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The cohomology of C_2-equivariant $\mathcal{A}(1)$ and the homotopy of ko_{C_2}

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We compute the cohomology of the subalgebra $\mathcal{A}^{C_2}(1)$ of the C_2-equivariant Steenrod algebra $\mathcal{A}^{C_2}$. This serves as the input to the C_2-equivariant Adams spectral sequence converging to the completed $\mathrm{RO}(C_2)$-graded homotopy groups of an equivariant spectrum ko_{C_2}. Our approach is to use simpler $\mathbb{C}$-motivic and $\mathbb{R}$-motivic calculations as stepping stones.

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1. Introduction

The $\mathrm{RO}(G)$-graded homotopy groups are among the most fundamental invariants of the stable G-equivariant homotopy category. This article is a first step towards systematic application of the equivariant Adams spectral sequence to calculate these groups.

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Araki and Iriye [1982; Iriye 1982] computed much information about the C_2-equivariant stable homotopy groups using EHP-style techniques in the spirit of Toda [1962]. Our approach is entirely independent from theirs.

We work only with the two-element group C_2 because it is the most elementary nontrivial case. In order to compute C_2-equivariant stable homotopy groups of the C_2-equivariant sphere spectrum using the Adams spectral sequence, one needs to work with the full C_2-equivariant Steenrod algebra $\mathcal{A}^{C_2}$ for the constant Mackey functor $\underline{\mathbb{F}}_2$. As the C_2-equivariant Eilenberg–Mac Lane spectrum for $\underline{\mathbb{F}}_2$ is flat [Hu and Kriz 2001, Corollary 6.45] the E_2-term of the Adams spectral sequence is given by the cohomology of the equivariant Steenrod algebra. In this article, we tackle a computationally simpler situation by working over the subalgebra $\mathcal{A}^{C_2}(1)$. This means that we are computing the C_2-equivariant stable homotopy groups not of the sphere but of the C_2-equivariant analogue of connective real K-theory ko. We will explicitly construct this C_2-equivariant spectrum ko_{C_2} in Section 10.

Our calculational program is carried out for $\mathcal{A}^{C_2}(1)$ in this article as a warmup for the full Steenrod algebra $\mathcal{A}$ to be studied in future work. Roughly speaking, $\mathcal{A}$ contains Steenrod squaring operations Sq^i with the expected properties, and $\mathcal{A}^{C_2}(1)$ is the subalgebra generated by Sq^1 and Sq^2. A key point is that our program works just as well in theory for $\mathcal{A}^{C_2}$ as for $\mathcal{A}^{C_2}(1)$, except that the details are even more complicated. It remains to be seen how far this can be carried out in practice.

Our strategy is to build up to the complexity of the C_2-equivariant situation by first studying the $\mathbb{C}$-motivic and $\mathbb{R}$-motivic situations. The relevant stable homotopy categories are related by functors as in the diagram

$$\begin{tikzcd} \mathrm{Ho}(\mathbf{Sp}^{\mathbb{R}}) \arrow[r, "-\otimes_{\mathbb{R}}\mathbb{C}"] \arrow[d, "\mathbf{Re}"'] & \mathrm{Ho}(\mathbf{Sp}^{\mathbb{C}}) \arrow[d, "\mathbf{Re}"] \\ \mathrm{Ho}(\mathbf{Sp}^{C_2}) \arrow[r, "\iota^*"] & \mathrm{Ho}(\mathbf{Sp}) \end{tikzcd}$$

The vertical functors are Betti realization (see [Heller and Ormsby 2016, Section 4.4]). The functor ι^* restricts an equivariant spectrum to the trivial subgroup, yielding the underlying spectrum.

The $\mathbb{C}$-motivic cohomology of a point is equal to $\mathbb{F}_2[\tau]$ [Voevodsky 2003a] (see also [Dugger and Isaksen 2010, Section 2.1]). The $\mathbb{C}$-motivic Steenrod algebra $\mathcal{A}^{\mathbb{C}}$ is very similar to the classical Steenrod algebra, but there are some small complications related to τ. In particular, these complications allow the element h_1 in the cohomology of $\mathcal{A}^{\mathbb{C}}$ to be nonnilpotent, detecting the nonnilpotence of the motivic Hopf map $\eta_{\mathbb{C}}$ [Morel 2004, Corollary 6.4.5]. In the cohomology of $\mathcal{A}^{\mathbb{C}}(1)$, the nonnilpotence of h_1 is essentially the only difference to the classical case.

The $\mathbb{R}$-motivic cohomology of a point is equal to $\mathbb{F}_2[\tau, \rho]$ [Voevodsky 2003a] (again, see the discussion in [Dugger and Isaksen 2010, Section 2.1]). Now an additional complication enters because $\mathrm{Sq}^1(\tau) = \rho$. The computation of the cohomology of the $\mathbb{R}$-motivic Steenrod algebra $\mathcal{A}^{\mathbb{R}}$ becomes more difficult because the cohomology of a point is a nontrivial $\mathcal{A}^{\mathbb{R}}$-module. In addition, the $\mathbb{R}$-motivic Steenrod algebra $\mathcal{A}^{\mathbb{R}}$ has additional complications associated with terms involving higher powers of ρ [Voevodsky 2003b, Theorem 12.6].

A natural way to avoid this problem is to filter by powers of ρ. In the associated graded object, $\mathrm{Sq}^1(\tau)$ becomes zero and the associated graded Hopf algebroid is simply the $\mathbb{C}$-motivic Hopf algebra with an adjoined polynomial generator ρ. Therefore, the ρ-Bockstein spectral sequence starts from the cohomology of $\mathcal{A}^{\mathbb{C}}$ and converges to the cohomology of $\mathcal{A}^{\mathbb{R}}$.

This ρ-Bockstein spectral sequence has lots of differentials and hidden extensions. Nevertheless, a complete calculation for $\mathcal{A}^{\mathbb{R}}(1)$ is reasonable. A key point is to first carry out the ρ-inverted calculation. This turns out to be much simpler. With a priori knowledge of the ρ-inverted calculation in hand, there is just one possible pattern of ρ-Bockstein differentials.

Relying on our experience from the $\mathbb{R}$-motivic situation, we are now ready to tackle the C_2-equivariant situation. The C_2-equivariant cohomology of a point contains $\mathbb{F}_2[\tau, \rho]$, but there is an additional "negative cone" that is infinitely divisible by both τ and ρ [Hu and Kriz 2001, Proposition 6.2]. Except for the complications in the cohomology of a point, the C_2-equivariant Steenrod algebra $\mathcal{A}^{C_2}$ is no more complicated than the $\mathbb{R}$-motivic one [Hu and Kriz 2001, pp. 386–387].

Again, a ρ-Bockstein spectral sequence allows us to compute the cohomology of $\mathcal{A}^{C_2}(1)$. Because of infinite τ-divisibility, the starting point of the spectral sequence is more complicated than just the cohomology of $\mathcal{A}^{\mathbb{C}}(1)$. Once identified, this issue presents only a minor difficulty.

The ρ-inverted calculation determines the part of the cohomology of $\mathcal{A}^{C_2}(1)$ that supports infinitely many ρ multiplications. Dually, it is also helpful to determine in advance the part of the cohomology of $\mathcal{A}^{C_2}(1)$ that is infinitely ρ-divisible, i.e., the inverse limit of an infinite tower of ρ-multiplications. We anticipate that this approach via infinitely ρ-divisible classes will be essential in the more complicated calculation over the full Steenrod algebra $\mathcal{A}^{C_2}$, to be studied in future work.

As for the $\mathbb{R}$-motivic case, the ρ-Bockstein spectral sequence is manageable, even though it does have lots of differentials and hidden extensions.

All of these calculations lead to a thorough understanding of the cohomology of $\mathcal{A}^{C_2}(1)$. The charts in Section 12 display the calculation graphically.

The next step is to consider the C_2-equivariant Adams spectral sequence. For degree reasons, there are no nonzero Adams differentials. The same simple situation occurs in the classical, $\mathbb{C}$-motivic, and $\mathbb{R}$-motivic cases.

However, it turns out that there are many hidden extensions to be analyzed. The presence of so many hidden extensions suggests that the Adams filtration may not be optimal for equivariant purposes. Unfortunately, we do not have an alternative to propose.

The final description of the homotopy groups is complicated. Nevertheless, our computation establishes that the homotopy of ko_{C_2} is nearly periodic (see Theorem 11.15). We refer to Section 11 and the charts in Section 12 for details.

1A. *Organization.* In Section 2, we provide the basic algebraic input to our calculation by thoroughly describing the C_2-equivariant cohomology of a point and the C_2-equivariant Steenrod algebra $\mathcal{A}^{C_2}$. In Section 3, we set up the ρ-Bockstein spectral sequence, which is our main tool for computing the cohomology of $\mathcal{A}^{C_2}(1)$. In Sections 4 and 5, we carry out the ρ-inverted and the infinitely ρ-divisible calculations. In Section 6, we carry out the $\mathbb{R}$-motivic ρ-Bockstein spectral sequence as a warmup for the C_2-equivariant ρ-Bockstein spectral sequence in Section 7. Section 8 provides some information about Massey products in the C_2-equivariant cohomology of $\mathcal{A}(1)$, which is used in Section 9 to determine multiplicative structure that is hidden by the ρ-Bockstein spectral sequence. Section 10 gives the construction of the C_2-equivariant spectrum whose homotopy groups are computed by the cohomology of $\mathcal{A}^{C_2}(1)$, and Section 11 analyzes multiplicative structure in these homotopy groups that is hidden by the Adams spectral sequence. Finally, Section 12 includes a series of charts that graphically describe our calculation.

1B. *Notation.* We employ notation as follows:

(1) $\mathbb{M}_2^{\mathbb{C}} = \mathbb{F}_2[\tau]$ is the motivic cohomology of $\mathbb{C}$ with $\mathbb{F}_2$ coefficients, where τ has bidegree $(0, 1)$.

(2) $\mathbb{M}_2^{\mathbb{R}} = \mathbb{F}_2[\tau, \rho]$ is the motivic cohomology of $\mathbb{R}$ with $\mathbb{F}_2$ coefficients, where τ and ρ have bidegrees $(0, 1)$ and $(1, 1)$, respectively.

(3) $\mathbb{M}_2^{C_2}$ is the bigraded equivariant cohomology of a point with coefficients in the constant Mackey functor $\underline{\mathbb{F}}_2$. See Section 2A for a description of this algebra.

(4) NC is the "negative cone" part of $\mathbb{M}_2^{C_2}$. See Section 2A for a precise description.

(5) $H_{C_2}^{*,*}(X)$ is the C_2-equivariant cohomology of X, with coefficients in the constant Mackey functor $\underline{\mathbb{F}}_2$.

(6) $\mathcal{A}^{\mathrm{cl}}$, $\mathcal{A}^{\mathbb{C}}$, $\mathcal{A}^{\mathbb{R}}$, and $\mathcal{A}^{C_2}$ are the classical, $\mathbb{C}$-motivic, $\mathbb{R}$-motivic, and C_2-equivariant mod 2 Steenrod algebras.

(7) $\mathcal{A}^{\mathrm{cl}}(n)$, $\mathcal{A}^{\mathbb{C}}(n)$, $\mathcal{A}^{\mathbb{R}}(n)$, and $\mathcal{A}^{C_2}(n)$ are the classical, $\mathbb{C}$-motivic, $\mathbb{R}$-motivic, and C_2-equivariant subalgebras generated by $\mathrm{Sq}^1, \mathrm{Sq}^2, \mathrm{Sq}^4, \ldots, \mathrm{Sq}^{2^n}$.

(8) $\mathcal{E}^{C_2}(1)$ is the subalgebra of $\mathcal{A}^{C_2}$ generated by

$$Q_0 = \mathrm{Sq}^1 \quad \text{and} \quad Q_1 = \mathrm{Sq}^1\mathrm{Sq}^2 + \mathrm{Sq}^2\mathrm{Sq}^1.$$

(9) $\mathrm{Ext}_{\mathrm{cl}}$ is the bigraded ring $\mathrm{Ext}_{\mathcal{A}^{\mathrm{cl}}}(\mathbb{F}_2, \mathbb{F}_2)$, i.e., the cohomology of $\mathcal{A}^{\mathrm{cl}}$.

(10) $\mathrm{Ext}_{\mathbb{C}}$ is the trigraded ring $\mathrm{Ext}_{\mathcal{A}^{\mathbb{C}}}(\mathbb{M}_2^{\mathbb{C}}, \mathbb{M}_2^{\mathbb{C}})$, i.e., the cohomology of $\mathcal{A}^{\mathbb{C}}$.

(11) $\mathrm{Ext}_{\mathbb{R}}$ is the trigraded ring $\mathrm{Ext}_{\mathcal{A}^{\mathbb{R}}}(\mathbb{M}_2^{\mathbb{R}}, \mathbb{M}_2^{\mathbb{R}})$, i.e., the cohomology of $\mathcal{A}^{\mathbb{R}}$.

(12) Ext_{C_2} is the trigraded ring $\mathrm{Ext}_{\mathcal{A}^{C_2}}(\mathbb{M}_2^{C_2}, \mathbb{M}_2^{C_2})$, i.e., the cohomology of $\mathcal{A}^{C_2}$.

(13) $\mathrm{Ext}_{\mathrm{NC}}$ is the $\mathrm{Ext}_{\mathcal{A}^{\mathbb{R}}}$-module $\mathrm{Ext}_{\mathcal{A}^{\mathbb{R}}}(\mathrm{NC}, \mathbb{M}_2^{\mathbb{R}})$.

(14) $\mathrm{Ext}_{\mathrm{cl}}(n)$ is the bigraded ring $\mathrm{Ext}_{\mathcal{A}^{\mathrm{cl}}(n)}(\mathbb{F}_2, \mathbb{F}_2)$, i.e., the cohomology of $\mathcal{A}^{\mathrm{cl}}(n)$.

(15) $\mathrm{Ext}_{\mathbb{C}}(n)$ is the trigraded ring $\mathrm{Ext}_{\mathcal{A}^{\mathbb{C}}(n)}(\mathbb{M}_2^{\mathbb{C}}, \mathbb{M}_2^{\mathbb{C}})$, i.e., the cohomology of $\mathcal{A}^{\mathbb{C}}(n)$.

(16) $\mathrm{Ext}_{\mathbb{R}}(n)$ is the trigraded ring $\mathrm{Ext}_{\mathcal{A}^{\mathbb{R}}(n)}(\mathbb{M}_2^{\mathbb{R}}, \mathbb{M}_2^{\mathbb{R}})$, i.e., the cohomology of $\mathcal{A}^{\mathbb{R}}(n)$.

(17) $\mathrm{Ext}_{C_2}(n)$ is the trigraded ring $\mathrm{Ext}_{\mathcal{A}^{C_2}(n)}(\mathbb{M}_2^{C_2}, \mathbb{M}_2^{C_2})$, i.e., the cohomology of $\mathcal{A}^{C_2}(n)$.

(18) $\mathrm{Ext}_{\mathrm{NC}}(n)$ is the $\mathrm{Ext}_{\mathbb{R}}(n)$-module $\mathrm{Ext}_{\mathcal{A}^{\mathbb{R}}(n)}(\mathrm{NC}, \mathbb{M}_2^{\mathbb{R}})$.

(19) E^+ is the ρ-Bockstein spectral sequence

$$\mathrm{Ext}_{\mathbb{C}}(1)[\rho] \Rightarrow \mathrm{Ext}_{\mathbb{R}}(1).$$

See Section 3.

(20) E^- is the ρ-Bockstein spectral sequence that converges to $\mathrm{Ext}_{\mathrm{NC}}(1)$. See Section 3.

(21) $\frac{\mathbb{F}_2[x]}{x^\infty}\{y\}$ is the infinitely x-divisible module $\mathrm{colim}_n\, \mathbb{F}_2[x]/x^n$, consisting of elements of the form $\frac{y}{x^k}$ for $k \geq 1$. See Remark 2.1.

(22) ko_{C_2} is a C_2-equivariant spectrum such that $H^{*,*}_{C_2}(\mathrm{ko}_{C_2}) \cong \mathcal{A}^{C_2}/\!/\mathcal{A}^{C_2}(1)$. See Section 10.

(23) $\pi_{*,*}(X)$ are the bigraded C_2-equivariant stable homotopy groups of X, completed at 2 so that the equivariant Adams spectral sequence converges.

(24) $\Pi_n(X)$ is the Milnor–Witt n-stem $\bigoplus_p \pi_{p+n,p}$.

We use grading conventions that are common in motivic homotopy theory but less common in equivariant homotopy theory. In equivariant homotopy theory, $\mathrm{RO}(C_2) \cong \mathbb{Z}[\sigma]/(\sigma^2 - 1)$ is the real representation ring of C_2, where σ is the 1-dimensional sign representation. The main points of translation are:

(1) Equivariant degree $p + q\sigma$ will be expressed, according to the motivic convention, as $(p+q, q)$, where $p+q$ is the total degree and q is the weight.

(2) The element τ in $\mathbb{M}_2^{\mathbb{R}}$ maps to u [Hill et al. 2016, Definition 3.12] under the realization map from $\mathbb{R}$-motivic to C_2-equivariant homotopy theory. We use the symbol τ in both cases.

(3) Similarly, realization takes the $\mathbb{R}$-motivic homotopy class $\rho : S^{-1,-1} \to S^{0,0}$ to a in $\pi_{-1,-1}$ [Hill et al. 2016, Definition 3.11]. We use the symbol ρ for both of these homotopy classes, and also for the corresponding elements of $\mathbb{M}_2^{\mathbb{R}}$ and $\mathbb{M}_2^{C_2}$.

We grade Ext groups in the form (s, f, w), where s is the stem, i.e., the total degree minus the homological degree; f is the Adams filtration, i.e., the homological degree; and w is the weight. We will also refer to the Milnor–Witt degree, which equals $s - w$.

2. Ext groups

2A. ***The equivariant cohomology of a point.*** The purpose of this section is to carefully describe the structure of the equivariant cohomology ring $\mathbb{M}_2^{C_2}$ of a point from a perspective that will be useful for our calculations. This section is a reinterpretation of results from [Hu and Kriz 2001, Proposition 6.2].

Additively, $\mathbb{M}_2^{C_2}$ equals

(1) $\mathbb{F}_2$ in degree (s, w) if $s \geq 0$ and $w \geq s$,

(2) $\mathbb{F}_2$ in degree (s, w) if $s \leq 0$ and $w \leq s - 2$,

(3) 0 otherwise.

This additive structure is represented by the dots in Figure 1. The nonzero element in degree $(0, 1)$ is called τ, and the nonzero element in degree $(1, 1)$ is called ρ. We remind the reader that we are here employing cohomological grading. Thus the class ρ has degree $(-1, -1)$ when considered as an element of the homology ring $\pi_{*,*} H\underline{\mathbb{F}}_2$.

The "positive cone" refers to the part of $\mathbb{M}_2^{C_2}$ in degrees (s, w) with $w \geq 0$. The positive cone is isomorphic to the $\mathbb{R}$-motivic cohomology ring $\mathbb{M}_2^{\mathbb{R}}$ of a point. Multiplicatively, the positive cone is just a polynomial ring on two variables, ρ and τ.

The "negative cone" NC refers to the part of $\mathbb{M}_2^{C_2}$ in degrees (s, w) with $w \leq -2$. Multiplicatively, the product of any two elements of NC is zero, so $\mathbb{M}_2^{C_2}$ is a square-zero extension of $\mathbb{M}_2^{\mathbb{R}}$. Also, multiplications by ρ and τ are nonzero in NC whenever they make sense. Thus, the elements of NC are infinitely divisible by both ρ and τ.

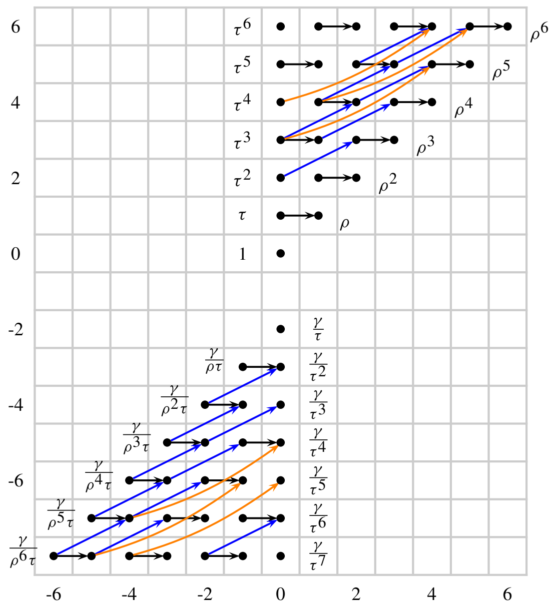


Figure 1. $\mathbb{M}_2^{C_2}$, with action by Sq^1, Sq^2, and Sq^4.

We use the notation $\frac{\gamma}{\rho^j\tau^k}$ for the nonzero element in degree $(-j, -1-j-k)$. This is consistent with the multiplicative properties described in the previous paragraph. So $\tau \cdot \frac{\gamma}{\rho^j\tau^k}$ equals $\frac{\gamma}{\rho^j\tau^{k-1}}$ when $k \geq 2$, and $\rho \cdot \frac{\gamma}{\rho^j\tau^k}$ equals $\frac{\gamma}{\rho^{j-1}\tau^k}$ when $j \geq 2$.

The symbol γ, which does not correspond to an actual element of $\mathbb{M}_2^{C_2}$, has degree $(0, -1)$.

The $\mathbb{F}_2[\tau]$-module structure on $\mathbb{M}_2^{C_2}$ is essential for later calculations, since we will filter by powers of ρ. Therefore, we explore further the $\mathbb{F}_2[\tau]$-module structure on NC.

Remark 2.1. Recall that $\mathbb{F}_2[\tau]/\tau^\infty$ is the $\mathbb{F}_2[\tau]$-module $\operatorname{colim} \mathbb{F}_2[\tau]/\tau^k$, which consists entirely of elements that are divisible by τ. We write $\frac{\mathbb{F}_2[\tau]}{\tau^\infty}\{x\}$ for the infinitely divisible $\mathbb{F}_2[\tau]$-module consisting of elements of the form $\frac{x}{\tau^k}$ for $k \geq 1$. Note that x itself is not an element of $\frac{\mathbb{F}_2[\tau]}{\tau^\infty}\{x\}$. The idea is that x represents the infinitely many relations $\tau^k \cdot \frac{x}{\tau^k} = 0$ that define $\frac{\mathbb{F}_2[\tau]}{\tau^\infty}\{x\}$.

With this notation in place, $\mathbb{M}_2^{C_2}$ is equal to

$$\mathbb{M}_2^{\mathbb{R}} \oplus \mathrm{NC} = \mathbb{M}_2^{\mathbb{R}} \oplus \bigoplus_{s \geq 0} \frac{\mathbb{F}_2[\tau]}{\tau^\infty} \left\{ \frac{\gamma}{\rho^s} \right\} \tag{2-1}$$

as an $\mathbb{F}_2[\tau]$-module.

2B. ***The equivariant Steenrod algebra.*** As a Hopf algebroid, the equivariant dual Steenrod algebra can be described [Ricka 2015, Proposition 6.10(2)] as

$$\mathcal{A}_*^{C_2} \cong \mathbb{M}_2^{C_2} \otimes_{\mathbb{M}_2^{\mathbb{R}}} \mathcal{A}_*^{\mathbb{R}}. \tag{2-2}$$

Recall [Voevodsky 2003b] that

$$\mathcal{A}_*^{\mathbb{R}} \cong \mathbb{M}_2^{\mathbb{R}}[\tau_0, \tau_1, \dots, \xi_1, \xi_2, \dots]/(\tau_i^2 = \rho\tau_{i+1} + \tau\xi_{i+1} + \rho\tau_0\xi_{i+1}),$$

with $\eta_R(\rho) = \rho$ and $\eta_R(\tau) = \tau + \rho\tau_0$. The formula for the right unit η_R on the negative cone given in [Hu and Kriz 2001, Theorem 6.41] appears in our notation as

$$\eta_R\left(\frac{\gamma}{\rho^j\tau^k}\right) = \frac{\gamma}{\rho^j\tau^k}\left[\sum_{i\geq 0}\left(\frac{\rho}{\tau}\tau_0\right)^i\right]^k. \tag{2-3}$$

Note that the sum is finite because $\frac{\gamma}{\rho^j\tau^k} \cdot \rho^n = 0$ if $n > j$.

We have quotient Hopf algebroids

$$\mathcal{A}_*^{\mathbb{R}}(n) := \mathbb{M}_2^{\mathbb{R}}[\tau_0, \dots, \tau_n, \xi_1, \dots, \xi_n]/(\xi_i^{2^{n-i+1}}, \tau_i^2 = \rho\tau_{i+1} + \tau\xi_{i+1} + \rho\tau_0\xi_{i+1}).$$

and

$$\mathcal{E}_*^{\mathbb{R}}(n) := \mathbb{M}_2^{\mathbb{R}}[\tau_0, \dots, \tau_n]/(\tau_i^2 = \rho\tau_{i+1}, \tau_n^2)$$

and their equivariant analogues

$$\mathcal{A}_*^{C_2}(n) := \mathbb{M}^{C_2} \otimes_{\mathbb{M}^{\mathbb{R}}} \mathcal{A}_*^{\mathbb{R}}(n), \qquad \mathcal{E}_*^{C_2}(n) := \mathbb{M}^{C_2} \otimes_{\mathbb{M}^{R}} E_*^{\mathbb{R}}(n) \tag{2-4}$$

Their duals are the subalgebras $\mathcal{A}^{C_2}(n) \subseteq \mathcal{A}^{C_2}$ and $\mathcal{E}^{C_2}(n) \subseteq \mathcal{A}^{C_2}$.

The relationship between the equivariant and $\mathbb{R}$-motivic Steenrod algebras leads to an analogous relationship between Ext groups.

Proposition 2.2. *Suppose that Γ is a Hopf algebroid over A and that $B \cong A \oplus M$ is a Γ-comodule which is a square-zero extension of A, meaning that the product of any two elements in M is zero. Then the A-module splitting of B induces a splitting*

$$\mathrm{Ext}_{B\otimes_A\Gamma}(B, B) \cong \mathrm{Ext}_\Gamma(A, A) \oplus \mathrm{Ext}_\Gamma(M, A)$$

of $\mathrm{Ext}_\Gamma(A, A)$*-modules. Furthermore, this is an isomorphism of* $\mathrm{Ext}_\Gamma(A, A)$*-algebras, if the right-hand side is taken to be a square-zero extension of* $\mathrm{Ext}_\Gamma(A, A)$.

Proof. We may express the cobar complex as:

$$\begin{aligned}\mathrm{coB}^s(B, B\otimes_A\Gamma) = B\otimes_B(\Gamma)^{\otimes s} &\cong B\otimes_B(B\otimes_A\Gamma)^{\otimes s}\\ &\cong B\otimes_A(\Gamma)^{\otimes s}.\end{aligned}$$

As the splitting of B is a splitting as Γ-comodules, there results a splitting

$$\mathrm{coB}^s(A, \Gamma) \oplus \mathrm{coB}^s(M, \Gamma)$$

of the cobar complex. This splitting is square-zero, in the sense that the product of any two elements in the second factor is equal to zero. This observation follows from the fact that the product of any two elements of M is zero.

