
Dieudonné modules for abelian Hopf algebras

Notas de matematica y simposia ; iv. 1.

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By Abelian Hopf algebra we mean graded connected biassociative strictly bi-commutative Hopf algebra of finite type over a perfect field \( k \) of characteristic \( p \). Let \( \mathcal{A} \) denote the category of such objects. \( \mathcal{A} \) is known to be abelian ([12]) and our purpose here is to show that it is isomorphic to a certain category of modules. An analogous theorem for the nongraded case was proved long ago by Dieudonné, and the modules that he used have been studied extensively (see [1], Chapter V, and [4]). I am grateful to Bill Singer for first bringing this work to my attention and suggesting the problem of carrying it over to the graded case.

The ring \( D \) in question is a noncommutative power series over \( W(k) \) (the Witt ring of \( k \)) in two variables \( F \) and \( V \) subject to the relations

\[
FV = VF = p
\]

\[
F^p = V^p, \quad V^p = W
\]

(*) Research partially supported by N.S.F.
for \( w \in W(\mathbb{k}) \), where \( \mathbf{w} \) denotes the action of the Frobenius automorphism of \( \mathbb{k} \) lifted to \( W(\mathbb{k}) \).

In our case we will obtain modules over a commutative graded ring \( E = W(\mathbb{k})[[F,V]]/(FV-p) \) where \( \dim F = 1, \dim V = -1 \). \( F \) will be seen to correspond to the Frobenius endomorphism of a Hopf algebra \( A \) which sends \( x \in A \) to \( x^p \), while \( V \) corresponds to the dual of \( F \), commonly known as the Verschiebung.

The relation between abelian Hopf algebras and \( E \)-modules will be described in Theorem 3 below, which is our main result.

Our first result is a decomposition theorem.

**Definition.** Let \( n \) be an integer prime to \( p \). An Abelian Hopf algebra is of type \( n \) if each of its primitives and generators has dimension \( np^i \) for some \( i \). Let \( T_A \) denote the full subcategory of type \( n \) Abelian Hopf algebras.

**Theorem 1.** There is a canonical categorical splitting \( A \cong \bigotimes_{(n,p)=1} T_A \), i.e.

- a) Every Abelian Hopf algebra is canonically a direct product of type \( n \) Abelian Hopf algebras.
- b) There are no nontrivial maps between a type \( n \) Hopf algebra and a type \( m \) Hopf algebra for \( m \neq n \).
- c) Moreover, \( T_A \cong T_n \bigvee T_m \)

Such a decomposition is well-known for the Hopf algebra \( H_b(\mathbb{B}U;\mathbb{k}) \) (see [3] for example). The general decomposition is established by showing that the endomorphism ring of \( H_b(\mathbb{B}U;\mathbb{k}) \) acts canonically on any abelian Hopf algebra. Part (b) follows from the fact that a Hopf algebra map sends primitives to primitives. Part (c) is trivial.

We now construct a set of projective generators for \( T_A \).

Let \( B_n \in A \) be \( k[b_1, b_2, \ldots, b_n] \) with \( \dim b_i = 1 \) and coproduct \( b_1 = \sum_{s+t=n} b_s \otimes b_t \) where \( b_0 = 1 \). Let \( W_n \) be the type 1 factor of \( B_n^p \). It is a polynomial algebra \( k[w_0, w_1, \ldots, w_n] \) with \( \dim w_i = p^i \). The coproduct is obtained lifting to \( W(k) \) and defining the Witt polynomials \( \mathbf{w}(w) = \sum_{i=0}^{p^i-1} p^{i-1} w_i \), \( 0 \leq n \leq n \), to be primitive.

**Theorem 2.** \( W_n \) is a projective object in \( A \), and its dual \( W_n^* \) is therefore injective.

**Proof.** Let \( S \) be the simple object \( k[x]/x^p \), \( \dim x = r \). Any Abelian Hopf algebra can be built up out of these simple objects by multiple extensions, so it suffices to show \( \text{Ext}^1_A(W_n, S) = 0 \forall r \), which is a simple calculation.

