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What is an ∞-category?

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Abstract

This is an expository paper on ∞-categories, also known as $(\infty, 1)$-categories. It was originally given as a talk at the Regensburg conference of August, 2023. I thank Sadok Kallel for inviting me to publish it here and Siddharth Gurumurthy for some useful discussions.

MSC 2020: 55U40, 55P60, 18G30

1 Relevant Literature

The main references for this topic are two remarkable books by Jacob Lurie:

- *Higher Topos Theory* published in 2009 (949 pages), which we denote by [Lur09].
- *Higher Algebra* last edited in 2017 (1553 pages), which we denote by [Lur17].

While these (along with Joyal's unpublished [Joy08a] and [Joy08b]) are the primary sources in the subject, they are not the most accessible to the beginner. Some easier introductions are the following.

- Moritz Groth's 2010 *A short course on ∞-categories* [Gro10] (53 pages) introduces the basic definitions and gets into the nuts and bolts of a symmetric monoidal structure on an ∞-category and the ∞-category of spectra.
- Omar Antolín-Camarena's 2014 *A Whirlwind Tour of the World of $(\infty, 1)$-categories* [AC14] (46 pages) is written in a similar spirit and includes the following helpful slogan:

  > The intuition that $(\infty, 1)$-categories have spaces of morphisms and that these spaces only matter up to (weak) homotopy equivalence usually leads to useful definitions and correct statements.

- Charles Rezk's 2022 course notes [Rez22] (194 pages) is a more thorough introduction that includes the relevant background on simplicial sets among other things.
- Julie Bergner's 2018 book *The homotopy theory of $(\infty, 1)$-categories* [Ber18] (273 pages) treats the subject from a model category perspective.
- Markus Land's 2021 textbook *Introduction to Infinity-Categories* [Lan21] (300 pages) is, according to its MathSciNet reviewer Gij Heuts, "a concise and rather comprehensive introduction to the theory of ∞-categories, aimed at a wide audience."

We will adhere to the following color convention:

- Ordinary categories will be written in green.
- ∞-categories (that are not ordinary categories) will be written in cyan.

2 Introduction

Before defining ∞-categories (see Definition 2 below), we note some of their general features.

An ∞-category is a generalization of an ordinary category, also known as a 1-category. Like an ordinary category, it has objects and morphisms (also known as 1-morphisms), but composition of morphisms is not well defined. It also has higher structures called k-morphisms for $k > 1$, to be spelled out later. We will describe these explicitly for the ∞-category of topological spaces in Section 7.

∞-categories provide a convenient setting for doing homotopy theory.

There is nothing easy about ∞-categories. Most concepts and results from ordinary category theory have ∞-categorical analogs, but the definitions are less obvious and the proofs are harder. For example, the definition of a symmetric monoidal ∞-category $\mathcal{C}$ requires far more than a functor $\mathcal{C} \times \mathcal{C} \to \mathcal{C}$ with the expected properties. See the discussion at the beginning of [Lur17, Chapter 2].

For objects W, X and Y in an ordinary category C, one has a morphism sets $C(X,Y)$, $C(W,Y)$ and $C(W,X)$, with a composition map

$$\begin{array}{ccc} C(X,Y) \times C(W,X) & \longrightarrow & C(W,Y) \\ (g,f) & \longmapsto & gf. \end{array}$$

In an ∞-category $\mathcal{C}$, these three sets are topological spaces or simplicial sets, more precisely Kan complexes. Given morphisms $f : W \to X$ and $g : X \to Y$, instead of a well defined composite $gf \in \mathcal{C}(W,Y)$, we get a contractible subspace of $\mathcal{C}(W,Y)$. All morphisms in this subspace are homotopic to each other, meaning that they all lie in the same path component.

Many definitions involve weak equivalences of morphism spaces rather than isomorphisms of morphism sets. For example, an initial object X in $\mathcal{C}$ is one for which $\mathcal{C}(X,Y)$ is contractible (rather than a one point set) for all Y.

In an ∞-category, homotopy limits and colimits are the same as ordinary limits and colimits when they exist. We will see a simple example of this in Section 8.

In an ∞-category one need not worry about a model structure, but concepts of model category theory are needed to develop the theory of ∞-categories.

An ∞-category is a certain kind of simplicial set (but not generally a Kan complex), so it is sort of like a topological space. There is a model structure on the category of simplicial sets due to Joyal in which the fibrant objects are the ∞-categories, see [Lur09, Theorem 2.4.6.1]. Hence one can speak of limits of ∞-categories, and certain functors between them are Joyal fibrations, also known as *inner fibrations* [Lur09, Definition 2.0.0.3].

3 Review of simplicial sets

The simplicial category Δ is that of finite ordered sets and order preserving maps. For each integer $n \geq 0$, let $[n]$ denote the ordered set $\{0, 1, \ldots, n\}$.