In $\mathrm{Ext}_{B\otimes_A \Gamma}$, this yields

$$\mathrm{Ext}_{B\otimes_A \Gamma} \cong \mathrm{Ext}_\Gamma(A, A) \oplus \mathrm{Ext}_\Gamma(M, A).$$

The multiplication on $\mathrm{Ext}_\Gamma(M, A)$ is zero since this is already true in the cobar complex $\mathrm{coB}^s(M, \Gamma)$. □

Employing notation given in Section 1B, Proposition 2.2 applies to give isomorphisms

$$\mathrm{Ext}_{C_2} \cong \mathrm{Ext}_{\mathbb{R}} \oplus \mathrm{Ext}_{\mathrm{NC}}$$

and

$$\mathrm{Ext}_{C_2(n)} \cong \mathrm{Ext}_{\mathbb{R}(n)} \oplus \mathrm{Ext}_{\mathrm{NC}(n)} .$$

Thus from the point of view of $\mathbb{R}$-motivic homotopy theory, the cohomology of the negative cone is the only new feature in $\mathrm{Ext}_{\mathcal{A}^{C_2}}$ or $\mathrm{Ext}_{\mathcal{A}^{C_2}(n)}$.

3. The ρ-Bockstein spectral sequence

Our tool for computing $\mathbb{R}$-motivic or C_2-equivariant Ext is the ρ-Bockstein spectral sequence [Hill 2011; Dugger and Isaksen 2017a]. The ρ-Bockstein spectral sequence arises by filtering the cobar complex by powers of ρ. More precisely, we can define an $\mathcal{A}^{\mathbb{R}}$-module filtration on $\mathbb{M}_2^{C_2}$, where $F_p(\mathbb{M}_2^{C_2})$ is the part of $\mathbb{M}_2^{C_2}$ concentrated in degrees (s, w) with $s \geq p$. Dualizing, we get a filtration of comodules over the dual Steenrod algebra, which induces a filtration on the cobar complex that computes Ext_{C_2}.

Recall that the $\mathbb{C}$-motivic cohomology of a point is $\mathbb{M}_2^{\mathbb{C}} = \mathbb{F}_2[\tau]$, and the $\mathbb{C}$-motivic Steenrod algebra is $\mathcal{A}^{\mathbb{C}} = \mathcal{A}^{\mathbb{R}}/\rho$ [Voevodsky 2003a; 2003b]. For convenience, we write $\mathrm{Ext}_{\mathbb{C}}$ for $\mathrm{Ext}_{\mathcal{A}^{\mathbb{C}}}(\mathbb{M}_2^{\mathbb{C}}, \mathbb{M}_2^{\mathbb{C}})$.

Proposition 3.1. *There is a ρ-Bockstein spectral sequence*

$$E_1 = \mathrm{Ext}_{\mathrm{gr}_\rho \mathcal{A}^{C_2}}(\mathrm{gr}_\rho \mathbb{M}_2^{C_2}, \mathrm{gr}_\rho \mathbb{M}_2^{C_2}) \Rightarrow \mathrm{Ext}_{C_2}$$

such that a Bockstein differential d_r takes a class x of degree (s, f, w) to a class $d_r(x)$ of degree $(s-1, f+1, w)$. Under the splitting of Proposition 2.2, this decomposes as

$$E_1^+ = \mathrm{Ext}_{\mathbb{C}}[\rho] \Rightarrow \mathrm{Ext}_{\mathbb{R}}$$

and

$$E_1^- \Rightarrow \mathrm{Ext}_{\mathrm{NC}},$$

where E_1^- belongs to a split short exact sequence

$$\bigoplus_{s\geq 0} \frac{\mathbb{M}_2^{\mathbb{C}}}{\tau^\infty}\left\{\frac{\gamma}{\rho^s}\right\} \otimes_{\mathbb{M}_2^{\mathbb{C}}} \mathrm{Ext}_{\mathbb{C}} \to E_1^- \to \bigoplus_{s\geq 0} \mathrm{Tor}_{\mathbb{M}_2^{\mathbb{C}}}\left(\frac{\mathbb{M}_2^{\mathbb{C}}}{\tau^\infty}\left\{\frac{\gamma}{\rho^s}\right\}, \mathrm{Ext}_{\mathbb{C}}\right).$$

Remark 3.2. Beware that the short exact sequence for E_1^- does not split canonically.

Remark 3.3. The same spectral sequences occur in the same form when $\mathcal{A}^{C_2}$, $\mathcal{A}^{\mathbb{R}}$, and $\mathcal{A}^{\mathbb{C}}$ are replaced by $\mathcal{A}^{C_2}(n)$, $\mathcal{A}^{\mathbb{R}}(n)$, and $\mathcal{A}^{\mathbb{C}}(n)$.

Proof. See [Hill 2011, Proposition 2.3] (or [Dugger and Isaksen 2017a, Section 3]) for the description of E_1^+.

For E_1^-, the associated graded of NC is

$$\mathrm{gr}_\rho \mathrm{NC} \cong \bigoplus_{s\geq 0} \frac{\mathbb{M}_2^{\mathbb{C}}}{\tau^\infty}\left\{\frac{\gamma}{\rho^s}\right\},$$

as described in Section 2A. It follows that the Bockstein spectral sequence begins with

$$E_0 \cong \bigoplus_{s\geq 0} \frac{\mathbb{M}_2^{\mathbb{C}}}{\tau^\infty}\left\{\frac{\gamma}{\rho^s}\right\} \otimes_{\mathbb{M}_2^{\mathbb{C}}} \mathrm{coB}(\mathbb{M}_2^{\mathbb{C}}, \mathcal{A}_*^{\mathbb{C}}).$$

The ring $\mathbb{M}_2^{\mathbb{C}} \cong \mathbb{F}_2[\tau]$ is a graded principal ideal domain (in fact, it is a graded local ring with maximal ideal generated by τ). Therefore, the Künneth split exact sequence gives

$$\left(\bigoplus_{s\geq 0} \frac{\mathbb{M}_2^{\mathbb{C}}}{\tau^\infty}\left\{\frac{\gamma}{\rho^s}\right\}\right) \otimes_{\mathbb{M}_2^{\mathbb{C}}} \mathrm{Ext}_{\mathbb{C}} \to E_1^- \to \mathrm{Tor}_{\mathbb{M}_2^{\mathbb{C}}}\left(\bigoplus_{s\geq 0} \frac{\mathbb{M}_2^{\mathbb{C}}}{\tau^\infty}\left\{\frac{\gamma}{\rho^s}\right\}, \mathrm{Ext}_{\mathbb{C}}\right).$$

The first and third terms of the short exact sequence may be rewritten as in the statement of the proposition because the direct sum in each case is a splitting of $\mathbb{M}_2^{\mathbb{C}}$-modules. □

We shall completely analyze the spectral sequence

$$E_1^+ = \mathrm{Ext}_{\mathbb{C}}(1)[\rho] \Rightarrow \mathrm{Ext}_{\mathbb{R}}(1)$$

in Section 6. While nontrivial, this part of our calculation is comparatively straightforward.

On the other hand, analysis of the spectral sequence

$$E_1^- \Rightarrow \mathrm{Ext}_{\mathrm{NC}}(1)$$

requires significantly more work. The first step is to compute E_1^- more explicitly. In particular, we must describe the Tor groups that arise.

Lemma 3.4. (1) $\mathrm{Tor}^*_{\mathbb{M}_2^{\mathbb{C}}}\big(\frac{\mathbb{M}_2^{\mathbb{C}}}{\tau^\infty}, \mathbb{M}_2^{\mathbb{C}}\big)$ *equals* $\frac{\mathbb{M}_2^{\mathbb{C}}}{\tau^\infty}$, *concentrated in homological degree zero.*

(2) $\mathrm{Tor}^*_{\mathbb{M}_2^{\mathbb{C}}}\big(\frac{\mathbb{M}_2^{\mathbb{C}}}{\tau^\infty}, \frac{\mathbb{M}_2^{\mathbb{C}}}{\tau^k}\big)$ *equals* $\frac{\mathbb{M}_2^{\mathbb{C}}}{\tau^k}$, *concentrated in homological degree one.*

Proof. (1) This is a standard fact about the vanishing of higher Tor groups for free modules.

(2) This follows from direct computation, using the resolution

$$\frac{\mathbb{M}_2^{\mathbb{C}}}{\tau^k} \longleftarrow \mathbb{M}_2^{\mathbb{C}} \xleftarrow{\tau^k} \mathbb{M}_2^{\mathbb{C}} \longleftarrow 0.$$

After tensoring with $\frac{\mathbb{M}_2^{\mathbb{C}}}{\tau^\infty}$, this gives the map

$$\frac{\mathbb{M}_2^{\mathbb{C}}}{\tau^\infty}\{x\} \longleftarrow \frac{\mathbb{M}_2^{\mathbb{C}}}{\tau^\infty}\{y\}$$

that takes $\frac{y}{\tau^a}$ to $\frac{x}{\tau^{a-k}}$ if $a > k$, and takes $\frac{y}{\tau^a}$ to zero if $a \le k$. This map is onto, and its kernel is isomorphic to $\mathbb{M}_2^{\mathbb{C}}/\tau^k$. □

Remark 3.5. Lemma 3.4 provides a practical method for identifying the E_1^- in Proposition 3.1. Copies of $\mathbb{M}_2^{\mathbb{C}}$ in $\mathrm{Ext}_{\mathbb{C}}(1)$ lead to copies of the negative cone in E_1^-. On the other hand, copies of $\mathbb{M}_2^{\mathbb{C}}/\tau$, such as the submodule generated by h_1^3, lead to copies of $\mathbb{M}_2^{\mathbb{C}}/\tau$ in E_1^- that are infinitely divisible by ρ. These copies of $\mathbb{M}_2^{\mathbb{C}}/\tau$ occur with a degree shift because they arise from Tor^1.

4. ρ-inverted $\mathrm{Ext}_{\mathbb{R}}(1)$

As a first step towards computing $\mathrm{Ext}_{C_2}(1)$, we will invert ρ in the $\mathbb{R}$-motivic setting and study $\mathrm{Ext}_{\mathbb{R}}(1)[\rho^{-1}]$. This gives partial information about $\mathrm{Ext}_{\mathbb{R}}(1)$ and also about $\mathrm{Ext}_{C_2}(1)$. Afterwards, it remains to compute ρ^k torsion, including infinitely ρ-divisible elements.

We write $\mathcal{A}^{\mathrm{cl}}$ for the classical Steenrod algebra. For convenience, we write $\mathrm{Ext}_{\mathrm{cl}}$ and $\mathrm{Ext}_{\mathrm{cl}}(n)$ for $\mathrm{Ext}_{\mathcal{A}^{\mathrm{cl}}}(\mathbb{F}_2, \mathbb{F}_2)$ and $\mathrm{Ext}_{\mathcal{A}^{\mathrm{cl}}(n)}(\mathbb{F}_2, \mathbb{F}_2)$ respectively.

Proposition 4.1. *There is an injection* $\mathrm{Ext}_{\mathrm{cl}}(n-1)[\rho^{\pm 1}] \hookrightarrow \mathrm{Ext}_{\mathbb{R}}(n)[\rho^{-1}]$ *such that*:

(1) *The map is highly structured*, *i.e.*, *preserves products*, *Massey products*, *and algebraic squaring operations.*

(2) *The element h_i of* $\mathrm{Ext}_{\mathrm{cl}}(n-1)$ *corresponds to h_{i+1} of* $\mathrm{Ext}_{\mathbb{R}}(n)$.

(3) *The map induces an isomorphism*

$$\mathrm{Ext}_{\mathbb{R}}(n)[\rho^{-1}] \cong \mathrm{Ext}_{\mathrm{cl}}(n-1)[\rho^{\pm 1}] \otimes \mathbb{F}_2[\tau^{2^{n+1}}].$$

(4) *An element in* $\mathrm{Ext}_{\mathrm{cl}}(n-1)$ *of degree* (s, f) *corresponds to an element in* $\mathrm{Ext}_{\mathbb{R}}(n)$ *of degree* $(2s+f, f, s+f)$.

Proof. The proof is similar to the proof of [Dugger and Isaksen 2017a, Theorem 4.1]. Since localization is an exact functor, we may compute the cohomology of the Hopf algebroid $(\mathbb{M}_2^{\mathbb{R}}[\rho^{-1}], \mathcal{A}^{\mathbb{R}}(n+1)_*[\rho^{-1}])$ to obtain $\mathrm{Ext}_{\mathbb{R}}(n+1)[\rho^{-1}]$. After inverting ρ, we have

$$\tau_{k+1} = \rho^{-1}\tau_k^2 + \rho^{-1}\tau\xi_{k+1} + \tau_0\xi_{k+1},$$

and it follows that

$$\mathcal{A}^{\mathbb{R}}(n)_*[\rho^{-1}] \cong \mathbb{M}_2^{\mathbb{R}}[\rho^{-1}][\tau_0, \xi_1, \dots, \xi_n]/(\tau_0^{2^{n+1}}, \xi_1^{2^n}, \dots, \xi_n^2).$$

This splits as

$$\bigl(\mathbb{M}_2^{\mathbb{R}}[\rho^{-1}], \mathcal{A}(n)_*[\rho^{-1}]\bigr) \cong \bigl(\mathbb{M}_2^{\mathbb{R}}[\rho^{-1}], \mathcal{A}'(n)\bigr) \otimes_{\mathbb{F}_2} (\mathbb{F}_2, \mathcal{A}''(n)),$$

where

$$\mathcal{A}'(n) = \mathbb{M}_2^{\mathbb{R}}[\rho^{-1}][\tau_0]/\tau_0^{2^{n+1}}$$

and

$$\mathcal{A}''(n) = \mathbb{F}_2[\xi_1, \dots, \xi_n]/(\xi_1^{2^n}, \dots, \xi_n^2).$$

Because it is isomorphic to the classical Hopf algebra $(\mathbb{F}_2, \mathcal{A}(n-1))$ with altered degrees, the Hopf algebra $(\mathbb{F}_2, \mathcal{A}''(n))$ has cohomology $\mathrm{Ext}_{\mathrm{cl}}(n-1)$.

For the Hopf algebroid $\bigl(\mathbb{M}_2^{\mathbb{R}}[\rho^{-1}], \mathcal{A}'(n)\bigr)$, we have an isomorphism

$$(\mathbb{M}_2^{\mathbb{R}}[\rho^{-1}], \mathcal{A}'(n)) \cong \mathbb{F}_2[\rho^{\pm 1}] \otimes_{\mathbb{F}_2} (\mathbb{F}_2[\tau], \mathbb{F}_2[\tau][x]/x^{2^{n+1}})$$

with

$$\eta_L(\tau) = \tau, \quad \eta_R(\tau) = \tau + x.$$

An argument like that of [Dugger and Isaksen 2017a, Lemma 4.2] shows that the cohomology of this Hopf algebroid is $\mathbb{F}_2[\tau^{2^{n+1}}]$. $\square$

Corollary 4.2. $\mathrm{Ext}_{C_2}(1)[\rho^{-1}] \cong \mathrm{Ext}_{\mathbb{R}}(1)[\rho^{-1}] \cong \mathbb{F}_2[\rho^{\pm 1}, \tau^4, h_1]$.

Proof. The first isomorphism follows from Proposition 2.2, as $\mathrm{Ext}_{\mathrm{NC}}$ is ρ-torsion. The second isomorphism follows immediately from Proposition 4.1, given that $\mathrm{Ext}_{\mathrm{cl}}(0) \cong \mathbb{F}_2[h_0]$. $\square$

Remark 4.3. Corollary 4.2 implies that the products $\tau^4 \cdot h_1^k$ are nonzero in $\mathrm{Ext}_{\mathbb{R}}(1)$. But $\tau^4 h_1^k = 0$ in $\mathrm{Ext}_{\mathbb{C}}(1)$ when $k \geq 3$, so the products $\tau^4 \cdot h_1^k$ are hidden in the ρ-Bockstein spectral sequence for $k \geq 3$. We will sort this out in detail in Section 6.

5. Infinitely ρ-divisible elements of $\mathrm{Ext}_{\mathcal{A}^{C_2}(1)}$

Having computed the effect of inverting ρ in Section 4, we now consider the dual question and study infinitely ρ-divisible elements. This gives additional partial information about $\mathrm{Ext}_{C_2}(1)$. Afterwards, it remains only to compute the ρ^k torsion classes that are not infinitely ρ-divisible.

In fact, this section is not strictly necessary to carry out the computation of $\mathrm{Ext}_{C_2}(1)$. Nevertheless, the infinitely ρ-divisible calculation works out rather nicely and provides some useful insight into the main computation. We also anticipate that this approach via infinitely ρ-divisible classes will be essential in the much more complicated calculation of Ext_{C_2}, to be studied in further work.

For a $\mathbb{F}_2[\rho]$-module M, the ρ-colocalization, or ρ-cellularization, is the limit $\lim_\rho M$ of the inverse system

$$\cdots \xrightarrow{\rho} M \xrightarrow{\rho} M.$$

While ρ-localization detects ρ-torsion-free elements, the ρ-colocalization detects infinitely ρ-divisible elements.

An alternative description is given by the isomorphism

$$\lim_\rho M \cong \mathrm{Hom}_{\mathbb{F}_2[\rho]}(\mathbb{F}_2[\rho^{\pm 1}], M)$$

because $\mathbb{F}_2[\rho^{\pm 1}]$ is isomorphic to $\mathrm{colim}_\rho\, \mathbb{F}_2[\rho]$. It follows that $\lim_\rho M$ is an $\mathbb{F}_2[\rho^{\pm 1}]$-module, and the functor $M \mapsto \lim_\rho M$ is right adjoint to the restriction

$$\mathrm{Mod}_{\mathbb{F}_2[\rho^{\pm 1}]} \to \mathrm{Mod}_{\mathbb{F}_2[\rho]}.$$

Lemma 5.1. (1) *Let M be a cyclic $\mathbb{F}_2[\rho]$-module $\mathbb{F}_2[\rho]$ or $\mathbb{F}_2[\rho]/\rho^k$. Then* $\lim_\rho M$ *is zero.*

(2) *Let M be the infinitely divisible $\mathbb{F}_2[\rho]$-module $\mathbb{F}_2[\rho]/\rho^\infty$. Then* $\lim_\rho M$ *is isomorphic to $\mathbb{F}_2[\rho^{\pm 1}]$.*

Proof. If M is cyclic, then no nonzero element is infinitely ρ-divisible, which implies the first statement. For the case $M = \mathbb{F}_2[\rho]/\rho^\infty$, a (homogeneous) element of the limit is either of the form

$$\left(\frac{1}{\rho^k}, \frac{1}{\rho^{k+1}}, \ldots\right)$$

or of the form

$$\left(0, \ldots, 0, 1, \frac{1}{\rho}, \frac{1}{\rho^2}, \ldots\right).$$

For $k \geq 0$, the isomorphism $\mathbb{F}_2[\rho^{\pm 1}] \to \lim_\rho M$ sends ρ^k to the tuple

$$\left(0, \ldots, 0, 1, \frac{1}{\rho}, \ldots\right)$$

having $k-1$ zeroes and sends $\frac{1}{\rho^k}$ to $\left(\frac{1}{\rho^k}, \frac{1}{\rho^{k+1}}, \ldots\right)$. $\square$

We will now compute the ρ-colocalization of $\mathrm{Ext}_{C_2(1)}$.

Proposition 5.2.

$$\lim_{\rho} \mathrm{Ext}_{C_2}(1) \cong \bigoplus_{k\geq 1} \mathbb{F}_2[\rho^{\pm 1}, h_1]\left\{\frac{\gamma}{\tau^{4k}}\right\} \cong \mathbb{F}_2[\rho^{\pm 1}, h_1] \otimes \frac{\mathbb{F}_2[\tau^4]}{\tau^{\infty}}\{\gamma\}.$$

Recall that γ itself is not an element of $\lim_\rho \mathrm{Ext}_{C_2}(1)$, as described in Remark 2.1. The main point of Proposition 5.2 is that the elements $\frac{\gamma}{\tau^{4k}} h_1^j$ are infinitely ρ-divisible classes in $\mathrm{Ext}_{C_2}(1)$, and there are no other infinitely ρ-divisible families in $\mathrm{Ext}_{C_2}(1)$.

Proof. Since the cobar complex $\mathrm{coB}^*(\mathbb{M}_2^{C_2}, A^{C_2}(1))$ is finite-dimensional in each tridegree, the inverse systems

$$\cdots \xrightarrow{\rho} \mathrm{coB}^*(\mathbb{M}_2^{C_2}, A^{C_2}(1)) \xrightarrow{\rho} \mathrm{coB}^*(\mathbb{M}_2^{C_2}, A^{C_2}(1))$$

and

$$\cdots \xrightarrow{\rho} \mathrm{Ext}_{C_2}(1) \xrightarrow{\rho} \mathrm{Ext}_{C_2}(1)$$

satisfy the Mittag-Leffler condition, so that [Weibel 1994, Theorem 3.5.8]

$$\lim_\rho \mathrm{Ext}_{C_2}(1) \cong \mathrm{H}^*\big(\lim_\rho \mathrm{coB}^*(\mathbb{M}_2^{C_2}, A^{C_2}(1))\big).$$

Now we compute

$$\begin{aligned}\lim_{\rho} \mathrm{coB}^s(\mathbb{M}_2^{C_2}, A^{C_2}(1)) &= \lim_{\rho}\big(\mathbb{M}_2^{C_2} \otimes_{\mathbb{M}_2^{C_2}} A^{C_2}(1)^{\otimes s}\big)\\ &\cong \lim_{\rho}\big(\mathbb{M}_2^{C_2} \otimes_{\mathbb{M}_2^{\mathbb{R}}} A^{\mathbb{R}}(1)^{\otimes s}\big).\end{aligned}$$

The splitting $\mathbb{M}_2^{C_2} = \mathbb{M}_2^{\mathbb{R}} \oplus \mathrm{NC}$ yields a splitting

$$\big(\mathbb{M}_2^{\mathbb{R}} \otimes_{\mathbb{M}_2^{\mathbb{R}}} \mathcal{A}^{\mathbb{R}}(1)^{\otimes s}\big) \oplus \big(\mathrm{NC} \otimes_{\mathbb{M}_2^{\mathbb{R}}} \mathcal{A}^{\mathbb{R}}(1)^{\otimes s}\big)$$

of $\mathbb{M}_2^{C_2} \otimes_{\mathbb{M}_2^{\mathbb{R}}} \mathcal{A}^{\mathbb{R}}(1)^{\otimes s}$ as an $\mathbb{F}_2[\rho]$-module. The first piece of the splitting contributes nothing to the ρ-colocalization by Lemma 5.1(1) because $\mathbb{M}_2^{\mathbb{R}}$ is free as an $\mathbb{F}_2[\rho]$-module.

On the other hand, the $\mathbb{F}_2[\rho]$-module NC is a direct sum of copies of $\mathbb{F}_2[\rho]/\rho^{\infty}$. By Lemma 5.1(2), we have that $\lim_\rho\big(\mathrm{NC} \otimes_{\mathbb{M}_2^{\mathbb{R}}} \mathcal{A}^{\mathbb{R}}(1)^{\otimes s}\big)$ is isomorphic to

$$\left(\frac{\mathbb{M}_2^{\mathbb{R}}[\rho^{-1}]}{\tau^{\infty}}\{\gamma\}\right) \otimes_{\mathbb{M}_2^{\mathbb{R}}} \mathcal{A}^{\mathbb{R}}(1)^{\otimes s}.$$

Now the argument of Proposition 4.1 provides a splitting

$$\begin{aligned}&\mathrm{coB}^*_{\mathbb{M}_2^{\mathbb{R}}}\left(\frac{\mathbb{M}_2^{\mathbb{R}}[\rho^{-1}]}{\tau^{\infty}}\{\gamma\}, A^{\mathbb{R}}(1)\right)\\ &\qquad\simeq \mathrm{coB}^*_{\mathbb{F}_2[\tau]}\left(\frac{\mathbb{F}_2[\tau]}{\tau^{\infty}}\{\gamma\}, \frac{\mathbb{F}_2[\tau, x]}{x^4}\right)[\rho^{\pm 1}] \otimes_{\mathbb{F}_2} \mathrm{coB}^*_{\mathbb{F}_2}(\mathbb{F}_2, \mathbb{F}_2[\xi_1]/\xi_1^2),\end{aligned}$$

where $x = \rho\tau_0$. The cohomology of the second factor is $\mathbb{F}_2[h_1]$.

It remains to show that the cohomology of

$$\mathrm{coB}^*_{\mathbb{F}_2[\tau]}\left(\frac{\mathbb{F}_2[\tau]}{\tau^\infty}\{\gamma\}, \frac{\mathbb{F}_2[\tau,x]}{x^4}\right)$$

is equal to $\frac{\mathbb{F}_2[\tau^4]}{\tau^\infty}\{\gamma\}$. As in Formula (2-3), the comodule structure on $\frac{\mathbb{F}_2[\tau]}{\tau^\infty}\{\gamma\}$ is given by

$$\eta_R\left(\frac{\gamma}{\tau^k}\right) = \frac{\gamma}{\tau^k}\left(1 + \frac{x}{\tau} + \frac{x^2}{\tau^2} + \frac{x^3}{\tau^3}\right)^k.$$

Now we filter $\mathrm{coB}^*_{\mathbb{F}_2[\tau]}\big(\frac{\mathbb{F}_2[\tau]}{\tau^\infty}\{\gamma\}, \frac{\mathbb{F}_2[\tau,x]}{x^4}\big)$ by powers of x. We then have

$$E_1 \cong \frac{\mathbb{F}_2[\tau]}{\tau^\infty}\{\gamma\} \otimes_{\mathbb{F}_2} \mathbb{F}_2[v_0, v_1],$$

where $v_0 = [x]$ and $v_1 = [x^2]$. The differential

$$d_1\left(\frac{\gamma}{\tau^{2k-1}}\right) = \frac{\gamma}{\tau^{2k}} v_0$$

gives

$$E_2 \cong \frac{\mathbb{F}_2[\tau^2]}{\tau^\infty}\{\gamma\} \otimes_{\mathbb{F}_2} \mathbb{F}_2[v_1].$$

Finally, the differential

$$d_2\left(\frac{\gamma}{\tau^{4k-2}}\right) = \frac{\gamma}{\tau^{4k}} v_1$$

gives

$$E_3 = E_\infty \cong \frac{\mathbb{F}_2[\tau^4]}{\tau^\infty}\{\gamma\}. \qquad \square$$

6. The cohomology of $\mathcal{A}^{\mathbb{R}}(1)$

Our next step in working towards the calculation of $\mathrm{Ext}_{C_2}(1)$ is to describe the simpler $\mathbb{R}$-motivic $\mathrm{Ext}_{\mathbb{R}}(1)$. The reader is encouraged to consult the charts on pages 616–619 to follow along with the calculations described in this section. This calculation was originally carried out in [Hill 2011]. We include the details of the $\mathbb{R}$-motivic ρ-Bockstein spectral sequence, but we take the approach of [Dugger and Isaksen 2017a], rather than [Hill 2011], in establishing ρ-Bockstein differentials. The point is that there is only one pattern of differentials that is consistent with the ρ-inverted calculation of Corollary 4.2. This observation avoids much technical work with Massey products that would otherwise be required to establish relations that then imply differentials.

