COMPLEX COBORDISM THEORY FOR NUMBER THEORISTS

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§1. Elliptic cohomology theory

The purpose of this paper is to give the algebraic topological background for elliptic cohomology theory and to pose some number theoretic problems suggested by these concepts.

Algebraic topologists study functors from the category of spaces to various algebraic categories. In particular there are functors to the category of graded rings called multiplicative generalized cohomology theories. (All rings are assumed to be commutative and unital. Graded rings are commutative subject to the usual sign conventions, i.e., odd-dimensional elements anticommute with each other.) These functors satisfy all of the Eilenberg-Steenrod axioms but the dimension axiom. In other words they have the same formal properties as ordinary cohomology except that the cohomology of a point may be more complicated.

For a discussion of these axioms the interested reader should consult [S] or [ES]; for generalized cohomology theories a good reference is Part III of [A1].

Among these cohomology theories the following examples are mentioned elsewhere in this volume:

Examples 1.1

(i)	$H^*(\cdot; R)$	ordinary cohomology with coefficients in a ring R.
(ii)	$K^*(\cdot)$	complex K-theory.
(iii)	$KO^*(\cdot)$	real K-theory.
(iv)	$MU^*(\cdot)$	complex cobordism theory.
(v)	$MSO^*(\cdot)$	oriented cobordism theory.
(vi)	$MSpin^*(\cdot)$	Spin cobordism theory.

A comprehensive reference for cobordism theory is [St]; a very brief account can be found in Sections 4.1 and 4.2 of [R1].

A cohomology theory E is said to be **complex-oriented** if it behaves well on infinite dimensional complex projective space $\mathbb{C}P^\infty$, namely if

$$E^*(\mathbb{C}P^\infty) = E^*(pt.)[[x]] \text{ with } x \in E^2(\mathbb{C}P^\infty).$$

Here $E^*(pt.)$ means the cohomology of a point and x is such that its restriction to $E^*(pt.)$ is zero. This is a graded ring which will be abbreviated by E^*. For any space X, $E^*(X)$

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is an algebra over E^*. Of the examples given above, all but (iii) and (vi) are complex-oriented. Complex-oriented theories are studied in detail in Part II of [A1] and in Sections 4.1 and 4.2 of [R1].

When the theory is complex-oriented, there is a formal group law associated with it.

Definition 1.2. *A* **formal group law** *over a commutative ring with unit R is a power series $F(x, y)$ over R that satisfies*

(i) $F(x,0) = F(0,x) = x$ *(identity),*
(ii) $F(x,y) = F(y,x)$ *(commutativity) and*
(iii) $F(F(x,y),z) = F(x,F(y,z))$ *(associativity).*

(The existence of an inverse is automatic. It is the power series $i(x)$ determined by the equation $F(x,i(x)) = 0$.)

Examples 1.3

(i) $F(x,y) = x + y$. This is called the additive formal group law.
(ii) $F(x,y) = x+y+xy = (1+x)(1+y)-1$. This is called the multiplicative formal group law.
(iii)

$$F(x,y) = \frac{x\sqrt{R(y)} + y\sqrt{R(x)}}{1 - \varepsilon x^2 y^2}$$

where

$$R(x) = 1 - 2\delta x^2 + \varepsilon x^4.$$

This is the formal group law associated with the elliptic curve

$$y^2 = R(x),$$

a Jacobi quartic. It is defined over $\mathbb{Z}[1/2][\delta, \varepsilon]$. This curve is nonsingular mod p (for p odd) if the discriminant $\Delta = \varepsilon(\delta^2 - \varepsilon)^2$ is invertible. This example figures prominently in elliptic cohomology theory; see [L1] for more details.

The theory of formal group laws from the power series point of view is treated comprehensively in [Ha]. A short account containing all that is relevant for the current discussion can be found in Appendix 2 of [R1].

If E^* is an oriented cohomology theory then we have

$$E^*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty) = E^*(pt.)[[x \otimes 1, 1 \otimes x]].$$

There is a map from $\mathbb{C}P^\infty \times \mathbb{C}P^\infty$ to $\mathbb{C}P^\infty$ inducing the tensor product of complex line bundles. The induced map in cohomology goes the other way since cohomology is a contravariant functor. Thus we get a map

$$E^*(pt.)[[x]] \to E^*(pt.)[[x \otimes 1, 1 \otimes x]].$$

Definition 1.4. *Let $F_E(x \otimes 1, 1 \otimes x)$ denote the image of x under this map. It is easy to verify that this F_E satisfies the conditions of 1.2, so we have a formal group law over the ring E^*.*

Now we need a result from the theory of formal group laws first proved by Lazard.