A *simplicial set* X is a contravariant $\mathcal{S}et$ valued functor on Δ. Its value on $[k]$, its set of k-simplices, is denoted by X_k. X comes equipped with families of maps $X_k \to X_{k-1}$ (called face maps) and $X_k \to X_{k+1}$ (degeneracies), each indexed by i for $0 \leq i \leq k$. The ith such maps are induced respectively by

- the order preserving monomorphism $[k-1] \to [k]$ whose image does not contain i and
- the order preserving epimorphism $[k+1] \to [k]$ sending both i and $i+1$ to i.

A simplex is *degenerate* if it is in the image of a degeneracy map. Otherwise it is *nondegenerate*.

The simplicial set Δ^n, the *standard n-simplex*, is defined by

$$(\Delta^n)_k = \Delta([k],[n]).$$

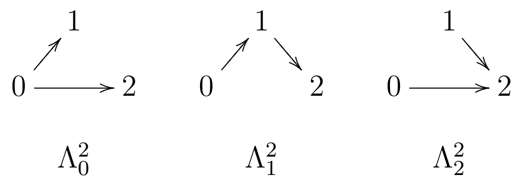
In its *boundary* $\partial\Delta^n$, the set of k-simplices is the set of such morphisms in Δ which are not surjective.

In its *ith face*, the set of k-simplices is the set of such morphisms whose image does not contain i.

In the *ith horn* $\Lambda^n_i \subseteq \partial\Delta^n$ for $0 \le i \le n$, the set of k-simplices is the set of nonsurjective morphisms whose image does contain i.

The *inner faces and horns* are those for which $0 < i < n$. The other two are *outer*.

Here are the three horns of a 2-simplex. Only the middle one is inner.



(3.1)

In the ith horn, the missing face is the one opposite the ith vertex.

A *Kan complex* is a simplicial set X for which every map from a horn $\Lambda^n_i \to X$ extends to Δ^n.

The *topological n-simplex* Δ^n_{top} is the space

$$\left\{(x_0, x_1, \dots, x_n) \in \mathbb{R}^{n+1} : x_i \ge 0 \text{ and } \sum x_i = 1\right\}.$$

The *geometric realization* $|X|$ of a simplicial set X is the colimit of the Top-valued functor

$$[k] \mapsto X_k \times \Delta^k_{\text{top}}.$$

This space turns out to be the union of geometric realizations of the nondegenerate topological simplices of X, meaning ones not in the image of any degeneracy map. The data given by the face maps determine how they are glued together. In particular, $|\Delta^n| = \Delta^n_{\text{top}} \approx D^n$, $|\partial\Delta^n| \approx S^{n-1}$, and $|\Lambda^n_i| \approx D^{n-1}$.

Given simplicial sets X and Y, one can define a simplicial set $X \times Y$ in which

$$(X \times Y)_n = \coprod_{0 \le i \le n} X_i \times Y_{n-i} \quad \text{and} \quad |X \times Y| = |X| \times |Y|.$$

The category of simplicial sets is denoted by $\mathcal{S}\text{et}_\Delta$.

A simplicial map $X \to Y$ is a natural transformation of contravariant functors on Δ. The set of such maps is $\mathcal{S}\text{et}_\Delta(X, Y)$. This can be thickened up to a simplicial set $\underline{\mathcal{S}\text{et}_\Delta(X, Y)}$ in which the set of k-simplices is $\mathcal{S}\text{et}_\Delta(X \times \Delta^k, Y)$.

Hence $\mathcal{S}\text{et}_\Delta$ is enriched over itself.

4 Of all the nerve!

The nerve NC of a small category C is the simplicial set in which the set of n-simplices NC_n is the set of diagrams

$$X_0 \to X_1 \to \cdots \to X_n$$

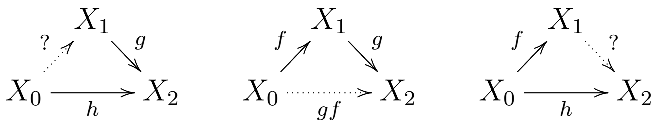
in C. Face and degeneracy maps are defined by composing adjacent morphisms and inserting identity maps. Equivalently we can regard $[n]$ as the category

$$0 \to 1 \to \cdots \to n$$

and define NC_n to be the set of functors from $[n]$ to C.

This simplicial set has the following property: *Any simplicial map $\Lambda_i^n \to NC$ for $0 < i < n$ extends uniquely to Δ^n.*

The following is an illustration for $n = 2$. The X_i are objects in C. The three diagrams with the dotted arrows removed indicate C-valued functors from the three diagrams of (3.1), that is maps from the three horns of a 2-simplex to NC. Extending these maps to all of Δ^2 means identifying the dotted arrow. There is a unique way to do this for the inner horn Λ_1^2, but there may or may not be such an arrow for the two outer horns.