For $\mathcal{A}^{\mathbb{R}}(1)$, the $\mathbb{R}$-motivic ρ-Bockstein spectral sequence takes the form

$$\mathrm{Ext}_{\mathbb{C}}(1)[\rho] \Rightarrow \mathrm{Ext}_{\mathbb{R}}(1),$$

where

$$\mathrm{Ext}_{\mathbb{C}}(1) \cong \mathbb{M}_2^{\mathbb{C}}[h_0, h_1, a, b]/h_0h_1, \tau h_1^3, h_1a, a^2 + h_0^2 b.$$

Proposition 6.1. *In the* $\mathbb{R}$*-motivic* ρ*-Bockstein spectral sequence, we have differentials*

$$d_1(\tau) = \rho h_0, \tag{1}$$

$$d_2(\tau^2) = \rho^2 \tau h_1, \tag{2}$$

$$d_3(\tau^3 h_1^2) = \rho^3 a. \tag{3}$$

All other differentials on multiplicative generators are zero, and E_4 *equals* E_∞*.*

Proof. By Corollary 4.2, the infinite ρ-towers that survive the ρ-Bockstein spectral sequence occur in the Milnor–Witt $4k$-stem. All other infinite ρ-towers are either truncated by a differential or support a differential.

For example, the permanent cycle h_0 must be ρ-torsion in $\mathrm{Ext}_{\mathbb{R}}(1)$, which forces the Bockstein differential

$$d_1(\tau) = \rho h_0.$$

Next, the ρ-tower on τh_1 cannot survive, and the only possibility is that there is a differential

$$d_2(\tau^2) = \rho^2 \tau h_1.$$

Note that these differentials also follow easily from the right unit formula given in Section 2B. The ρ-tower on $\tau^3 h_1^2$ cannot survive, and we conclude that it must support a differential

$$d_3(\tau^3 h_1^2) = \rho^3 a.$$

There is no room for further nonzero differentials, so $E_4 = E_\infty$. □

Proposition 6.1 leads to an explicit description of the $\mathbb{R}$-motivic ρ-Bockstein E_∞-page. However, there are hidden multiplications in passing from E_∞ to $\mathrm{Ext}_{\mathbb{R}}(1)$.

Theorem 6.2. $\mathrm{Ext}_{\mathcal{A}^{\mathbb{R}}(1)}$ *is the* $\mathbb{F}_2$*-algebra on generators given in Table 1 with relations given in Table 2.*

The horizontal lines in Table 2 group the relations into families. The first family describes the ρ^k-torsion. The remaining families are associated to the classical products h_0^2, h_0h_1, h_1^3, h_0a, h_1a, and $a^2 + h_0^2 b$ respectively.

Proof. The family of ρ^k-torsion relations follows from the ρ-Bockstein differentials of Proposition 6.1.

Many relations follow immediately from the ρ-Bockstein E_∞-page because there are no possible additional terms.

mw	(s, f, w)	generator
0	$(-1, 0, -1)$	ρ
0	$(0, 1, 0)$	h_0
0	$(1, 1, 1)$	h_1
1	$(1, 1, 0)$	τh_1
2	$(0, 1, -2)$	$\tau^2 h_0$
2	$(4, 3, 2)$	a
4	$(4, 3, 0)$	$\tau^2 a$
4	$(8, 4, 4)$	b
4	$(0, 0, -4)$	τ^4

Table 1. Generators for $\mathrm{Ext}_{\mathbb{R}}(1)$

mw	(s, f, w)	relation
0	$(-1, 1, -1)$	ρh_0
2	$(-1, 1, -3)$	$\rho \cdot \tau^2 h_0$
1	$(-1, 1, -2)$	$\rho^2 \cdot \tau h_1$
2	$(1, 3, -1)$	$\rho^3 a$
4	$(0, 2, -4)$	$(\tau^2 h_0)^2 + \tau^4 h_0^2$
0	$(1, 2, 1)$	$h_0 h_1$
1	$(1, 2, 0)$	$h_0 \cdot \tau h_1 + \rho h_1 \cdot \tau h_1$
2	$(1, 2, -1)$	$\tau^2 h_0 \cdot h_1 + \rho(\tau h_1)^2$
3	$(1, 2, -2)$	$\tau^2 h_0 \cdot \tau h_1$
1	$(3, 3, 2)$	$h_1^2 \cdot \tau h_1$
2	$(3, 3, 1)$	$h_1(\tau h_1)^2 + \rho a$
3	$(3, 3, 0)$	$(\tau h_1)^3$
4	$(3, 3, -1)$	$\tau^4 \cdot h_1^3 + \rho \cdot \tau^2 a$
4	$(4, 4, 0)$	$\tau^2 h_0 \cdot a + h_0 \cdot \tau^2 a$
6	$(4, 4, -2)$	$\tau^2 h_0 \cdot \tau^2 a + \tau^4 h_0 a$
2	$(5, 4, 3)$	$h_1 a$
3	$(5, 4, 2)$	$\tau h_1 \cdot a$
4	$(5, 4, 1)$	$h_1 \cdot \tau^2 a + \rho^3 b$
5	$(5, 4, 0)$	$\tau h_1 \cdot \tau^2 a$
4	$(8, 6, 4)$	$a^2 + h_0^2 b$
6	$(8, 6, 2)$	$a \cdot \tau^2 a + \tau^2 h_0 \cdot h_0 b$
8	$(8, 6, 0)$	$(\tau^2 a)^2 + \tau^4 h_0^2 b + \rho^2 \tau^4 h_1^2 b$

Table 2. Relations for $\mathrm{Ext}_{\mathbb{R}}(1)$.

mw	(s, f, w)	$x \in \mathrm{Ext}_{\mathcal{A}^{\mathbb{R}}(1)}$	$q_* x \in \mathrm{Ext}_{\mathcal{E}^{\mathbb{R}}(1)}$
0	$(-1, 0, -1)$	ρ	ρ
0	$(0, 1, 0)$	h_0	h_0
0	$(1, 1, 1)$	h_1	0
1	$(1, 1, 0)$	τh_1	ρv_1
2	$(0, 1, -2)$	$\tau^2 h_0$	$\tau^2 h_0$
2	$(4, 3, 2)$	a	$h_0 v_1^2$
4	$(4, 3, 0)$	$\tau^2 a$	$\tau^2 h_0 v_1^2$
4	$(8, 4, 4)$	b	v_1^4
4	$(0, 0, -4)$	τ^4	τ^4

Table 3. The homomorphism $\mathrm{Ext}_{\mathcal{A}^{\mathbb{R}}(1)} \to \mathrm{Ext}_{\mathcal{E}^{\mathbb{R}}(1)}$.

Corollary 4.2 implies that $\tau^4 \cdot h_1^3$, is nonzero in $\mathrm{Ext}_{\mathbb{R}}(1)$. It follows that there must be a hidden relation

$$\tau^4 \cdot h_1^3 = \rho \cdot \tau^2 a.$$

Similarly, there is a hidden relation

$$h_1 \cdot \tau^2 a = \rho^3 b$$

because $\tau^4 \cdot h_1^4$ is nonzero in $\mathrm{Ext}_{\mathbb{R}}(1)$. This last relation then gives rise to the extra term $\rho^2 \tau^4 h_1^2 b$ in the relation for $(\tau^2 a)^2 + \tau^4 h_0^2 b$.

Shuffling relations for Massey products imply the remaining three relations, namely

$$h_0 \cdot \tau h_1 = h_0 \langle h_1, h_0, \rho \rangle = \langle h_0, h_1, h_0 \rangle \rho = \rho h_1 \cdot \tau h_1,$$

$$\tau^2 h_0 \cdot h_1 = \langle \rho \tau h_1, \rho, h_0 \rangle h_1 = \rho \tau h_1 \langle \rho, h_0, h_1 \rangle = \rho (\tau h_1)^2,$$

and

$$\rho a = \rho \langle h_0, h_1, \tau h_1 \cdot h_1 \rangle = \langle \rho, h_0, h_1 \rangle \tau h_1 \cdot h_1 = h_1 (\tau h_1)^2.$$

See Table 6 in Section 8 for more details on these Massey products, whose indeterminacies are all zero. □

Remark 6.3. For comparison purposes, we recall from [Hill 2011, Theorem 3.1] that

$$\mathrm{Ext}_{\mathcal{E}^{\mathbb{R}}(1)} \cong \mathbb{F}_2[\rho, \tau^4, h_0, \tau^2 h_0, v_1]/(\rho h_0, \rho^3 v_1, (\tau^2 h_0)^2 + \tau^4 h_0^2).$$

The ρ-torsion is created by the Bockstein differentials $d_1(\tau) = \rho h_0$ and $d_3(\tau^2) = \rho^3 v_1$. The class v_1 is in degree $(s, f, w) = (2, 1, 1)$.

Proposition 6.4. *The ring homomorphism $q_* : \mathrm{Ext}_{\mathcal{A}^{\mathbb{R}}(1)} \to \mathrm{Ext}_{\mathcal{E}^{\mathbb{R}}(1)}$ induced by the quotient $q : \mathcal{A}^R(1)_* \to \mathcal{E}^{\mathbb{R}}(1)_*$ of Hopf algebroids is given as in Table 3.*

Proof. Many of the values $q_*(x)$ are already true over $\mathbb{C}$ and follow easily from their descriptions in the May spectral sequence. For instance, b is represented by $h_{2,1}^4$, and v_1 is represented by $h_{2,1}$. On the other hand, the value $q_*(\tau h_1)$ is most easily seen using the cobar complex. The class τh_1 in $\mathrm{Ext}_{\mathcal{A}^{\mathbb{R}}(1)}$ is represented by $\tau\xi_1 + \rho\tau_1$. This maps to $\rho\tau_1$ in the cobar complex for $\mathcal{E}^{\mathbb{R}}(1)$ and represents the class ρv_1 there. □

7. Bockstein differentials in the negative cone

We finally come to the key step in our calculation of $\mathrm{Ext}_{C_2}(1)$. We are now ready to analyze the ρ-Bockstein differentials associated to the negative cone, i.e., to the spectral sequence E^- of Proposition 3.1. We already analyzed the spectral sequence E^+ in Section 6.

7A. ***The structure of E_1^-.*** First, we need some additional information about the algebraic structure of E_1^-. Since $E_1 = E_1^+ \oplus E_1^-$ is defined in terms of Ext groups, it is a ring and has higher structure in the form of Massey products. The subobject E_1^- is a module over E_1^+, and it possesses Massey products of the form $\langle x_1, \dots, x_n, y\rangle$, where $x_1, \dots, x_n$ belong to E_1^+ and y belongs to E_1^-.

Definition 7.1. Suppose that x is a nonzero element of $\mathrm{Ext}_{\mathbb{C}}(1)$ such that τx is zero. According to Remark 3.5, for each $s \geq 0$, the element x gives rise to a copy of $\mathbb{M}_2^{\mathbb{C}}/\tau$ in $\mathrm{Tor}_{\mathbb{M}_2^{\mathbb{C}}}\big(\frac{\mathbb{M}_2^{\mathbb{C}}}{\tau^\infty}, \mathrm{Ext}_{\mathbb{C}}(1)\big)\big\{\frac{\gamma}{\rho^s}\big\}$ that is infinitely divisible by ρ. In particular, it gives a nonzero element of the Tor group. Let $\frac{Q}{\rho^s}x$ be any lift to E_1^- of this nonzero element.

Remark 7.2. There is indeterminacy in the choice of Qx which arises from the first term of the short exact sequence for E_1^- in Proposition 3.1.

Lemma 7.3. *The element Qx of E_1^- is contained in the Massey product $\langle x, \tau, \frac{\gamma}{\tau}\rangle$.*

Proof. If $d(u) = \tau \cdot x$ in the cobar complex for $\mathrm{Ext}_{\mathbb{C}}(1)$, then $\frac{\gamma}{\tau}u$ is a cycle, since $\tau\frac{\gamma}{\tau} = 0$. This cycle $\frac{\gamma}{\tau}u$ represents both the Massey product as well as Qx. □

Remark 7.4. The most important example is the element Qh_1^3, which is defined because τh_1^3 equals zero in $\mathrm{Ext}_{\mathbb{C}}(1)$. Another possible name for Qh_1^3 is $\frac{\gamma}{\tau}v_1^2$, since v_1^2 is the element of the May spectral sequence that creates the relation τh_1^3.

Remark 7.5. Beware that the Massey product description for Qx holds in E_1^-, not in $\mathrm{Ext}_{C_2}(1)$. In fact, we have already seen in Section 6 that τ is not a permanent cycle in the ρ-Bockstein spectral sequence.

Nevertheless, minor variations on these Massey products may exist in $\mathrm{Ext}_{C_2}(1)$. For example, $\langle h_1^2, \tau h_1, \frac{\gamma}{\tau}\rangle$ equals Qh_1^3 in $\mathrm{Ext}_{C_2}(1)$.

We can now deduce a specific computational property of E_1^- that we will need later.

mw	(s, f, w)	element
0	$(-1, 0, -1)$	ρ
0	$(0, 1, 0)$	h_0
0	$(1, 1, 1)$	h_1
1	$(0, 0, -1)$	τ
2	$(4, 3, 2)$	a
4	$(8, 4, 4)$	b
0	$(4, 2, 4)$	Qh_1^3
$-k-1$	$(0, 0, k+1)$	$\frac{\gamma}{\tau^k}$

Table 4. Generators for the Bockstein E_1-page.

Lemma 7.6. *In E_1^-, there is a relation $h_0 \cdot Qh_1^3 = \frac{\gamma}{\tau} a$.*

Proof. Use Lemma 7.3 and the Massey product shuffle

$$h_0 \cdot Qh_1^3 = h_0 \left\langle h_1^3, \tau, \frac{\gamma}{\tau} \right\rangle = \langle h_0, h_1^3, \tau \rangle \frac{\gamma}{\tau} = \frac{\gamma}{\tau} a. \qquad \square$$

Table 4 gives multiplicative generators for the Bockstein E_1-page. The elements above the horizontal line are multiplicative generators for E_1^+. The elements below the horizontal generate E_1^- in the following sense. Every element of E_1^- can be formed by starting with one of the these listed elements, multiplying by elements of E_1^+, and then dividing by ρ. The elements in Table 7 are not multiplicative generators for $\mathrm{Ext}_{C_2}(1)$ in the usual sense, because we allow for division by ρ. The point of this notational approach is that the elements of E_1^- and of $\mathrm{Ext}_{\mathrm{NC}}$ are most easily understood as families of ρ-divisible elements.

7B. ***ρ-Bockstein differentials in E^-.*** Our next goal is to analyze the ρ-Bockstein differentials in E^-. We will rely heavily on the ρ-Bockstein differentials for E^+ established in Section 6, using that E^- is an E^+-module.

As an E_1^+-module, E_1^- is generated by the elements $\frac{\gamma}{\rho^j \tau^k}$ and $\frac{Q}{\rho^j} h_1^3$. This arises from the observation that the τ torsion in $\mathrm{Ext}_{\mathbb{C}}(1)$ is generated as an $\mathrm{Ext}_{\mathbb{C}}(1)$-module by h_1^3.

Proposition 7.7 gives the values of the ρ-Bockstein d_1 differential on these generators of E_1^-. All other d_1 differentials can then be deduced from the Leibniz rule and the E_1^+-module structure.

All of the differentials in E^- are infinitely divisible by ρ, in the following sense. When we claim that $d_r(x) = y$, we also have differentials $d_r\left(\frac{x}{\rho^j}\right) = \frac{y}{\rho^j}$ for all $j \geq 0$. For example, in Proposition 7.7, the formula $d_1\left(\frac{\gamma}{\rho\tau}\right) = \frac{\gamma}{\tau^2} h_0$ implies that

$$d_1\left(\frac{\gamma}{\rho^{j+1}\tau}\right) = \frac{\gamma}{\rho^j \tau^2} h_0 \quad \text{for all } j \geq 0.$$

Proposition 7.7. *For all $k \geq 0$,*

$$d_1\left(\frac{\gamma}{\rho\tau^{2k+1}}\right) = \frac{\gamma}{\tau^{2k+2}} h_0, \tag{1}$$

$$d_1\left(\frac{Q}{\rho} h_1^3\right) = \frac{\gamma}{\tau^2} a. \tag{2}$$

These differentials are infinitely divisible by ρ.

Proof. We give three proofs for the first formula. First, it follows from

$$\mathrm{Sq}^1\left(\frac{\gamma}{\rho\tau^{2k+1}}\right) = \frac{\gamma}{\tau^{2k+2}},$$

using the relationship between d_1 and the left and right units of the Hopf algebroid. Second, we have

$$\begin{aligned} 0 = d_1\left(\tau^{2k+1}\frac{\gamma}{\rho\tau^{2k+1}}\right) &= \tau^{2k+1} d_1\left(\frac{\gamma}{\rho\tau^{2k+1}}\right) + \frac{\gamma}{\rho\tau^{2k+1}}\rho\tau^{2k} h_0 \\ &= \tau^{2k+1} d_1\left(\frac{\gamma}{\rho\tau^{2k+1}}\right) + \frac{\gamma}{\tau} h_0. \end{aligned}$$

Third, we can use Proposition 5.2 to conclude that the infinitely ρ-divisible elements $\frac{\gamma}{\tau^{2k+1}}$ cannot survive the ρ-Bockstein spectral sequence. The only possibility is that they support a d_1 differential.

For the second formula, use the first formula to determine that $d_1\left(\frac{\gamma}{\rho\tau} a\right) = \frac{\gamma}{\tau^2} h_0 a$. Then use the relation of Lemma 7.6. Alternatively, this differential is also forced by Proposition 5.2. □

It is now straightforward to compute E_2^-, since the ρ-Bockstein d_1 differential is completely known. The charts in Section 12 depict E_2^- graphically.

Next, Proposition 7.8 gives a ρ-Bockstein d_2 differential in E_2^-. This is the essential calculation, in the sense that the d_2 differential is zero on all other E_2^+-module generators of E_2^-.

Proposition 7.8. $d_2\left(\frac{\gamma}{\rho^2\tau^{4k+2}}\right) = \frac{\gamma}{\tau^{4k+3}} h_1$ *for all $k \geq 0$. This differential is infinitely divisible by ρ.*

Proof. As for Proposition 7.7, we give three proofs. First, $\mathrm{Sq}^2\left(\frac{\gamma}{\rho^2\tau^{4k+2}}\right) = \frac{\gamma}{\tau^{4k+3}}$. Second, we have

$$\begin{aligned} 0 = d_2\left(\tau^{4k+2}\frac{\gamma}{\rho^2\tau^{4k+2}}\right) &= \tau^{4k+2} d_2\left(\frac{\gamma}{\rho^2\tau^{4k+2}}\right) + \rho^2\tau^{4k+1}\frac{\gamma}{\rho^2\tau^{4k+2}} h_1 \\ &= \tau^{4k+2} d_2\left(\frac{\gamma}{\rho^2\tau^{4k+2}}\right) + \frac{\gamma}{\tau} h_1. \end{aligned}$$

Third, use Proposition 5.2 to conclude that the infinitely ρ-divisible elements $\frac{\gamma}{\tau^{4k+1}}$ cannot survive the ρ-Bockstein spectral sequence. The only possibility is that they support a d_2 differential. □

At this point, the behavior of E^- becomes qualitatively different from E^+. For E^+, there are nonzero d_3 differentials, and then the E_4^+-page equals the E_∞^+-page.

For E^-, it turns out that the d_r differential is nonzero for infinitely many values of r. This does not present a convergence problem, because there are only finitely many nonzero differentials in any given degree. One consequence is that the orders of the ρ-torsion in $\mathrm{Ext}_{C_2}(1)$ are unbounded. In other words, for every s, there exists an element x of such that $\rho^s x$ is nonzero but $\rho^{s+t}x$ is zero for some $t > 0$. This is fundamentally different from $\mathrm{Ext}_{\mathbb{R}}(1)$, where $\rho^3 x$ is zero if x is not ρ-torsion free.

Proposition 7.9 makes explicit these higher differentials.

Proposition 7.9. *For all $k \geq 1$,*

(1) $$d_{4k}\left(\frac{Q}{\rho^{4k}}h_1^{4k}\right) = \frac{\gamma}{\tau^{4k}}b^k,$$

(2) $$d_{4k+1}\left(\frac{Q}{\rho^{4k+1}}h_1^{4k+3}\right) = \frac{\gamma}{\tau^{4k+2}}ab^k.$$

These differentials are infinitely divisible by ρ.

Proof. We know that $\frac{\gamma}{\tau^{4k}}$ and b are permanent cycles. On the other hand, in $\mathrm{Ext}_{C_2}(1)$ the relation $\tau^4 h_1^4 = \rho^4 b$ gives

$$\frac{\gamma}{\tau^{4k}}b^k = \rho^4\frac{\gamma}{\rho^4\tau^{4k}}b^k = \tau^4\frac{\gamma}{\rho^4\tau^{4k}}h_1^4 b^{k-1}.$$

Thus $\frac{\gamma}{\tau^{4k}}b^k$ is h_1-divisible, which implies that it must be zero in $\mathrm{Ext}_{C_2}(1)$, as there is no surviving class in the appropriate degree to support the h_1-multiplication. The only Bockstein differential that could hit $\frac{\gamma}{\tau^{4k}}b^k$ is the claimed one.

For the second formula, the classes $\frac{\gamma}{\tau^{4k+2}}a$ and b are permanent cycles, yet

$$\frac{\gamma}{\tau^{4k+2}}ab^k = \rho^4\frac{\gamma}{\rho^4\tau^{4k+2}}ab^k = \tau^4\frac{\gamma}{\rho^4\tau^{4k+2}}ah_1^4 b^{k-1}$$

in $\mathrm{Ext}_{C_2}(1)$. But $h_1 a = 0$, so $\frac{\gamma}{\tau^{4k+2}}ab^k$ must be zero in $\mathrm{Ext}_{C_2}(1)$, forcing the claimed differential.

Alternatively, one can use Proposition 5.2 to obtain both differentials. □

Table 5 summarizes the Bockstein differentials that we computed in Sections 6 and 7B. The differentials above the horizontal line occur in E^+, while the differentials below the horizontal line occur in E^- and are infinitely divisible by ρ.

The ρ-Bockstein differentials of Sections 6 and 7 allow us to completely compute the E_∞-page of the ρ-Bockstein spectral sequence for $\mathrm{Ext}_{C_2}(1)$.

mw	(s, f, w)	element	r	d_r	proof
1	$(0, 0, -1)$	τ	1	ρh_0	Prop. 6.1
2	$(0, 0, -2)$	τ^2	2	$\rho^2 \tau h_1$	Prop. 6.1
3	$(2, 2, -1)$	$\tau^3 h_1^2$	3	$\rho^3 a$	Prop. 6.1
$-2k-2$	$(1, 0, 2k+3)$	$\frac{\gamma}{\rho\tau^{2k+1}}$	1	$\frac{\gamma}{\tau^{2k+2}} h_0$	Prop. 7.7
0	$(5, 2, 5)$	$\frac{Q}{\rho} h_1^3$	1	$\frac{\gamma}{\tau^2} a$	Prop. 7.7
$-4k-3$	$(2, 0, 4k+5)$	$\frac{\gamma}{\rho^2\tau^{4k+2}}$	2	$\frac{\gamma}{\tau^{4k+3}} h_1$	Prop. 7.8
0	$(8k+1, 4k-1, 8k+1)$	$\frac{Q}{\rho^{4k}} h_1^{4k}$	$4k$	$\frac{\gamma}{\tau^{4k}} b^k$	Prop. 7.9
0	$(8k+5, 4k+2, 8k+5)$	$\frac{Q}{\rho^{4k+1}} h_1^{4k+3}$	$4k+1$	$\frac{\gamma}{\tau^{4k+2}} ab^k$	Prop. 7.9

Table 5. Bockstein differentials.

7C. *ρ-Bockstein differentials in E^- for $\mathcal{E}^{C_2}(1)$.* For comparison, we also carry out the analogous but easier computation over $\mathcal{E}^{C_2}(1)$ rather than $\mathcal{A}^{C_2}(1)$. Besides $d_1\left(\frac{\gamma}{\rho\tau^{2k+1}}\right) = \frac{\gamma}{\tau^{2k+2}} h_0$, the only other Bockstein differential is given in the following result.

Proposition 7.10. *$d_3\left(\frac{\gamma}{\rho^3\tau^{4k+2}}\right) = \frac{\gamma}{\tau^{4k+4}} v_1$ for all $k \geq 0$. This differential is infinitely divisible by ρ.*

Proof. The differential $d_3(\tau^2) = \rho^3 v_1$ of Remark 6.3 gives

$$0 = d_3\left(\tau^{4k+2} \frac{\gamma}{\rho^3\tau^{4k+2}}\right) = \tau^{4k+2} d_3\left(\frac{\gamma}{\rho^3\tau^{4k+2}}\right) + \rho^3\tau^{4k} \frac{\gamma}{\rho^3\tau^{4k+2}} v_1$$
$$= \tau^{4k+2} d_3\left(\frac{\gamma}{\rho^3\tau^{4k+2}}\right) + \frac{\gamma}{\tau^2} v_1. \qquad \square$$

8. Some Massey products

The final step in the computation of $\mathrm{Ext}_{C_2}(1)$ is to determine multiplicative extensions that are hidden in the ρ-Bockstein E_∞-page. In order to do this, we will need some Massey products in $\mathrm{Ext}_{C_2}(1)$. Table 6 summarizes the information that we will need.