Theorem 1.5. *(i) There is a universal formal group law defined over a ring L of the form*

$$G(x, y) = \sum_{i,j} a_{i,j} x^i y^j \quad \text{with } a_{i,j} \in L$$

such that for any formal group law F over R there is a unique ring homomorphism θ from L to R such that

$$F(x, y) = \sum_{i,j} \theta(a_{i,j}) x^i y^j.$$

(ii) L is a polynomial algebra $\mathbb{Z}[x_1, x_2, \ldots]$. If we put a grading on L such that $a_{i,j}$ has degree $2(1 - i - j)$ then x_i has degree $-2i$.

The relevance of this to algebraic topology is embodied in the following result of Quillen [Q] proved in 1969.

Theorem 1.6. *The formal group law F_{MU} associated with complex cobordism theory is isomorphic to Lazard's universal formal group law. In particular L is isomorphic to MU^*.*

This suggests that complex cobordism should be central to the study of oriented theories and their relation to formal group laws. On the other hand Ochanine's theorem [**O**] concerns oriented manifolds, which suggests using oriented cobordism theory MSO^*. This discrepancy is not a serious one. Every complex manifold is oriented, so there is a natural homomorphism from MU^* to MSO^*. Moreover if we localize away from the prime 2 i.e., if we tensor with $\mathbb{Z}[1/2]$, then MU^* and MSO^* are essentially equivalent theories, i.e., each one is functorially determined by the other.

Quillen's theorem (1.6) also raises the question of whether there is an oriented cohomology theory for each formal group law. A formal group law over R corresponds by 1.5 to a homomorphism from L to R, i.e., an L-module structure on R. One can ask if there is an MU-module spectrum E which is a ring spectrum such that $E^* = R$ with the desired L-module structure. There are no known counterexamples, but also no general theorems. Given such an R, a more precise question is whether the functor

$$R^*(X) = MU^*(X) \otimes_L R,$$

is a cohomology theory. It is if it satisfies certain criteria spelled out in the Landweber Exact Functor Theorem [L2]; a precise statement can be found in 4.2 of [R1].

The formal group law of 1.3 (ii) does satisfy Landweber's criteria while 1.3 (i) does not. In the case of (ii) the resulting cohomology theory is classical complex K-theory. We get elliptic cohomology from (iii) after inverting the prime 2 and any of $\delta^2 - \varepsilon$, ε or Δ. The proof in the latter case involves some deeper aspects of formal group law theory and elliptic curves; see [LRS]. In particular it uses the fact the the formal group law of an elliptic curve must have height 1 or 2; this is proved as Corollary 7.5 in Silverman's book [Si]. (The height of a formal group law is defined below in 2.4.)

Unfortunately this method of definition is too abstract to reveal the full power of the cohomology theory in question. In the case of K-theory there is the classical definition in terms of vector bundles, which bears little obvious relation with complex cobordism or formal group laws. Any 'physical' interpretation of K-theory, such as its relation to the Dirac operator, relies completely on the geometric definition and would not be possible without it.

It would be extremely desirable to have a comparable geometric definition of elliptic cohomology. In [Wi] Witten studies a possible geometric definition of the map from MSO^* to R used to define elliptic cohomology.

§2. Some curious group cohomology and the chromatic filtration

In studying complex cobordism theory, topologists have been led to the study of the cohomology (in the algebraic sense of the term) of certain groups which appear to be arithmetically interesting.

Definition 2.1. *Let Γ be the group of power series over $\mathbb{Z}$ having the form*

$$\gamma = x + b_1x^2 + b_2x^3 + \cdots$$

where the group operation is functional composition. Γ acts on the Lazard ring L of 1.5 as follows. Let $G(x, y)$ be the universal formal group law as above and let $\gamma \in \Gamma$. Then $\gamma^{-1}G(\gamma(x), \gamma(y))$ is another formal group law over L, and therefore is induced by a homomorphism from L to itself. Since γ is invertible, this homomorphism is an automorphism, giving the desired action of Γ on L.