Now let \( \mathcal{H} \subset T_A \) denote the full subcategory whose objects are the \( W_n \). Let \( \text{FM} \) denote the category of contravariant functors from \( \mathcal{H} \) to the category of finite \( W(k) \) modules. This category is abelian. We define a functor

\[
\mathcal{R} : T_A \to \text{FM}
\]
by
\[ \mathcal{D}(A)(W_n) = \text{Hom}_S(W_n, A). \]

Now we can state our main result:

**Theorem 3.** The functor \( \mathcal{D} \) defined above is an equivalence of abelian categories.

The proof is analogous to that of Theorem V, §1.4.3 of [1]. Theorem 3 can be described in a more useful way by analyzing the structure of \( W \). Let \( \mathcal{V}_n : W_n \rightarrow W_{n-1} \) be the inclusion and let \( \mathcal{F}_n : W_{n+1} \rightarrow W_n \) be defined by \( \mathcal{F}_n(w) = w^{-1}_{n+2} \). Note that \( \mathcal{V}_n \mathcal{F}_{n-1} = \mathcal{F}_n \mathcal{V}_{n-1} = p \). Then we have

**Lemma 4.** The endomorphism ring of \( W_n \) is \( \mathbb{W}(k)/p^{n+1} \) and these endomorphisms along with the \( \mathcal{F}_n \) and \( \mathcal{V}_n \) generate all of the morphisms of \( W \).

Hence Theorem 3 can be paraphrased as

**Theorem 3'.** A type 1 Abelian Hopf algebra is characterized by a sequence of \( \mathbb{W}(k) \) modules \( W_n(A) = \text{Hom}(W_n, A) \) and maps \( \mathcal{F}_n : W_n(A) \rightarrow W_{n+1}(A) \) and \( \mathcal{V}_n : W_n(A) \rightarrow W_{n-1}(A) \) where \( \mathcal{V}_n \mathcal{F}_{n-1} = \mathcal{F}_n \mathcal{V}_{n-1} = p \).

If we identify \( f \in W_n(A) \) with the element \( f(w) \in A \), we have \( (\mathcal{F}_n f)(w) = f(w)^{-1}_{n+2} \in A \), i.e. \( \mathcal{F}_n \) corresponds to the Frobenius endomorphism of \( A \), while \( \mathcal{V}_n \) corresponds similarly to the dual endomorphism, i.e. the Verschiebung.

To make this more concise let \( E^d_A \) denote the where \( A_0 \) is projective and \( A_1 \) is polynomial. (If \( A \) is not finitely generated, one can still construct and \( A_0 \) and \( A_1 \) but they need not be of finite type).

This is a consequence of

**Theorem 6.** \( \text{Ext}^2_A(B, A) = 0 \) for all \( A \) iff \( B \) is polynomial.

We will conclude by identifying some well-known Hopf algebra functors with standard functors from homological algebra. It is convenient at this point to embed \( E^d_A \) into \( E \), the full category of graded \( E \)-modules and maps of all degrees. Hence for \( M, N \in E \), \( \text{Hom}_E(M, N) \) is also an \( E \)-module. Moreover, if \( N \) is nonnegative and \( M \) does not have any generators in positive dimensions then \( \text{Hom}_E(M, N) \) will also be nonnegatively graded.

Define modules \( P = E/VE, R = E/FE \).

**Theorem 7.** Let \( A \in E \). Then \( \text{Hom}_E(P, C(A)) \) is isomorphic to the abelian restricted Lie algebra of primitives of \( A \) (where \( F \) corresponds to the restriction), and \( \text{Ext}^1_E(P, C(A)) \) is isomorphic to the abelian restrict Lie coalgebra (with \( V \) corresponding to the corestriction) of decomposable elements of \( A \).

The functors \( \text{Ext}^1_E(P, C(A)) \) and \( \text{Hom}_E(R, C(A)) \) are the functors \( \mathcal{D} \) and \( \mathcal{D} \) respectively defined in [6] and also in [5].
§3. Hence an extension in $\mathcal{T}$ induces six term exact sequences relating these functors as was shown in [6]. (Note
that $\text{Ext}^2_E(\mathcal{F}, -) = \text{Ext}^2_E(\mathcal{R}, -) = 0$). It is evident that the con-
necting homomorphisms of these sequences must be $E$-module maps,
i.e. they must preserve the restriction and corestriction res-
pectively. Hence the argument of 4.10 of [6] (which leads to
contradictions of Theorems 2 and 4) is incorrect.

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N.B. These results were also obtained by C. Schoeller, "Etude
de la Categorie des Algebres de Hopf Commutatives Connexes sur

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