Theorem 8.1. *Some Massey products in* $\mathrm{Ext}_{C_2}(1)$ *are given in Table 6. All have zero indeterminacy.*

Proof. For some Massey products in Table 6, a ρ-Bockstein differential is displayed in the last column. In these cases, May's convergence theorem [May 1969; Isaksen 2014, Chapter 2.2] applies, and the Massey product can be computed with the given differential. Roughly speaking, May's convergence theorem says that Massey products in $\mathrm{Ext}_{C_2}(1)$ can be computed with any ρ-Bockstein differential. Beware that

mw	(s, f, w)	bracket	contains	proof
1	$(1, 1, 0)$	$\langle \rho, h_0, h_1 \rangle$	τh_1	$d_1(\tau) = \rho h_0$
1	$(2, 2, 1)$	$\langle h_0, h_1, h_0 \rangle$	τh_1^2	classical
2	$(4, 3, 2)$	$\langle \tau h_1 \cdot h_1, h_1, h_0 \rangle$	a	classical
2	$(0, 1, -2)$	$\langle \rho\tau h_1, \rho, h_0 \rangle$	$\tau^2 h_0$	$d_2(\tau^2) = \rho^2 \tau h_1$
4	$(8, 5, 4)$	$\langle a, h_1, \tau h_1^2 \rangle$	$h_0 b$	classical
-4	$(0, 0, 4)$	$\langle \tau^2 h_0, \rho, \frac{\gamma}{\tau^6} \rangle$	$\frac{\gamma}{\tau^3}$	$d_1(\tau^3) = \rho\tau^2 h_0$
-4	$(0, 0, 4)$	$\langle h_0, \rho, \frac{\gamma}{\tau^4} \rangle$	$\frac{\gamma}{\tau^3}$	$d_1(\tau) = \rho h_0$
-3	$(1, 0, 4)$	$\langle \rho, \frac{\gamma}{\tau^4}, \tau h_1 \rangle$	$\frac{\gamma}{\rho\tau^2}$	$d_2\left(\frac{\gamma}{\rho^2\tau^2}\right) = \frac{\gamma}{\tau^3} h_1$
-3	$(0, 0, 3)$	$\langle \rho\tau h_1, \rho, \frac{\gamma}{\tau^4} \rangle$	$\frac{\gamma}{\tau^2}$	$d_2(\tau^2) = \rho^2 \tau h_1$
-2	$(4, 2, 6)$	$\langle \frac{\gamma}{\tau^3}, h_1, \tau h_1 \cdot h_1 \rangle$	$\frac{\gamma}{\rho^2\tau} h_1^2$	$d_2\left(\frac{\gamma}{\rho^2\tau^2}\right) = \frac{\gamma}{\tau^3} h_1$
-2	$(0, 0, 2)$	$\langle \tau^2 h_0, \rho, \frac{\gamma}{\tau^4} \rangle$	$\frac{\gamma}{\tau}$	$d_1(\tau^3) = \rho\tau^2 h_0$
-2	$(0, 0, 2)$	$\langle h_0, \rho, \frac{\gamma}{\tau^2} \rangle$	$\frac{\gamma}{\tau}$	$d_1(\tau) = \rho h_0$
-2	$(2, 1, 4)$	$\langle h_1, h_0, \frac{\gamma}{\tau^2} \rangle$	$\frac{\gamma}{\rho\tau} h_1$	$d_1\left(\frac{\gamma}{\rho\tau}\right) = \frac{\gamma}{\tau^2} h_0$
0	$(4,2,4) + (8k,4k,8k)$	$\langle \rho, \frac{\gamma}{\tau^{4k+2}}, ab^k \rangle$	$\frac{Q}{\rho^{4k}} h_1^{4k+3}$	$d_{4k+1}\left(\frac{Q}{\rho^{4k+1}} h_1^{4k+3}\right) = \frac{\gamma}{\tau^{4k+2}} ab^k$
0	$(8,3,8) + (8k,4k,8k)$	$\langle \rho, \frac{\gamma}{\tau^{4k+4}}, b^{k+1} \rangle$	$\frac{Q}{\rho^{4k+3}} h_1^{4k+4}$	$d_{4k+4}\left(\frac{Q}{\rho^{4k+4}} h_1^{4k+4}\right) = \frac{\gamma}{\tau^{4k+4}} b^{k+1}$

Table 6. Some Massey products in $\mathrm{Ext}_{C_2}(1)$.

May's Convergence Theorem requires technical hypotheses involving "crossing differentials" that are not always satisfied. Failure to check these conditions can lead to mistaken calculations.

The proofs for other Massey products in Table 6 are described as "classical". In these cases, the Massey product already occurs in $\mathrm{Ext}_{\mathrm{cl}}$. □

Remark 8.2. The eight Massey products in the middle Section of Table 6 are only the first examples of infinite families that are τ^4-periodic. For example, $\langle \tau^2 h_0, \rho, \frac{\gamma}{\tau^{4k+6}} \rangle$ equals $\frac{\gamma}{\tau^{4k+3}}$ for all $k \geq 0$, and $\langle \rho, \frac{\gamma}{\tau^{4k+4}}, \tau h_1 \rangle$ equals $\frac{\gamma}{\tau^{4k+3}}$ for all $k \geq 0$.

9. Hidden extensions

We now determine multiplicative extensions that are hidden in the ρ-Bockstein E_∞-page. We have already determined some of these hidden extensions in Section 6. In this section, we establish additional hidden relations on elements associated with the negative cone. We have not attempted a completely exhaustive analysis of the ring structure of $\mathrm{Ext}_{C_2}(1)$.

Recall that $\mathrm{Ext}_{C_2}(1)$ is a square-zero extension of $\mathrm{Ext}_{\mathbb{R}}(1)$. This eliminates many possible hidden extensions. For example, $(Qh_1^3)^2$ is zero in $\mathrm{Ext}_{C_2}(1)$.

Proposition 9.1. *For all $k \geq 0$,*

$$h_0 \cdot \frac{Q}{\rho^{4k}} h_1^{4k+3} = \frac{\gamma}{\tau^{4k+1}} ab^k, \tag{1}$$

$$a \cdot \frac{Q}{\rho^{4k}} h_1^{4k+3} = \frac{\gamma}{\tau^{4k+1}} h_0 b^{k+1}. \tag{2}$$

Proof. (1) $h_0 \left\langle \rho, \frac{\gamma}{\tau^{4k+2}}, ab^k \right\rangle = \left\langle h_0, \rho, \frac{\gamma}{\tau^{4k+2}} \right\rangle ab^k$.

(2) Using part (1), we have that

$$h_0 a \cdot \frac{Q}{\rho^{4k} h_1^{4k+3}} = a \cdot \frac{\gamma}{\tau^{4k+1}} ab^k = \frac{\gamma}{\tau^{4k+1}} h_0^2 b^{k+1},$$

which is nonzero. Therefore, $a \cdot \frac{Q}{\rho^{4k}} h_1^{4k+3}$ must also be nonzero, and $\frac{\gamma}{\tau^{4k+1}} h_0 b^{k+1}$ is the only nonzero class in the appropriate tridegree. □

Proposition 9.2. *For all $k \geq 1$,*

$$\tau^2 a \cdot \frac{Q}{\rho^{4k}} h_1^{4k+3} = \frac{\gamma}{\tau^{4k-1}} h_0 b^{k+1} + \frac{Q}{\rho^{4k-3}} h_1^{4k+2} b, \tag{1}$$

$$\tau^4 \cdot \frac{Q}{\rho^{4k}} h_1^{4k+3} = \frac{Q}{\rho^{4k-4}} h_1^{4k-1} b, \tag{2}$$

$$\tau^2 h_0 \cdot \frac{Q}{\rho^{4k}} h_1^{4k+3} = \frac{\gamma}{\tau^{4k-1}} ab^k. \tag{3}$$

Proof. (1) Using Proposition 9.1(1), we have that

$$h_0 \cdot \tau^2 a \cdot \frac{Q}{\rho^{4k}} h_1^{4k+3} = \tau^2 a \cdot \frac{\gamma}{\tau^{4k+1}} ab^k = \frac{\gamma}{\tau^{4k-1}} h_0^2 b^{k+1},$$

which is nonzero. Hence $\tau^2 a \cdot \frac{Q}{\rho^{4k}} h_1^{4k+3}$ is either $\frac{\gamma}{\tau^{4k-1}} h_0 b^{k+1}$ or $\frac{\gamma}{\tau^{4k-1}} h_0 b^{k+1} + \frac{Q}{\rho^{4k-3}} h_1^{4k+2} b$.

On the other hand,

$$h_1 \cdot \tau^2 a \cdot \frac{Q}{\rho^{4k}} h_1^{4k+3} = \rho^3 b \cdot \frac{Q}{\rho^{4k}} h_1^{4k+3} = \frac{Q}{\rho^{4k-3}} h_1^{4k+3} b.$$

Therefore, $\tau^2 a \cdot \frac{Q}{\rho^{4k}} h_1^{4k+3}$ must equal $\frac{\gamma}{\tau^{4k-1}} h_0 b^{k+1} + \frac{Q}{\rho^{4k-3}} h_1^{4k+2} b$.

(2) Using Proposition 9.1(1), we have that

$$h_0 \cdot \tau^4 \cdot \frac{Q}{\rho^{4k}} h_1^{4k+3} = \tau^4 \frac{\gamma}{\tau^{4k+1}} ab^k = \frac{\gamma}{\tau^{4k-3}} ab^k,$$

which is nonzero. This shows that $\tau^4 \cdot \frac{Q}{\rho^{4k}} h_1^{4k+3}$ is also nonzero, and there is just one possible value.

(3) $$\tau^2 h_0 \left\langle \rho, \frac{\gamma}{\tau^{4k+2}}, ab^k \right\rangle = \left\langle \tau^2 h_0, \rho, \frac{\gamma}{\tau^{4k+2}} \right\rangle ab^k.$$ □

Proposition 9.3. *For all $k \geq 0$,*

(1) $$h_0 \cdot \frac{Q}{\rho^{4k+3}} h_1^{4k+4} = \frac{\gamma}{\tau^{4k+3}} b^{k+1},$$

(2) $$\tau h_1 \cdot \frac{Q}{\rho^{4k+3}} h_1^{4k+4} = \frac{\gamma}{\rho \tau^{4k+2}} b^{k+1},$$

(3) $$\tau^2 h_0 \cdot \frac{Q}{\rho^{4k+3}} h_1^{4k+4} = \frac{\gamma}{\tau^{4k+1}} b^{k+1},$$

(4) $$a \cdot \frac{Q}{\rho^{4k+3}} h_1^{4k+4} = \frac{\gamma}{\rho^2 \tau^{4k+1}} h_1^2 b^{k+1},$$

(5) $$\tau^2 a \cdot \frac{Q}{\rho^{4k+3}} h_1^{4k+4} = \frac{Q}{\rho^{4k}} h_1^{4k+3} b.$$

Proof. (1) $h_0 \left\langle \rho, \frac{\gamma}{\tau^{4k+4}}, b^{k+1} \right\rangle = \left\langle h_0, \rho, \frac{\gamma}{\tau^{4k+4}} \right\rangle b^{k+1}$.

(2) $\rho \tau h_1 \left\langle \rho, \frac{\gamma}{\tau^{4k+4}}, b^{k+1} \right\rangle = \left\langle \rho \tau h_1, \rho, \frac{\gamma}{\tau^{4k+4}} \right\rangle b^{k+1}$.

(3) $\tau^2 h_0 \left\langle \rho, \frac{\gamma}{\tau^{4k+4}}, b^{k+1} \right\rangle = \left\langle \tau^2 h_0, \rho, \frac{\gamma}{\tau^{4k+4}} \right\rangle b^{k+1}$.

(4) Using part (1), we have that

$$h_0 a \cdot \frac{Q}{\rho^{4k+3}} h_1^{4k+4} = a \cdot \frac{\gamma}{\tau^{4k+3}} b^{k+1} = \frac{\gamma}{\tau^{4k+3}} ab^{k+1},$$

which is nonzero. Therefore, $a \cdot \frac{Q}{\rho^{4k+3}} h_1^{4k+4}$ must also be nonzero, and there is just one possibility.

(5) Using part (1), we have that

$$h_0 \cdot \tau^2 a \cdot \frac{Q}{\rho^{4k+3}} h_1^{4k+4} = \tau^2 a \cdot \frac{\gamma}{\tau^{4k+3}} b^{k+1} = \frac{\gamma}{\tau^{4k+1}} ab^{k+1},$$

which is nonzero. This shows that $\tau^2 a \cdot \frac{Q}{\rho^{4k+3}} h_1^{4k+4}$ is also nonzero, and there is just one possible value. □

Proposition 9.4. *For all $k \geq 0$,*

(1) $$h_0 \cdot \frac{\gamma}{\rho^2 \tau^{4k+1}} h_1^2 = \frac{\gamma}{\tau^{4k+3}} a,$$

(2) $$a \cdot \frac{\gamma}{\rho^2 \tau^{4k+1}} h_1^2 = \frac{\gamma}{\tau^{4k+3}} h_0 b,$$

(3) $$\tau^2 a \cdot \frac{\gamma}{\rho^2 \tau^{4k+1}} h_1^2 = \frac{\gamma}{\tau^{4k+1}} h_0 b.$$

Proof. (1) $\left\langle \frac{\gamma}{\tau^{4k+3}}, h_1, \tau h_1 \cdot h_1 \right\rangle h_0 = \frac{\gamma}{\tau^{4k+3}} \langle h_1, \tau h_1 \cdot h_1, h_0 \rangle$.

(2) Using part (1), we have that

$$h_0 a \cdot \frac{\gamma}{\rho^2 \tau^{4k+1}} h_1^2 = a \cdot \frac{\gamma}{\tau^{4k+3}} a,$$

which equals $\frac{\gamma}{\tau^{4k+3}} h_0^2 b$ modulo a possible error term involving higher powers of ρ. Using that $h_1 a = 0$, we conclude that the error term is zero.

(3) Using part (1), we have that

$$h_0 \cdot \tau^2 a \cdot \frac{\gamma}{\rho^2 \tau^{4k+1}} h_1^2 = \tau^2 a \cdot \frac{\gamma}{\tau^{4k+3}} a = \frac{\gamma}{\tau^{4k+1}} h_0^2 b,$$

which is nonzero. This shows that $\tau^2 a \cdot \frac{\gamma}{\rho^2 \tau^{4k+1}} h_1^2$ is also nonzero, and there is just one possible value. □

Proposition 9.5. *For all $k \geq 0$,*

(1) $$h_0 \cdot \frac{\gamma}{\rho \tau^{4k+1}} h_1 = \frac{\gamma}{\tau^{4k+1}} h_1^2,$$

(2) $$h_0 \cdot \frac{\gamma}{\rho \tau^{4k+2}} = \frac{\gamma}{\tau^{4k+2}} h_1.$$

Proof. All of these extensions follow from Massey product shuffles:

(1) $h_0 \left\langle h_1, h_0, \frac{\gamma}{\tau^{4k+2}} \right\rangle = \langle h_0, h_1, h_0 \rangle \frac{\gamma}{\tau^{4k+2}}$.

(2) $h_0 \left\langle \rho, \frac{\gamma}{\tau^{4k+4}}, \tau h_1 \right\rangle = \left\langle h_0, \rho, \frac{\gamma}{\tau^{4k+4}} \right\rangle \tau h_1$. □

Proposition 9.6. *For all $k \geq 0$,*

(1) $$h_1 \cdot \frac{\gamma}{\rho \tau^{4k+4}} h_1^2 = \frac{\gamma}{\tau^{4k+6}} a,$$

(2) $$h_1 \cdot \frac{\gamma}{\rho^3 \tau^{4k+6}} a = \frac{\gamma}{\tau^{4k+8}} b.$$

Proof. (1) $\tau h_1 \cdot h_1 \left\langle h_1, h_0, \frac{\gamma}{\tau^{4k+6}} \right\rangle = \langle \tau h_1 \cdot h_1, h_1, h_0 \rangle \frac{\gamma}{\tau^{4k+6}}$. Alternatively, this h_1 extension is forced by Lemma 5.1.

(2) We have

$$h_1 \cdot \frac{\gamma}{\rho^3 \tau^{4k+6}} a = \frac{\gamma}{\rho^3 \tau^{4k+8}} h_1 \cdot \tau^2 a = \frac{\gamma}{\rho^3 \tau^{4k+8}} \rho^3 b = \frac{\gamma}{\tau^{4k+8}},$$

where the second equality follows from Table 2. □

Over $\mathcal{E}^{C_2}(1)$, the only hidden multiplication is

Proposition 9.7. *In* $\mathrm{Ext}_{\mathcal{E}^{C_2}(1)}$, *we have* $h_0 \cdot \frac{\gamma}{\rho^2 \tau^{4k+2}} v_1^n = \frac{\gamma}{\tau^{4k+3}} v_1^{n+1}$. *for all $k, n \geq 0$.*

Proof. $$h_0 \cdot \frac{\gamma}{\rho^2 \tau^2} = h_0 \left\langle \rho, \frac{\gamma}{\tau^4}, v_1 \right\rangle = \left\langle h_0, \rho, \frac{\gamma}{\tau^4} \right\rangle v_1 = \frac{\gamma}{\tau^3} v_1.$$ □

mw	(s, f, w)	element
0	$(-1, 0, -1)$	ρ
0	$(0, 1, 0)$	h_0
0	$(1, 1, 1)$	h_1
1	$(1, 1, 0)$	τh_1
2	$(0, 1, -2)$	$\tau^2 h_0$
2	$(4, 3, 2)$	a
4	$(0, 0, -4)$	τ^4
4	$(4, 3, 0)$	$\tau^2 a$
4	$(8, 4, 4)$	b
$-k-1$	$(0, 0, k+1)$	$\frac{\gamma}{\tau^k}$
0	$(4, 2, 4)$	Qh_1^3

Table 7. Generators for $\mathrm{Ext}_{C_2}(1)$.

9A. $\mathrm{Ext}_{C_2}(1)$. The charts in Section 12 depict $\mathrm{Ext}_{C_2}(1)$ graphically. Table 7 gives generators for $\mathrm{Ext}_{C_2}(1)$. The elements above the horizontal line are multiplicative generators for $\mathrm{Ext}_{\mathbb{R}}(1)$. The elements below the horizontal generate $\mathrm{Ext}_{\mathrm{NC}}$ in the following sense. Every element of $\mathrm{Ext}_{\mathrm{NC}}$ can be formed by starting with one of these listed elements, multiplying by elements of $\mathrm{Ext}_{\mathbb{R}}(1)$, and then dividing by ρ.

The elements in Table 7 are not multiplicative generators for $\mathrm{Ext}_{C_2}(1)$ in the usual sense, because we allow for division by ρ. For example, $\frac{\gamma}{\rho^2\tau}h_1^2$ is indecomposable in the usual sense, yet it does not appear in Table 7 because $\rho^2 \cdot \frac{\gamma}{\rho^2\tau}h_1^2 = \frac{\gamma}{\tau}h_1^2$ is decomposable.

The point of this notational approach is that the elements of $\mathrm{Ext}_{\mathrm{NC}}$ are most easily understood as families of ρ-divisible elements.

9B. *The ring homomorphism $q_* : \mathrm{Ext}_{\mathcal{A}^{C_2}(1)} \to \mathrm{Ext}_{\mathcal{E}^{C_2}(1)}$.* It is worthwhile to consider the comparison to $\mathrm{Ext}_{\mathcal{E}^{C_2}(1)}$. We already described the map on the summand arising from the positive cone in Proposition 6.4. The map on the summand for the negative cone is given as follows.

Proposition 9.8. *The homomorphism $q_* : \mathrm{Ext}_{\mathcal{A}^{\mathbb{R}}(1)}(\mathrm{NC}, \mathbb{M}_2^{\mathbb{R}}) \to \mathrm{Ext}_{\mathcal{E}^{\mathbb{R}}(1)}(\mathrm{NC}, \mathbb{M}_2^{\mathbb{R}})$ induced by the quotient $q : \mathcal{A}^R(1)_* \to \mathcal{E}^{\mathbb{R}}(1)$ of Hopf algebroids is given as in Table 8.*

Proof. For the classes of the form $\frac{\gamma}{\rho^j\tau^k}$, this is true on the cobar complex. For the classes of the form $\frac{Q}{\rho^j}h_1^n$, this follows from the h_0-extension given in Proposition 9.1 and the value $q_*(a) = h_0 v_1^2$. Similarly, the value on $\frac{\gamma}{\rho^2\tau}h_1^2$ is obtained by combining

mw	(s, f, w)	$x \in \mathrm{Ext}_{\mathcal{A}^{\mathbb{R}}(1)}\,\mathrm{NC}$	$q_*x \in \mathrm{Ext}_{\mathcal{E}^{\mathbb{R}}(1)}\,\mathrm{NC}$
0	$(4,2,4)+k(8,4,4)$	$\frac{Q}{\rho^{4k}}h_1^{4k+3}$	$\frac{\gamma}{\tau}v_1^{4k+2}$
0	$(8,3,8)+k(8,4,4)$	$\frac{Q}{\rho^{4k+4}}h_1^{4k+3}$	$\frac{\gamma}{\rho^2\tau^2}v_1^{4k+3}$
-2	$(0,0,2)$	$\frac{\gamma}{\tau}$	$\frac{\gamma}{\tau}$
-2	$(2,1,4)$	$\frac{\gamma}{\rho\tau}h_1$	$\frac{\gamma}{\tau^2}v_1$
-2	$(4,2,6)$	$\frac{\gamma}{\rho^2\tau}h_1^2$	$\frac{\gamma}{\tau^3}v_1^2$
-3	$(1,0,4)$	$\frac{\gamma}{\rho\tau^2}$	$\frac{\gamma}{\rho\tau^2}$
-5	$(0,0,5)$	$\frac{\gamma}{\tau^4}$	$\frac{\gamma}{\tau^4}$

Table 8. The homomorphism $\mathrm{Ext}_{\mathcal{A}^{\mathbb{R}}(1)}(\mathrm{NC}) \to \mathrm{Ext}_{\mathcal{E}^{\mathbb{R}}(1)}(\mathrm{NC})$.

Proposition 9.4 with the value of $q_*(a)$. Lastly, the value on $\frac{\gamma}{\rho\tau}h_1$ follows from $q_*(\tau h_1) = \rho v_1$. □

Remark 9.9. Note that, on the other hand, the hidden h_0-extensions on classes in $\mathrm{Ext}_{\mathcal{A}^{C_2}(1)}$, such as Qh_1^3, can also be deduced from the homomorphism q_* if its values are determined by other means.

10. The spectrum $\mathbf{ko}_{C_2}$

Let **Sp** denoted the category of spectra, and let $\mathbf{Sp}^{C_2}$ denote the category of "genuine" C_2-spectra [May 1996, Chapter XII], obtained from the category of based C_2-spaces by inverting suspension with respect to the one-point compactification $S^{2,1}$ of the regular representation $(\mathbb{C}, z \mapsto \bar{z})$. There are restriction and fixed-point functors

$$\iota^* : \mathrm{Ho}(\mathbf{Sp}^{C_2}) \to \mathrm{Ho}(\mathbf{Sp}), \quad (-)^{C_2} : \mathrm{Ho}(\mathbf{Sp}^{C_2}) \to \mathrm{Ho}(\mathbf{Sp})$$

which detect the homotopy theory of C_2-spectra, meaning that a map f in $\mathrm{Ho}(\mathbf{Sp}^{C_2})$ is an equivalence if and only if $\iota^*(f)$ and f^{C_2} are equivalences in $\mathrm{Ho}(\mathbf{Sp})$. Moreover, a sequence $X \to Y \to Z$ is a cofiber sequence in $\mathrm{Ho}(\mathbf{Sp}^{C_2})$ if and only if applying both functors ι^* and $(-)^{C_2}$ yield cofiber sequences. Both statements follow from the fact [Schwede and Shipley 2003, Example 3.4(i)] that the pair of C_2-spectra $\{\Sigma^\infty_{C_2} S^0, \Sigma^\infty_{C_2} C_{2\,+}\}$ give a compact generating set for $\mathrm{Ho}(\mathbf{Sp}^{C_2})$. Beware that we are discussing categorical fixed-point spectra here, not geometric fixed-point spectra.

Recall (see [Lewis 1995, Proposition 3.3]) that for a C_2-spectrum X, the equivariant connective cover $X\langle 0\rangle \xrightarrow{q} X$ is a C_2-spectrum such that:

(1) $\iota(q)$ is the connective cover of the underlying spectrum X, and

(2) q^{C_2} is the connective cover of X^{C_2}.

Recall that KO_{C_2} is the C_2-spectrum representing the $\mathbb{K}$-theory of C_2-equivariant real vector bundles [May 1996, Chapter XIV].

Definition 10.1. Let ko_{C_2} be the equivariant connective cover $\mathrm{KO}_{C_2}\langle 0\rangle$ of KO_{C_2}.

We also have a description from the point of view of equivariant infinite loop space theory.

Theorem 10.2 [Merling 2017, Theorem 7.1]. $\mathrm{ko}_{C_2} \simeq \mathbb{K}_{C_2}(\mathbb{R})$, *where $\mathbb{R}$ is considered as a topological ring with trivial C_2-action.*

The underlying spectrum of ko_{C_2} is ko.

Lemma 10.3. *The fixed-point spectrum of* ko_{C_2} *is* $(\mathrm{ko}_{C_2})^{C_2} \simeq \mathrm{ko} \vee \mathrm{ko}$.

Proof. This is a specialization of the statement that, if X is any space equipped with a trivial G-action, then $\mathrm{KO}_G(X)$ is isomorphic to $\mathrm{RO}(G)\otimes \mathrm{KO}(X)$ [May 1996, Section XIV.2]. Alternatively, from the point of view of algebraic $\mathbb{K}$-theory, we have $\mathbb{K}_{C_2}(\mathbb{R})^{C_2} \simeq \mathbb{K}(\mathbb{R}[C_2])$ [Merling 2017, Theorem 1.2], and $\mathbb{R}[C_2] \cong \mathbb{R}\times\mathbb{R}$. It follows that

$$(\mathrm{ko}_{C_2})^{C_2} \simeq \mathbb{K}_{C_2}(\mathbb{R})^{C_2} \simeq \mathbb{K}(\mathbb{R})\times\mathbb{K}(\mathbb{R}) \simeq \mathrm{ko}\vee\mathrm{ko}. \qquad \square$$

We are working towards a description of the C_2-equivariant cohomology of ko_{C_2} as the quotient $\mathcal{A}^{C_2}/\!/\mathcal{A}^{C_2}(1)$. This will allow us to express the E_2-page of the Adams spectral sequence for ko_{C_2} in terms of the cohomology of $\mathcal{A}^{C_2}(1)$. The main step will be to establish the cofiber sequence of Proposition 10.13. In preparation, we first prove some auxiliary results.

Definition 10.4. Let ρ be the element of $\pi_{-1,-1}$ determined by the inclusion $S^{0,0} \hookrightarrow S^{1,1}$ of fixed points.

Note that the element $\rho \in \pi_{-1,-1}$ induces multiplication by ρ in cohomology under the Hurewicz homomorphism.

Recall that the real C_2-representation ring $\mathrm{RO}(C_2)$ is a rank two free abelian group. Generators are given by the trivial one-dimensional representation 1 and the sign representation σ. Let $A(C_2)$ denote the Burnside ring of C_2, defined as the Grothendieck group associated to the monoid of finite C_2-sets. This is also a rank two free abelian group, with generators the trivial one-point C_2-set 1 and the free C_2-set C_2. As a ring, $A(C_2)$ is isomorphic to $\mathbb{Z}[C_2]/(C_2^2-2C_2)$.

The linearization map $A(C_2)\to \mathrm{RO}(C_2)$ sending a C_2-set to the induced permutation representation is an isomorphism, sending the free orbit C_2 to the regular representation $1\oplus\sigma$. Recall that the Euler characteristic moreover gives an isomorphism from $A(C_2)$ to $\pi_0(S^{0,0})$ [Segal 1971, Corollary to Proposition 1].

Lemma 10.5. *The C_2-fixed point spectrum of* $\Sigma^{1,1}\mathrm{ko}_{C_2}$ *is equivalent to* ko.