The cohomology group in question is $H^*(\Gamma; L)$. It is of topological interest because it is closely related to the homotopy groups of spheres. More precisely, it is the E_2-term of the Adams-Novikov spectral sequence, which converges to the stable homotopy groups of spheres. For more discussion of this point, see Sections 1.3 and 1.4 of [R1]. The group is bigraded since L itself is graded. It is known that $H^0(\Gamma; L) = \mathbb{Z}$ (concentrated in dimension 0) and that $H^s(\Gamma; L)$ is locally finite for $s > 0$.

This group is difficult to compute and a lot of machinery has been developed for doing so. $H^{1,*}$ and $H^{2,*}$ are completely known. The problem can be be studied locally at each prime p. For $s > 2$, the p-component of $H^{s,t}$ is known for $t < 2(p-1)p^3$ for p odd and for $t < 40$ for $p = 2$. The description of H^1 is very suggestive.

Theorem 2.2. *$H^{1,t}$ is trivial if t is odd and is cyclic of order 2 if t is a positive odd multiple of 2. When $t = 4k, H^{1,t}$ is cyclic of order a_{2k} (for $k = 1$ its order is half this number or 12), where a_{2k} is the denominator of $B_{2k}/4k$, and B_{2k} is the $2k^{th}$ Bernoulli number.*

The values of a_{2k} for small k are displayed in the following table.

k	1	2	3	4	5	6	7	8	9	10	11	12
a_{2k}	24	240	504	480	264	65520	24	16320	28728	13200	552	131040

These numbers can be described in at least two different ways. $B_{2k}/4k$ is the value of the Riemann zeta function $\zeta(1-2k)$. As such it appears as a coefficient in an Eisenstein series associated with the Weierstrass $\wp$-function. An alternate description which is more useful for our purposes is that

$$a_{2k} = \gcd n^L(n^{2k}-1) \tag{2.3}$$

where n ranges over all the integers and L is as large as necessary. (This is proved is [A2].) For $2k = 2$, the square of any odd integer is congruent to 1 mod 8, so whenever n is even we take $L \geq 3$ and this makes a_2 divisible by 8. The square of any integer prime to 3 is congruent to 1 mod 3, so when n is divisible by 3 we take $L \geq 1$ and a_2 is divisible by 3. There are no other such congruences for larger primes so $a_2 = 24$ as indicated. For k odd 2.3 gives $a_k = 2$, which is consistent with 2.2.

The groups $H^{2,t}$ are also known. They are far more complicated than $H^{1,t}$; for example there is no upper bound on the number of summands the p-component can have. Moreover there is no known arithmetic description comparable to 2.2. Given the fact that the group has such a complicated structure, a number theoretic interpretation could lead to a great deal of new information.

There is a deeper way to view $H^*(\Gamma; L)$ due to Jack Morava. *Spec*(L), which is an infinite-dimensional affine space, can be though of as a moduli space for formal group laws over $\mathbb{Z}$. Then the orbits under Γ are isomorphism classes of formal group laws. The classification of formal group laws becomes quite manageable if we replace $\mathbb{Z}$ by $\overline{\mathbb{F}}_p$, the algebraic closure of the field with p elements. We lose little information about cohomology since there is an isomorphism

$$H^*(\Gamma; L \otimes \mathbb{Z}/p) \otimes \overline{\mathbb{F}}_p \cong H^*(\Gamma_{\overline{\mathbb{F}}_p}; L \otimes \overline{\mathbb{F}}_p)$$

where $\Gamma_{\overline{\mathbb{F}}_p}$ is the group of power series over $\overline{\mathbb{F}}_p$ similar to Γ.

Formal group laws over $\overline{\mathbb{F}}_p$ are determined up to isomorphism by an invariant called the **height**. To define it we introduce some power series associated with a formal group law. For each integer n define $[n](x)$ (called the n-**series**) by

$$\begin{aligned} [1](x) &= x, \\ [n](x) &= F(x, [n-1](x)) \qquad \text{for } n > 1 \text{ and} \\ [-n](x) &= i([n](x)). \end{aligned} \tag{2.4}$$

These satisfy

$$\begin{aligned} [n](x) &\equiv nx \mod (x^2), \\ [m+n](x) &= F([m](x), [n](x)) \qquad \text{and} \\ [mn](x) &= [m]([n](x)). \end{aligned}$$

In characteristic p the p-series always has leading term ax^q where $q = p^h$. The height is defined to be h. For the additive formal group law we have $[p](x) = 0$ and the height is said to be ∞. The multiplicative formal group law has height 1 since $[p](x) = x^p$. The mod p reduction (for p odd) of the elliptic formal group law of 1.3(iii) has height one or two depending on the values of δ and ε. For example if $\delta = 0$ and $\varepsilon = 1$ then the height is one for $p \equiv 1$ mod 4 and 2 for $p \equiv 3$ mod 4. (See pp.373-374 of [R1].)

