2}(u_{2}^\alpha x) = (a_{\sigma} f_1, a_{\alpha}, \text{tr}_{2}(x)) \text{ with } 2a_{\alpha}[u_{2\alpha} \text{tr}_{2}(x)] = 0. \]
(11.5)

Now, as before, let \( G = C_{4} \) and \( G' = C_{2} \subseteq G \). We need to translate the \( d_3 \) above in the slice spectral sequence for \( MU_{U} \) into a statement about the one for \( k_{[2]} \) as a \( G' \)-spectrum. We have an equivariant multiplication map \( m \) of \( G' \)-spectra,
\[
\begin{array}{c c c}
\text{MU}^{(G)} & \eta & \text{MU}_{R} \\
\text{MU}_{R} & m & \text{MU}_{R} \\
\end{array}
\]
\[
\begin{array}{c c c c}
\tau_{1}^{G'} & \eta_{G} & \tau_{1,0}^{G} & \tau_{1,1}^{G} \\
\tau_{1}^{G'} & a_{3}^{G}(\tau_{1,0}^{G}, \tau_{1,1}^{G}) & a_{3}^{G}(\tau_{1,0}^{G}, \tau_{1,1}^{G}) & a_{3}^{G}(\tau_{1,0}^{G}, \tau_{1,1}^{G}) \\
\tau_{3}^{G'} & 5\tau_{1,0}^{G}, \tau_{1,1}^{G} & a_{3}^{G}(\tau_{1,0}^{G}, \tau_{1,1}^{G}) & a_{3}^{G}(\tau_{1,0}^{G}, \tau_{1,1}^{G}) \\
\end{array}
\]
where the elements lie in \( \tau_{i}^{G'}(\cdot)(G'/G) \) and \( a_{3}^{G}(\cdot)(G'/G) \). In the slice spectral sequence for \( MU^{(G)} \) as a \( G' \)-spectrum, \( d_{3}(u_{2\alpha}) \) and \( d_{7}(u_{2\alpha}^2) \) must be \( G \)-invariant since \( u_{2\alpha} \) is, and they must map respectively to \( a_{3}^{G}\tau_{i}^{G'} \) and \( a_{7}^{G}\tau_{3}^{G'} \), so we have
\[
\begin{align*}
d_{3}(u_{2\alpha}) &= d_{3}(\rho_{\alpha}) = a_{3}^{G}(\rho_{1,0}^{G}, \rho_{1,1}^{G}) = a_{3}^{G}(\eta_{0} + \eta_{1}) \\
d_{7}(u_{2\alpha}^2) &= d_{7}(\rho_{2}^{G}) = a_{7}^{G}(5\rho_{1,0}^{G}, \rho_{1,1}^{G} + \rho_{1,1}^{G}) + (\rho_{1,1}^{G})^3 + \cdots \\
&= a_{7}^{G}(\rho_{1,0}^{G})^3 + \cdots \text{ since } a_{7}^{G}(\rho_{1,0}^{G}, \rho_{1,1}^{G}) = 0 \text{ in } E_{4}, \\
\end{align*}
\]
(11.6)

We get similar differentials in the slice spectral sequence for \( k_{[2]} \) as a \( C_{2} \)-spectrum in which the missing terms in \( d_{7}(\rho_{2}^{G}) \) vanish.

Pulling back along the isomorphism \( E_{4}^{G} \) gives
\[
\begin{align*}
d_{3}(\text{res}_{2}^{G}(u_{\alpha})) &= d_{3}(\rho_{\alpha}) = a_{3}^{G}(\eta_{0} + \eta_{1}) = \text{res}_{2}^{G}(a_{\lambda} \eta), \\
d_{7}(\text{res}_{2}^{G}(u_{\alpha}^2)) &= d_{7}(\rho_{2}^{G}) = \text{res}_{2}^{G}(a_{3}^{G}(\eta_{0}^3) = \text{res}_{2}^{G}(a_{\alpha}^2 v). \\
\end{align*}
\]
(11.7)

These imply that
\[ d_{3}(u_{\lambda}) = a_{\lambda} \eta \quad \text{and} \quad d_{5}(u_{\alpha}^2) = a_{\alpha}^2 v. \]

The differential on \( u_{\lambda} \) leads to the following Massey products, the second two of which are permanent cycles:
\[
\begin{align*}
[u_{\lambda}^2] &= \langle a_{\lambda}, \eta, a_{\lambda}, \eta \rangle \in E_{4}^{G, 4-2\lambda}(G/G), \\
[2u_{\alpha}] &= \langle 2, \eta, a_{\alpha} \rangle \in E_{4}^{G, 2-\lambda}(G/G), \\
\nu : = [a_{\alpha} u_{\alpha}] &= \langle a_{\sigma}, \eta, a_{\lambda} \rangle \in E_{4}^{G, 3-3\lambda}(G/G), \\
\end{align*}
\]
(11.8)
where $\mathbb{V}$ satisfies

\[
\begin{align*}
\alpha_2^3\mathbb{V} &= \langle \alpha_2^3, \eta, \alpha_1 \rangle = q_0[\alpha_2^3 u_1] \\
&= q_0[2a_1 u_2o] = [2a_0 a_1 u_2o] = 0,
\end{align*}
\]

\[
\begin{align*}
\text{res}_2^1(\mathbb{V}) &= [\alpha_2^2 \tilde{r}_{1,1}] u_0 \in E_3^{1,5-\sigma^1(G/G^*)} \\
2\mathbb{V} &= \text{tr}_3^2(\text{res}_2^1(\mathbb{V})) = \text{tr}_3^2(u_0[\alpha_2^3 \tilde{r}_{1,1}]) \\
&= \eta^1 a_1 \in E_3^{3,5-\sigma^1(G/G^*)} \\
\end{align*}
\]

( exotic restriction with filtration jump 2 by Theorem 4.4 (i)),

\[
\begin{align*}
\text{tr}_3^2(x \mathbb{V}) &= \text{tr}_3^2(x \cdot \text{res}_2^1(\mathbb{V})) = \text{tr}_3^2(x[\alpha_2^3 \tilde{r}_{1,0}] u_0), \\

\eta^1 \mathbb{V} &= \text{tr}_3^2([\alpha_2^3 \tilde{r}_{1,1}] \mathbb{V}) \\
&= \text{tr}_3^2([\alpha_2^3 \tilde{r}_{1,0} \tilde{r}_{1,1}] u_0) = \alpha_1^2 \tilde{d}_1 \text{tr}_3^2(u_0^2) \\
&= a_1^2 \tilde{d}_1 (2, a_0, a_0 f_1) = \langle 2, a_0, f_1 \rangle, \\

\eta^1 \mathbb{V} &= \alpha_1^2 \tilde{d}_1 \text{tr}_3^2(u_0^2) = 0,
\end{align*}
\]

\[
\begin{align*}
d_7^v([\mathbb{V}]_1) &= [\alpha_2^2 \tilde{r}_{1,0}^1] \\
&= \text{res}_2^1(\mathbb{V}) \text{res}_2^1(\alpha_1^2 \tilde{d}_1) \\
&= \text{res}_2^1(\tilde{\eta} \alpha_1^2 \tilde{d}_1) \\
&= \text{res}_2^1(d_7(\alpha_1^2)), \\
d_5([u_2^1]) &= \tilde{\eta} \alpha_1^2 \tilde{d}_1 = \eta^1 v, \\
d_7([u_2^1]) &= (2\tilde{\eta}) \alpha_1^2 \tilde{d}_1 = \eta^1 \tilde{d}_1.
\end{align*}
\]

Note that $v = \tilde{\eta} \tilde{d}_1$, with the exotic restriction and group extension on $\mathbb{V}$ being consistent with those on $v$.

The differential on $[u_2^1]$ yields Massey products

\[
\begin{align*}
[a_2^3 u_2^1] &= \langle a_2^3, \mathbb{V}, a_1^2 \tilde{d}_1 \rangle, \\
[\eta^1 u_2^1] &= \langle \eta^1, \mathbb{V}, a_1^2 \tilde{d}_1 \rangle. \\
\end{align*}
\]

\[
\text{Theorem 11.10 (Normed up slice differentials for $k_{[2]}$ and $K_{[2]}$). In the slice spectral sequences for $k_{[2]}$ and $K_{[2]}$, we have}
\]

\[
\begin{align*}
d_5([a_0 u_2^1]) &= 0 \quad \text{and} \quad d_{13}([a_0 u_2^1]) = a_1^2 [u_2^2 \tilde{d}_1].
\end{align*}
\]

\[
\text{Proof. The two slice differentials over $G'$ are}
\]

\[
\begin{align*}
d_5(u_{2o}) &= \alpha_2^2 \tilde{r}_{1,0}^2 = \alpha_2^2 (\tilde{r}_{1,0} + \tilde{r}_{1,1}), \\
d_7([u_2^2]) &= \alpha_2^2 \tilde{r}_3^2 = \alpha_2^2 (5\tilde{r}_{1,0} \tilde{r}_{1,1} + 5\tilde{r}_{1,0}^2 \tilde{r}_{1,1} + \tilde{r}_{1,1}^2).
\end{align*}
\]

We need to find the norms of both sources and targets. Lemma 4.9 tells us that

\[
\begin{align*}
N_2^h(a_2^k) &= a_1^k, \\
N_2^h(u_{2o}^k) &= u_{2o}^k / u_{2o} \text{ in } E_\pm(G/G).
\end{align*}
\]