Proof. Recall the cofiber sequence $C_{2+} \xrightarrow{\pi} S^{0,0} \xrightarrow{\rho} S^{1,1}$ of C_2-spaces. This yields a cofiber sequence

$$C_{2+} \wedge \mathrm{ko}_{C_2} \xrightarrow{\pi} \mathrm{ko}_{C_2} \xrightarrow{\rho} \Sigma^{1,1}\mathrm{ko}_{C_2}$$

of equivariant spectra. Passing to fixed point spectra gives the cofiber sequence

$$\mathrm{ko} \xrightarrow{\pi^{C_2}} \mathrm{ko} \vee \mathrm{ko} \xrightarrow{\rho^{C_2}} (\Sigma^{1,1}\mathrm{ko}_{C_2})^{C_2}.$$

In the analogous sequence for the sphere $S^{0,0}$, the map π^{C_2} is induced by the split inclusion $\mathbb{Z} \to A(C_2)$ sending 1 to the free orbit C_2. It follows that the map π^{C_2} is induced by the split inclusion $\mathbb{Z} \to \mathrm{RO}(C_2)$ that takes 1 to the regular representation ρ_{C_2}, and this induces a splitting of the cofiber sequence. Therefore, $(\Sigma^{1,1}\mathrm{ko}_{C_2})^{C_2}$ is equivalent to ko. □

Recall that $k\mathbb{R}$ denotes the equivariant connective cover $K\mathbb{R}\langle 0\rangle$ of Atiyah's K-theory "with reality" spectrum $K\mathbb{R}$ [Atiyah 1966]. The latter theory classifies complex vector bundles equipped with a conjugate-linear action of C_2. The underlying spectrum of $k\mathbb{R}$ is ku, and its fixed-point spectrum is ko.

Theorem 10.6 [Merling 2017, Theorem 7.2]. $k\mathbb{R} \simeq \mathbb{K}_{C_2}(\mathbb{C})$, *where* $\mathbb{C}$ *is considered as a topological ring with* C_2*-action given by complex conjugation.*

Definition 10.7. The C_2-equivariant Hopf map η is

$$\mathbb{C}^2 - \{0\} \to \mathbb{CP}^1 : (x, y) \mapsto [x : y],$$

where both source and target are given the complex conjugation action.

As $\mathbb{C} \cong \mathbb{R}[C_2]$, the punctured representation $\mathbb{C}^2 - \{0\}$ is homotopy equivalent to $S^{3,2}$, and $\mathbb{CP}^1$ is homeomorphic to $S^{2,1}$. It follows that η gives rise to a stable homotopy class in $\pi_{1,1}$.

Remark 10.8. The element η only defines a specific element of $\pi_{1,1}$ after choosing isomorphisms $\mathbb{C}^2 - \{0\} \cong S^{3,2}$ and $\mathbb{CP}^1 \cong S^{2,1}$ in the homotopy category. We follow the choices of [Dugger and Isaksen 2013, Example 2.12]. By Proposition C.5 of [Dugger and Isaksen 2013], with these choices, the induced map $\eta^{C_2} : S^1 \to S^1$ on fixed points is a map of degree -2.

Lemma 10.9. *The element* $\rho\eta$ *in* $\pi_{0,0}$ *corresponds to the element* $C_2 - 2$ *of* $A(C_2)$.

Proof. In $\pi_{0,0}$, we have $(\eta\rho)^2 = -2\eta\rho$ [Morel 2004, Lemma 6.1.2]. The nonzero solutions to $x^2 = -2x$ in $A(C_2)$ are $x = -2$, $x = C_2 - 2$, and $x = -C_2$. The only such solution which restricts to zero at the trivial subgroup is $x = C_2 - 2$. □

Lemma 10.10. *The induced map* $\eta^{C^2} : (\Sigma^{1,1}\mathrm{ko}_{C_2})^{C_2} \to (\mathrm{ko}_{C_2})^{C_2}$ *is equivalent to* $\mathrm{ko} \xrightarrow{(-1,1)} \mathrm{ko} \vee \mathrm{ko}$.

Proof. To determine the fixed map η^{C_2}, we use that a map $X \xrightarrow{\varphi} Y$ of C_2-spectra induces a commutative diagram

$$\begin{tikzcd} X^{C_2} \arrow[r, "\varphi^{C_2}"] \arrow[d] & Y^{C_2} \arrow[d] \\ X^e \arrow[r, "\varphi^e"'] & Y^e \end{tikzcd}$$

in which the vertical maps are the inclusions of fixed points. In the case of η on ko_{C_2}, this gives the diagram

$$\begin{tikzcd} \mathrm{ko} \simeq (\Sigma^{1,1}\mathrm{ko}_{C_2})^{C_2} \arrow[r, "\eta^{C_2}"] \arrow[d, "0"'] & \mathrm{ko} \vee \mathrm{ko} \simeq (\mathrm{ko}_{C_2})^{C_2} \arrow[d, "\nabla"] \\ \Sigma^1\mathrm{ko} \arrow[r, "\iota^*\eta"'] & \mathrm{ko} \end{tikzcd}$$

where ∇ is the fold map, as both the sign representation σ and the trivial representation 1 of C_2 restrict to the 1-dimensional trivial representation of the trivial group. This shows that η^{C_2} factors through the fiber of ∇, so that η^{C_2} must be of the form $(k, -k)$ for some integer k. On the other hand, we have the commutative diagram

$$\begin{tikzcd} \mathrm{ko} \otimes \mathrm{RO}(C_2) \arrow[r] & \mathrm{ko} \arrow[r] & \mathrm{ko} \otimes \mathrm{RO}(C_2) \\ (\mathrm{ko}_{C_2})^{C_2} \arrow[r, "\rho^{C_2}"] \arrow[u, "\cong"] & (\Sigma^{1,1}\mathrm{ko}_{C_2})^{C_2} \arrow[r, "\eta^{C_2}"] \arrow[u, "\cong"] & (\mathrm{ko}_{C_2})^{C_2} \arrow[u, "\cong"] \\ (S^{0,0})^{C_2} \arrow[r, "\rho^{C_2}"] \arrow[u] & (S^{1,1})^{C_2} \arrow[r, "\eta^{C_2}"] \arrow[u] & (S^{0,0})^{C_2} \arrow[u] \end{tikzcd}$$

According to Lemma 10.9, on the sphere $\eta\rho$ induces multiplication by $(C_2 - 2)$ under the isomorphism $\pi_{0,0} \cong A(C_2)$. The outer vertical compositions induce the linearization isomorphism $A(C_2) \cong \mathrm{RO}(C_2)$ on π_0. It follows that the top row induces multiplication by $(\sigma - 1)$ on homotopy. We conclude that η^{C_2} is $(-1, 1)$. $\square$

Definition 10.11. The complexification map $\mathrm{KO}_{C_2} \xrightarrow{c} K\mathbb{R}$ assigns to an equivariant real bundle $E \to X$ the associated bundle $\mathbb{C} \otimes_{\mathbb{R}} E \to X$, where C_2 acts on $\mathbb{C}$ via complex conjugation. We denote by $\mathrm{ko}_{C_2} \xrightarrow{c} k\mathbb{R}$ the associated map on connective covers.

Remark 10.12. Alternatively, from the point of view algebraic $\mathbb{K}$-theory, the complexification map can be described as $\mathbb{K}_{C_2}(\iota)$, where $\mathbb{R} \xrightarrow{\iota} \mathbb{C}$ is the inclusion of C_2-equivariant topological rings.

Proposition 10.13. *The Hopf map η induces a cofiber sequence*

$$\Sigma^{1,1}\mathrm{ko}_{C_2} \xrightarrow{\eta} \mathrm{ko}_{C_2} \xrightarrow{c} k\mathbb{R}. \tag{10-1}$$

Proof. On underlying spectra, this is the classical cofiber sequence

$$\Sigma\mathrm{ko} \xrightarrow{\eta} \mathrm{ko} \to ku.$$

On fixed points, according to Lemma 10.5 the sequence (10-1) induces a sequence

$$\mathrm{ko} \xrightarrow{\eta^{C_2}} \mathrm{ko} \vee \mathrm{ko} \xrightarrow{c^{C_2}} \mathrm{ko}.$$

By Lemma 10.10, the map η^{C_2} is of the form $(-1, 1)$. For any real C_2-representation V, the construction $\mathbb{C} \otimes_{\mathbb{R}} V$ only depends on the dimension of V, which implies that c^{C_2} is the fold map. So the fixed points sequence is also a cofiber sequence. □

Remark 10.14. From the point of view of spectral Mackey functors [Guillou and May 2011; Barwick 2017], the cofiber sequence (10-1) is the cofiber sequence of Mackey functors

$$\begin{tikzcd}
\mathrm{ko} \arrow[r, "{(1,-1)}"] \arrow[d, bend right, "0"'] & \mathrm{ko} \vee \mathrm{ko} \arrow[r, "\nabla"] \arrow[d, bend right, "\nabla"'] & \mathrm{ko} \arrow[d, bend right, "c"'] \\
\Sigma^{1,1}\mathrm{ko} \arrow[r, "\eta"] \arrow[u, bend right, "\eta"'] \arrow[loop below, "\mathrm{sign}"] & \mathrm{ko} \arrow[r, "c"] \arrow[u, bend right, "\Delta"'] \arrow[loop below, "\mathrm{triv}"] & ku \arrow[u, bend right, "r"'] \arrow[loop below, "\mathrm{conj}"]
\end{tikzcd}$$

where $ku \xrightarrow{r} \mathrm{ko}$ considers a rank n complex bundle as a rank $2n$ real bundle.

Theorem 10.15. *The C_2-equivariant cohomology of* ko_{C_2}*, as a module over* $\mathcal{A}^{C_2}$*, is*

$$H^{*,*}_{C_2}(\mathrm{ko}_{C_2}; \mathbb{F}_2) \cong \mathcal{A}^{C_2} /\!/ \mathcal{A}^{C_2}(1).$$

Proof. According to [Ricka 2015, Corollary 6.19], we have $H^{*,*}_{C_2}(k\mathbb{R}) \cong \mathcal{A}^{C_2} /\!/ \mathcal{E}^{C_2}(1)$. Since η induces the trivial map on equivariant cohomology, the sequence (10-1) induces a short exact sequence

$$0 \to H^{*-2,*-1}_{C_2}(\mathrm{ko}_{C_2}) \xrightarrow{i} \mathcal{A}^{C_2} /\!/ \mathcal{E}^{C_2}(1) \xrightarrow{j} H^{*,*}_{C_2}(\mathrm{ko}_{C_2}) \to 0 \tag{10-2}$$

of A^{C_2}-modules.

The cofiber $C\eta$ is a 2-cell complex that supports a Sq^2 in cohomology. It follows that the composition

$$k\mathbb{R} \simeq \mathrm{ko}_{C_2} \wedge C(\eta) \to \Sigma^{2,1}\mathrm{ko}_{C_2} \hookrightarrow \Sigma^{2,1}\mathrm{ko}_{C_2} \wedge C(\eta)$$

induces the map

$$\mathcal{A}^{C_2} /\!/ \mathcal{E}^{C_2}(1) \xrightarrow{ij} \mathcal{A}^{C_2} /\!/ \mathcal{E}^{C_2}(1) : 1 \mapsto \mathrm{Sq}^2.$$

In particular, the composition $\mathcal{A}^{C_2} \to \mathcal{A}^{C_2}/\!/\mathcal{E}^{C_2}(1) \xrightarrow{j} H^{*,*}_{C_2}(\mathrm{ko}_{C_2})$ factors through $\mathcal{A}^{C_2}/\!/\mathcal{A}^{C_2}(1)$. Given the right $\mathcal{E}^{C_2}(1)$-module decomposition

$$\mathcal{A}^{C_2}(1) \cong \mathcal{E}^{C_2}(1) \oplus \Sigma^{2,1}\mathcal{E}^{C_2}(1),$$

it follows that the sequence (10-2) sits in a diagram

$$\begin{tikzcd}
0 \arrow[r] & H^{*-2,*-1}_{C_2}(\mathrm{ko}_{C_2}) \arrow[r] & \mathcal{A}^{C_2}/\!/\mathcal{E}^{C_2}(1) \arrow[r] & H^{*,*}_{C_2}(\mathrm{ko}_{C_2}) \arrow[r] & 0 \\
0 \arrow[r] & \Sigma^{2,1}\mathcal{A}^{C_2}/\!/\mathcal{A}^{C_2}(1) \arrow[r] \arrow[u, hook] & \mathcal{A}^{C_2}/\!/\mathcal{E}^{C_2}(1) \arrow[r] \arrow[u, equal] & \mathcal{A}^{C_2}/\!/\mathcal{A}^{C_2}(1) \arrow[r] \arrow[u, two heads] & 0
\end{tikzcd}$$

The outer two maps agree up to suspension, so they are both isomorphisms. □

Corollary 10.16. *The E_2-page of the Adams spectral sequence for* ko_{C_2} *is*

$$E_2 \cong \mathrm{Ext}_{\mathcal{A}^{C_2}}(H^{*,*}_{C_2}(\mathrm{ko}_{C_2}), \mathbb{M}_2^{C_2}) \cong \mathrm{Ext}_{C_2}(1).$$

Proof. This is a standard change of rings isomorphism [Ravenel 1986, Theorem A1.3.12], using that $H^{*,*}_{C_2}(\mathrm{ko}_{C_2})$ is isomorphic to $\mathcal{A}^{C_2}/\!/\mathcal{A}^{C_2}(1)$. Note that the change of rings theorem applies by [Ricka 2015, Corollary 6.15]. □

Remark 10.17. Working in the 2-complete category, it is also possible to build ko_{C_2} using the "Tate diagram" approach. See, for example, [Greenlees 2018] for a nice description of this approach. According to this approach, one specifies a C_2-spectrum X by giving three pieces of data:

(1) an underlying spectrum X^e with C_2-action,

(2) a geometric fixed points spectrum X^{gC_2}, and

(3) a map $X^{gC_2} \to (X^e)^{tC_2}$ from the geometric fixed points to the Tate construction.

In our case, the underlying spectrum is ko with trivial C_2-action. The rest of the Tate diagram information is given by the following result.

Proposition 10.18. *The geometric fixed points of* ko_{C_2} *is* $\bigvee_{k\geq 0} \Sigma^{4k} H\hat{\mathbb{Z}}_2$, *and the map* $(\mathrm{ko}_{C_2})^{gC_2} \to \mathrm{ko}^{tC_2}$ *is the connective cover.*

Proof. The Tate construction ko^{tC_2} was computed by Davis and Mahowald [1984, Theorem 1.4] to be $\bigvee_{n\in\mathbb{Z}} \Sigma^{4n} H\hat{\mathbb{Z}}_2$. For the interpretation of the Davis–Mahowald calculation in terms of the Tate construction, see [May 1996, Section XXI.3].

The geometric fixed points sit in a cofiber sequence

$$\mathrm{ko} \wedge \mathbb{RP}^\infty_+ \simeq \mathrm{ko}_{hC_2} \to (\mathrm{ko}_{C_2})^{C_2} \to (\mathrm{ko}_{C_2})^{gC_2},$$

which we can write as

$$\mathrm{ko} \vee (\mathrm{ko} \wedge \mathbb{RP}^\infty) \to \mathrm{ko} \vee \mathrm{ko} \to (\mathrm{ko}_{C_2})^{gC_2}.$$

The left map is a map of ko-modules, and we consider the simpler cofiber sequence

$$\mathrm{ko} \wedge \mathbb{RP}^\infty \xrightarrow{\mathrm{ko} \wedge t} \mathrm{ko} \to (\mathrm{ko}_{C_2})^{gC_2},$$

where $t : \mathbb{RP}^\infty \to S^0$ is the Kahn–Priddy transfer. As in [Ravenel 1986, Section 1.5], we write R for the cofiber of t, so that $(\mathrm{ko}_{C_2})^{gC_2} \simeq \mathrm{ko} \wedge R$. As Adams explained in [Adams 1974], the cohomology of R has a filtration as $\mathcal{A}^{\mathrm{cl}}(1)$-modules in which the associated graded object is $\bigoplus_{k \geq 0} \Sigma^{4k} \mathcal{A}^{\mathrm{cl}}(1)//\mathcal{A}^{\mathrm{cl}}(0)$. It follows that $\mathrm{ko} \wedge R \simeq \bigvee_{k \geq 0} \Sigma^{4k} H\hat{\mathbb{Z}}_2$.

Similarly, the associated graded for $\mathrm{colim}_n\, \mathrm{H}^*(\Sigma \mathbb{RP}^\infty_{-n})$ is

$$\bigoplus_{k \in \mathbb{Z}} \Sigma^{4k} \mathcal{A}^{\mathrm{cl}}(1)//\mathcal{A}^{\mathrm{cl}}(0).$$

The map $R \to \mathrm{holim}_n\, \Sigma \mathbb{RP}^\infty_{-n}$ is surjective on cohomology, and the same is true for the induced map $R \wedge \mathrm{ko} \to \mathrm{holim}_n (\mathbb{RP}^\infty_{-n} \wedge \Sigma \mathrm{ko})$. We conclude that the map

$$\bigvee_{k \geq 0} \Sigma^{4k} H\hat{\mathbb{Z}}_2 \simeq (\mathrm{ko}_{C_2})^{gC_2} \to \mathrm{ko}^{tC_2} \simeq \operatorname*{holim}_n (\mathbb{RP}^\infty_{-n} \wedge \Sigma \mathrm{ko})$$

is a split inclusion in homotopy and therefore a connective cover. □

Remark 10.19. Note that the description of geometric fixed points given here is confirmed by Corollary 4.2. That is, the geometric fixed points of a C_2-spectrum X are given by the categorical fixed points of $S^{\infty,\infty} \wedge X$, where

$$S^{\infty,\infty} = \mathrm{colim}(S^{n,n} \xrightarrow{\rho} S^{n+1,n+1}).$$

Thus the geometric fixed points are computed by the ρ-inverted Adams spectral sequence. As we recall in the next section, the homotopy element 2 is detected by the element $h_0 + \rho h_1$ in Ext. Thus the element $\rho^k h_1^k \tau^{4j}$ of Corollary 4.2 detects 2^k in the $4j$-stem of the geometric fixed points.

11. The homotopy ring

In this section, we will describe the bigraded homotopy ring $\pi_{*,*}(\mathrm{ko}_{C_2})$ of ko_{C_2}. We are implicitly completing the homotopy groups at 2 so that the Adams spectral sequence converges [Hu and Kriz 2001, Corollary 6.47].

It turns out that the Adams spectral sequence collapses, so that $\mathrm{Ext}_{C_2}(1)$ is an associated graded object of $\pi_{*,*}(\mathrm{ko}_{C_2})$. Nevertheless, the Adams spectral sequence hides much of the multiplicative structure.

Recall that the Milnor–Witt stem of X is defined (see [Dugger and Isaksen 2017a]) as the direct sum

$$\Pi_n(X) \cong \bigoplus_i \pi_{n+i,i}(X).$$

Proposition 11.1. *There are no nonzero differentials in the Adams spectral sequence for* ko_{C_2}.

Proof. This follows by inspection of the E_2-page, shown in the charts in Section 12.

Adams d_r differentials decrease the stem by 1, increase the filtration by r, and preserve the weight. It follows that Adams differentials decrease the Milnor–Witt stem by 1. Every class in Milnor–Witt stem congruent to 3 modulo 4 is infinitely ρ-divisible. As there are no infinitely ρ-divisible classes in Milnor–Witt stem congruent to 2 modulo 4, it follows that there are no nonzero differentials supported in the Milnor–Witt $(4k+3)$-stem.

Every class in Milnor–Witt stem $4k$ supports an infinite tower of either h_0-multiples or h_1-multiples, while there are no such towers in Milnor–Witt stem $4k+1$. It follows that there cannot be any nonzero differentials emanating from the $(4k+1)$-Milnor–Witt-stem. Finally, direct inspection shows there cannot be any nonzero differentials starting in the Milnor–Witt $(4k+2)$ or $4k$-stems. □

The structure of the Milnor–Witt n-stem $\Pi_n(\mathrm{ko}_{C_2})$ of course depends on n. The description of these Milnor–Witt stems naturally breaks into cases, depending on the value of $n \pmod 4$.

The notation that we will use for specific elements of $\pi_{*,*}(\mathrm{ko}_{C_2})$ is summarized in Table 9. The definition of each element is discussed in detail in the following sections.

11A. ***The Milnor–Witt 0-stem.*** Our first task is to describe the Milnor–Witt 0-stem $\Pi_0(\mathrm{ko}_{C_2})$. The other Milnor–Witt stems are modules over $\Pi_0(\mathrm{ko}_{C_2})$, and we will use this module structure heavily in order to understand them.

Proposition 11.2. *Let X be a C_2-equivariant spectrum, and let α belong to $\pi_{n,k}(X)$. The element α is divisible by ρ if and only if its underlying class $\iota^*(\alpha)$ in $\pi_n(\iota^* X)$ is zero.*

Proof. The C_2-equivariant cofiber sequence

$$C_{2\,+} \to S^{0,0} \xrightarrow{\rho} S^{1,1}$$

induces a long exact sequence

$$\cdots \to \pi_{n+1,k+1}(X) \xrightarrow{\rho} \pi_{n,k}(X) \xrightarrow{\iota^*} \pi_n(\iota^* X) \to \pi_{n+2,k+1}(X) \xrightarrow{\rho} \cdots. \qquad \square$$

Corollary 11.3. *There is a hidden ρ extension from Qh_1^3 to h_1^3 in the Adams spectral sequence.*

mw	(s, w)	element	detected by	defining relation
0	$(-1,-1)$	ρ	ρ	
0	$(1,1)$	η	h_1	
0	$(4,4)$	α	Qh_1^3	$\rho\alpha = \eta^3$
0	$(0,0)$	ω	h_0	$\omega = \eta\rho + 2$
4	$(0,-4)$	τ^4	τ^4	
0	$(8,8)$	β	$\frac{Qh_1^4}{\rho^3}$	$4\beta = \alpha^2$
2	$(0,-2)$	$\tau^2\omega$	$\tau^2 h_0$	$(\tau^2\omega)^2 = 2\omega\cdot\tau^4$
-2	$(0,2)$	$\tau^{-2}\omega$	$\frac{\gamma}{\tau}$	$\tau^4\cdot\tau^{-2}\omega = \tau^2\omega$
-4	$(0,4)$	$\tau^{-4}\omega$	$\frac{\gamma}{\tau^3}$	$\tau^4\cdot\tau^{-4}\omega = \omega$
$-5-4k$	$(0,5+4k)$	$\frac{\Gamma}{\tau^{4+4k}}$	$\frac{\gamma}{\tau^{4+4k}}$	$\tau^4\cdot\frac{\Gamma}{\tau^{4+4k}} = \frac{\Gamma}{\tau^{4+4(k-1)}}$
1	$(1,0)$	$\tau\eta$	τh_1	
2	$(4,2)$	$\tau^2\alpha$	a	$2\tau^2\alpha = \alpha\cdot\tau^2\omega$

Table 9. Notation for $\pi_{*,*}(\mathrm{ko}_{C_2})$.

Proof. Recall that η^3 is zero in $\pi_3(\mathrm{ko})$. Proposition 11.2 implies that η^3 in $\pi_{3,3}(\mathrm{ko}_{C_2})$ is divisible by ρ. The only possibility is that there is a hidden extension from Qh_1^3 to h_1^3. □

Proposition 11.4. *The element η in $\pi_{1,1}(\mathrm{ko}_{C_2})$ is detected by h_1.*

Proof. The restriction $\iota^*(\eta)$ of η is the classical η, which is detected by the classical element h_1. As all other elements of $\mathrm{Ext}_{\mathcal{A}^{C_2}(1)}$ in the 1-stem and weight 1 all live in higher filtration, the result follows. □

Definition 11.5. Let α be an element in $\pi_{4,4}(\mathrm{ko}_{C_2})$ detected by Qh_1^3 such that $\rho\alpha = \eta^3$.

Corollary 11.3 guarantees that such an element α exists.

There are many elements of $\pi_{4,4}$ detected by Qh_1^3 because of the presence of elements in higher Adams filtration. The condition $\rho\alpha = \eta^3$ narrows the possibilities, but still does not determine a unique element because of the elements $\frac{\gamma}{\tau}h_0^k a$ in higher Adams filtration. For our purposes, this remaining choice makes no difference.

Definition 11.6. Let ω be the element $\eta\rho + 2$ of $\pi_{0,0}^{C_2}(\mathrm{ko}_{C_2})$.

As for ρ and η, the element ω comes from the homotopy groups of the equivariant sphere spectrum. Strictly speaking, there is no need for the notation ω since it can be written in terms of other elements. Nevertheless, it is convenient because ω plays a central role. According to Lemma 10.9, ω corresponds to the element C_2 of the Burnside ring $A(C_2)$.

Note that ω is detected by h_0, while 2 is detected by $h_0+\rho h_1$. For this reason, ω, rather than 2, plays the role of the zeroth Hopf map in the equivariant (and $\mathbb{R}$-motivic) context. Also note that ω equals $1-\epsilon$, where ϵ is the twist

$$S^{1,1}\wedge S^{1,1}\to S^{1,1}\wedge S^{1,1}.$$

Proposition 11.7. *The homotopy class η^5 is divisible by* 2.

Proof. The relation $\omega\eta=0$ was established by Morel [2004] in the $\mathbb{R}$-motivic stable stems, and the equivariant stems agree with the $\mathbb{R}$-motivic ones in the relevant degrees [Dugger and Isaksen 2017b, Theorem 4.1]. (See also [Dugger and Isaksen 2013] for a geometric argument for this relation given in the motivic context. This geometric argument works just as well equivariantly.)

Using the defining relation for α, it follows that

$$-2\eta\alpha=\rho\eta^2\alpha=\eta^5. \qquad \square$$

Proposition 11.7 was already known to be true in the homotopy of the C_2-equivariant sphere spectrum [Bredon 1968]. The divisibility of the elements η^k is very much related to work of Landweber [1969].

Definition 11.8. Let τ^4 be an element of $\pi_{0,-4}(\mathrm{ko}_{C_2})$ that is detected by τ^4.

The element τ^4 is not uniquely determined because of elements in higher Adams filtration. For our purposes, we may choose an arbitrary such element.

Proposition 11.9. (1) *There is a hidden τ^4 extension from Qh_1^3 to $\tau^2 a$.*

(2) *There is a hidden τ^4 extension from $\frac{Q}{\rho^3}h_1^4$ to b.*

Proof. (1) The product $\rho\alpha\cdot\tau^4$ equals $\tau^4\cdot\eta^3$, which is detected by $\tau^4\cdot h_1^3$. This last expression equals $\rho\cdot\tau^2 a$ in Ext.

(2) Part (1) implies that there is a hidden τ^4 extension from Qh_1^4 to $\rho^3 b$, since $h_1\cdot\tau^2 a$ equals $\rho^3 b$ in Ext. This means that there is a hidden τ^4 extension from $\frac{Q}{\rho^3}h_1^4$ to b, since $\rho^3\cdot\frac{Q}{\rho^3}h_1^4$ equals Qh_1^4 in Ext. $\square$

Lemma 11.10. *The class α^2 in $\pi_{8,8}(\mathrm{ko}_{C_2})$ is divisible by* 4.