Λ_0^2 Λ_1^2 Λ_2^2

It is known that the category C is determined by its nerve, and that any simplicial set with the property above is the nerve of some small category.

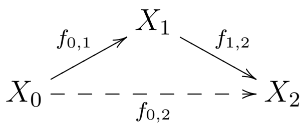
A small category is thus defined by a simplicial set (its nerve) in which each map from an inner horn Λ_i^n extends uniquely to a map from Δ^n. *An ∞-category is defined to be a simplicial set in which this uniqueness condition is dropped.*

5 The main definition

Definition 2. An *∞-category* (also called a quasicategory) $\mathcal{C}$ is a simplicial set for which each simplicial map $\Lambda_i^n \to \mathcal{C}$ for $0 < i < n$ extends to a map $\Delta^n \to \mathcal{C}$. A functor $F : \mathcal{C} \to \mathcal{C}'$ from one ∞-category to another is a simplicial map.

There are some other equivalent definitions of an ∞-category in the literature, but this is the one used by Lurie. There are several features of it worth noting.

- We are not requiring extensions of maps from Λ_0^n and Λ_n^n (known as the left and right outer horns) as in the definition of a Kan complex. Boardman and Vogt [BV73, Definition 4.8] called this *the restricted Kan condition.*
- The extension of each map from an inner horn is not required to be unique, as it is in the nerve of an ordinary category. This means that this notion is more general than that of an ordinary category as seen through its nerve. Hence an ordinary category is a special case of an ∞-category.
- Given such a simplicial set $\mathcal{C}$, we can think of elements of the sets $\mathcal{C}_0$ and $\mathcal{C}_1$ as objects and morphisms. The two face maps $\mathcal{C}_1 \rightrightarrows \mathcal{C}_0$ define the source and target (aka domain and codomain) of each morphism. Elements in the sets $\mathcal{C}_k$ for $k > 1$ can be thought of as *higher morphisms* in $\mathcal{C}$.
- A diagram

  without the dashed arrow is equivalent to a map $\Lambda_1^2 \to \mathcal{C}$. Choosing a dashed arrow (in which the diagram is *not* required to commute) is equivalent to extending this map to $\partial\Delta^2$. Choosing a homotopy between $f_{1,2}f_{0,1}$ and $f_{0,2}$ is equivalent to extending this map to all of Δ^2. Such an extension is guaranteed to exist, but it is not unique. In the nerve of an ordinary category this extension is unique and identifies the composite $f_{1,2}f_{0,1}$. In an ∞-category this extension is only unique up to homotopy, so *composition of morphisms in an ∞-category is not well defined.*

- The simplicial set $\mathcal{S}\mathrm{et}_\Delta(K, \mathcal{D})$ of simplicial maps from a simplicial set K to an ∞-category $\mathcal{D}$ is itself an ∞-category.
- K above could be an ∞-category $\mathcal{C}$, in particular it could be NC for an ordinary category C. In other words, the collection of functors $\mathcal{C} \to \mathcal{D}$ is an ∞-category $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$.

To a topological space X we can associate an ∞-category X (also known as $\mathrm{Sing}\, X$, the singular simplicial set of X) in which X_n is the set of continuous maps $|\Delta^n| \to X$. X is also a Kan complex since a map $|\Lambda_i^n| \to X$, for any horn Λ_i^n, extends to $|\Delta^n|$ using a retraction $|\Delta^n| \to |\Lambda_i^n|$.

Such an ∞-category is called an ∞-*groupoid* because all morphisms, i.e., paths in X, are invertible up to homotopy.

6 Set-theoretic technicalities

Here we summarize the discussion in [Lur09, 1.2.15]. In ordinary category theory it is common to allow the collection of objects to be a proper class rather than a set (as in the categories of sets, groups, topological spaces and so on) but to require the collections of morphisms from one object to another to be a set. In the theory of ∞-categories this dichotomy will not do because the collections of both objects and morphisms are part of the data defining a simplicial set.

One can avoid the perils of Russell's paradox by limiting one's attention to sets with cardinality less than a strongly inaccessible cardinal κ. Recall that such a κ is a cardinal number that is uncountable and regular, meaning that it cannot be expressed as the union of fewer than the cardinality of itself many sets, each of cardinality less than that of κ. A strongly inaccessible cardinal κ exceeding the cardinality of the real line is big enough so that we are unlikely to miss any mathematical objects with cardinality not bounded by it. To quote [Lur09, 1.2.15],

> We then let $\mathcal{U}(\kappa)$ denote the collection of all sets having rank $< \kappa$, so that $\mathcal{U}(\kappa)$ is a Grothendieck universe: in other words, $\mathcal{U}(\kappa)$ satisfies all of the usual axioms of set theory. We will refer to a mathematical object as small if it belongs to $\mathcal{U}(\kappa)$ (or is isomorphic to such an object), and essentially small if it is equivalent (in whatever relevant sense) to a small object. Whenever it is convenient to do so, we will choose another strongly inaccessible cardinal $\kappa' > \kappa$ to obtain a larger Grothendieck universe $\mathcal{U}(\kappa')$ in which $\mathcal{U}(\kappa)$ becomes small.