Previous calculations give

\[
\begin{align*}
N_2^h(\tilde{r}_{1,0} + \tilde{r}_{1,1}) &= -\tilde{d}_2 \\
N_2^h(5\tilde{r}_{1,0} \tilde{r}_{1,1} + 5\tilde{r}_{1,0}^2 \tilde{r}_{1,1} + \tilde{r}_{1,1}^2) &= -\tilde{d}_1 (5\tilde{d}_2^2 - 20\tilde{d}_2 \tilde{d}_1 + 11\tilde{d}_1^2) \\
\end{align*}
\]

by (9.10),

\[
\begin{align*}
N_2^h(5\tilde{r}_{1,0}^2 \tilde{r}_{1,1} + 5\tilde{r}_{1,0} \tilde{r}_{1,1}^2 + \tilde{r}_{1,1}^2) &= -\tilde{d}_1 (5\tilde{d}_2^2 - 20\tilde{d}_2 \tilde{d}_1 + 11\tilde{d}_1^2) \text{ by (9.11).}
\end{align*}
\]

For the first differential, Corollary 4.8 tells us that

\[
\begin{align*}
\alpha_1^2 \tilde{d}_2 &= d_5(\tilde{d}_2) \\
&= \tilde{d}_2 (\tilde{d}_2) / u_{2o} - a_0 \tilde{d}_2^2 d_5(u_{2o}) / [u_{2o}^2] \\
&= \tilde{d}_2 (\tilde{d}_2) / u_{2o} - a_0 \tilde{d}_2^2 a_1^2 a_0 \tilde{d}_1 / [u_{2o}^2].
\end{align*}
\]
We can use this to find the differential on \(u^2_{20}\) since \(a_4\eta\) is killed by \(d_3\). It follows that \(d_5(a_\sigma u^2_{1\sigma}) = a^2_{\lambda}[u_{20}\tilde{t}_2] = 0\), as claimed.

For the second differential we have
\[
d_{13}((a_\sigma u^2_{1\sigma}))[u^2_{20}]= a^2_{\lambda}[u_{20}\tilde{t}_2]- 5t^2_{2\tilde{t}_2} + 9\tilde{t}^2_{1}\tilde{t}_2,
\]
\[
d_{13}((a_\sigma u^2_{1\sigma}))[u_{20}] = a^2_{\lambda}[u_{20}\tilde{t}_2] - 9\tilde{t}^2_{1}\tilde{t}_2 + 9\tilde{t}^2_{1}
\]
since \(a_\lambda\) has order 4. As we saw above, \(a^2_{\lambda}[u_{20}\tilde{t}_2]\) vanishes in \(E_5\), so \(d_{13}((a_\sigma u^2_{1\sigma}))\) is as claimed.

We can use this to find the differential on \([u^2_{1\sigma}]\). We have
\[
d((u^2_{1\sigma})) = [2u^2_{1\sigma}]d((u^2_{1\sigma})) = [2u^2_{1\sigma}][\varpi a^2_{\lambda}\tilde{t}_1] = (2\varpi)a^2_{\lambda}[u^2_{1\sigma}][\tilde{t}_1]
\]
\[
\eta' a^2_{\lambda}[u^2_{1\sigma}][\tilde{t}_1] = [\eta' u^2_{1\sigma}][a^2_{\lambda}\tilde{t}_1] = (\eta', \varpi, a^2_{\lambda}[\tilde{t}_1])
\]
\[
(11.11)
\]
The differential on \(u_{20}\) yields
\[
[xu_{20}] = \langle x, a^2_{\sigma}, f_1 \rangle
\]
for any permanent cycle \(x\) killed by \(a^2_{\sigma}\). Possible values of \(x\) include 2, \(\eta, \eta'\) (each of which is killed by \(a_\sigma\) as well) and \(\varpi\). For the last of these we write
\[
\xi := [\varpi u_{20}] = \langle \varpi, a^2_{\sigma}, f_1 \rangle = \langle a_\sigma u_{1\sigma}, a^2_{\sigma}, f_1 \rangle \in E_6^{1.5-3\sigma-\lambda}(G/G),
\]
\[
(11.12)
\]
which satisfies
\[
\text{res}_{u_{20}}^2(\xi) = a^3_0 u^3_0 [u_{1\sigma}, e] \in E_4^{3.7-3\sigma-\lambda}(G/G')
\]
(exotic restriction with jump 2),
\[
2\xi = \text{tr}_r^2(\text{res}_u^2(\xi)) = \eta' a_\sigma u_{20} \in E_6^{3.7-3\sigma-\lambda}(G/G)
\]
(exotic group extension with jump 2),
\[
d_5([u_{20}u^2_{1\sigma}]) = a^3_0 a_\sigma u^3_0 \tilde{t}_1 + \varpi a^2_0 u_{20} \tilde{t}_1 = (a^3_0 a^2_0 + \varpi u_{20})a^3_0 \tilde{t}_1
\]
\[
= (a^3_0 a_\sigma u_{1\sigma} + \varpi a^2_0) a^3_0 \tilde{t}_1
\]
\[
d_7([2u^2_{1\sigma}]) = 2\xi \cdot a^3_0 \tilde{t}_1 = \eta' a^3_0 u_{20} \tilde{t}_1,
\]
\[
\text{res}^2_{u_{20}}(d_5([u_{20}u^2_{1\sigma}])) = u_{20} a^3_0 [u_{1\sigma}, e] \text{res}^2_{u_{20}}(a^3_0 \tilde{t}_1) = u_{20} a^3_0 [u_{1\sigma}, e] \text{res}^3_{u_{20}}(a^2_0 \tilde{t}_1, 0, 0, 0) = u_{20} a^3_0 \tilde{t}_1.
\]

**Theorem 11.13** (The differentials on powers of \(u_{1\sigma}\) and \(u_{20}\)). The following differentials occur in the slice spectral sequence for \(k_{[2]}\); here \(\overline{u}_\sigma\) denotes \(\text{res}^2_{u_{20}}(u_{1\sigma})\):
\[
d_{3}(u_{1\sigma}) = a_\sigma \eta = \text{tr}_r^2(a^2_0 \tilde{t}_{1,e}),
\]
\[
d_{3}(u_{20}) = \text{res}^2_{u_{20}}(a_\sigma)(\eta_0 + \eta_1) = [a^3_0, (\tilde{t}_{1,0} + \tilde{t}_{1,1})],
\]
\[
d_{5}(u_{20}) = a^3_0 a_\sigma \tilde{t}_1,
\]
\[
d_{5}(u^2_{1\sigma}) = a^3_0 a_\sigma u_{1\sigma} \tilde{t}_1 = a^3_0 \varpi \tilde{t}_1 = a^3_0 \varpi
\]
\[
\text{for } \varpi \text{ as in (11.8)},
\]
\[
d_{5}([u_{20}u^2_{1\sigma}]) = a^3_0 a_\sigma u^3_0 \tilde{t}_1 + \varpi a^2_0 u_{20} \tilde{t}_1 = (a^3_0 a^2_0 + \varpi u_{20})a^3_0 \tilde{t}_1 = \xi a^3_0 \tilde{t}_1 \text{ for } \xi \text{ as in (11.12)},
\]
\[
d_{7}([2u^2_{1\sigma}]) = 2\xi \cdot a^3_0 \tilde{t}_1 = \eta' a^3_0 u_{20} \tilde{t}_1,
\]
\[
d_{7}([u^2_{1\sigma}]) = \text{res}^2_{u_{20}}(a^3_0 \tilde{t}_{1,0}) = a^3_0 \tilde{t}_{1,0},
\]
\[
d_{7}([u^2_{1\sigma}]) = [\eta' u^2_{1\sigma}][a^3_0 \tilde{t}_1] = (\eta', \varpi, a^3_0 \tilde{t}_1) a^3_0 \tilde{t}_1.
\]
The elements
\[ u_{\sigma}, \quad [2u_{\sigma}] = \langle 2, a_0^2, f_1 \rangle = \text{tr}_2^2(u_{\sigma}^2), \quad [2u_{\lambda}] = \langle 2, \eta, a_\lambda \rangle, \quad [4u_{\lambda}^3] = \langle 2, \eta', a_\lambda^3 \tilde{\delta}_1 \rangle = \text{tr}_1^4(u_{\lambda}^2), \]
\[ [u_{\sigma}^2] = \langle a_0^2, f_1, a_0^2, f_1 \rangle \quad [2u_{\lambda}^2] = \langle 2, a_0^6, a_\sigma^2 \tilde{\delta}_1^0 \rangle = \text{tr}_3^4(u_{\lambda}^2), \]
\[ [2u_{2\sigma}u_{\lambda}] = \langle 2u_{2\sigma}, \eta, a_\lambda \rangle, \quad [u_{\lambda}^4] = \langle a_0^3, \tilde{\tau}_{1,0}, a_0^7, \tilde{\tau}_{1,0} \rangle, \quad [u_{\lambda}^8] = \langle \eta' u_{\lambda}^2, a_\lambda^3 \tilde{\delta}_1, [\eta' u_{\lambda}^2], a_\lambda^3 \tilde{\delta}_1 \rangle \]
are permanent cycles.