Proof. By Proposition 11.9, the multiplication map

$$\tau^4:\pi_{8,8}(\mathrm{ko}_{C_2})\xrightarrow{\cong}\pi_{8,4}(\mathrm{ko}_{C_2})$$

is an isomorphism. By considering the effect of multiplication by τ^4 in Ext, we see that

$$\tau^4:\pi_{8,4}(\mathrm{ko}_{C_2})\xrightarrow{\cong}\pi_{8,0}(\mathrm{ko}_{C_2})$$

is also an isomorphism. Thus it suffices to show that $(\tau^4)^2\alpha^2$ is 4-divisible in $\pi_{8,0}(\mathrm{ko}_{C_2})$. But $(\tau^4)^2\cdot\alpha^2$ is detected by $(\tau^2 a)^2$ by Proposition 11.9 (1), which equals $(h_0+\rho h_1)^2\tau^4 b$ in Ext. Finally, observe that $h_0+\rho h_1$ detects 2. $\square$

Definition 11.11. Let β be the element of $\pi_{8,8}(\mathrm{ko}_{C_2})$ detected by $\frac{Q}{\rho^3}h_1^4$ and satisfying $4\beta = \alpha^2$.

Note that β is uniquely determined by α, even though there are elements of higher Adams filtration, because there is no 2-torsion in $\pi_{8,8}(\mathrm{ko}_{C_2})$.

Proposition 11.12. $\rho^3\beta = \eta\alpha$.

Proof. The defining relation for β implies that $4\rho^3\beta$ equals $\rho^3\alpha^2$, which equals $\rho^2\eta^3\alpha$ by the defining relation for α. Using the relation $(\eta\rho+2)\eta = 0$, this element equals $4\eta\alpha$. Finally, there is no 2-torsion in $\pi_{5,5}(\mathrm{ko}_{C_2})$. □

Proposition 11.13. *The* (2*-completed*) *Milnor–Witt* 0*-stem of* ko_{C_2} *is*

$$\Pi_0(\mathrm{ko}_{C_2}) \cong \mathbb{Z}_2[\eta, \rho, \alpha, \beta]/(\rho(\eta\rho+2), \eta(\eta\rho+2), \rho\alpha - \eta^3, \rho^3\beta - \eta\alpha, \alpha^2 - 4\beta),$$

where the generators have degrees $(1, 1)$, $(-1, -1)$, $(4, 4)$, *and* $(8, 8)$ *respectively. These homotopy classes are detected by* h_1, ρ, Qh_1^3, *and* $\frac{Qh_1^4}{\rho^3}$ *in the Adams spectral sequence.*

Proof. The relations $\rho(\eta\rho+2)$ and $\eta(\eta\rho+2)$ are already true in the sphere [Morel 2004; Dugger and Isaksen 2013]. The third and fifth relations are part of the definitions of α and β, while the fourth relation is Proposition 11.12.

It remains to show that β^k is detected by $\frac{Q}{\rho^{4k-1}}h_1^{4k}$ and that $\alpha\beta^k$ is detected by $\frac{Q}{\rho^{4k-1}}h_1^{4k+4}$.

We assume for induction on k that β^k is detected by $\frac{Q}{\rho^{4k-1}}h_1^{4k}$. We have the relation $h_0 \cdot \frac{Q}{\rho^{4k-1}}h_1^{4k} = \frac{\gamma}{\tau^{4k-1}}b^k$ in Ext, so $\omega\beta^k$ is detected by $\frac{\gamma}{\tau^{4k-1}}b^k$ in Ext. Now b detects $\tau^4 \cdot \beta$ by Proposition 11.9 (2), so $\omega\beta^{k+1}$ is detected by $\frac{\gamma}{\tau^{4k-1}}b^{k+1}$. Finally, $\frac{\gamma}{\tau^{4k-1}}b^{k+1}$ equals $\tau^4 \cdot \frac{\gamma}{\tau^{4k+3}}b^{k+1}$ in Ext, which equals $\tau^4 \cdot h_0 \cdot \frac{Q}{\rho^{4k+3}}h_1^{4k+4}$.

We have now shown that $\tau^4 \cdot h_0 \cdot \frac{Q}{\rho^{4k+3}}h_1^{4k+4}$ detects $\tau^4 \cdot \omega\beta^{k+1}$. It follows that $\frac{Q}{\rho^{4k+3}}h_1^{4k+4}$ detects β^{k+1}.

A similar argument handles the case of $\alpha\beta^k$. □

11B. τ^4*-periodicity.* Before analyzing the other Milnor–Witt stems of ko_{C_2}, we will explore a piece of the global structure involving the element τ^4 of $\pi_{0,-4}(\mathrm{ko}_{C_2})$.

Proposition 11.14. *There are hidden* τ^4 *extensions*

(1) *from* $\frac{\gamma}{\tau}$ *to* $\tau^2 h_0$,

(2) *from* $\frac{\gamma}{\rho^2\tau}h_1^2$ *to* a,

(3) *from* $\frac{\gamma}{\tau^3}$ *to* h_0,

(4) *from* $\frac{\gamma}{\rho\tau^2}$ *to* τh_1.

Proof. (1) Recall that $\frac{\gamma}{\tau} \cdot a$ equals $h_0 \cdot Qh_1^3$ in Ext, so the hidden τ^4 extension on Qh_1^3 from Proposition 11.9(1) implies that there is a hidden τ^4 extension from $\frac{\gamma}{\tau} \cdot a$ to $\tau^2 h_0 a$. It follows that there is a hidden τ^4 extension from $\frac{\gamma}{\tau}$ to $\tau^2 h_0$.

(2) Using that $h_1^2 \cdot \tau^2 h_0$ equals $\rho^2 a$ in Ext, part (1) implies that there is a hidden τ^4 extension from $\frac{\gamma}{\tau} h_1^2$ to $\rho^2 a$.

(3) Recall that $\frac{\gamma}{\tau^3} \cdot b$ equals $h_0 \cdot \frac{Q}{\rho^3} h_1^4$ in Ext, so the hidden τ^4 extension on $\frac{Q}{\rho^3} h_1^4$ from Proposition 11.9(2) implies that there is a hidden τ^4 extension from $\frac{\gamma}{\tau^3} \cdot b$ to $h_0 b$. It follows that there is a hidden τ^4 extension from $\frac{\gamma}{\tau^3}$ to h_0.

(4) Using that ρa equals $h_1(\tau h_1)^2$ in Ext, part (2) implies that there is a hidden τ^4 extension from $\frac{\gamma}{\rho\tau} h_1^2$ to $h_1(\tau h_1)^2$. Now $\frac{\gamma}{\rho\tau} h_1^2$ equals $\frac{\gamma}{\rho\tau^2} h_1 \cdot \tau h_1$, so there is also a hidden τ^4 extension on $\frac{\gamma}{\rho\tau^2}$. □

The homotopy of ko_{C_2} is nearly τ^4-periodic, in the following sense.

Theorem 11.15. *Multiplication by τ^4 gives a homomorphism on Milnor–Witt stems*

$$\Pi_n(\mathrm{ko}_{C_2}) \to \Pi_{n+4}(\mathrm{ko}_{C_2})$$

which is

(1) *injective if $n = -4$,*

(2) *surjective (and zero) if $n = -5$,*

(3) *bijective in all other cases.*

Proof. (1) This is already true in Ext, except in the 0-stem. But the 0-stem is handled by Proposition 11.14(3).

(2) There is nothing to prove here, given that $\Pi_{-1}(\mathrm{ko}_{C_2}) = 0$.

(3) We give arguments depending on the residue of n modulo 4.

- $n \equiv 0 \pmod 4$: If $n < -4$, this is already true in Ext. For $n \geq 0$, this follows from the relation $\rho\alpha = \eta^3$ and the hidden τ^4 extensions on α and β given in Proposition 11.9.
- $n \equiv 1 \pmod 4$: For $n < -3$, this is already true in Ext. For $n \geq -3$, this follows from Proposition 11.14(4).
- $n \equiv 2 \pmod 4$: For $n < -2$, this is already true in Ext. For $n \geq -2$, this follows from Proposition 11.14(1) and (2).
- $n \equiv 3 \pmod 4$: This is already true in Ext. □

Remark 11.16. Another way to view the τ^4-periodicity is via the Tate diagram (Proposition 10.18). We have a cofiber sequence

$$EC_{2+} \wedge \mathrm{ko} \to \mathrm{ko}_{C_2} \to S^{\infty,\infty} \wedge \mathrm{ko}_{C_2}.$$

The homotopy orbit spectrum therefore captures the ρ-torsion. If $x \in \pi_{*,*}\mathrm{ko}_{C_2}$ is ρ-torsion, then so is $\tau^4 \cdot x$. But multiplication by τ^4 is an equivalence on underlying spectra and therefore gives an equivalence on homotopy orbits. This implies the τ^4-periodicity in the ρ-torsion.

11C. ***The Milnor–Witt n-stem with $n \equiv 0 \pmod 4$.*** Theorem 11.15 indicates that τ^4 multiplications are useful in describing the structure of the homotopy groups of ko_{C_2}. Therefore, our next task is to build on our understanding of $\Pi_0(\mathrm{ko}_{C_2})$ and to describe the subring $\bigoplus_{k\in\mathbb{Z}} \Pi_{4k}(\mathrm{ko}_{C_2})$ of $\pi_{*,*}\mathrm{ko}_{C_2}$.

The Ext charts indicate that the behavior of these groups differs for $k \geq 0$ and for $k < 0$.

Proposition 11.17. *$\bigoplus_{k\geq 0} \Pi_{4k}(\mathrm{ko}_{C_2})$ is isomorphic to $\Pi_0(\mathrm{ko}_{C_2})[\tau^4]$.*

Proof. This follows immediately from Theorem 11.15. □

Definition 11.18. Define $\tau^2\omega$ to be an element in $\pi_{0,-2}(\mathrm{ko}_{C_2})$ that is detected by $\tau^2 h_0$ such that $(\tau^2\omega)^2 = 2\omega\cdot\tau^4$.

An equivalent way to specify a choice of $\tau^2\omega$ is to require that the underlying map $\iota^*(\tau^2\omega)$ equals 2 in $\pi_0(\mathrm{ko})$.

Definition 11.19. For $k \geq 1$, let $\frac{\Gamma}{\tau^k}$ be an element of $\pi_{0,k+1}$ detected by $\frac{\gamma}{\tau^k}$ such that

(1) $\tau^4 \cdot \frac{\Gamma}{\tau} = \tau^2\omega$,

(2) $\tau^4 \cdot \frac{\Gamma}{\tau^3} = \omega$,

(3) $\tau^4 \cdot \frac{\Gamma}{\tau^k} = \frac{\Gamma}{\tau^{k-4}}$ when $k \geq 5$.

According to Theorem 11.15, the elements $\frac{\Gamma}{\tau^k}$ are uniquely determined by the stated conditions. Proposition 11.14 (1) and (3) allow us to choose $\frac{\Gamma}{\tau}$ and $\frac{\Gamma}{\tau^3}$ with the desired properties. As suggested by the defining relations for these elements, we will often write $\tau^{-2-4k}\omega$ for $\frac{\Gamma}{\tau^{1+4k}}$ and $\tau^{-4-4k}\omega$ for $\frac{\Gamma}{\tau^{3+4k}}$.

Proposition 11.20. *As a $\pi_0(\mathrm{ko}_{C_2})[\tau^4]$-module, $\bigoplus_{k\in\mathbb{Z}} \Pi_{4k}(\mathrm{ko}_{C_2})$ is isomorphic to the $\pi_0(\mathrm{ko}_{C_2})[\tau^4]$-module generated by 1 and the elements $\tau^{-4-4k}\omega$ for $k \geq 0$, subject to the relations*

$$\tau^4 \cdot \tau^{-4-4k}\omega = \tau^{-4k}\omega, \tag{1}$$

$$\rho \cdot \tau^{-4-4k}\omega = 0, \tag{2}$$

$$\eta \cdot \tau^{-4-4k}\omega = 0, \tag{3}$$

$$\tau^4 \cdot \tau^{-4}\omega = \omega. \tag{4}$$

Proof. This follows by inspection of the Ext charts, together with the defining relations for $\tau^{-4-4k}\omega$. □

11D. ***The Milnor–Witt n-stem with $n \equiv 1 \pmod 4$.***

Definition 11.21. Denote by $\tau\eta$ an element of $\pi_{1,0}(\mathrm{ko}_{C_2})$ that is detected by τh_1.

Note that $\tau\eta$ is not uniquely determined because of elements in higher Adams filtration, but the choice makes no practical difference. One way to specify a choice of $\tau\eta$ is to use the composition

$$S^{1,0} \to S^{0,0} \to \mathrm{ko}_{C_2},$$

where the first map is the image of the classical Hopf map $\eta : S^1 \to S^0$, and the second map is the unit.

Proposition 11.22. *As a* $\Pi_0(\mathrm{ko}_{C_2})[\tau^4]$*-module, there is an isomorphism*

$$\bigoplus_{k\in\mathbb{Z}} \Pi_{1+4k}(\mathrm{ko}_{C_2}) \cong \bigl(\Pi_0(\mathrm{ko}_{C_2})[(\tau^4)^{\pm 1}]/(2, \rho^2, \eta^2, \alpha)\bigr)\{\tau\eta\}.$$

Proof. This follows from inspection of the Ext charts, together with Theorem 11.15. □

11E. ***The Milnor–Witt n-stem with n ≡ 2 (mod 4).*** Recall from Definition 11.18 that $\tau^2\omega$ is an element of $\pi_{0,-2}(\mathrm{ko}_{C_2})$ that is detected by $\tau^2 h_0$.

Lemma 11.23. *The product* $\alpha \cdot \tau^2\omega$ *in* $\pi_{4,2}(\mathrm{ko}_{C_2})$ *is detected by* $h_0 a$.

Proof. The product $\tau^4 \cdot \alpha \cdot \tau^2\omega$ is detected by $\tau^4 h_0 a$ by Proposition 11.9(1). □

Definition 11.24. Define $\tau^2\alpha$ to be an element of $\pi_{4,2}(\mathrm{ko}_{C_2})$ that is detected by a such that $2 \cdot \tau^2\alpha$ equals $\alpha \cdot \tau^2\omega$.

Proposition 11.25. *As a* $\Pi_0(\mathrm{ko}_{C_2})[\tau^4]$*-module,* $\bigoplus_{k\in\mathbb{Z}} \Pi_{2+4k}(\mathrm{ko}_{C_2})$ *is isomorphic to the free* $\Pi_0(\mathrm{ko}_{C_2})[(\tau^4)^{\pm 1}]$*-module generated by* $\tau^2\omega$, $(\tau\eta)^2$, *and* $\tau^2\alpha$, *subject to the relations*

(1) $$\rho \cdot \tau^2\omega = 0,$$

(2) $$\alpha \cdot \tau^2\omega = 2 \cdot \tau^2\alpha,$$

(3) $$\rho(\tau\eta)^2 = \eta \cdot \tau^2\omega,$$

(4) $$2(\tau\eta)^2 = 0,$$

(5) $$\eta(\tau\eta)^2 = \rho \cdot \tau^2\alpha,$$

(6) $$\alpha(\tau\eta)^2 = 0,$$

(7) $$\eta \cdot \tau^2\alpha = 0,$$

(8) $$\alpha \cdot \tau^2\alpha = 2\beta \cdot \tau^2\omega.$$

Proof. Except for the last relation, this follows from inspection of the Ext charts, together with Theorem 11.15.

For the last relation, use that $2\alpha \cdot \tau^2\alpha$ equals $\tau^2\omega \cdot \alpha^2$ by the definition of $\tau^2\alpha$, and that $\tau^2\omega \cdot \alpha^2$ equals $4\beta \cdot \tau^2\omega$ by the defining relation for β. As there is no 2-torsion in this degree, relation (8) follows. □

11F. *The Milnor–Witt n-stem with n ≡ 3 (mod 4).* The structure of

$$\bigoplus_{k\in\mathbb{Z}} \Pi_{4k+3}(\mathrm{ko}_{C_2})$$

is qualitatively different than the other cases because it contains elements that are infinitely divisible by ρ. The Ext charts show that $\bigoplus_{k\in\mathbb{Z}} \Pi_{4k+3}(\mathrm{ko}_{C_2})$ is concentrated in the range $k \le -2$.

The elements $\frac{\Gamma}{\tau^{4k}}$ are infinitely divisible by both ρ and τ^4. We write $\frac{\Gamma}{\rho^j\tau^{4k}}$ for an element such that $\rho^j \cdot \frac{\Gamma}{\rho^j\tau^{4k}}$ equals $\frac{\Gamma}{\tau^{4k}}$.

By inspection of the Ext charts, we see that $\bigoplus_{k\le 0} \Pi_{4k-5}(\mathrm{ko}_{C_2})$ is generated as an abelian group by the elements $\frac{\Gamma}{\rho^j\tau^{4k}}$. The $\Pi_0(\mathrm{ko}_{C_2})[\tau^4]$-module structure on $\bigoplus_{k\le 0} \Pi_{4k-5}(\mathrm{ko}_{C_2})$ is then governed by the orders of these elements, together with the relations

$$\alpha \cdot \frac{\Gamma}{\tau^{4k}} = -8\frac{\Gamma}{\rho^4\tau^{4k}}$$

and

$$\beta \cdot \frac{\Gamma}{\tau^{4k}} = 16\frac{\Gamma}{\rho^8\tau^{4k}}.$$

The first relation follows from the calculation

$$\alpha \cdot \frac{\Gamma}{\tau^{4k}} = \rho\alpha \cdot \frac{\Gamma}{\rho\tau^{4k}} = \eta^3 \cdot \frac{\Gamma}{\rho\tau^{4k}} = (\eta\rho)^3 \cdot \frac{\Gamma}{\rho^4\tau^{4k}} = (-2)^3 \cdot \frac{\Gamma}{\rho^4\tau^{4k}} = -8\frac{\Gamma}{\rho^4\tau^{4k}}.$$

The second relation follows from a similar argument, using that $\rho^3\beta = \eta\alpha$.

Proposition 11.26. *The order of* $\frac{\Gamma}{\tau^{4k}\rho^j}$ *is* $2^{\varphi(j)+1}$, *where* $\varphi(j)$ *is the number of positive integers* $0 < i \le j$ *such that* $i \equiv 0, 1, 2$ *or* $4 \pmod 8$.

Proof. Since $h_0 + \rho h_1$ detects the element 2, the result is represented by the chart on page 625, in stems zero to sixteen. As the top edge of the region is (8, 4)-periodic, this gives the result in higher stems as well. □

Remark 11.27. Proposition 11.26 is an independent verification of a well-known calculation. We follow the argument given in [Dugger 2005, Appendix B].

Let $\mathbb{R}^{q,q}$ be the antipodal C_2-representation on $\mathbb{R}^q$. Consider the cofiber sequence

$$S(q,q) \to D(q,q) \to S^{q,q},$$

where $S(q,q) \subset D(q,q) \subset \mathbb{R}^{q,q}$ are the unit sphere and unit disk respectively. Since $D(q,q)$ is equivariantly contractible, this gives the exact sequence

$$\pi_{m,0}(\mathrm{ko}_{C_2}) \leftarrow \pi_{m+q,q}(\mathrm{ko}_{C_2}) \leftarrow \mathrm{ko}_{C_2}^{-m-1,0}(S(q,q)) \leftarrow \pi_{m+1,0}(\mathrm{ko}_{C_2}).$$

If $m \le -2$, the outer groups vanish. Moreover, C_2 acts freely on $S(q,q)$, and the orbit space is $S(q,q)/C_2 \cong \mathbb{RP}^{q-1}$. It follows [May 1996, Section XIV.1] that

$$\mathrm{ko}_{C_2}^{-m-1}(S(q,q)) \cong \mathrm{ko}^{-m-1}(\mathbb{RP}^{q-1})$$

when $m \leq -2$ and $q \geq 1$. In particular,

$$\pi_{j,j+5}(\mathrm{ko}_{C_2}) \cong \mathrm{ko}^4(\mathbb{RP}^{j+4}),$$

and the latter groups are known (see [Davis and Mahowald 1979, Section 2]) to be cyclic of order $\varphi(j)$.

Having described all of the Milnor–Witt stems as $\Pi_0(\mathrm{ko}_{C_2})[\tau^4]$-modules, it remains only to understand products of the various $\Pi_0(\mathrm{ko}_{C_2})[\tau^4]$-module generators.

Proposition 11.28. *In the homotopy groups of* ko_{C_2}, *we have the relations*

$$(\tau^2\omega)^2 = 2\omega \cdot \tau^4, \tag{1}$$

$$\tau^2\omega \cdot \tau^2\alpha = \tau^4 \cdot \omega\alpha, \tag{2}$$

$$(\tau^2\alpha)^2 = 2\tau^4 \cdot \omega\beta. \tag{3}$$

Proof. The first relation is part of the definition of $\tau^2\omega$.

For the second relation, use the definitions of $\tau^2\alpha$ and of $\tau^2\omega$ to see that

$$2 \cdot \tau^2\omega \cdot \tau^2\alpha = (\tau^2\omega)^2\alpha = 2\tau^4 \cdot \omega\alpha.$$

The group $\pi_{4,0}(\mathrm{ko}_{C_2})$ has no 2-torsion, so it follows that $\tau^2\omega \cdot \tau^2\alpha$ equals $\tau^4 \cdot \omega\alpha$.

The proof of the third relation is similar. Use the definitions of $\tau^2\alpha$ and β and part (2) to see that

$$2(\tau^2\alpha)^2 = \tau^2\omega \cdot \tau^2\alpha \cdot \alpha = \tau^4 \cdot \omega\alpha^2 = 4\tau^4 \cdot \omega\beta.$$

The group $\pi_{8,4}(\mathrm{ko}_{C_2})$ has no 2-torsion. □

11G. *The homotopy ring of $k\mathbb{R}$.* We may similarly describe the homotopy of $k\mathbb{R}$. Since this has already appeared in the literature (see [Greenlees and Meier 2017, Section 11]), we do not give complete details.

We use the forgetful exact sequence of Proposition 11.2 to define the homotopy classes listed in Table 10. In each case, the forgetful map is injective, and we stipulate that τ^4 restricts to 1, that v_1 and $\tau^{-4}v_1$ restrict to the Bott element, and that $\tau^2\omega$, $\tau^{-2}\omega$, and $\tau^{-4}\omega$ all restrict to 2.

Proposition 11.29. *There are* τ^4*-extensions*

$$\tau^4 \cdot \tau^{-2}\omega = \tau^2\omega, \quad \tau^4 \cdot \tau^{-4}\omega = 2, \quad \tau^4 \cdot \tau^{-4}v_1 = v_1.$$

Proof. These all follow from the definition of these classes using the forgetful exact sequence of Proposition 11.2. Since the forgetful map is a ring homomorphism, we get that

$$\iota^*(\tau^4 \cdot \tau^{-2}\omega) = \iota^*(\tau^4) \cdot \iota^*(\tau^{-2}\omega) = 1 \cdot 2 = 2.$$

Since the forgetful map is injective in this degree, we conclude that $\tau^4 \cdot \tau^{-2}\omega = \tau^2\omega$. The same argument handles the other relations just as well. □

In order to describe the Milnor–Witt 0-stem of $k\mathbb{R}$, it is convenient to write $\alpha = \tau^{-2}\omega v_1^2$ and $\beta = \tau^{-4}v_1 \cdot v_1^3$.

Proposition 11.30. *The* (2*-completed*) *Milnor–Witt* 0*-stem of* $k\mathbb{R}$ *is*

$$\Pi_0(k\mathbb{R}) \cong \mathbb{Z}_2[\rho, \alpha, \beta]/(2\rho, \rho\alpha, \rho^3\beta, \alpha^2 - 4\beta),$$

where the generators have degrees $(-1, -1)$, $(4, 4)$, *and* $(8, 8)$ *respectively. These homotopy classes are detected by* ρ, $\frac{\gamma}{\tau}v_1^2$, *and* $\frac{\gamma}{\rho^2\tau^2}v_1^3$ *in the Adams spectral sequence.*

The other Milnor–Witt stems, aside from those in degree $-5-4k$, can all be described cleanly as ideals in $\Pi_0(k\mathbb{R})$. The τ^4-periodicities asserted in the following results all hold already on the level of Ext, except for the τ^4-multiplications from $\mathrm{Ext}_{\mathrm{NC}}$ to $\mathrm{Ext}_{\mathcal{E}(1)}$. Those are handled by Proposition 11.29. We recommend the reader to consult the diagram on page 630 in order to visualize the following results.

Proposition 11.31. *The map* $\Pi_{-4}(k\mathbb{R}) \xrightarrow{\tau^4} \Pi_0(k\mathbb{R})$ *is a monomorphism and identifies* $\Pi_{-4}(k\mathbb{R})$ *with the ideal generated by* 2, α, *and* β. *If* $k \neq -1$, *then multiplication by* τ^4 *is an isomorphism* $\Pi_{4k}(k\mathbb{R}) \cong \Pi_{4(k+1)}(k\mathbb{R})$.

Thus the Milnor–Witt stems of degree $4k$ break up into two families, which are displayed as the first two rows of the diagram on page 630.

Proposition 11.32. *The map* $\Pi_{-1}(k\mathbb{R}) \xrightarrow{v_1} \Pi_0(k\mathbb{R})$ *is a monomorphism and identifies* $\Pi_{-1}(k\mathbb{R})$ *with the ideal generated by* α *and* β. *Multiplication by* τ^4 *is a split epimorphism*

$$\frac{\mathbb{F}_2[\rho]}{\rho^\infty} \to \Pi_{-5}(k\mathbb{R}) \xrightarrow{\tau^4} \Pi_{-1}(k\mathbb{R}).$$

If $k \neq -1$, *then multiplication by* τ^4 *is an isomorphism* $\Pi_{-1+4k}(k\mathbb{R}) \cong \Pi_{3+4k}(k\mathbb{R})$.

Proposition 11.33. *The map* $\Pi_{-2}(k\mathbb{R}) \xrightarrow{v_1} \Pi_{-1}(k\mathbb{R})$ *is an isomorphism. Multiplication by* τ^4 *is an isomorphism* $\Pi_{4k-2}(k\mathbb{R}) \cong \Pi_{4k+2}(k\mathbb{R})$ *for all* $k \in \mathbb{Z}$.

Proposition 11.34. *The map* $\Pi_{-3}(k\mathbb{R}) \xrightarrow{v_1^3} \Pi_0(k\mathbb{R})$ *is a monomorphism and identifies* $\Pi_{-3}(k\mathbb{R})$ *with the ideal generated by* β. *Multiplication by* τ^4 *is an isomorphism* $\Pi_{4k-3}(k\mathbb{R}) \cong \Pi_{4k+1}(k\mathbb{R})$ *for all* $k \in \mathbb{Z}$.

Combining the information from Table 3 and Table 8 yields the induced homomorphism on homotopy groups as described in Table 11. Note that all values $c_*(x)$ are to be interpreted as correct modulo higher powers of 2.