Theorem 2.5. *(i) For each prime p there are polynomial generators v_i of of $L \otimes \mathbb{Z}_{(p)}$ having degree $2p^i - 2$ such that the height of a formal group law induced by θ is n if and only if*

$$\theta(v_i) = 0 \qquad \text{for } i < n \text{ and } \theta(v_n) \neq 0.$$

(ii) Two formal group laws over $\overline{\mathbb{F}}_p$ are isomorphic if and only if they have the same height.
(iii) Let $I_n = (p, v_1, ...v_{n-1}) \subset L$. Then

$$H^*(\Gamma; v_n^{-1}L/I_n) \otimes \overline{\mathbb{F}}_p \cong H^*(S_n; \overline{\mathbb{F}}_p)$$

where S_n is the automorphism group of a height n formal group law over $\overline{\mathbb{F}}_p$.

The structure of S_n is known and is described in Chapter 6 of [R1]. It is a group of units in a division algebra D_n (with Hasse invariant $1/n$) over the p-adic numbers $\mathbb{Q}_p$ of rank n^2. It is known that each degree n extension of $\mathbb{Q}_p$ embeds as a subfield of D_n.

The isomorphism 2.5(iii) is very useful since the cohomology of S_n is much easier to compute than that of Γ. This can be used to get information about $H^*(\Gamma; L)$ with the help of the chromatic spectral sequence, which is described in Chapter 5 of [R1]. The cohomology groups of 2.5(iii) are "v_n-periodic" in the sense that they are modules over the ring

$$K(n)^* = \mathbb{Z}/p[v_n, v_n^{-1}]. \tag{2.6}$$

This notion can be generalized in such a way that each element in $H^*(\Gamma; L)$ is v_n-periodic, and this leads to a chromatic filtration of $H^*(\Gamma; L)$.

We know now that this filtration can be defined on the homotopy category itself, not just on $H^*(\Gamma; L)$. The author made several conjectures concerning this in [R2], many of which have been proved by Devinatz, Hopkins and Smith in [DHS]. Expository accounts can be found in [R3] and [Ho]. We will describe some of this material now.

The ring $K(n)^*$ of 2.6 is the coefficient ring of a cohomology theory called the n^{th} **Morava K-theory**. We use $K(0)^*$ to denote rational cohomology. $K(1)^*$ is mod p complex K-theory. Every graded module over $K(n)^*$ is free, so we can regard $K(n)^*$ as a graded field. In particular there is a very convenient Künneth isomorphism

$$K(n)^*(X \times Y) \cong K(n)^*(X) \otimes_{K(n)^*} K(n)^*(Y).$$

Ordinary cohomology with coefficients in a field (such as $\mathbb{Z}/p$ or $\mathbb{Q}$) enjoys a similar property. In [DHS] it is shown that ordinary cohomology with field coefficients and the various Morava K-theories are the only theories with such a Künneth isomorphism. If we tensor with the field with p^n elements, then we get a functorial action of the group S_n of 2.5(iii). For $n = 1$ this group is the p-adic units and we get the p-adic Adams operations.

If G is a finite group then we can define a $K(n)$-theoretic generalization of group cohomology by considering $K(n)^*(BG)$, where BG is the classifying space of G. (Recall that the usual Eilenberg-Mac Lane cohomology of G is $H^*(BG)$.) Unlike $H^*(BG), K(n)^*(BG)$ is known to have finite rank. Atiyah showed that $K^*(BG)$ is the completion of the complex representation ring of G at its augmentation ideal, so $K(1)^*(BG)$ can be described in similar terms. In particular its rank is the number of conjugacy classes of elements in G whose order is a power of p. In [HKR] we generalize this result as follows. The Euler characteristic (and presumably the rank) of $K(n)^*(BG)$ is the number of conjugacy classes of n-tuples of elements in G which commute with each other and have order a power of p.

Morava K-theory is also useful for studying the general homotopy theory of finite complexes. If X is a finite complex it is known that

$$rk\ K(n)^*(X) \leq rk\ K(n+1)^*(X).$$

These numbers are all finite and not all zero (unless X is contractible). After fixing a prime p we say that a finite complex has **type** n if n is the smallest integer such that $K(n)^*(X) \neq 0$. Equivalently n is the smallest integer such that $MU^*(X)$ is not annihilated by the invariant prime ideal I_n of 2.5(iii).