We also have the following exotic restriction and transfers:

\[ \text{res}_{2\lambda}^2((a_\sigma u_{\lambda}) \eta_\sigma) = u_{\sigma} \text{res}_{2\lambda}^2(a_\lambda) \eta_\sigma = u_{\sigma} a_\sigma^2 \tilde{\tau}_{1,e} \quad (\text{filtration jump 2}), \]
\[ \text{tr}_2^k(u_{\sigma}^k) = \begin{cases} a_\sigma^2 a_0^2 \tilde{\tau}_{1,e} u_{2\sigma}^{(k-1)/2} = a_\sigma a_{\sigma}^1 u_{2\sigma}^{(k-1)/2} & \text{for } k \equiv 1 \text{ mod 4} \quad (\text{filtration jump 4}), \\ 2u_{2\sigma}^k & \text{for even } k, \\ 0 & \text{for } k \equiv 3 \text{ mod 4}, \end{cases} \]
\[ \text{tr}_3^k(u_{\sigma}^k) = \begin{cases} a_\sigma^2 \tilde{\tau}_{1,e} u_{2\sigma}^{(k+1)/2} = a_\sigma^2 a_{\sigma}^3 \tilde{\tau}_{1,e} u_{2\sigma}^{(k+1)/2} & \text{for } k \equiv 1 \text{ mod 4} \quad (\text{filtration jump 2}), \\ 2\tilde{u}_{\lambda}^k & \text{for even } k, \\ a_\sigma^2 \tilde{\tau}_{1,e} \tilde{u}_{\lambda}^{(k-3)/2} & \text{for } k \equiv 3 \text{ mod 4} \quad (\text{filtration jump 6}). \end{cases} \]

Proof. All differentials were established above.

The differential on \( u_{\lambda}^4 \) does not lead to an exotic transfer because neither \( \tilde{u}_{\lambda}^2 \) nor \( [u_{\lambda}, a_\lambda^3 \tilde{\delta}_1] \) is a permanent cycle as required by Theorem 4.4.

We need to discuss the element \( [2u_{2\sigma}u_{\lambda}] = \langle 2u_{2\sigma}, \eta, a_\lambda \rangle \). To see that this Toda bracket is defined, we need to verify that \( [2u_{2\sigma}] \eta = 0 \). For this we have
\[ [2u_{2\sigma}] \eta = [2u_{2\sigma}] \text{tr}_2^2(\eta_0) = \text{tr}_2^2(2u_{2\sigma}^2 \eta_0) = \text{tr}_2^2(0) = 0. \]

The exotic restriction and transfers are applications of Theorem 4.4 to the differentials on \( u_{\lambda} \) and on \( u_{2\sigma}^{(k+1)/2} \) and \( \tilde{u}_{\lambda}^{(k+1)/2} \) for odd \( k \). For even \( k \) we have
\[ \text{tr}_2^k(u_{\sigma}^k) = \text{tr}_2^k(\text{res}_{2\lambda}^2(\eta_{2\sigma}^k)) = [2u_{2\sigma}^k] \quad \text{since } \text{tr}_2^k(\text{res}_{2\lambda}^2(x)) = (1 + y)x, \]
and similarly for even powers of \( u_{\sigma} \).

As remarked above, we lose no information by inverting the class \( D \), which is divisible by \( \tilde{\delta}_1 \). It is shown in [6, Theorem 9.16] that inverting the latter makes \( u_{\lambda}^2 \) a permanent cycle. One can also see this from (11.3). Since \( d_3(u_{2\sigma}) = a_\sigma^3 a_{\sigma}^1 \tilde{\delta}_1 \), we have \( d_3(u_{2\sigma} \tilde{\delta}_1^{-1}) = a_\sigma^2 a_{\sigma}^1 \tilde{\delta}_3 \). This means that \( d_3([u_{2\sigma}]) = a_\sigma^2 a_{\sigma}^3 \tilde{\delta}_3 \) is trivial in \( E_2(G/G) \). It turns out that there is no possible target for a higher differential.

\[ \square \]

12 \( k_{[2]} \) as a \( C_2 \)-spectrum

Before studying the slice spectral sequence for the \( C_4 \)-spectrum \( k_{[2]} \) further, it is helpful to explore its restriction to \( G' = C_2 \), for which the \( Z \)-bigraded portion
\[ E_2^{*,*} k_{[2]}(G'/G') \cong E_2^{*,*} k_{[2]}(G/G) \cong E_2^{*,*} k_{[2]}(G'/G') \]

(see Theorem 2.13 for these isomorphisms) is the isomorphic image of the subring of Corollary 10.3. In the following we identify \( \Sigma_{2,e}, \tilde{\delta}_1 \) and \( \tilde{\tau}_{1,e} \) (see Table 3) with their images under \( r_0^k \). From the differentials of (11.6) we get
\[
\begin{align*}
    d_3(\Sigma_{2,e}) &= \eta_{0}^2 \eta_0 + \eta_{1}^2 = a_0^3 \tilde{\tau}_{1,e}^2 (\tilde{\tau}_{1,0} + \tilde{\tau}_{1,1}), \\
    d_3(\tilde{\delta}_1) &= \eta_{0}^2 \eta_0 + \eta_{1} = a_0^2 \tilde{\tau}_{1,0} \tilde{\tau}_{1,1} (\tilde{\tau}_{1,0} + \tilde{\tau}_{1,1}), \\
    d_3([\tilde{\delta}_1]) &= d_3(u_{2\sigma} \tilde{\tau}_{1,0} \tilde{\tau}_{1,1}) = a_0^2 \tilde{\tau}_{1,0} \tilde{\tau}_{1,1}^2 = a_0^2 (7 \tilde{\tau}_{1,0} \tilde{\tau}_{1,1} + 5 \tilde{\tau}_{1,0} \tilde{\tau}_{1,1}^2 + \tilde{\tau}_{1,0} \tilde{\tau}_{1,1}^2, \tilde{\tau}_{1,0} \tilde{\tau}_{1,1}^2). 
\end{align*}
\]
The differentials $d_1$ above make all monomials in $\eta_0$ and $\eta_1$ of any given degree $\geq 3$ the same in $E_4(G/G')$ and $E_4(G'/G')$, so $d_1(\delta_1^2) = \eta_0^7$. Similar calculations show that

$$d_1((\Sigma^2_{\varepsilon,\ell})) = \eta_0^7 = a_0^7\tau_{1,0}^7.$$ 

The image of the periodicity element $D$ here is as in (7.4).

We have the following values of the transfer on powers of $u_0$:

$$\text{tr}_2^i(u_0) = \begin{cases} 
[2u_0^{i/2}] & \text{for } i \text{ even}, \\
[a^{(i-1)/2}_0u_0^{(i-2)/2}](\bar{r}_{1,0} + \bar{r}_{1,1}) & \text{for } i \equiv 1 \mod 4, \\
[u_0^i]^{(i-3)/8}a_0^3\bar{r}_{1,0}^3 & \text{for } i \equiv 3 \mod 8, \\
0 & \text{for } i \equiv 7 \mod 8.
\end{cases}$$

This leads to the following, for which Figure 8 is a visual aid.

**Theorem 12.2** (The slice spectral sequence for $k[2]$ as a $C_2$-spectrum). Using the notation of Table 1 and Definition 5.3, we have

$$E_2^{s,t}(G'/\{e\}) = \mathbb{Z}[r_{1,0}, r_{1,1}] \quad \text{with } r_{1,e} \in E_2^{0,2}(G'/\{e\}),$$

$$E_2^{s,t}(G'/G') = \mathbb{Z}[\delta_1, \Sigma_2, \eta_e; e = 0, 1]/(2\eta_e, \delta_1^3 - \Sigma_2, \eta_{e+1} + \eta_{e-1}),$$

so

$$E_2^{s,t} = \begin{cases} 
\boxplus \boxplus \boxplus & \text{for } (s, t) = (0, 4\ell) \text{ with } \ell \geq 0, \\
\boxplus \boxplus \boxplus & \text{for } (s, t) = (0, 4\ell + 2) \text{ with } \ell \geq 0, \\
\boxplus \boxplus \boxplus & \text{for } (s, t) = (2u, 4\ell + 4u) \text{ with } \ell \geq 0 \text{ and } u > 0, \\
\boxplus \boxplus \boxplus & \text{for } (s, t) = (2u - 1, 4\ell + 4u - 2) \text{ with } \ell \geq 0 \text{ and } u > 0, \\
0 & \text{otherwise}.
\end{cases}$$

The first set of differentials and determined by

$$d_3(\Sigma_2, \ell) = \eta_0^2(\eta_0 + \eta_1) \quad \text{and} \quad d_3(\delta_1) = \eta_0\eta_1(\eta_0 + \eta_1)$$

and there is a second set of differentials determined by

$$d_7(\Sigma_2, \ell) = d_7(\delta_1^2) = \eta_0^7.$$ 

**Corollary 12.3** (Some nontrivial permanent cycles). The elements listed below in $E_4,_{\text{even}}^{8,8i+2s} k[2]_2(G/G')$ are nontrivial permanent cycles. Their transfers in $E_4,_{\text{even}}^{8,8i+2s} k[2]_2(G/G)$ are also permanent cycles.