Remark 11.35. Note that the results of this section provide another means of demonstrating the τ^4-periodicity in ko_{C_2} established in Section 11B. More specifically, the τ^4-extensions given in Proposition 11.29, together with the homomorphism c_* as described in Table 11, imply the τ^4-extensions given in Proposition 11.14.

mw	(s, w)	element	detected by	definition
0	$(-1, -1)$	ρ	ρ	
1	$(2, 1)$	v_1	v_1	$\iota^*(v_1) = v_1$
4	$(0, -4)$	τ^4	τ^4	$\iota^*(\tau^4) = 1$
2	$(0, -2)$	$\tau^2\omega$	$\tau^2 h_0$	$\iota^*(\tau^2\omega) = 2$
-2	$(0, 2)$	$\tau^{-2}\omega$	$\frac{\gamma}{\tau}$	$\iota^*(\tau^{-2}\omega) = 2$
-4	$(0, 4)$	$\tau^{-4}\omega$	$\frac{\gamma}{\tau^3}$	$\iota^*(\tau^{-4}\omega) = 2$
-3	$(2, 5)$	$\tau^{-4}v_1$	$\frac{\gamma}{\rho^2\tau^2}$	$\iota^*(\tau^{-4}v_1) = v_1$
-5	$(0, 5)$	$\frac{\Gamma}{\tau^4}$	$\frac{\gamma}{\tau^4}$	

Table 10. Notation for $\pi_{*,*}(k\mathbb{R})$.

mw	(s, w)	$x \in \pi_{*,*}(\mathrm{ko}_{C_2})$	$c_* x \in \pi_{*,*}(k\mathbb{R})$
0	$(-1, -1)$	ρ	ρ
0	$(1, 1)$	η	0
0	$(4, 4)$	α	$\tau^{-2}\omega \cdot v_1^2$
0	$(0, 0)$	ω	2
4	$(0, -4)$	τ^4	τ^4
0	$(8, 8)$	β	$\tau^{-4}v_1 \cdot v_1^3$
2	$(0, -2)$	$\tau^2\omega$	$\tau^2\omega$
-2	$(0, 2)$	$\tau^{-2}\omega$	$\tau^{-2}\omega$
-4	$(0, 4)$	$\tau^{-4}\omega$	$\tau^{-4}\omega$
-5	$(j, j+5)$	$\frac{\Gamma}{\rho^j\tau^4}$	$\frac{\Gamma}{\rho^j\tau^4}$
1	$(1, 0)$	$\tau\eta$	ρv_1
2	$(4, 2)$	$\tau^2\alpha$	$2v_1^2$

Table 11. The homomorphism $\pi_{*,*}(\mathrm{ko}_{C_2}) \xrightarrow{c_*} \pi_{*,*}(k\mathbb{R})$, modulo higher powers of 2.

12. Charts

12A. ***Bockstein E^+ and $\mathrm{Ext}_{\mathcal{A}^{\mathbb{R}}(1)}$ charts.*** The charts on pages 616–619 depict the Bockstein E^+ spectral sequence that converges to $\mathrm{Ext}_{\mathcal{A}^{\mathbb{R}}(1)}$. The details of this calculation are described in Section 6.

The E_2^+-page is too complicated to present conveniently in one chart, so this page is separated into two parts by Milnor–Witt stem modulo 2. Similarly, the E_3^+-page is separated into four parts by Milnor–Witt stem modulo 4. The E_4^+-page in Milnor–Witt stems 0 or 1 modulo 4 is not shown, since it is identical to the E_3^+-page in those Milnor–Witt stems. The E_4^+-page in Milnor–Witt stems 3 modulo 4 is not shown because it is zero.

Here is a key for reading the Bockstein charts:

(1) Gray dots and green dots indicate groups as displayed on the charts.

(2) Horizontal lines indicate multiplications by ρ.

(3) Vertical lines indicate multiplications by h_0.

(4) Diagonal lines indicate multiplications by h_1.

(5) Horizontal arrows indicate infinite sequences of multiplications by ρ.

(6) Vertical arrows indicate infinite sequences of multiplications by h_0.

(7) Diagonal arrows indicate infinite sequences of multiplications by h_1.

Here is a key for the charts of $\mathrm{Ext}_{\mathcal{A}^{\mathbb{R}}(1)}$:

(1) Gray dots indicate copies of $\mathbb{F}_2[\tau^4]$ that arise from a copy of $\mathbb{F}_2[\tau^4]$ in the E_∞^+-page.

(2) Green dots indicate copies of $\mathbb{F}_2[\tau^4]$ that arise from a copy of $\mathbb{F}_2$ and a copy of $\mathbb{F}_2[\tau^4]$ in the E_∞^+-page, connected by a τ^4 extension that is hidden in the Bockstein spectral sequence. For example, the green dot at $(3, 3)$ arises from a hidden τ^4 extension from h_1^3 to $\rho \cdot \tau^2 a$.

(3) Blue dots indicate copies of $\mathbb{F}_2[\tau^4]$ that arise from two copies of $\mathbb{F}_2$ and one copy of $\mathbb{F}_2[\tau^4]$ in the E_∞^+-page, connected by τ^4 extensions that are hidden in the Bockstein spectral sequence. For example, the blue dot at $(7, 7)$ arises from hidden τ^4 extensions from h_1^7 to $\rho^4 h_1^3 b$, and from $\rho^4 h_1^3 b$ to $\rho^5 \cdot \tau^2 a \cdot b$.

(4) Horizontal lines indicate multiplications by ρ.

(5) Vertical lines indicate multiplications by h_0.

(6) Diagonal lines indicate multiplications by h_1.

(7) Dashed lines indicate extensions that are hidden in the Bockstein spectral sequence.

(8) Orange horizontal lines indicate ρ multiplications that equal τ^4 times a generator. For example, $\rho \cdot \tau^2 a$ equals $\tau^4 \cdot h_1^3$.

(9) Horizontal arrows indicate infinite sequences of multiplications by ρ.

(10) Vertical arrows indicate infinite sequences of multiplications by h_0.

(11) Diagonal arrows indicate infinite sequences of multiplications by h_1.

12B. ***Bockstein E^- and* $\mathrm{Ext}_{\mathrm{NC}}$ *charts for* $\mathcal{A}^{C_2}(1)$.** The charts on pages 620–624 depict the Bockstein E^- spectral sequence that converges to $\mathrm{Ext}_{\mathrm{NC}}$. The details of this calculation are described in Section 7.

The E_2^--page is too complicated to present conveniently in one chart, so this page is separated into two parts by Milnor–Witt stem modulo 2. Similarly, the E_3^--page is separated into four parts by Milnor–Witt stem modulo 4. The E_4^--page in Milnor–Witt stems 0 or 3 modulo 4 is not shown, since it is identical to the E_3^--page in those Milnor–Witt stems. The E_5^--page and E_6^--page in Milnor–Witt stems 1 or 2 modulo 4 is not shown, since it is identical to the E_4^--page in those Milnor–Witt stems.

Here is a key for reading the Bockstein charts:

(1) Gray dots and green dots indicate groups as displayed on the charts.

(2) Horizontal lines indicate multiplications by ρ.

(3) Vertical lines indicate multiplications by h_0.

(4) Diagonal lines indicate multiplications by h_1.

(5) Horizontal rightward arrows indicate infinite sequences of divisions by ρ, i.e., infinitely ρ-divisible elements.

(6) Vertical arrows indicate infinite sequences of multiplications by h_0.

(7) Diagonal arrows indicate infinite sequences of multiplications by h_1.

The structure of $\mathrm{Ext}_{\mathrm{NC}}$ is too complicated to present conveniently in one chart, so it is separated into parts by Milnor–Witt stem modulo 4. Unfortunately, the part in positive Milnor–Witt stems 0 modulo 4 alone is still too complicated to present conveniently in one chart. Instead, we display Ext_{C_2}, including both $\mathrm{Ext}_{\mathcal{A}^{\mathbb{R}}(1)}$ and $\mathrm{Ext}_{\mathrm{NC}}$, for the Milnor–Witt 0-stem and the Milnor–Witt 4-stem.

Here is a key for the charts of $\mathrm{Ext}_{\mathrm{NC}}$:

(1) Gray dots indicate copies of $\mathbb{F}_2[\tau^4]/\tau^\infty$.

(2) Horizontal lines indicate multiplications by ρ.

(3) Vertical lines indicate multiplications by h_0.

(4) Diagonal lines indicate multiplications by h_1.

(5) Dashed lines indicate extensions that are hidden in the Bockstein spectral sequence.

(6) Dashed lines of slope -1 indicate ρ extensions that are hidden in the Adams spectral sequence.

(7) Horizontal rightward arrows indicate infinite sequences of divisions by ρ, i.e., infinitely ρ-divisible elements.

(8) Vertical arrows indicate infinite sequences of multiplications by h_0.

(9) Diagonal arrows indicate infinite sequences of multiplications by h_1.

12C. ***Bockstein and* Ext *charts for* $\mathcal{E}^{C_2}$(1).** The Bockstein E^+ and E^- spectral sequences that converge to $\mathrm{Ext}_{\mathcal{E}^{\mathbb{R}}(1)}$ and $\mathrm{Ext}_{\mathcal{E}^{\mathbb{R}}(1)}(\mathrm{NC}, \mathbb{M}_2^{\mathbb{R}})$, respectively, are shown in the charts on page 627. The details of this calculation are described in Remark 6.3 and Section 7C. For legibility, we have split each of the E_∞^+, E_4^-, and $\mathrm{Ext}_{\mathrm{NC}}$ pages into a pair of charts, organized by families of v_1-multiples rather than by Milnor–Witt stems.

Here is a key for reading the Bockstein and $\mathrm{Ext}_{\mathrm{NC}}$ charts:

(1) Gray dots indicate groups as displayed on the charts.

(2) Horizontal lines indicate multiplications by ρ.

(3) Vertical lines indicate multiplications by h_0. Dashed vertical lines denote h_0-multiplications that are hidden in the Bockstein spectral sequence

(4) Horizontal rightward arrows indicate infinite sequences of divisions by ρ, i.e., infinitely ρ-divisible elements.

(5) Vertical arrows indicate infinite sequences of multiplications by h_0.

12D. ***Milnor–Witt stems.*** The diagrams on pages 629 and 630 depict the Milnor–Witt stems for ko_{C_2} and $k\mathbb{R}$ in families as described in Section 11.

The top figure on page 629 represents the Milnor–Witt $4k$-stem, where $k \geq 0$. The middle three figures represent the τ^4-periodic classes, as in Theorem 11.15. The bottom figure represents the Milnor–Witt stem Π_n, where $n \equiv 3 \pmod 4$ and $n \leq -5$.

Here is a key for reading the Milnor–Witt charts:

(1) Black dots indicate copies of $\mathbb{F}_2$.

(2) Hollow circles indicate copies of $\mathbb{Z}_2^2$.

(3) Circled numbers indicate cyclic groups of given order. For instance, the 1-stem of Π_{-5} is $\mathbb{Z}/4$.

(4) Blue lines indicate multiplications by η.

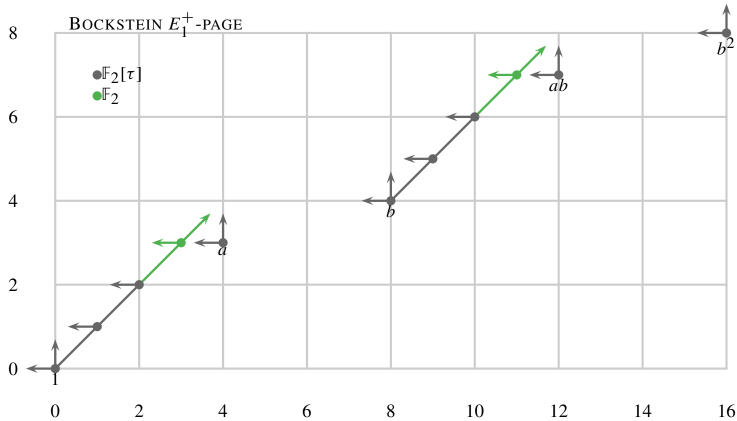
(5) Red lines indicate multiplications by ρ.

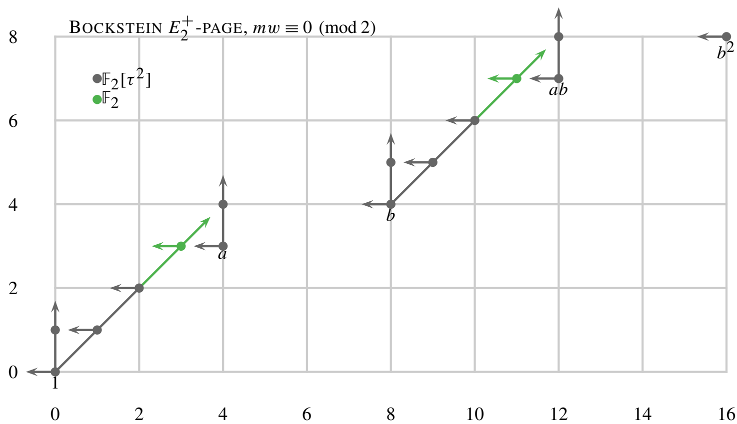
(6) Curved green lines denote multiplications by α.

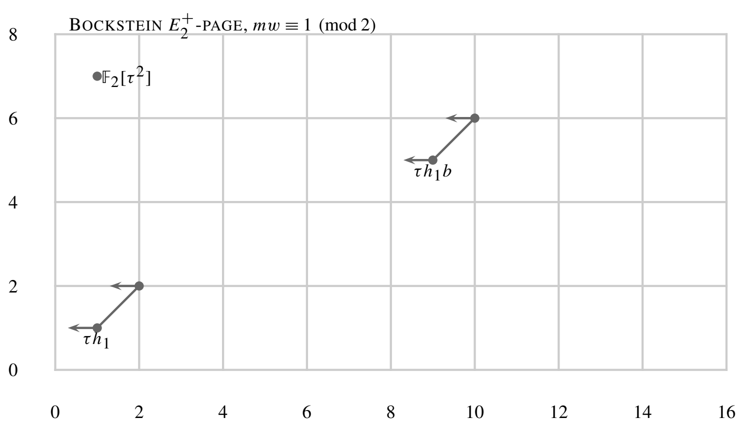
(7) Lines labeled with numbers indicate that a multiplication equals a multiple of an additive generator. For example, $\alpha \cdot \eta^4$ equals $4\eta\rho\beta$ in Π_0.

For clarity, some α multiplications are not shown in the first and last diagrams of page 629. For example, the α multiplication on η is not shown in the first diagram.

Bockstein charts for $\mathcal{A}^{\mathbb{R}}(1)$

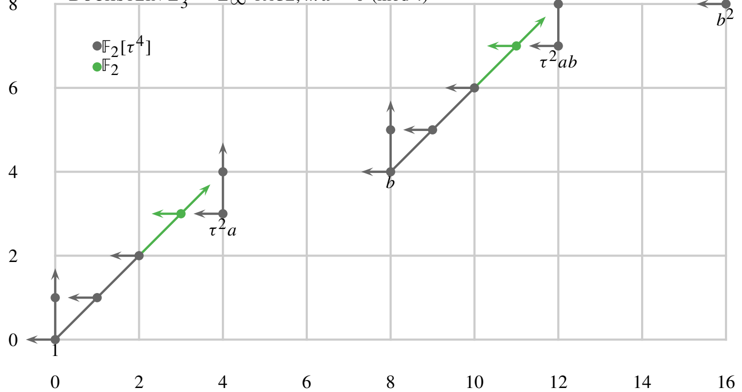




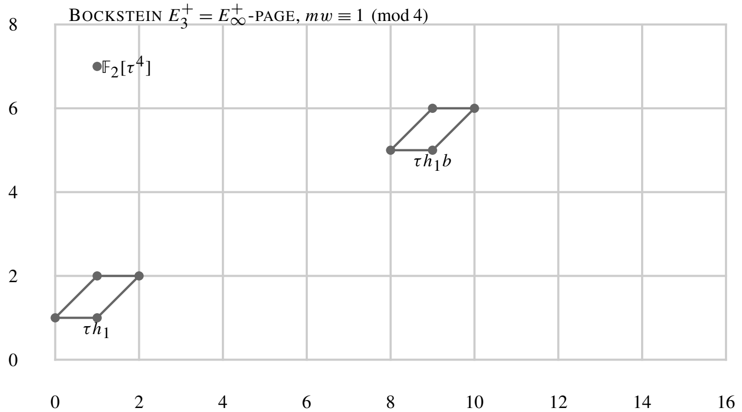


Bockstein charts for $\mathcal{A}^{\mathbb{R}}(1)$

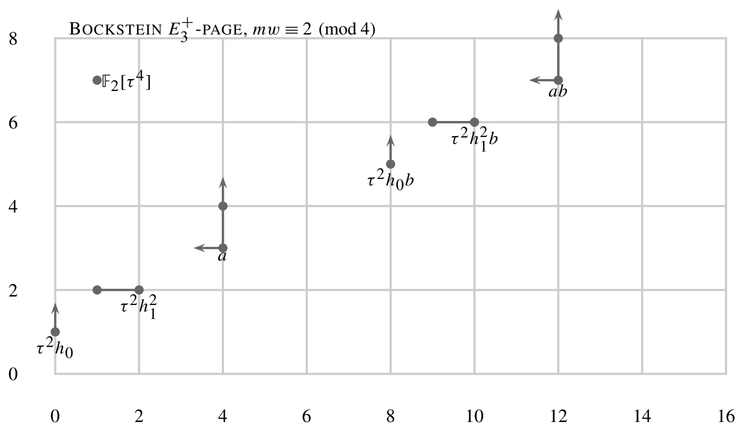
BOCKSTEIN $E_3^+ = E_\infty^+$-PAGE, $mw \equiv 0 \pmod 4$



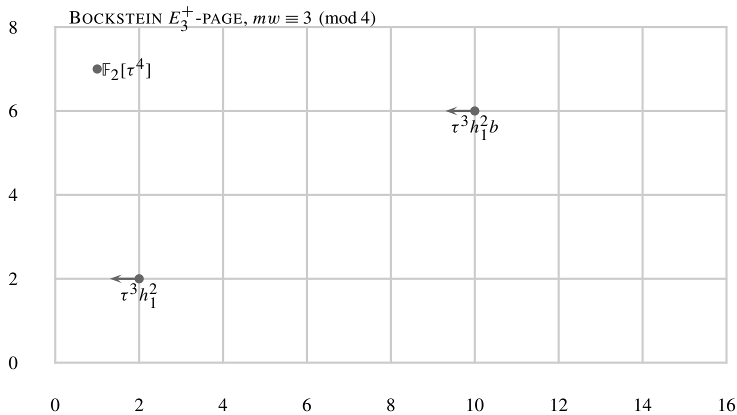
BOCKSTEIN $E_3^+ = E_\infty^+$-PAGE, $mw \equiv 1 \pmod 4$

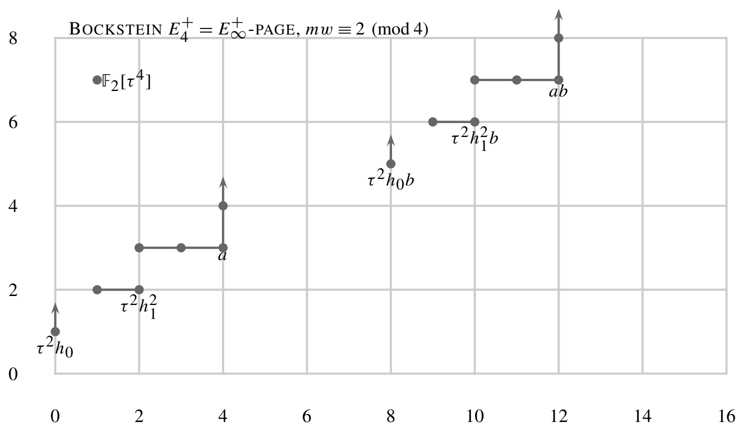


BOCKSTEIN E_3^+-PAGE, $mw \equiv 2 \pmod 4$

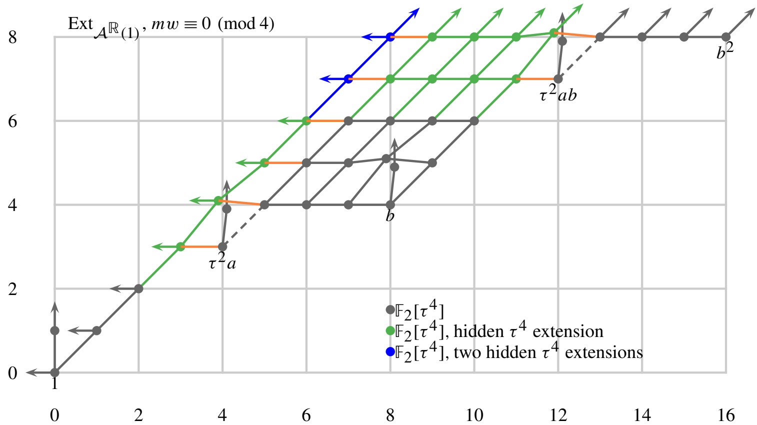


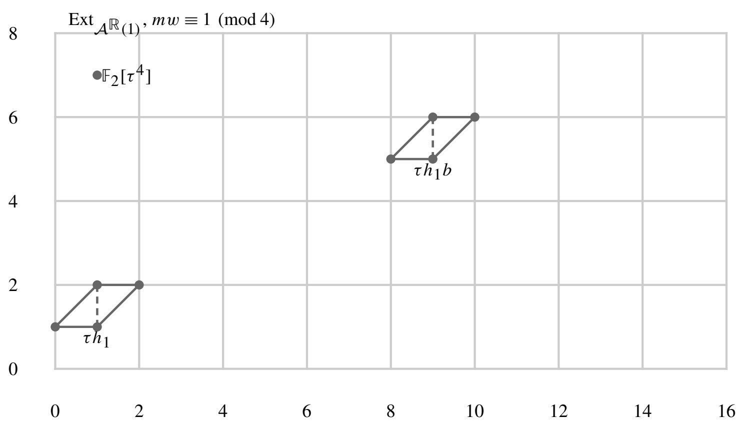
Bockstein charts for $\mathcal{A}^{\mathbb{R}}(1)$

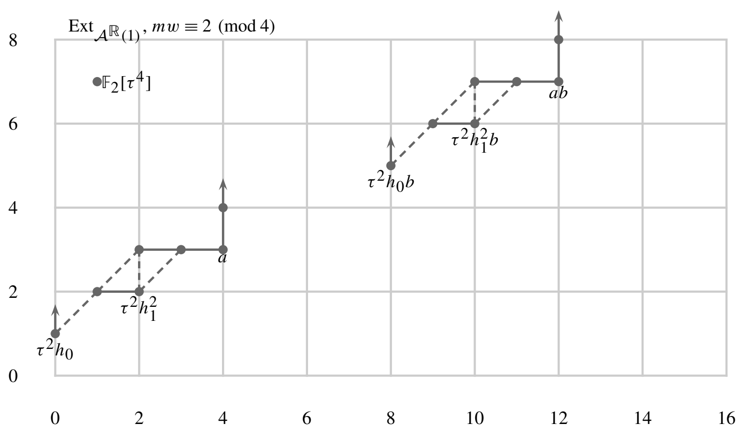




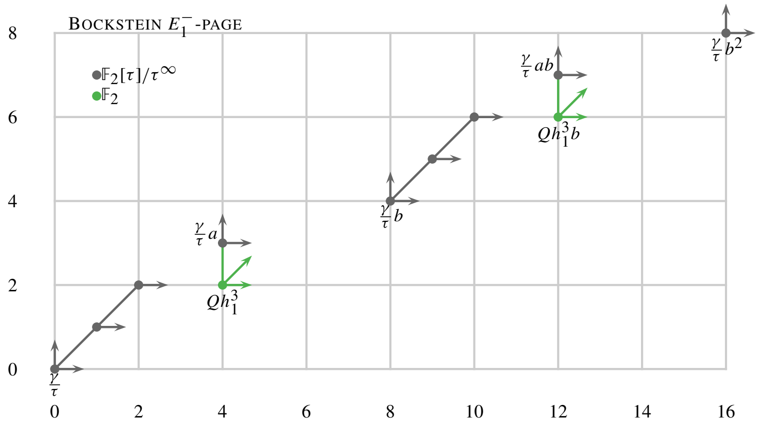
Ext charts for $\mathcal{A}^{\mathbb{R}}(1)$

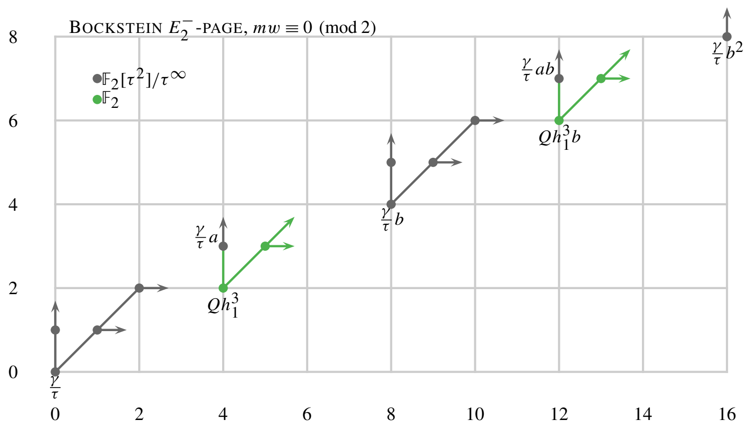


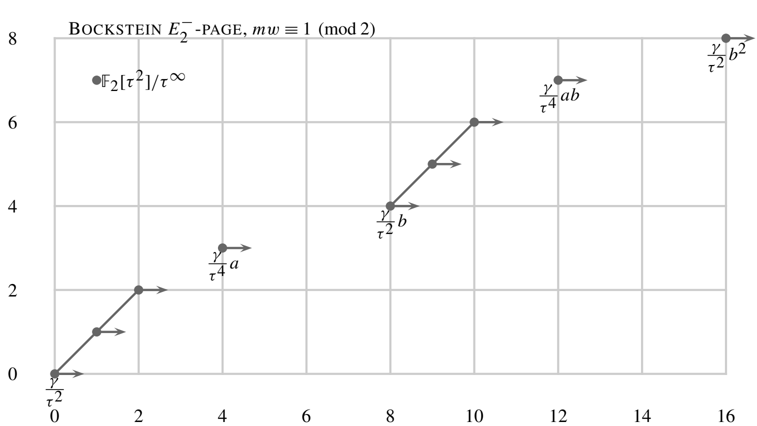




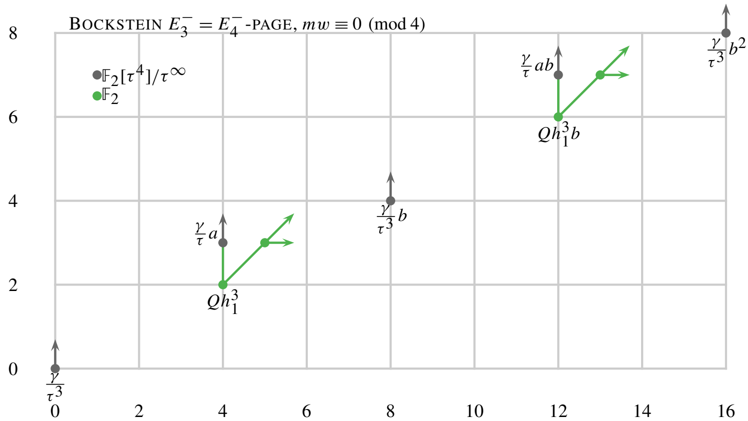
Bockstein E^- charts for $\mathcal{A}^{C_2}(1)$