7 The ∞-category of topological spaces

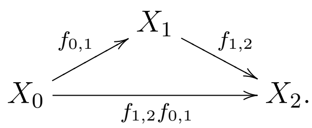
Let Top denote the category of compactly generated weak Hausdorff spaces *with cardinality less than* κ, where κ is a sufficiently large regular cardinal. This version of the category of topological spaces is small, so we can consider its nerve.

There is another construction called the *homotopy coherent nerve* whose definition [Lur09, Definition 1.1.5.5] baffled me for several years. Rather than giving it here, I will describe the ∞-category $\mathcal{S}$ (Lurie's notation of [Lur09, Definition 1.2.16.1]) one gets by applying it to Top. This is the ∞-category of topological spaces. *We will see that it comes equipped with a large collection of higher morphisms not present in the ordinary category of topological spaces.*

Lurie's $\mathcal{S}$ is actually the homotopy coherent nerve of the category $\mathcal{K}\mathrm{an}$ of Kan complexes, which is equivalent to the category of CW-complexes. The distinction between CW-complexes and more general spaces does not matter in what follows in this section.

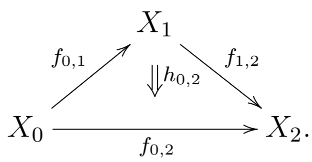
As in Definition 2, $\mathcal{S}$ is a simplicial set. Its vertices and edges are objects and morphisms in Top, meaning spaces (as described above) and continuous maps.

The set of 2-simplices is more interesting. In the subcategory NTop (the ordinary nerve), it is the set of commutative diagrams of the form



The top two edges can be viewed as a map $\Lambda^1_2 \to N\mathsf{Top}$, with the full diagram being its unique extension to Δ^2.

The set of 2-simplices $\mathcal{S}_2$ consists of similar diagrams in which the bottom arrow is replaced by any map $f_{0,2}$ homotopic to $f_{1,2}f_{0,1}$, with the homotopy $h_{0,2}$ being part of the datum. Thus we have a diagram



(7.3)

The homotopy is a map

$$I \times X_0 \xrightarrow{h_{0,2}} X_2$$

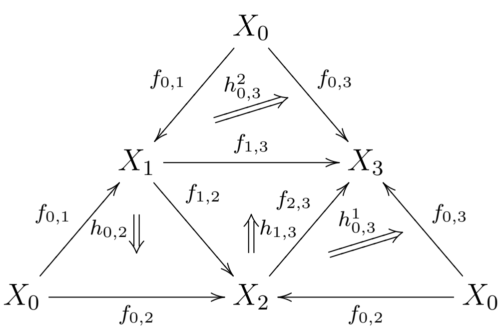
with certain properties. It is adjoint to a path

$$\begin{array}{rcl} I & \xrightarrow{\widehat{h}_{0,2}} & \mathsf{Top}(X_0, X_2) \\ 0 & \longmapsto & f_{1,2}f_{0,1} \\ 1 & \longmapsto & f_{0,2} \end{array}$$

Here $\mathsf{Top}(X_0, X_2)$, the set of continuous maps from X_0 to X_2, is given the compact-open topology and the map $\widehat{h}_{0,2}$ is required to be continuous.

As in the ordinary case, the top two edges of the diagram (7.3) can be viewed as a map $\Lambda^2_1 \to \mathcal{S}$. Now there is an extension of it to Δ^2 for each path $\widehat{h}_{0,2}$ in $\mathsf{Top}(X_0, X_2)$ starting at the point $f_{1,2}f_{0,1}$. *The space of such paths is contractible*, as is the space of paths starting at a given point in any topological space.

The set of 3-simplices in $\mathcal{S}$. The following diagram shows four 2-simplices with their homotopies.



(7.4)

Our convention for labeling homotopies is as follows. The subscripts correspond to the first and third vertices of the triangle while the super script corresponds to the second one. The latter is omitted when it is uniquely determined by the subscripts.