- $\Sigma_2^{2i-j} \delta_1^j$ for $0 \leq j \leq 2i$ ($4i + 1$ elements of infinite order including $\delta_1^2$), $i$ even and $s = 0$.
- $\eta_0^i\Sigma_2^{2i-j} \delta_1^j$ for $0 \leq j < 2i$ and $\eta_0^i\delta_1^j (4i + 2$ elements or order 2) for $i$ even and $s = 1$.
- $\eta_0^i\Sigma_2^{2i-j} \delta_1^j$ for $0 \leq j < 2i$ and $\delta_1^j (\eta_0^2\eta_0^2, \eta_0^2, \eta_0^2) (4i + 3$ elements or order 2) for $i$ even and $s = 2$.
- $\eta_0^i\delta_1^j$ for $3 \leq s \leq 6$ (four elements or order 2) and $i$ even.
- $\Sigma_2^{2i-j} \delta_1^j + \delta_1^j$ for $0 \leq j < 2i$ ($4i + 1$ elements of infinite order including $2\delta_1^2$), i odd and $s = 0$.
- $\eta_0^i\Sigma_2^{2i-j} \delta_1^j + \delta_1^j$ for $0 \leq j < 2i - 1$ and $\eta_0^i\delta_1^2(\Sigma_2, \delta_1 + \delta_1) = \eta_0^i\delta_1^2(\Sigma_2, 0 + \delta_1)$ $4i + 1$ elements of order 2), i odd and $s = 1$.
- $\eta_0^i\Sigma_2^{2i-j} \delta_1^j + \delta_1^j$ for $0 \leq j < 2i - 1$, $\eta_0^i\delta_1^2(\Sigma_2, \delta_1 + \delta_1) = \eta_0^i\delta_1^2(\Sigma_2, 0 + \delta_1)$ and $\eta_0^i\delta_1^2(\Sigma_2, 0 + \delta_1) = \eta_0^i\delta_1^2(\Sigma_2, 0 + \delta_1)$ $4i + 2$ elements of order 2) for i odd and $s = 2$.

In $E_2^{0,8i+4} k[2](G/G')$ we have $2\Sigma_2^{2i-j} \delta_1^j$ for $0 \leq j \leq 2i$ and $2\delta_1^j$, $4i + 3$ elements of infinite order, each in the image of the transfer $\text{tr}_2^i$. 


On the other hand, we now give the Poincaré series computation analogous to the one following Remark 8.7, using the notation of (8.8). In RO(G')-graded slice spectral sequence for $k_{[2]}$ we have

$$E_2(G'/G') = \mathbb{Z}[a_\sigma, u_{2\sigma}, \bar{r}_{1,0}, \bar{r}_{1,2}]/(2a_\sigma),$$

Remark 12.4. In the RO(G)-graded slice spectral sequence for $k_{[2]}$ one has

$$d_3(u_{2\sigma}) = a_\sigma^2(\bar{r}_{1,0} + \bar{r}_{1,1}) \quad \text{and} \quad d_7([u_{2\sigma}^2]) = a_\sigma^7 \bar{r}_{3}^3 = a_\sigma r_{1,0}^3,$$

but $a^7$ itself, and indeed all higher powers of $a$, survive to $E_\infty = E_{\infty0}$. Hence the $E_{\infty0}$-term of this spectral sequence does not have the horizontal vanishing line that we see in $E_{3\infty}$-term of Figure 7. However when we pass from $k_{[2]}$ to $k_{[3]}$, $\bar{r}_{3}^3 = 5\bar{r}_{1,0}^2 \bar{r}_{1,1} + 5\bar{r}_{1,0} \bar{r}_{1,1}^2 + \bar{r}_{1,1}^3$ becomes invertible and we have

$$d_7((\bar{r}_{3}^3)^{-1}[u_{2\sigma}^2]) = d_7(\bar{r}_{1,0}^3[u_{2\sigma}^2]) = a^7.$$

On the other hand, $\bar{r}_{1,0} + \bar{r}_{1,1}$ is not invertible, so we cannot divide $u_{2\sigma}$ by it.

We now give the Poincaré series computation analogous to the one following Remark 8.7, using the notation of (8.8). In RO(G')-graded slice spectral sequence for $k_{[2]}$ we have

$$E_2(G'/G') = \mathbb{Z}[a_\sigma, u_{2\sigma}, \bar{r}_{1,0}, \bar{r}_{1,2}]/(2a_\sigma),$$
so
\[ g(E_r(G'/G')) = \left( \frac{1}{1 - t} + \frac{\bar{a}}{1 - \bar{a}} \right) \frac{1}{(1 - \bar{u})(1 - \bar{r})^2}, \]
\[ g(E_{r+1}(G'/G')) = g(E_r(G'/G')) - \frac{\bar{u} + \bar{r}\bar{a}^3}{(1 - \bar{u})(1 - \bar{u})^2(1 - \bar{r})^2} \]
\[ = \frac{1 + \bar{u}u}{(1 - t)(1 - \bar{u})(1 - \bar{r})^2} + \frac{\bar{a} + \bar{a}^2}{(1 - \bar{a})(1 - \bar{u})(1 - \bar{r})^2} + \frac{\bar{a}^3}{(1 - \bar{a})(1 - \bar{u})(1 - \bar{r})} \]
as before. The next differential leads to
\[ \frac{\bar{u}^2 + \bar{r}^3\bar{a}^7}{(1 - \bar{a})(1 - \bar{u})^6(1 - \bar{r})} \]
\[ = \frac{\bar{u}^2}{1 + \bar{u}u} - \frac{\bar{a} + \bar{a}^2}{\bar{a}^3} \]
\[ = \frac{1 + \bar{u}u}{(1 - t)(1 - \bar{u})(1 - \bar{r})^2} + \frac{(\bar{a} + \bar{a}^2)(1 + \bar{u}^2 - \bar{u}^2(1 - \bar{r}))}{(1 - \bar{u}^2)(1 - \bar{r})^2} + \frac{\bar{a}^3(1 + \bar{u}^2 - \bar{a}^2\bar{a}^3 - \bar{r}^3\bar{a}^7)}{(1 - \bar{a})(1 - \bar{u})(1 - \bar{r})} \]
\[ = \frac{1 + \bar{u}u + (t + \bar{r} - \bar{t})\bar{u}^2 + \bar{u}^3}{(1 - t)(1 - \bar{u}^2)(1 - \bar{r})^2} + \frac{(\bar{a} + \bar{a}^2)(1 + \bar{u}^2\bar{r}^2)}{(1 - \bar{u}^4)(1 - \bar{r})^2} + \frac{\bar{a}^3 - \bar{a}^2\bar{a}^3 - \bar{r}^3\bar{a}^7}{(1 - \bar{a})(1 - \bar{u})(1 - \bar{r})} \]
\[ = \frac{1 + \bar{u}u + (t + \bar{r} - \bar{t})\bar{u}^2 + \bar{u}^3}{(1 - t)(1 - \bar{u}^2)(1 - \bar{r})^2} + \frac{(\bar{a} + \bar{a}^2)(1 + \bar{u}^2\bar{r}^2)}{(1 - \bar{u}^4)(1 - \bar{r})^2} + \frac{\bar{a}^3 - \bar{a}^2\bar{a}^3 - \bar{r}^3\bar{a}^7}{(1 - \bar{a})(1 - \bar{u})(1 - \bar{r})}. \]
The fourth term of this expression represents the elements with filtration above six, and the first term represents the elements of filtration 0. The latter include
\[ [2u_{2,0}] \in \langle 2, a_0^2, a_0T_{1,0} + T_{1,1} \rangle, \]
\[ [2u_{2,0}^2] \in \langle 2, a_0, a_0^3 \rangle, \]
\[ [T_{1,0} + T_{1,1}]u_{2,0}^2 \in \langle a_0^6, a_0T_{1,0} + T_{1,1} \rangle \quad \text{with } \langle T_{1,0} + T_{1,1} \rangle [2u_{2,0}^2] = 2(T_{1,0} + T_{1,1})u_{2,0}^2 \]
\[ [2u_{2,0}] \in \langle 2, a_0^2T_{1,0} + T_{1,1}, a_0^2, a_0^6T_{1,0} \rangle, \]
\[ [u_{2,0}^2] \in \langle a_0^6, a_0^3T_{1,0}a_0, a_0^3T_{1,0} \rangle \]
with notation as in Remark 4.1.

As indicated in Remark 12.4, we can get rid of them by formally adjoining \( w := (\bar{r}^3)^{-1}u_{2,0} \) to \( E_2(G'/G') \). As before we denote the enlarged spectral sequence terms by \( E'_r(G'/G') \) This time let \( \bar{w} = \bar{r}^{-1}\bar{u}^2 = xy^{-7} \). Then we have
\[ E'_r(G'/G') = \left( \frac{1 - \bar{u}^2}{1 - \bar{w}} \right) E_r(G'/G') \quad \text{for } r = 2 \text{ and } r = 4 \]
and
\[ g(E'_r(G'/G')) = g(E'_r(G'/G')) - \frac{w + \bar{a}^7}{(1 - \bar{a})(1 - w^2)(1 - \bar{r})} \]
\[ = \frac{w}{1 - w^2(1 - \bar{r})} - \frac{(\bar{a} + \bar{a}^2)w}{1 - (1 - \bar{a})(1 - \bar{w}^2)(1 - \bar{r})} - \frac{\bar{a}^3w + \bar{a}^7}{1 - (1 - \bar{a})(1 - \bar{w}^2)(1 - \bar{r})} \]
\[ = \frac{1 + \bar{u}u}{(1 - t)(1 - \bar{w})(1 - \bar{r})^2} + \frac{\bar{a} + \bar{a}^2}{\bar{a}^3w + \bar{a}^7} \]
\[ = \frac{(1 + \bar{u}u)(1 + \bar{w} - (1 - \bar{t})\bar{w})}{(1 - t)(1 - \bar{w}^2)(1 - \bar{r})^2} + \frac{(\bar{a} + \bar{a}^2)(1 - (1 - \bar{t})\bar{w})}{(1 - (1 - \bar{w}^2)(1 - \bar{r})^2} + \frac{\bar{a}^3(1 + \bar{w} - \bar{a}^2\bar{w} - \bar{a}^7)}{(1 - \bar{a})(1 - \bar{w}^2)(1 - \bar{r})} \]
\[ = \frac{1 + \bar{u}u + (t + \bar{r} - \bar{t})\bar{w} + \bar{w}u}{(1 - t)(1 - \bar{w}^2)(1 - \bar{r})^2} + \frac{(\bar{a} + \bar{a}^2)(1 + \bar{w}\bar{r})}{(1 - \bar{w}^2)(1 - \bar{r})^2} + \frac{\bar{a}^3 + \bar{a}^4 + \bar{a}^5 + \bar{a}^6}{(1 - \bar{w}^2)(1 - \bar{r})}. \]
Again the first term represents the elements of filtration 0. These include:

\[ [2u_{2\sigma}] \in \langle 2, a_2^0, a_0(\bar{r}_{1,0} + \bar{r}_{1,1}) \rangle, \]
\[ [2w] \in \langle 2, a_0, a_0^5 \rangle, \]
\[ [(\bar{r}_{1,0} + \bar{r}_{1,1})w] \in \langle a_0^5, a_0^3, \bar{r}_{1,0} + \bar{r}_{1,1} \rangle \]

with \((\bar{r}_{1,0} + \bar{r}_{1,1})[2w] = 2[(\bar{r}_{1,0} + \bar{r}_{1,1})w], \]
\[ [2u_{2\sigma}w] \in \langle 2, a_0^7(\bar{r}_{1,0} + \bar{r}_{1,1}), a_2^2, a_0^5 \rangle \]

and \([w^2] \in \langle a_0^5, a_0^3, a_0^5 \rangle \)

where, as indicated above, \(w = (\bar{r}_{1,1}^G)^{-1}u_{2\sigma}^2\).

### 13 The effect of the first differentials over \(C_4\)

Theorem 10.2 lists elements in the slice spectral sequence for \(k[2]\) over \(C_4\) in terms of

\(r_1, \bar{r}_2, \bar{d}_1; \ \eta, a_0, a_3; \ \sigma, u_1, u_0, u_{2\sigma}\).

All but the \(u\) are permanent cycles, and the action of \(d_3\) on \(u_1, u_0, u_{2\sigma}\) is described above in Theorem 11.13.

**Proposition 13.1** (\(d_3\) on elements in Theorem 10.2). *We have the following \(d_3s, subject to the conditions on \(i, j, k\ and \(l\) of Theorem 10.2:*

- **On** \(X_{2\ell, 2\ell}:

\[
d_3(a_0^i u_{2\sigma}^j u_0^k \delta_1^{2\ell}) = \begin{cases} 
  a_0^{i+1} u_0^{j+1} u_0^{2\ell-1} \delta_1^{2\ell} 
  & \text{if } j \text{ odd,} \\
  0 
  & \text{if } j \text{ even,}
\end{cases}
\]

\[
d_3(a_0^i u_0^{2\ell} a_0^k u_{2\sigma}^j \delta_1^{2\ell}) = 0.
\]

- **On** \(X_{2\ell+1, 2\ell+1}:

\[
d_3(\eta u_{2\sigma}^{2\ell+1} \delta_1^{2\ell+1}) = \eta u_{2\sigma}^{2\ell+1} \delta_1^{2\ell+1} 
\]

\[
d_3(u_0^{2\ell+1} \delta_1^{2\ell+1}) = \begin{cases} 
  \eta u_0^{2\ell+1} \delta_1^{2\ell+1} 
  & \text{if } j \text{ even,} \\
  0 
  & \text{if } j \text{ odd,}
\end{cases}
\]

\[
d_3(\eta u_0^{2\ell} a_0^i u_0^{2\ell-1} u_0^k \delta_1^{2\ell}) = \begin{cases} 
  \eta u_0^{2\ell} a_0^i u_0^{2\ell-1} u_0^k \delta_1^{2\ell} 
  & \text{if } j \text{ even,} \\
  0 
  & \text{if } j \text{ odd,}
\end{cases}
\]

\[
d_3(a_0^i u_0^{2\ell} a_0^k u_{2\sigma}^{2\ell+1} \delta_1^{2\ell}) = 0.
\]

- **On** \(X_{1, 2\ell-1}:

\[
d_3(u_0^{2\ell} \delta_2^{2\ell-1} \delta_1^{2\ell}) = \begin{cases} 
  \eta u_0^{2\ell} \delta_2^{2\ell-1} \delta_1^{2\ell} 
  & \text{if } \ell \text{ odd,} \\
  0 
  & \text{if } \ell \text{ even,}
\end{cases}
\]

\[
d_3(\eta u_0^{2\ell} \delta_2^{2\ell-1} \delta_1^{2\ell}) = \begin{cases} 
  \eta u_0^{2\ell} \delta_2^{2\ell-1} \delta_1^{2\ell} 
  & \text{if } \ell \text{ even,} \\
  0 
  & \text{if } \ell \text{ odd,}
\end{cases}
\]

- **On** \(X_{1, 2\ell+1-1}:

\[
d_3(r_1 \delta_2^{2\ell-1} \delta_1^{2\ell}) = 0,
\]

\[
d_3(\eta u_0^{2\ell} \delta_2^{2\ell-1} \delta_1^{2\ell}) = \begin{cases} 
  \eta u_0^{2\ell} \delta_2^{2\ell-1} \delta_1^{2\ell} 
  & \text{if } \ell \text{ even,} \\
  0 
  & \text{if } \ell \text{ odd,}
\end{cases}
\]

Note that in each case the first index of \(X\) is unchanged by the differential, and the second one is increased by one. Since \(X_{m,n}\) is a summand of the \(2(m + n)\)th slice, each \(d_3\) raises the slice degree by 2 as expected.
Remark 13.2 (The spectra $y_m$ and $Y_m$ of Corollaries 10.5 and 10.6). Similar statements can be proved for the case $\ell < 0$. We leave the details to the reader, but illustrate the results in Figures 11 and 12.

The source of each differential in Proposition 13.1 is the product of some element in $\pi_*HZ$ with a power of $\delta^1$ or $\delta_1$. The target is the product of a different element in $\pi_*HZ$ with the same power. This means they are differentials in the slice spectral sequence for the spectra $y_m$ of Corollary 10.5.

Similar differentials occur when we replace $\delta^1$ by any homogeneous polynomial of degree $i$ in $\delta^1$ and $\delta_2$ in which the coefficient of $\delta^1$ is odd. This means they are also differentials in the slice spectral sequence for the spectra $Y_m$ of Corollary 10.6.

These differentials are illustrated in the upper charts in Figures 9–12. In order to pass to $E_4$, we need the following exact sequences of Mackey functors:

$$
0 \longrightarrow \bullet \longrightarrow \circ \overset{d_2}{\longrightarrow} \ast \longrightarrow \ast \longrightarrow 0,
0 \longrightarrow \ast \longrightarrow \ast \overset{d_1}{\longrightarrow} \ast \longrightarrow \ast \longrightarrow 0,
0 \longrightarrow \ast \longrightarrow \ast \overset{d_1}{\longrightarrow} \ast \longrightarrow \ast \longrightarrow 0,
0 \longrightarrow \ast \longrightarrow \ast \overset{d_1}{\longrightarrow} \ast \longrightarrow \ast \longrightarrow 0.
$$

The resulting subquotients of $E_4$ are shown in the lower charts of Figures 9–12 and described below in Theorem 13.3. In the latter the slice summands are organized as shown in the Figures rather than by orbit type as in Theorem 10.2.

Figure 9. The subquotient of the slice $E_2$ and $E_4$ terms for $k_{12}$ for the slice summands $X_{4,n}$ for $n \geq 4$. Exotic transfers are shown in blue and differentials are in red. The symbols are defined in Table 2. This is also the slice spectral sequence for $y_4$ as in Corollary 10.5 and $Y_4$ (after tensoring with $R$) as in Corollary 10.6.
Figure 10. The subquotient of the slice $E_2$ and $E_4$ terms for $k_{21}$ for the slice summands $X_{5,n}$ for $n \geq 5$. Exotic restrictions and transfers are shown in dashed green and solid blue lines respectively. This is also the slice spectral sequence for $Y_5$ as in Corollary 10.5 and for $Y_3$ (after tensoring with $R$) as in Corollary 10.6.

Figure 11. The subquotient of the slice $E_2$ and $E_4$ terms for $k_{21}$ for the slice summands $X_{4,n}$ for $n \geq -4$. This is also the slice spectral sequence for $Y_{-4}$ (after tensoring with $R$) as in Corollary 10.6.
Theorem 13.3 (The slice $E_2$-term for $k_{12}$). The elements of Theorem 10.2 surviving to $E_3'$, which live in the appropriate subquotients of $\pi_* X_{m,n}$, are as follows:

(i) In $\pi_* X_{2\ell,2\ell}$ (see the leftmost diagonal in Figure 9), on the 0-line we still have a copy of $\Box$ generated under fixed point restrictions by $\Delta_\ell' \in E_3^{0,8\ell}$. In positive filtrations we have

$$
\circ \subseteq E_4^{2j,8\ell} \quad \text{generated by} \quad \begin{cases} 
& a_0^j u_0^{2j-1-2\ell} \delta_0^{1-2\ell} \in E_4^{2j,8\ell}(G/G'), \\
& 2a_0^j u_0^{2j-1-2\ell} \delta_0^{1-2\ell} = a_0^j u_0^{2j-1-2\ell} \delta_0^{1-2\ell} \in E_4^{2j,8\ell}(G/G'), \\
\end{cases}
$$

for $0 < j < 2\ell$, $j$ even, $0 < j \leq 2\ell$, $j$ odd, $0 < j \leq 2\ell$,

(ii) In $\pi_* X_{2\ell,2\ell+1}$ (see the second leftmost diagonal in Figure 9), in filtration 0 we have $\Box$, generated (under transfers and the group action) by

$$
r_1 \res_1(u_0^{2\ell} \res_1^2(u_0^{2\ell} \delta_1^{2\ell})) \in E_2^{0,8\ell+2}(G/G').
$$

In positive filtrations we have

$$
\circ \subseteq E_4^{1,8\ell+2} \quad \text{generated (under transfers and the group action) by} \quad \eta u_0^{2\ell} \res_2(u_0^1)^{2\ell} = E_4^{1,8\ell+2}(G/G'),
$$

$$
\star \subseteq E_4^{2k+1,8\ell+4} \quad \text{for} \quad 0 < k \leq \ell \text{ generated by} \quad x = \eta^{2k+1} u_0^{2\ell} \res_2^4(u_0^1)^{2\ell} \delta_0^{2\ell} \in E_4^{2k+1,8\ell+4}(G/G'),
$$

with $(1 - y)x = \tau_{\ell/2}(x) = 0$.