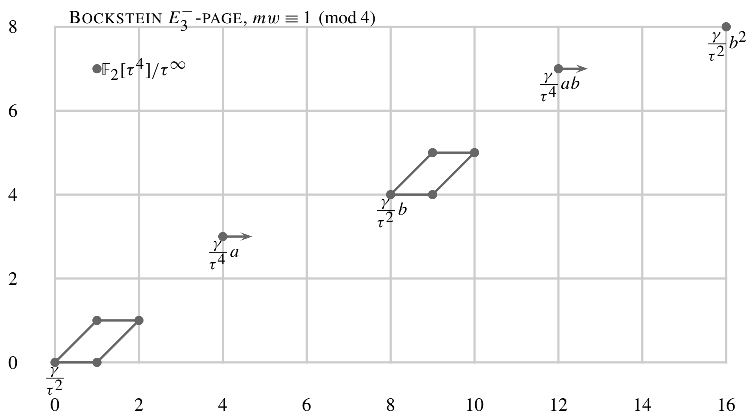


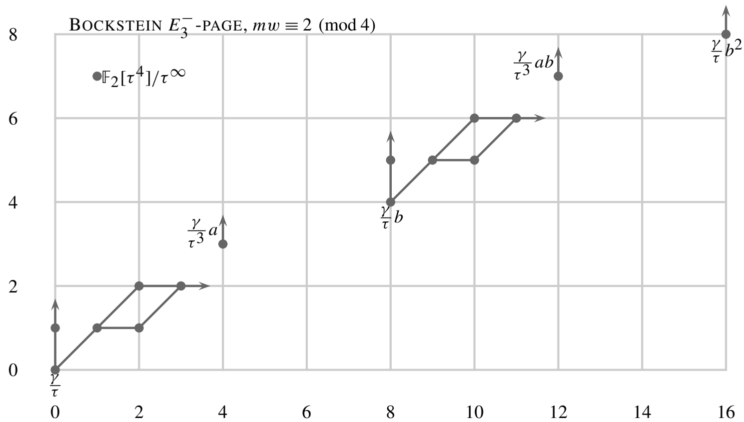




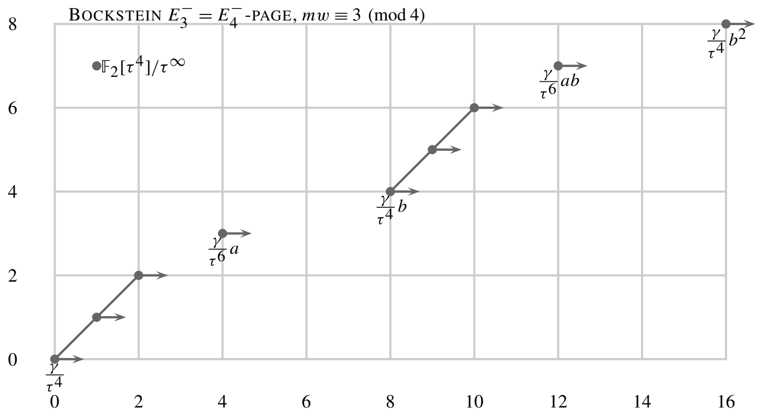
Bockstein E^- charts for $\mathcal{A}^{C_2}(1)$

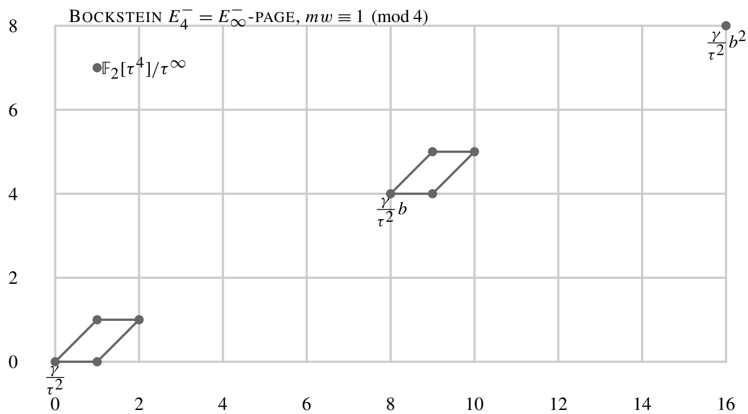


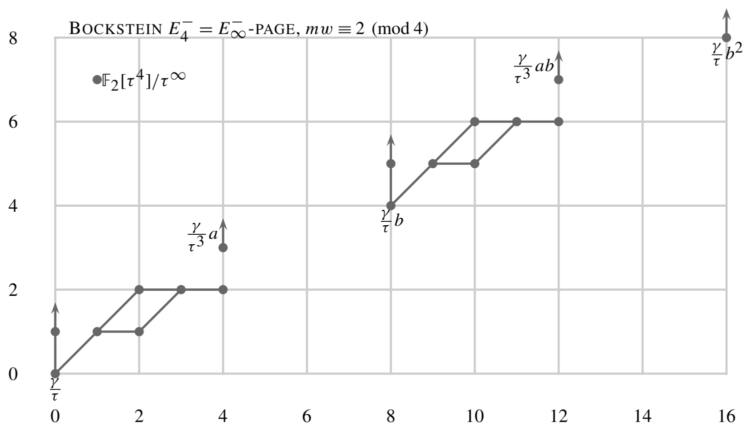




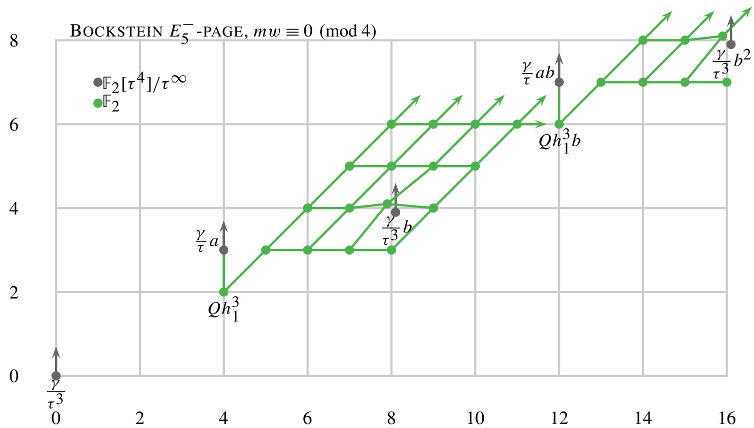
Bockstein E^- charts for $\mathcal{A}^{C_2}(1)$

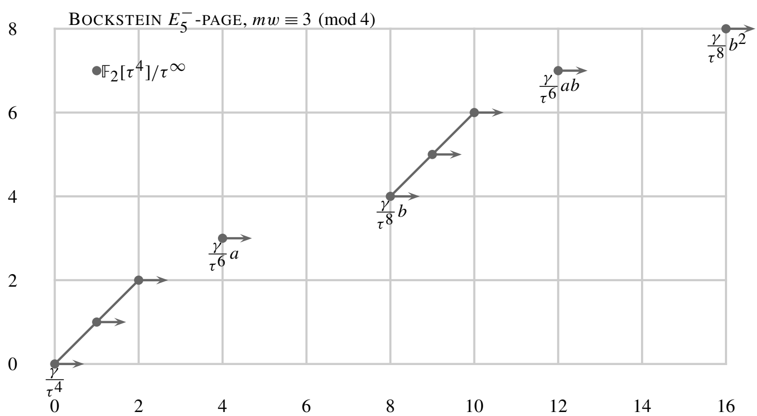




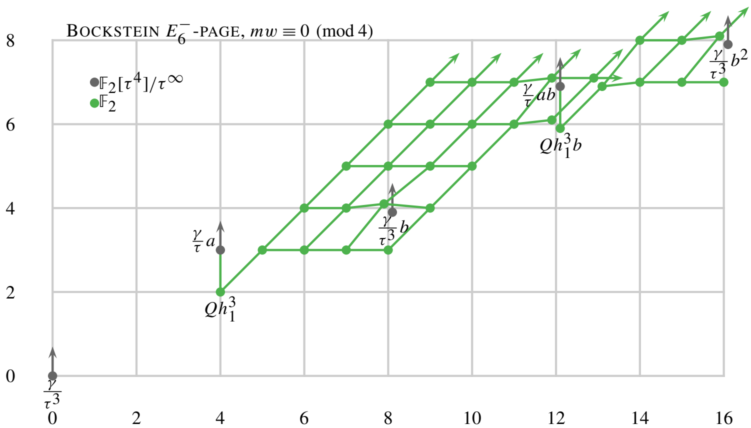


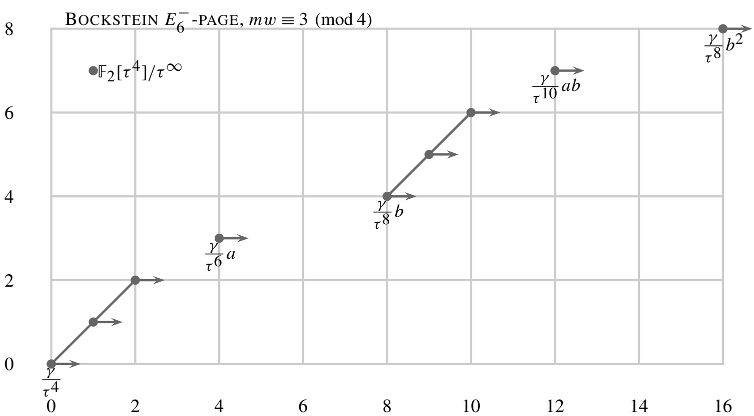
Bockstein E^- charts for $\mathcal{A}^{C_2}(1)$



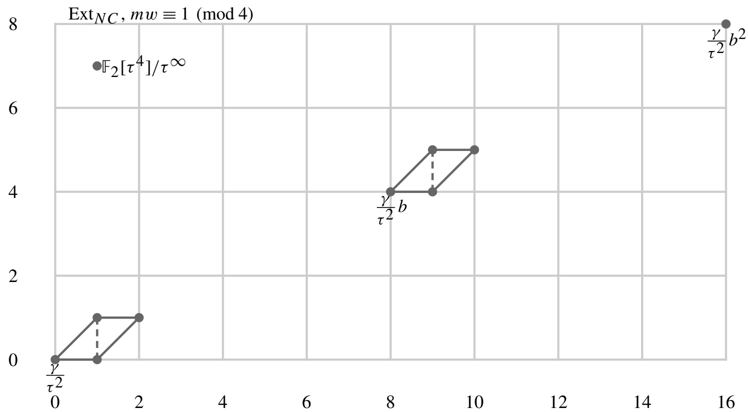


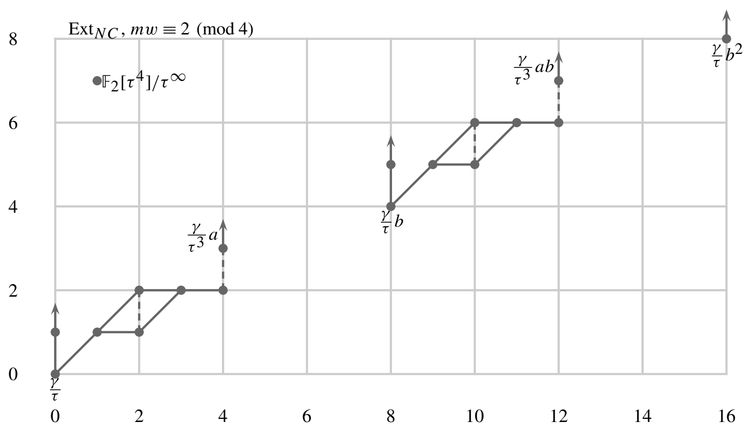
Bockstein E^- charts for $\mathcal{A}^{C_2}(1)$

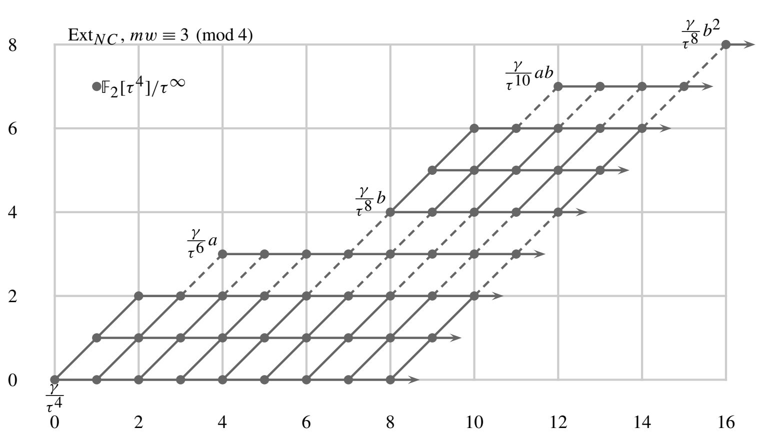




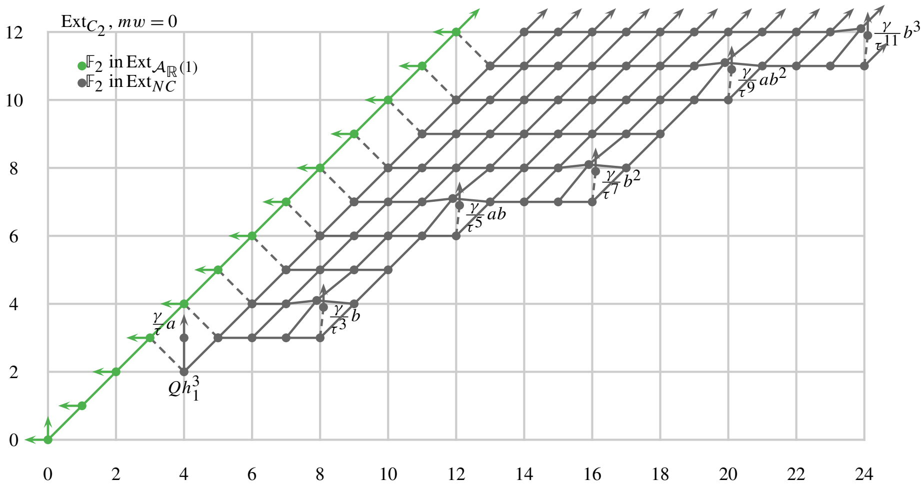
$\mathrm{Ext}_{\mathrm{NC}}$ charts for $\mathcal{A}^{C_2}(1)$

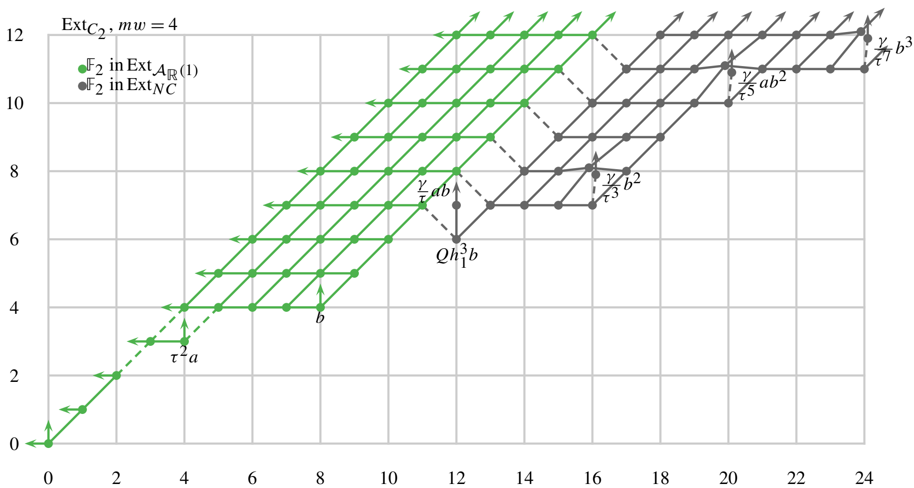




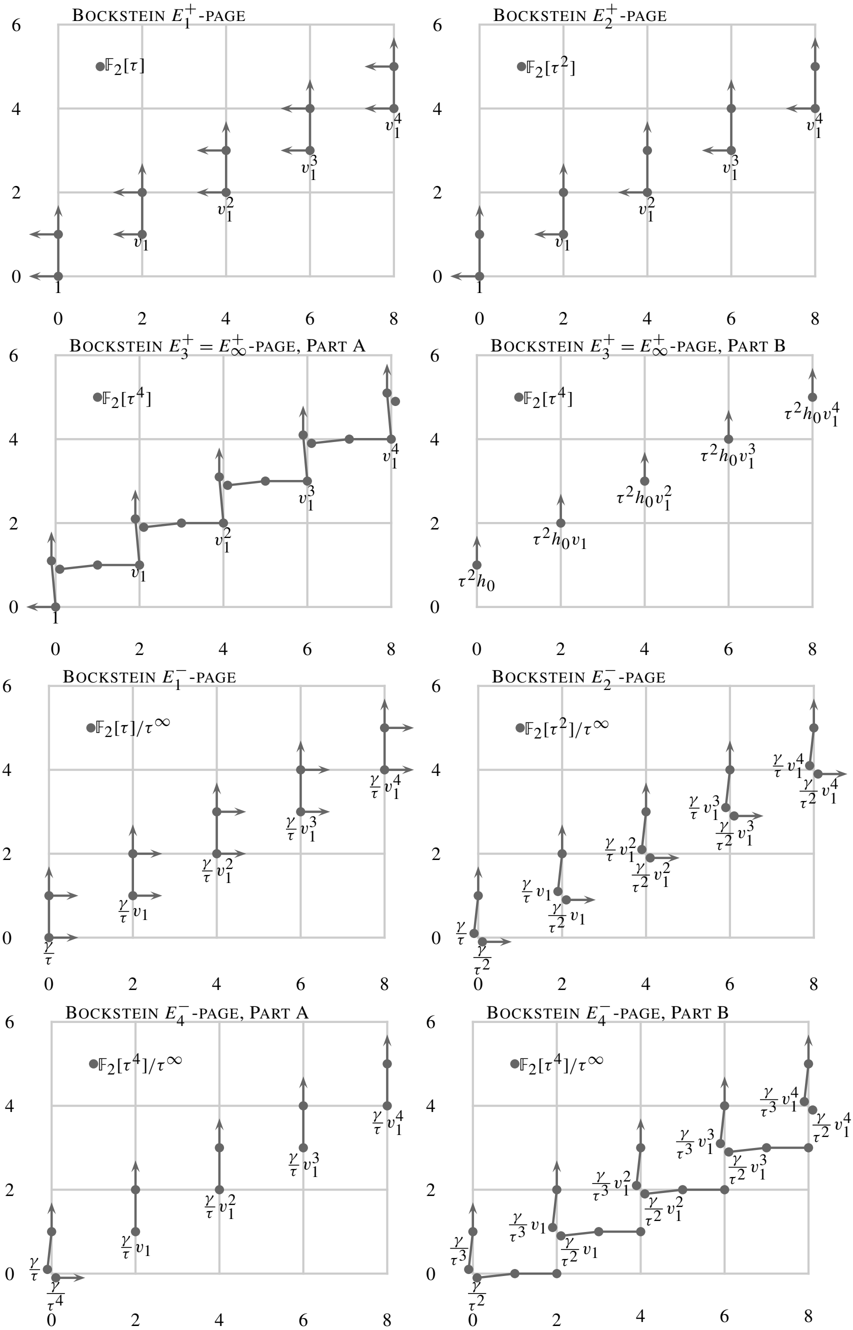


Ext charts for $\mathcal{A}^{C_2}(1)$ in $mw = 0$ and $mw = 4$

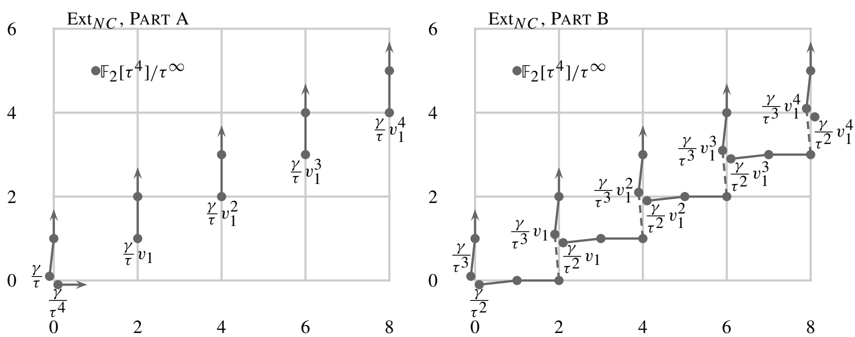




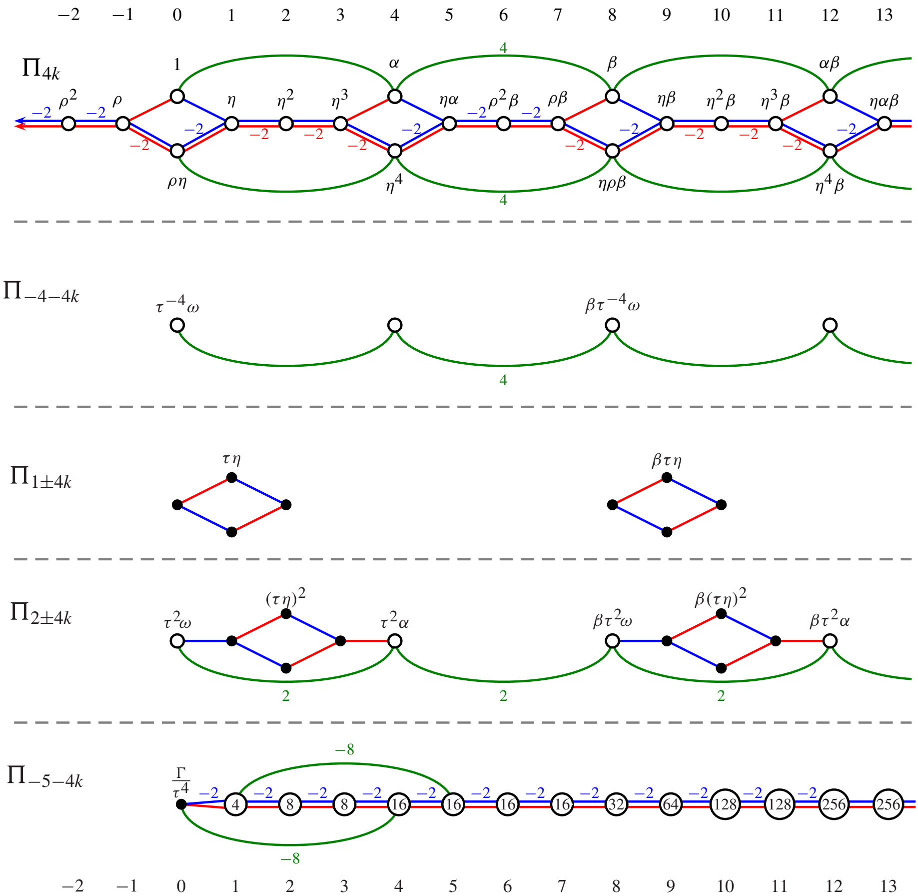
Bockstein charts for $\mathcal{E}^{C_2}(1)$



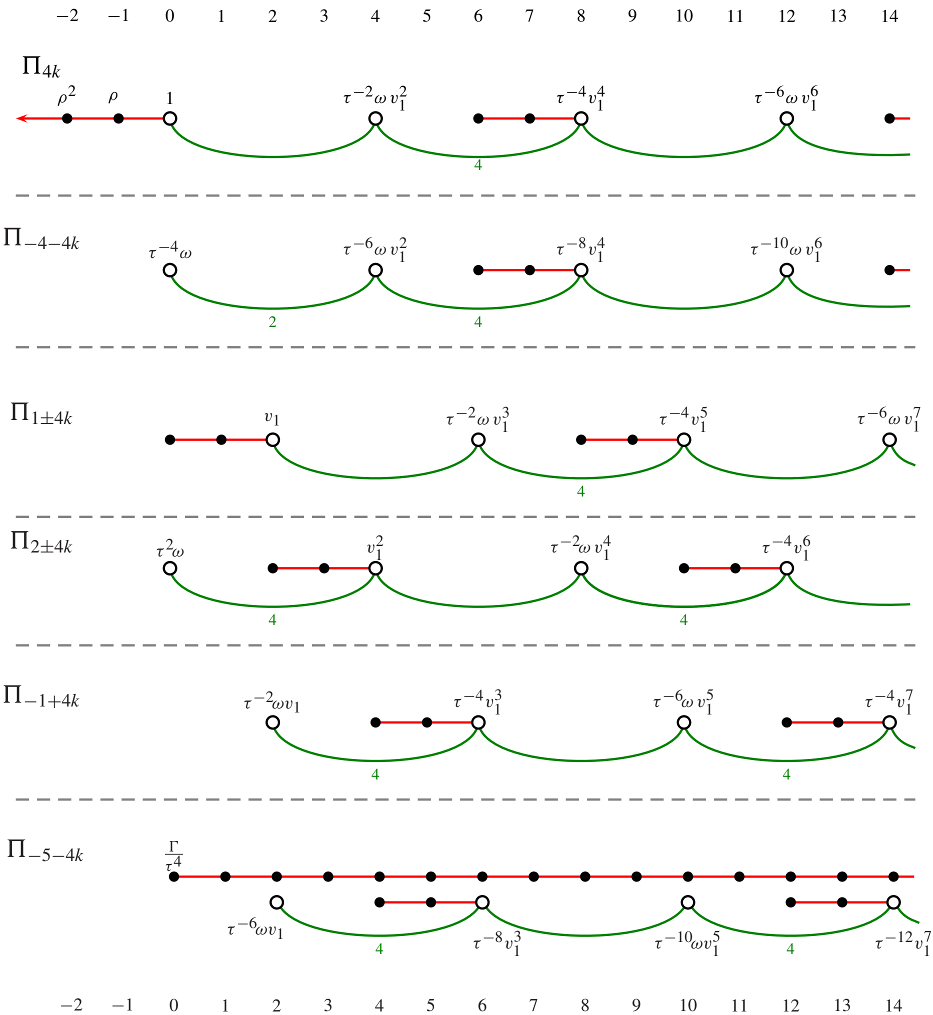
$\mathrm{Ext}_{\mathrm{NC}}$ charts for $\mathcal{E}^{C_2}(1)$



Milnor–Witt modules for ko_{C_2}



Milnor–Witt modules for $k\mathbb{R}$



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