algebra structure does not seem to give us this generality. When k is small, we can prove Theorem 1 in these cases by using unstable secondary operations coming from Adem relations of the form $Sq^{r+1} = \Sigma a_i b_i$ and using them to detect Dyer-Lashof operations.

To make this precise, we state the following proposition which does not seem to be generally known. Let φ_r be an unstable secondary cohomology operation coming from the Adem relation $Sq^{r+1} = \Sigma a_i b_i$.

PROPOSITION 3. Let $x \in H_r(\Omega X; Z_2)$, and $x^* \in H^r(\Omega X; Z_2)$ be such that $\langle x^*, x\rangle \neq 0$. Assume x^* is the loop of an element in $H^{r+1}(X; Z_2)$. Further assume that $\varphi_r(x^*)$ is defined. Then $\langle\varphi_r(x^*), Q_0(x)\rangle = \langle\varphi_r(x^*), x^2\rangle = \langle x^*, x\rangle$ mod indeterminacy.

This proposition implies that φ_r can be used to detect higher Dyer-Lashof operations in iterated loop spaces.

PROPOSITION 4. Let $x \in H_{r-k}(\Omega^{k+1}X; Z_2)$, and $x^* \in H^{r-k}(\Omega^{k+1}X; Z_2)$ be such that $\langle x^*, x\rangle \neq 0$. Assume x^* is the iterated loop of an element $y \in H^{r+1}(X; Z_2)$. Further, assume that $\varphi_r(y)$ is defined. Then $\langle\varphi_r(x^*), Q_k(x)\rangle = \langle x^*, x\rangle$ mod indeterminacy.

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Morava K-theories and Finite Groups

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The purpose of this note is to prove

<u>Theorem</u>. Let G be a finite group and $K(n)^*$ the n^{th} Morava K-theory associated with a prime number p. Then $K(n)^*(BG)$ is finitely generated as a module over $K(n)^*(pt)$. □

We will recall the basic properties of Morava K-theories below. The result above was obtained for abelian groups in [RW]. Since $K(1)^*$ is a summand of mod(p) complex K-theory, the result for $n = 1$ follows from Atiyah's description ([A]) of $K^*(BG)$ in terms the complex representation ring of G. In particular, if G is a p-group his result implies that the rank of $K(1)^*(BG)$ is the number of conjugacy classes of elements in G. The group theoretic significance of the rank of $K(n)^*(BG)$ is an intriguing question which we have no idea how to answer. For G abelian this number is the n^{th} power of the order of the p-Sylow subgroup of G.

The $K(n)$'s were invented by Jack Morava about ten years ago as a tool for studying complex bordism and BP-theory. $BP^*(X)$ is often very complicated and hard to compute; $K(n)^*(X)$ is simpler, more accessible, and often contains the essential information about X that one hoped to get by computing $BP^*(X)$. The best published references for Morava K-theories are [JW] and [RW] which include proofs of most of the statements below.

Recall that BP is a minimal summand of $MU_{(p)}$, the Thom spectrum for the unitary group localized at a prime p. We have

$$\pi_*(BP) = BP_* = Z_{(p)}[v_1, v_2, \ldots]$$

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with $\dim v_n = 2(p^n-1)$. Given any prime ideal $I \subset BP_*$, the Sullivan-Baas construction gives us a BP-module spectrum BPI with $\pi_*(BPI) = BP_*/I$. The ideal we want is $(p, v_1, v_2, \ldots v_{n-1}, v_{n+1}, v_{n+2} \ldots)$. The resulting spectrum is denoted by $k(n)$ and the corresponding theory $k(n)^*$ is the n^{th} connective Morava K-theory. There is a map

$$\Sigma^{2(p^n-1)} k(n) \to k(n)$$

inducing multiplication by v_n in homotopy. We get a directed system of spectra by iterating this map and $K(n)$ is the resulting direct limit.

Hence, we have $\pi_*(K(n)) = K(n)_* = Z/(p)[v_n, v_n^{-1}]$ for $n > 0$ and by convention $K(0)^*$ is ordinary rational cohomology. $K(n)_*$ is a graded field in the sense that every graded module over it is free. This means the Morava K-theories enjoy many of the computational advantages of ordinary cohomology with coefficients in a field. One has, for example, a Künneth isomorphism and linear duality between homology and cohomology.

We will prove the theorem by reducing to the case where G is a p-group and arguing by induction on its order. For the first step, let H be a p-Sylow subgroup of G and let $BH \to BG$ be the inclusion map. Stably one has a transfer map $BG \to BH$ such that the composite $BG \to BH \to BG$ induces multiplication by the index of H in G in ordinary cohomology. Since this index is prime to p the map becomes a stable equivalence after localizing at p. Hence, $BG_{(p)}$ is a retract of $BH_{(p)}$ and it suffices to show $K(n)^*(BH)$ is finite.