(iii) In $\pi_* X_{2\ell+1,2\ell+1}$ (see the leftmost diagonal in Figure 10), on the 0-line we have a copy of $\Box$ generated under fixed point $\Delta_\ell^{2\ell+1/2} \in E_3^{2,8\ell+4}$. In positive filtrations we have

$$
\star \subseteq E_2^{2j,8\ell+4} \quad \text{generated by} \quad a_{2j} u_{2j}^{2j-1-2\ell} \delta_1^{1-2\ell} \delta_0^{1-2\ell} \in E_2^{2j,8\ell+4}(G/G') \quad \text{for} \quad 0 < j < 2\ell + 1,
$$

$$
\star \subseteq E_2^{2j+1,8\ell+4} \quad \text{generated by} \quad a_{2j} u_{2j}^{2j-1-2\ell} \delta_1^{1-2\ell} \delta_0^{1-2\ell} \in E_2^{2j+1,8\ell+4}(G/G') \quad \text{for} \quad 0 < j < 2\ell + 1,
$$

$$
\star \subseteq E_2^{2k+3,8\ell+4} \quad \text{generated by} \quad a_{2k+1} u_{2k+1}^{2k+1} \delta_0^{1-2\ell} \delta_1^{1-2\ell} \in E_2^{2k+3,8\ell+4}(G/G') \quad \text{for} \quad 0 < k < 2\ell + 1.
$$
(iv) In $\pi_* X_{2^r+1,2^r+2}$ (see the second leftmost diagonal in Figure 10), in filtration 0 we have $\tilde{\mathfrak{C}}_\ast$, generated (under transfers and the group action) by

$$r_1 \text{res}_1^*(u_0^{2^r+1} r_1^* u_1^{2^r+1} \delta_1^{2^r+1}) \in E^{0,8r+6}_4(G/\{e\}).$$

In positive filtrations we have

$$\nabla \triangleright E^{0,k,8r+6}_4$$

for $0 \leq k \leq \ell$ generated under transfer by $x = \eta^{a+k,3,8r+6}_1 \in E^{0,k,8r+6}_4(G/G')$ with $(1 - y)x = 0$.

The generator of $E^{0,k,8r+6}_4(G/G')$ is the exotic restriction of the one in $E^{0,k+1,8r+6}_4(G/G)$.

(v) In $\pi_* X_{n,m+1}$ for $i \geq 2$ (see the rest of Figures 9 and 10), in filtration 0 we have

$$\tilde{\mathfrak{C}}_\ast \subseteq E^{0,4m+6i+2}_4$$

generated under transfers and group action by

$$r_1 \text{res}_1^*(u_0^{4m+6i+2}) \text{res}_1^*(u_0^{4m+6i+2}) \in E^{0,4m+6i+2}_4(G/\{e\})$$

for $j \geq 0$,

$$\tilde{\mathfrak{C}}_\ast \subseteq E^{0,4r+4}_4$$

generated under transfers and group action by

$$r_1 \text{res}_1^*(u_0^{4r+4}) \text{res}_1^*(u_0^{4r+4}) \in E^{0,4r+4}_4(G/\{e\})$$

for $\ell \geq m/2$,

$$\tilde{\mathfrak{C}}_\ast \subseteq E^{0,8r}_4$$

generated under transfers, restriction and group action by

$$x_{4r,m} = \Sigma^{2m-\ell} \delta_1^m + \ell \delta_1^\ell,$$

where

$$\Sigma_{2,\epsilon} = u_0, \delta_2, \epsilon \text{ and } \delta_1 = u_0, \text{ res}_1^*(\delta_1) \in E^{0,8r}_4(G/G') \text{ for } 0 \leq m \leq 2\ell - 1.$$

In positive filtrations we have

$$\nabla \triangleright E^{2,8r+4}_4$$

generated under transfers and group action by

$$\eta_0^2 \text{res}_2^*(\Delta_1^4) = \eta_0^2 \delta_1^{2\ell} = \eta_0^2 \text{res}_2^*(u_0^{2\ell}) \in E^{2,8r+4}_4(G/G'),$$

$$\nabla \triangleright E^{4,8r+2\ell}_4$$

generated under transfers and group action by

$$\eta_s x_{8r,m} \in E^{4,8r+2\ell}_4(G/G') \text{ for } s = 1, 2 \text{ and } 0 \leq m \leq 2\ell - 1.$$

Each generator of $E^{2,8r+4}_4(G/G')$ is an exotic transfer of one in $E^{0,8r+2}_4(G/\{e\}).$

**Proposition 13.4** (Some nontrivial permanent cycles). The elements listed in Theorem 13.3(v) other than $\eta_s^2 \delta_1^{2\ell}$ are all nontrivial permanent cycles.

**Proof.** Each such element is either in the image of $E_4^{0,4*}(G/\{e\})$ under the transfer and therefore a nontrivial permanent cycle, or it is one of the ones listed in Corollary 12.3.$\Box$

In subsequent discussions and charts, starting with Figure 14, we will omit the elements in Proposition 13.4. These elements all occur in $E_4^{0,4\ell}$ for $0 \leq s \leq 2$.

Analogous statements can be made about the slice spectral sequence for $K_2$. Each of its slices is a certain infinite wedge spelled out in Corollary 10.4. Their homotopy groups are determined by the chain complex calculations of Section 6 and illustrated in Figures 2 (with Mackey functor induction $\uparrow_3^1$ applied) and 3. Analogous of Figures 9–10 are shown in Figures 11–12. In each figure, exotic transfers and restrictions are indicated by blue and dashed green lines respectively. As in the $k_2$ case, most of the elements shown in this chart can be ignored for the purpose of calculating higher differentials. In the third quadrant the elements we are ignoring all occur in $E_4^{0,4\ell}$ for $-2 \leq s \leq 0$.

The resulting reduced $E_4$ for $K_2$ is shown in Figure 16. The information shown there is very useful for computing differentials and extensions. The periodicity theorem tells us that $\pi_{n-32} K_2$ and $\pi_{n-32} K_2$ are isomorphic. For $0 \leq n < 32$ these groups appear in the first and third quadrants respectively, and the information visible in the spectral sequence can be quite different.

For example, we see that $\pi_{n} K_2$ has summand of the form $\mathfrak{C}$, while $\pi_{n-32} K_2$ has a subgroup isomorphic to $\mathfrak{C}$. The quotient $\mathfrak{C}/\mathfrak{C}$ is isomorphic to $\mathfrak{C}$. This leads to the exotic restrictions and transfer in dimension 32 shown in Figure 16. Information that is transparent in dimension 0 implies subtle information in dimension 32. Conversely, we see easily that $\pi_{n-4} K_2 = \mathfrak{C}$ while $\pi_{28} K_2$ has a quotient isomorphic to $\mathfrak{C}$. This leads to the “long transfer” (which raises filtration by 12) in dimension 28.
14 Higher differentials and exotic Mackey functor extensions

We can use the results of the Section 12 to study higher differentials and extensions. The $E_2$-term implied by them is illustrated in Figure 13. For each $\ell$, $s \geq 0$ there is a generator

$$y_{8\ell + s, s} := \eta_0^s \delta_2^{2\ell} \in E_2^0,8\ell + 2s(G/G')$$

with

$$d_7(y_{16k+s+8, s}) = y_{16k+s+7, s+7}.$$  

Recall that

$$\delta_1 = \overline{u}_A \overline{\rho}_{1,0} \overline{\rho}_{1,1} \in E_2^{0,4}k_2[1](G/G') \cong E_2^{0,0}(G', s)k_2[1](G/G),$$

and in the latter group we denote $\overline{u}_A$ by $u_{2\ell}$. We have

$$d_3(\delta_1) = d_3(\overline{u}_A) \overline{\rho}_{1,0} \overline{\rho}_{1,1} \equiv d_3(u_{2\ell}) \overline{\rho}_{1,0} \overline{\rho}_{1,1} = a_3^1(\overline{\rho}_{1,0} + \overline{\rho}_{1,1}) \overline{\rho}_{1,0} \overline{\rho}_{1,1}.$$ 

If the source has the form $\text{res}_j(x_{16k+s+8, s})$, then such an $x$ must support a nontrivial $d_r$ for $r \leq 7$. If it has a nontrivial transfer $x'_{16k+s+8, s}$, then such an $x'$ cannot support an earlier differential, and we must have

$$d_r(x'_{16k+s+8, s}) = \text{tr}^s_2(d_7(y_{16k+s+8, s})) = \text{tr}^s_2(y_{16k+s+7, s+7}) \text{ for some } r \geq 7.$$ 

We could get a higher differential (meaning $r > 7$) if $y_{16k+s+7, s+7}$ supports an exotic transfer.