These four 2-simplices form the boundary of a 3-simplex in $\mathcal{S}$ iff there is a certain double homotopy adjoint to a map $\widehat{h}_{0,3} : I^2 \to \mathrm{Top}(X_0, X_3)$ of the following form.

$$\begin{array}{ccc} f_{2,3}f_{1,2}f_{0,1} & \xrightarrow{\mathrm{Top}(X_0,f_{2,3})\widehat{h}_{0,2}} & f_{2,3}f_{0,2} \\ {\scriptstyle \mathrm{Top}(f_{0,1},X_3)\widehat{h}_{1,3}}\downarrow & & \downarrow{\scriptstyle \widehat{h}^1_{0,3}} \\ f_{1,3}f_{0,1} & \xrightarrow[\widehat{h}^2_{0,3}]{} & f_{0,3} \end{array} \tag{7.5}$$

This is a picture rather than a diagram. Each vertex of the square is not an object but a point in $\mathrm{Top}(X_0, X_3)$, while the upper and left edges are not morphisms but the indicated paths. The other edges are paths adjoint to the homotopies shown in (7.4).

A comment is in order about the maps

$$\mathrm{Top}(X_0, f_{2,3}) : \mathrm{Top}(X_0, X_2) \to \mathrm{Top}(X_0, X_3)$$

and

$$\mathrm{Top}(f_{0,1}, X_3) : \mathrm{Top}(X_1, X_3) \to \mathrm{Top}(X_0, X_3)$$

appearing in (7.5).

For a category $\mathcal{C}$ with an object X and a morphism $f : Y \to Y'$, we can compose any morphism $X \to Y$ with f to get a morphism $X \to Y'$. This means that we can use X to define a $\mathcal{S}\mathrm{et}$-valued functor on $\mathcal{C}$,

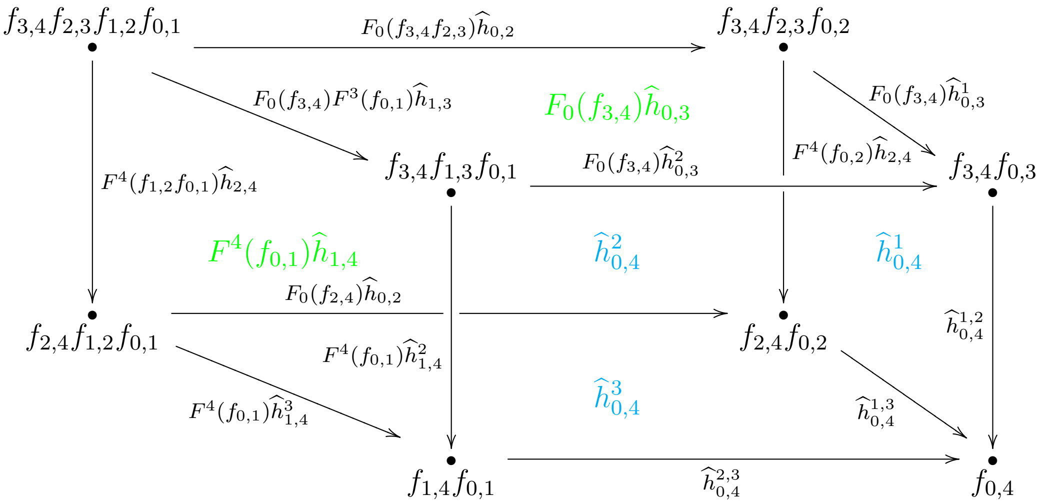
$$\begin{array}{rcl} \mathcal{C} & \xrightarrow{\mathcal{C}(X,-)} & \mathcal{S}\mathrm{et} \\ Y & \longmapsto & \mathcal{C}(X, Y) \\ (f : Y \to Y') & \longmapsto & (f_* : \mathcal{C}(X, Y) \to \mathcal{C}(X, Y')) . \end{array}$$

Dually we can precompose any morphism $W \to X$ with $g : W' \to W$ to get a morphism $W' \to X$. This leads to a contravariant functor,

$$\begin{array}{rcl} \mathcal{C} & \xrightarrow{\mathcal{C}(-,X)} & \mathcal{S}\mathrm{et} \\ W & \longmapsto & \mathcal{C}(W, X) \\ (g : W' \to W) & \longmapsto & (g^* : \mathcal{C}(W, X) \to \mathcal{C}(W', X)) . \end{array}$$

If the morphism sets of $\mathcal{C}$ come equipped with natural topologies, then these functors are Top-valued. Thus in the case $\mathcal{C} = \mathrm{Top}$, they are *endofunctors.*

The set of 4-simplices in $\mathcal{S}$. For each 4-simplex, the additional datum is a map $\widehat{h}_{0,4} : I^3 \to \mathrm{Top}(X_0, X_4)$ of the form:



where F_i and F^i denote the endofunctors $\mathrm{Top}(X_i, -)$ and $\mathrm{Top}(-, X_i)$.