Now let G be a finite p-group. By elementary group theory G has a normal subgroup of index p which we denote by H. We assume inductively that $K(n)^*(BH)$ is finite dimensional. We have a fibre sequence

$$BH \to BG \to BZ/(p) \qquad (*)$$

which we will use to prove the finiteness of $K(n)^*(BG)$. Our basic tool is

Lemma A. Let $F \to E \xrightarrow{f} B$ be a fibre sequence with B a CW-complex. There is a natural spectral sequence converging to $K(n)^*(E)$ with

$$E_2 = H^*(B; K(n)^*(F)),$$

possible having twisted coefficients if B is not simply connected.

Proof: This spectral sequence generalizes those of Serre, where $K(n)^*$ is replaced by ordinary cohomology, and Atiyah-Hirzelruch, in which F is a point. The construction is the same in all three cases: take the filtration of E induced by the skeletal filtration of B and apply the appropriate functor. If one is after $h^*(E)$ for a generalized cohomology theory h^*, there may be a convergence problem since what the spectral sequence actually computes is $\varprojlim h^*(f^{-1}(B^n))$. However, the corresponding homology spectral sequence computes $\varinjlim h^*(f^{-1}(B^n))$ which is indeed $h_*(E)$ since homology commutes with direct limits. In our case convergence in cohomology follows from the duality cited above. □

Rather than applying Lemma A to (*) directly we study the induced extension $\hat{G}$ of H by Z and get a fibre sequence

$$BH \to B\hat{G} \to S^1 \qquad (**)$$

and a map of fibre sequences

$$\begin{array}{ccccc} B\hat{G} & \longrightarrow & BG & \longrightarrow & CP^\infty \\ \downarrow & & \downarrow & & \| \\ S^1 & \longrightarrow & BZ/(p) & \longrightarrow & CP^\infty \end{array} \qquad (***)$$

Applying Lemma A to (**) shows that $K(n)^*(B\hat{G})$ is finite. To proceed further, we need

Lemma B. In the spectral sequence for the lower fibre sequence in (***), x^{p^n} is killed by a differential, where x is the generator of $H^2(CP^\infty; Z/(p))$.

Proof: In [RW] it is shown that $K(n)^*(BZ/(p))$ has rank p^n, and this forces the spectral sequence to behave as indicated. (The computation of $K(n)^*(BZ/(p))$ is done by means of a short exact sequence

$$0 \to K(n)^*(CP^\infty) \xrightarrow{[p]} K(n)^*(CP^\infty) \to K(n)^*(BZ/(p)) \to 0$$

where $[p]$ is induced by the degree p map. One has $K(n)^*(CP^\infty) = K(n)^*(pt.)[[x]]$ with $[p](x) = v_n x^{p^n}$.) □

To complete the proof of the theorem, the naturality of the spectral sequence and Lemma B guarantee that x^{p^n} dies in the spectral sequence for $K(n)^*(BG)$, so the latter some subquotient of $K(n)^*(B\hat{G})[x]/(x^{p^n})$ and is therefore finite.

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INVARIANT FRAMINGS OF QUOTIENTS OF $SL_2(\mathbb{R})$ BY DISCRETE SUBGROUPS

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If G is a Lie group of dimension n, then a choice of a basis for its Lie algebra $\mathfrak{g} \equiv T_eG$ defines, by right translation, a trivialization R of its tangent bundle. If G is compact and if we use the orientation of $\mathfrak{g}$ to orient G, then the pair (G, R) represents an element in the cobordism group Ω_n^{fr} of stably-framed n-manifolds and so an element in the stable homotopy group of the spheres $\pi_n^S \equiv \pi_{n+k}(S^k)$, k large. (By the Pontryagin-Thom construction.) For example, the sphere S^3 represents, with its right invariant framing R, the generator ν of the 3-stem $\pi_3^S = \mathbb{Z}_{24}$ (see [12]). However, negative results of various authors, e.g., [3, 6, 9], show that not many homotopy elements arise in this way, so it has become necessary to look for different ways of constructing manifolds with natural framings.

In [10] B. Steer and myself considered the following, more general, situation: Let us allow the group G to be, possibly, non-compact, and let Γ be a discrete subgroup of G with compact quotient $\Gamma\backslash G$. Then the projection

$$p : G \longrightarrow \Gamma\backslash G$$

is a local diffeomorphism, and a choice of a basis for the Lie algebra $\mathfrak{g}$ of G defines, as above, a right invariant trivialization of the tangent bundle of $\Gamma\backslash G$, and so an element $(\Gamma\backslash G, R)$ in the stable n-stem π_n^S. The element in π_n^S so obtained depends only on the orientation of $\mathfrak{g}$; if we choose the opposite orientation we obtain the inverse element in π_n^S.

In [10] we studied the cases when G was either the sphere S^3 or the non-compact group $SL_2(\mathbb{R})$ of real 2×2-matrices with determinant 1. Both of these groups are 3-dimensional, semi-simple Lie groups, with centre ± 1. If we divide S^3 by its centre we obtain the group $SO(3)$

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