We have seen (Figure 14 and Theorem 13.3) that for $s \geq 3$ and $k \geq 0$,

$$E_5^{16k + 8 + 2s} = \begin{cases} \circ & \text{for } s \equiv 0 \mod 4, \\ \bullet & \text{for } s \equiv 1, 2 \mod 4, \\ \nabla & \text{for } s \equiv 3 \mod 4. \end{cases} \quad (14.1)$$

For $s = 1, 2$, $E_5^{16k + 8 + 2s}$ has $\square$ as a direct summand. For $s = 0$ it has $\square$ as a summand, and the differentials on it factor through its quotient $\circ$; see (5.2).

The corresponding statement in the third quadrant is

$$E_5^{8 - 16k - 2s - 24} = \begin{cases} \circ & \text{for } s \equiv 3 \mod 4, \\ \bullet & \text{for } s \equiv 1, 2 \mod 4, \\ \nabla & \text{for } s \equiv 0 \mod 4. \end{cases}$$
for \( s \geq 3 \) and \( k \geq 0 \). For \( s = 1, 2 \) the groups have similar summands, and for \( s = 0 \) there is a summand of the form \( \mathbb{Z} \), which has \( \mathbb{Z} \) as a subgroup; again see (5.2). This is illustrated in Figure 16.

**Theorem 14.2** (Differentials for \( \mathbb{C}_4 \) related to the \( d_1 \) for \( \mathbb{C}_2 \)). The differential

\[ d_1(y_{16k+s+8,s}) = y_{16k+s+7,s+7} \quad \text{with} \quad s \geq 3 \]

has the following implications for the congruence classes of \( s \) modulo 4.

(i) For \( s \equiv 0 \mod 4 \), \( \varepsilon_{1,16k+8+2s} = 0 \) and \( \varepsilon_{1,16k+14+2s} = \mathbb{Z} \). Hence \( y_{16k+s+8,s} \) is a restriction with a nontrivial transfer, and

\[
\begin{align*}
    d_1(y_{16k+s+8,s}) &= y_{16k+s+7,s+7} \\
    d_1(2x_{16k+s+8,s}) &= 2d_1(y_{16k+s+7,s+7}) = 2x_{16k+s+7,s+7}.
\end{align*}
\]

(ii) For \( s \equiv 1 \),

\[
\begin{align*}
    d_1(y_{16k+s+8,s}) &= y_{16k+s+7,s+7} \\
    d_1(x_{16k+s+8,s+2}) &= x_{16k+s+7,s+3} \quad \text{with} \quad \text{res}^7_0(x_{16k+s+7,s+3}) = y_{16k+s+7,s+7}.
\end{align*}
\]

This leaves the fate of \( x_{16k+s+7,s+7} \) undecided; see below.

(iii) For \( s \equiv 2 \), \( \varepsilon_{2,16k+8+2s} = \mathbb{Z} \) and \( \varepsilon_{2,16k+14+2s} = \mathbb{Z} \). Neither the source nor target is a restriction or has a nontrivial transfer, so no additional differentials are implied.

(iv) For \( s \equiv 3 \), \( \varepsilon_3,\varepsilon_{16k+8+2s} = \mathbb{Z} \) and \( \varepsilon_{3,16k+14+2s} = \mathbb{Z} \). In this case the source is an exotic restriction; again see Figure 10. Thus we have

\[
\begin{align*}
    d_1(y_{16k+s+8,s}) &= y_{16k+s+7,s+7} \\
    d_1(x_{16k+s+8,s+2}) &= x_{16k+s+7,s+3} \quad \text{with} \quad \text{res}^7_0(x_{16k+s+7,s+3}) = y_{16k+s+7,s+7}.
\end{align*}
\]

Moreover, \( \text{tr}^4_1(y_{16k+s+8,s}) \) is nontrivial and it supports a nontrivial \( d_1 \) when \( 4k + s \equiv 3 \mod 8 \). The other case, \( 4k+s \equiv 7 \), will be discussed below.

**Proof.** (i) The target Mackey functor is \( \mathbb{Z} \) and \( y_{16k+s+7,s+7} \) is the exotic restriction of \( x_{16k+s+7,s+5} \); see Figure 10 and Theorem 13.3. The indicated \( d_5 \) and \( d_7 \) follow.

(ii) The differential is nontrivial on the \( G/G' \) component of

\[ a = \varepsilon_{2,16k+8+2s} - d_1 \]

Thus the target has a nontrivial transfer, so the source must have an exotic transfer. The only option is \( x_{16k+s+8,s+2} \), and the result follows.

(iii) We prove the statement about \( d_{11} \) by showing that

\[ y_{16k+s+7,s+7} = \eta_0^{s+7} \delta^k_1 \]

supports an exotic transfer that raises filtration by 4. First note that

\[ \text{tr}^4_1(\eta_0 \eta_1) = \text{tr}^4_1(a_2^2 r_1 r_1 a_0) = \text{tr}^4_2(u_0 \text{res}^7_0(a_1 \delta_1)) = \text{tr}^4_2(u_0 a_1 \delta_1) = a_0 a_1 \delta_1 a_1 \delta_1 \quad \text{by} \ (11.4). \]

Next note that the three elements

\[ y_{8,8} = \eta_0^4, \quad y_{20,4} = \eta_0^4 \delta^4 = \text{res}^7_0(\delta^4), \quad \text{and} \quad y_{32,0} = \delta^6_1 = \text{res}^7_0(\delta^6_1) \]

are all permanent cycles, so the same is true of all

\[ y_{16m+4\ell,4\ell} = \eta_0^{4 \ell} \delta^6_1 \quad \text{for} \ m, \ell \geq 0 \ \text{and} \ m + \ell \text{ even}. \]

It follows that for such \( \ell \) and \( m \),

\[ \eta_0 \eta_1 y_{16m+4\ell,4\ell} = \eta_0 \eta_1 \eta_0^{4 \ell} \delta^6_1 = \eta_0^{4 \ell+2} \delta^6_1 = y_{16m+4\ell+2,4\ell+2} = \eta_0 \eta_1 \text{res}^7_0(x_{16m+4\ell,4\ell}), \]

so

\[ \text{tr}^4_1(y_{16m+4\ell+2,4\ell+2}) = \text{tr}^4_1(\eta_0 \eta_1) x_{16m+4\ell,4\ell} = f^4_1 x_{16m+4\ell,4\ell}. \]

This is the desired exotic transfer.

\[ \square \]
We now turn to the unsettled part of Theorem 14.2 (iv).

**Theorem 14.3** (The fate of $x_{16k+s+8s}$ for $4k + s \equiv 7 \pmod{8}$ and $s \geq 7$). *Each of these elements is the target of a differential $d_7$ and hence a permanent cycle.*

*Proof.* Consider the element $\Delta_1^j \in E_2^{0,16}(G/G)$. We will show that

$$d_7(\Delta_1^j) = x_{15,7} = \tau_2^j(y_{15,7}).$$

This is the case $k = 0$ and $s = 7$. The remaining cases will follow via repeated multiplication by $c, r$ and $\Delta_1^j$.

We begin by looking at

$$\Delta_1 = u_{20}u_A^{2s} \delta_1^2.$$

From Theorem 11.13 we have

$$d_5(u_{20}) = a_\sigma^2 a_4 \delta_1 \quad \text{and} \quad d_5(u_1) = a_\sigma a_4^2 u_4 \delta_1.$$

Using the gold relation $a_\sigma u_1 = 2a_4 u_{20}$, we have

$$d_5(\Delta_1) = d_5(u_{20}u_1^2) \delta_1 = (a_\sigma^2 a_4 u_1^2 \delta_1 + a_\sigma a_4^2 u_{20} \delta_1) \delta_1$$

$$= a_\sigma a_4 u_A (a_\sigma u_1 + a_4 u_{20}) \delta_1^2$$

$$= a_\sigma a_4 u_A (2a_4 u_{20} + a_4 u_{20}) \delta_1^2$$

$$= a_\sigma a_4^2 u_{20} \delta_1^2 \quad \text{since} \quad 2a_4 = 0$$

$$= \nu x_4.$$

Since $\nu$ supports an exotic group extension, $2\nu = x_3$, we have

$$2d_5(\Delta_1) = d_7(2\Delta_1) = x_3 x_4.$$

From this it follows that

$$d_7(\Delta_1^2) = \Delta_1 d_7(2\Delta_1) = x_{15,7}$$

as claimed. \hfill \Box

The resulting reduced $E_{12}$-term is shown in Figure 15. It is sparse enough that the only possible remaining differentials are the indicated differentials $d_{13}$. In order to establish them we need the following.

The surviving class in $E_2^{20,3}(G/G)$ is

$$x_{17,3} = f_1 \Delta_1^7 = a_\sigma a_4 \delta_1 \cdot [u_{20}^2] u_A^4 \delta_1^4 = (a_\sigma u_A^4)(a_4 u_{20}^2) \delta_1^5.$$ 

The second factor is a permanent cycle, so Theorem 11.10 gives

$$d_{13}(f_1 \Delta_1) = (a_\sigma^7 [u_{20}^2] \delta_1^5)(a_4 [u_{20}^2] \delta_1^8) = a_{\sigma}^8 [u_{20}^8] \delta_1^8 = e^2 = x_A^4.$$ 

The surviving class in $E_2^{32,2}(G/G)$ is

$$x_{30,2} = a_\sigma^2 u_{20}^3 u_A^8 \delta_1^8 \in E_2^{32,2}(G/G)$$

and satisfies

$$\epsilon x_{30,2} = f_1 \bar{K} x_{17,3} = f_1^2 x_A \Delta_1^2,$$

so we have proved the following.