The restriction of $\widehat{h}_{0,4}$ to the left and top faces are the composite double homotopies indicated in green. The restrictions to the three faces abuting $f_{0,4}$ (the front lower right corner) are adjoint to the double homotopies $h^i_{0,4}$ indicated in cyan.

The restriction of $\widehat{h}_{0,4}$ to the back face (not labeled) is the composite

$$\begin{array}{ccc} I \times I & \xrightarrow{\widehat{h}_{2,4} \times \widehat{h}_{0,2}} & \mathrm{Top}(X_2, X_4) \times \mathrm{Top}(X_0, X_2) \\ & & \downarrow \text{comp} \\ & & \mathrm{Top}(X_0, X_4). \end{array}$$

The five labeled faces of the cube are associated with the five 3-dimensional faces of the corresponding 4-simplex in $\mathcal{S}$. These five tetrahedra fit together in a 3-dimensional analog of (7.4), with the central tetrahedron corresponding to the front face of the cube, on which the map restricts to $\widehat{h}^2_{0,4}$.

The set $\mathcal{S}_n$ for $n > 4$. For each n-simplex there is a sequence of spaces and continuous maps

$$X_0 \xrightarrow{f_{0,1}} X_1 \xrightarrow{f_{1,2}} \cdots \xrightarrow{f_{n-1,n}} X_n \tag{7.6}$$

and a map

$$\begin{array}{ccc} I^{n-1} & \xrightarrow{\widehat{h}_{0,n}} & \mathrm{Top}(X_0, X_n) \\ (0, \dots, 0) & \longmapsto & f_{n-1,n} \cdots f_{0,1} \\ (1, \dots, 1) & \longmapsto & f_{0,n} \end{array}$$

We refer to these two points as the left and right vertices of the $(n-1)$-cube, and the $n-1$ faces meeting each of them as the left and right faces.

The $n+1$ faces of the associated n-simplex correspond to the $n-1$ right faces of this cube, along with the two left faces

$$\{(t_1, \dots, t_{n-2}, 0)\} \qquad \text{and} \qquad \{(0, t_2, \dots, t_{n-1})\}.$$

To sum up, the ∞-category $\mathcal{S}$ of topological spaces is a simplicial set in which

- there is a vertex for each topological space in Top,
- there is an edge for each continuous map, and
- for $n > 0$, there is an n-simplex for each sequence of spaces and continuous maps

$$X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} \cdots \xrightarrow{f_{n-1}} X_n$$

and each map $\widehat{h}_{0,n-1} : I^n \to \mathrm{Top}(X_0, X_n)$ meeting certain boundary conditions described above.

To repeat, there is an n-simplex *for every suitable datum. This construction does not involve any choices.* In the simplicial set $N\mathrm{Top}$ there is exactly one n-simplex for each diagram of the form (7.6), while in $\mathcal{S}$ and n-simplex consists of the data of (7.6) *along with* the other maps and homotopies described above.

8 A colimit in $\mathcal{S}$

A pleasant feature of ∞-categories is the fact that limits and colimits are the same as homotopy limits and colimits. The "connective tissue" needed to pass from an ordinary colimit to a homotopy colimit is "built into" an ∞-category.

We will illustrate this with an elementary example taken from the highly recommended paper of Dwyer and Spalinski [DS95], a very friendly introduction to model categories. Consider the following pushout diagrams in Top.

$$\begin{array}{ccc} S^{n-1} & \longrightarrow & D^n \\ \downarrow & & \\ D^n & & \end{array} \qquad \text{and} \qquad \begin{array}{ccc} S^{n-1} & \longrightarrow & * \\ \downarrow & & \\ *, & & \end{array} \tag{8.7}$$

where the maps in the left diagram are each the inclusion of the boundary of the n-dimensional disk. The two diagrams are homotopy equivalent but have distinct pushouts, namely S^n and $*$. What to do?

One solution is to define a model structure on the category of pushout diagrams in Top, in which equivalences and fibrations are levelwise equivalences and fibrations, and cofibrations are defined in terms of lifting properties. This is described in [DS95]. It turns out that the left diagram in (8.7) is cofibrant, but the right one is not. The evident map from the left to the right is a cofibrant approximation. The colimit functor on such diagrams is homotopy invariant on cofibrant objects *but not in general.*

Another solution is to develop the theory of homotopy limits and colimits as Bousfield and Kan did in the "yellow monster" [BK72]. It turns out that the homotopy colimit of each diagram in (8.7) is S^n.