Theorem 2.7 (Mitchell [M]). *For each n there is a finite complex X_n of type n.*

It is known that for any finite complex X of type n there is an Γ-equivariant L-endomorphism α of $MU^*(X)$ which becomes an isomorphism after tensoring with $K^*(n)$. One such α is multiplication by an appropriate power of v_n. This is an algebraic property of L-modules with Γ-action. It raises the geometric question of the existence of an analogous endomorphism of X itself.

Periodicity Theorem 2.8 (Hopkins-Smith [HS]). *Any finite complex of type n admits an endomorphism α which induces a $K(n)^*$- isomorphism. Moreover some iterate of α is in the center of the endomorphism ring $End(X)$, which has Krull dimension one.*

It is also known that any other Γ-equivariant endomorphism of $MU^*(X)$ is nilpotent, i.e., some iterate of it is zero. The analogous geometric fact is the following.

Nilpotence Theorem 2.9 (Devinatz-Hopkins-Smith [DHS]). *An endomorphism of a finite complex that induces the trivial map in each Morava K-theory is nilpotent.*

The special case of 2.9 when X is the sphere spectrum is Nishida's Theorem, which says that each positive-dimensional element in the stable homotopy ring is nilpotent.

§3. Formal A-modules

Definition 3.1. *Let A be the ring of integers in a number field K (or a subring thereof) or in a finite extension of the p-adic numbers. A* **formal A-module** *over an A-algebra R is a formal group law over R equipped with power series $[a](x)$ for each $a \in A$ with similar properties to the $[n](x)$ of 2.4.*

Lubin and Tate [LT] used a formal A-module in the local case to construct the maximal totally ramified abelian extension of the local field K. This is a generalization of Kronecker's Jugendtraum, which concerns the case when K is an imaginary quadratic extension of $\mathbb{Q}$. There we have an elliptic curve E with complex multiplication whose endomorphism ring is A. The formal group law associated with E (over A) is a formal A-module. In both cases the abelian extensions are obtained by adjoining the roots of $[a](x)$.

The theory of formal A-modules is treated in Section 21 of [Ha].

It is possible to generalize the algebraic constructions of the previous section to formal A-modules. There is a generalization of Lazard's theorem (1.5). The resulting ring is denoted by L_A. There are inclusions

$$L \otimes A \subset L_A \subset L \otimes K \tag{3.2}$$

where K is the field of fractions of A. L_A is known to be a polynomial ring in the local case and when the class number of K is 1.

We define the group Γ_A to be the group of power series over A analogous to Γ (2.1). It acts on L_A in a similar way, so we can ask about $H^*(\Gamma_A; L_A)$. It is known that $H^0(\Gamma_A; L_A) = A$ (concentrated in dimension 0) and that $H^s(\Gamma_A; L_A)$ is locally finite for $s > 0$. The topological significance of this group is unclear. It could be the E_2-term of the Adams-Novikov spectral sequence for some generalization of the sphere, but this appears to be very hard to prove. Theorem 2.5 generalizes to formal A-modules in a satisfactory way as does the chromatic spectral sequence. More details can be found in [R4]. In the local case if q is the cardinality of the residue field then we have a sparseness result,

$$H^{s,t}(\Gamma_A; L_A) = 0 \quad \text{unless } t \text{ is divisible by } 2q-2. \tag{3.3}$$

$H^1(\Gamma_A; L_A)$ has been computed in the local case by Keith Johnson [Jo]. The two descriptions of this group (2.2 and 2.3) is the case $A = \mathbb{Z}$ generalize in different ways. In the global case one might generalize 2.2 by using the Dedekind zeta function for K, but this will not work since this function vanishes at negative integers unless K is totally real, while $H^1(\Gamma_A; L_A)$ is far from trivial.

However we can generalize 2.3 by defining ideals

$$J_k = \cap(a^N(a^k - 1)) \tag{3.4}$$

where the intersection is over all $a \neq 0 \in A$ and all natural numbers N. Then Johnson's result is that in the local case

$$H^{1,2k}(\Gamma_A; L_A) = A/J_k \tag{3.5}$$

except for certain small values of k and certain rings A. (Recall that for $A = \mathbb{Z}$ there was an exception for $k = 2$.)

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