**Theorem 14.4** (Differentials $d_{13}$ in the slice spectral sequence for $k_2$). *There are differentials

$$d_{13}(f_1^e x_m \Delta_1^{2m}) = f_1^{e-1} x_m \Delta_1^{2(n-1)}$$

for $e = 1, 2, m + n$ odd, $n \geq 1$ and $m \geq 1 - e$. The spectral sequence collapses from $E_{14}$.\hfill
Figure 14. The $E_4$-term of the slice spectral sequence for $k_{[2]}$ with elements of Proposition 13.4 removed. Differentials are shown in red. Exotic transfers and restrictions are shown as solid blue and dashed green lines respectively. The Mackey functor symbols are as in Table 2.

Figure 15. The $E_{22}$-term of the slice spectral sequence for $k_{[2]}$ with elements of Proposition 13.4 removed. Differentials are shown in red. Exotic transfers and restrictions are shown as solid blue and dashed green lines respectively. The Mackey functor symbols are as in Table 2.
To finish the calculation we have

**Theorem 14.5 (Exotic transfers from and restrictions to the 0-line).** In $\pi_k [2]$ for $i \geq 0$ we have

\[
\begin{align*}
\text{tr}_2^2(r_{1,0}^B 1_{1,1}) &= \eta_c r_{1,0}^B 1_{1,1} \in \pi_{32}^{32} + 2, \\
\text{tr}_1^2(r_{1,0}^B r_{1,1}^B) &= 2x_4 \Delta_{1}^4 \in \pi_{32}^{32} + 4, \\
\text{tr}_1^2((r_{1,0}^B r_{1,1}^B) r_{1,1}^B) &= \eta_{3} r_{1}^B \delta_1 \in \pi_{32}^{32} + 6, \\
\text{tr}_1^2(r_{1,0}^B r_{1,1}^B) &= 2x_4 \Delta_{1}^4 + 2 \in \pi_{32}^{32} + 20, \\
\text{tr}_1^2(r_{1,0}^B r_{1,1}^B) &= \eta_{3} r_{1}^B \delta_{1} \in \pi_{32}^{32} + 22, \\
\text{tr}_1^2(2\delta_{1}^4) &= x_4 \Delta_{1}^4 + 2 \in \pi_{32}^{32} + 28.
\end{align*}
\]

(filtration jump 2),

(filtration jump 4),

(filtration jump 6),

(filtration jump 4),

(filtration jump 6),

(filtration jump 12, the long transfer).

Let $M_k'$ denote the reduced value of $\pi_k [2]$, meaning the one obtained by removing the elements of Proposition 13.4. Its values are shown in purple in Figure 17, and each has at most two summands. For even $k$ one of them contains torsion free elements, and we denote it by $M_k'$. Its values depend on $k \mod 32$ and are as follows, with symbols as in Table 2.

<table>
<thead>
<tr>
<th>$k$</th>
<th>0</th>
<th>2</th>
<th>4</th>
<th>6</th>
<th>8</th>
<th>10</th>
<th>12</th>
<th>14</th>
<th>16</th>
<th>18</th>
<th>20</th>
<th>22</th>
<th>24</th>
<th>26</th>
<th>28</th>
<th>30</th>
</tr>
</thead>
<tbody>
<tr>
<td>$M_k'$</td>
<td>$\bigstar$</td>
<td>$\square$</td>
<td>$\bigstar$</td>
<td>$\bigstar$</td>
<td>$\bigstar$</td>
<td>$\square$</td>
<td>$\bigstar$</td>
<td>$\square$</td>
<td>$\square$</td>
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<td>$\square$</td>
</tr>
</tbody>
</table>

**Proof.** We have two tools at our disposal: the periodicity theorem and Theorem 4.4, which relates exotic transfers to differentials.

Figure 16 shows that $M_k'$ has the indicated value for $-8 \leq k \leq 0$ because the same is true of $E_\infty^{0,k}$ and there is no room for any exotic extensions. On the other hand $E_\infty^{0,k+32}$ does not have the same value for $k = -8$, $k = -6$ and $k = -4$. This comparison via periodicity forces

- the indicated $d_5$ and $d_7$ in dimension 24, which together convert $\square$ to $\bigstar$. These were also established in Theorem 14.5.
- the short transfer in dimension 26, which converts $\bigstar$ to $\square$. It also follows from the results of Section 12.
- the long transfer in dimension 28, which converts $\square$ to $\bigstar$. The differential corresponding to the long transfer is

\[
d_{13}([2u_A]) = a_6 u_2 a u_3 \delta_1,
\]

so

\[
d_{13}(a_6 u_2 a_3 u_2 a_3) = a_6 a_6 u_2 a_3 \delta_3 = a_7 [u_2 a_3] u_3 \delta_3.
\]

This compares well with the $d_{13}$ of Theorem 11.10, namely

\[
d_{13}(a_6 [u_2 a_3]) = a_7 [u_2 a_3] \delta_3.
\]

The statements in dimensions 4 and 20 have similar proofs, and we will only give the details for the former. It is based on comparing the $E_4$-term for $K[2]$ in dimensions $-28$ and 4. They must converge to the same thing by periodicity. From the slice $E_4$-term in dimension 4 we see there is a short exact sequence

\[
0 \to M_4' \to Z_2 \to 0
\]
while the \((-28)\)-stem gives

$$
\begin{array}{cccccccccc}
0 & \rightarrow & \mathbb{Z}/2 & \rightarrow & M_4' & \rightarrow & \mathbb{Z} & \rightarrow & 0 \\
\mathbb{Z}/2 & \rightarrow & \mathbb{Z}/2 & \rightarrow & 0 \\
\mathbb{Z} & \rightarrow & \mathbb{Z}/2 \oplus \mathbb{Z} & \rightarrow & \mathbb{Z}/2 \\
2 & \rightarrow & 0 & \rightarrow & \mathbb{Z} & \rightarrow & 0.
\end{array}
$$

The commutativity of the second diagram requires that

$$
a + b = c = 1
$$

and

$$
b + d = c + d = 0,
$$

giving

$$
(a, b, c, d) = (0, 1, 1, 1).
$$

The diagram for \(M_4\) is that of in Table 2.

In dimension 20 the short exact sequence of (14.6) is replaced by

$$
0 \rightarrow \mathbb{Z}/2 \rightarrow M_{20}' \rightarrow \mathbb{Z}/2 \rightarrow 0
$$

and the resulting diagram for \(M_{20}'\) is that of in Table 2.

Similar arguments can be made in dimensions 6 and 22.

We could prove a similar statement about exotic restrictions hitting the 0-line in the third quadrant in dimensions congruent to 0, 4, 6, 14, 16, 20 (where there is an exotic transfer) and 22. The problem is naming the elements involved.

In Table 4 we show short or 4-term exact sequences in the sixteen even-dimensional congruence classes.

In each case the value of \(M_k'\) is the symbol appearing in both rows of the diagram. For even \(k\) with \(0 \leq k < 32\), we typically have short exact sequences

$$
\begin{array}{cccccccccc}
0 & \rightarrow & E^{0,k-32}_4 & \rightarrow & M_k' & \rightarrow & \text{quotient} & \rightarrow & 0 \\
0 & \rightarrow & \text{subgroup} & \rightarrow & M_k' & \rightarrow & E^{0,k}_4 & \rightarrow & 0,
\end{array}
$$

where the quotient or subgroup is finite and may be spread over several filtrations. This happens for the quotient in dimensions \(-32\), \(-16\) and \(-12\), and for the subgroup in dimensions 6 and 22.

This is the situation in dimensions where no differential hits [originates on] the 0-line in the third [first] quadrant. When such a differential occurs, we may need a 4-term sequence, such as the one in dimension \(-22\).

In dimensions 8 and 24 there is more than one such differential, the targets being a quotient and subgroup of the Mackey functor \(\mathbb{Z} \rightarrow \mathbb{Z}\).

In dimension \(-18\) we have a \(d_7\) hitting the 0-line. Its source is written as \(\mathcal{O} \subseteq E_{17}^{7,-24}\) in Figure 16. Its generator supports a \(d_5\), leaving a copy of \(\mathcal{O}\) in \(E_{17}^{7,-24}\).

There is no case in which we have such differentials in both the first and third quadrants.

**Corollary 14.7** (The \(E_{\infty}\)-term of the slice spectral sequence for \(K_{[2]}\)). The surviving elements in the spectral sequence for \(K_{[2]}\) are shown in Figure 17.
Figure 16. The reduced $E_2$-term of the slice spectral sequence for the periodic spectrum $K_{[2]}$. Differentials are shown in red. Exotic transfers and restrictions are shown in solid blue and dashed green vertical lines respectively. The Mackey functor symbols are indicated in the table below Figure 17.
Figure 17. The reduced $E_{14} = E_{\infty}$ term of the slice spectral sequence for $K[\mathbb{J}]$. The exotic Mackey functor extensions lead to the Mackey functors shown in violet in the second and fourth quadrants. The Mackey functor symbols are indicated in the table on the right.
### References


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