In an ordinary category $\mathcal{C}$, the colimit of a diagram p is an initial object in the category of objects equipped with compatible maps from all the objects in p, which we denote by $\mathcal{C}_{p/}$, *the category of objects under* p. If p is a pushout diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle f'}\downarrow & & \\ B', & & \end{array}$$

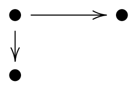
then an object in $\mathcal{C}_{p/}$ is a commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle f'}\downarrow & & \downarrow \\ B' & \longrightarrow & X. \end{array}$$

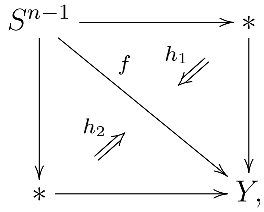
Similarly a limit is a terminal object in $C_{/p}$, *the category of objects under* p.

Now suppose we have an ∞-category $\mathcal{C}$ and a simplicial map $\widetilde{p} : K \to \mathcal{C}$ for a simplicial set K. Then we can define $\mathcal{C}_{\widetilde{p}/}$ ($\mathcal{C}_{\widetilde{p}/}$), *the* ∞*-category of objects under (over)* $\widetilde{p}$, and we can look for an initial (terminal) object in it.

In the case at hand, K is the nerve of the pushout category



Let p be the diagram on the right of (8.7), and choose a map $\widetilde{p} : K \to \mathcal{S}$ that does the right thing on the three vertices and two nondegenerate edges of K. *There are many such maps, and any one of them will do.* (To see that such maps exist, note that $|K|$ is contractible.) An object in $\mathcal{S}_{/\widetilde{p}}$ leads a diagram of the form



(compare with (7.3)) for some space Y. This is a pair of 2-simplices in $\mathcal{S}$ sharing a common edge. It amounts to a map $f : S^{n-1} \to Y$ *equipped with a pair of null homotopies* h_1 and h_2 that are determined by the choice of $\widetilde{p}$. These define extensions of f to the northern and southern hemispheres of S^n, meaning the diagram has the same information as a map $S^n \to Y$. It follows that S^n, which is the homotopy colimit of p in Top, *is the ordinary colimit of* $\widetilde{p}$ *(for any choice of* $\widetilde{p}$*!) in* $\mathcal{S}$.

More details can be found in [Lur09, 4.2.4].

9 Bousfield localization in ∞-categories

Bousfield localization may be the best construction in model category theory. It is essential to chromatic homotopy theory. We will indicate how a similar construction works in a suitable ∞-category.

We begin by recalling Bousfield's construction. We start with a model category $\mathcal{M}$, and try to alter the model structure in the following way. We enlarge the class of weak equivalences in some way without altering the class of cofibrations. This means there are more trivial cofibrations (cofibrations which are also weak equivalences) and hence fewer fibrations, since they must have the right lifting property with respect to all trivial cofibrations. However there are just as many trivial fibrations as before, since they must have the right lifting property with respect to all cofibrations. See [Lur09, A.3.7].

The hard part of this is verifying that the proposed new model structure (with more weak equivalences but fewer fibrations) satisfies the axiom requiring that each map factor as a trivial cofibration followed by a fibration. There is a theorem saying this can be done under mild hypotheses on $\mathcal{M}$. Thus we get a new model structure *with a much more interesting fibrant replacement functor* L.

For example, when we enlarge the class of weak equivalences in the category of spaces or spectra to those maps inducing an isomorphism of homotopy groups in dimensions up to a chosen integer m, but not necessarily in higher dimensions, *the resulting fibrant replacement functor is the* m*th Postnikov section.* This means attaching cells in dimensions above $m+1$ so as to kill of all the higher homotopy groups. The fibrant objects are those spaces or spectra with trivial homotopy groups above dimension m.

When we enlarge the class of weak equivalences in the category of spaces or spectra to all maps inducing an isomorphism in the nth Morava E-theory (or the nth Morava K-theory) for a fixed prime p and height n, *the resulting fibrant replacement functor is the* L_n *(or* $L_{K(n)}$*) of chromatic homotopy theory.* In this case there is no easy description of the fibrant objects.

We now turn to ∞-categories. [Lur09, Proposition 5.5.4.15] is a statement about an analog of Bousfield localization. The input is a presentable ∞-category $\mathcal{C}$ with a set of morphisms S that are meant to be made into equivalences. *Presentable* means that $\mathcal{C}$ has small colimits and every object is a colimit of small objects. An object is *small* if the mapping space from it to each filtered colimit is equivalent to the colimit of the mapping spaces.

For example, in the ordinary category of abelian groups, which is presentable, every object is the colimit of its finitely generated subgroups, which are the small objects. Any homomorphism from a finitely generated abelian group A to a filtered colimit of same factors through one of the groups in the diagram, making the functor $\mathrm{Hom}(A, -)$ commute with filtered colimits.

One might enlarge the morphism set of $\mathcal{C}$ by formally inverting the morphisms in S, forming what has been called a category of fractions $S^{-1}\mathcal{C}$, but experience has shown that this leads to technical problems. As Lurie says in [Lur09, page 364],

> Under suitable hypotheses on S, there is a drastically simpler approach: we can find the desired ∞-category $S^{-1}\mathcal{C}$ *inside* $\mathcal{C}$ as the full subcategory of S-local objects of $\mathcal{C}$.

In [Lur09, Definition 5.5.4.1] an object Z is said to be S-*local* if each morphism $s : X \to Y$ in S induces a weak equivalence $\mathcal{C}(Y, Z) \to \mathcal{C}(X, Z)$. A morphism $s : A \to B$ is an S-*equivalence* if it induces a weak equivalence $\mathcal{C}(B, Z) \to \mathcal{C}(A, Z)$ for each S-local object Z. Such definitions could (and have been) also be made in a model category. For example, see [HHR21, Chapter 6].

Let $\overline{S}$ be the set of all S-equivalences, which can be explicitly constructed from S. Let $\mathcal{C}'$ be the full subcategory of S-local objects. Then

(i) For each object $X \in \mathcal{C}$, there exists an S-equivalence $s : X \to X'$ where X' is S-local.

(ii) The ∞-category $\mathcal{C}'$ is presentable.

(iii) The inclusion functor $I : \mathcal{C}' \to \mathcal{C}$ has a left adjoint L. (It is a left adjoint as a functor to $\mathcal{C}'$, so it sends colimits in $\mathcal{C}$ to colimits in $\mathcal{C}'$.) *The composition IL (which need not be either a left or right adjoint) is the analog of Bousfield's fibrant replacement functor in model category theory.*

10 The ∞-category of spectra

The ∞-categorical approach to stable homotopy theory is discussed in [Lur17, §1.4], [Gro10, §5] and [Bar22]. We first need some definitions.

Definition 8. An ∞-category $\mathcal{C}$ is *pointed* if it has a zero object 0 which is both initial and final, meaning that the spaces $\mathcal{C}(X, 0)$ and $\mathcal{C}(0, Y)$ are contractible in all cases.

The zero object need not be unique, but all such objects are equivalent.

For an object X in a pointed ∞-category $\mathcal{C}$ with finite limits and colimits (such as $\mathcal{S}_*$, the ∞-category of pointed spaces), one has pushout and pullback diagrams

$$\begin{array}{ccc} \Omega_{\mathcal{C}} X & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ 0' & \longrightarrow & X. \end{array} \qquad \text{and} \qquad \begin{array}{ccc} X & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ 0' & \longrightarrow & \Sigma_{\mathcal{C}} X \end{array}$$

In the case $\mathcal{C} = \mathcal{S}_*$, we omit the subscripts, these being the usual loop and suspension functors Ω and Σ, and Ω is the right adjoint of Σ.

There is a functor $F : \mathcal{S} \to \mathcal{S}_*$ defined by adding a disjoint base point which is left adjoint to the forgetful functor.

The following is [Lur17, Definition 1.1.1.9].

Definition 9. An ∞-category $\mathcal{C}$ is *stable* if

(i) It is pointed.

(ii) For each morphism $f : X \to Y$ there are pullback and pushout diagrams

$$\begin{array}{ccc} W & \longrightarrow & X \\ \downarrow & & \downarrow f \\ 0 & \longrightarrow & Y \end{array} \qquad \text{and} \qquad \begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow & & \downarrow \\ 0' & \longrightarrow & Z, \end{array}$$

the *fiber* and *cofiber* sequences of f.

(iii) A diagram of the above form is a pushout if and only if it is a pullback, i.e., fiber sequences and cofiber sequences are the same.

Colimits, such as pushouts, in an ∞-category were explained in Section 8, and limits, such as pullbacks, have a dual description. They exist in the ∞-category of pointed spaces as well as that of spectra. The key point of Definition 9 is the third condition, that fiber and cofiber conditions coincide. Thus in a diagram

$$\begin{array}{ccc} X & \longrightarrow & * \\ \downarrow & & \downarrow \\ * & \longrightarrow & Y \end{array}$$

which is both a pushout and a pullback, $Y \simeq \Sigma_{\mathcal{C}} X$ and $X \simeq \Omega_{\mathcal{C}} Y$. It follows that $\Sigma_{\mathcal{C}}\Omega_{\mathcal{C}}$ and $\Omega_{\mathcal{C}}\Sigma_{\mathcal{C}}$ are both equivalent to the identity functor, so both $\Sigma_{\mathcal{C}}$ and $\Omega_{\mathcal{C}}$ are invertible.

To pass from $\mathcal{S}_*$, the ∞-category of pointed spaces, to Sp, the ∞-category of spectra, here is an informal definition from [Lur17, §1.4].

Definition 10. Sp is the homotopy limit (which is the same as the limit in the ∞-category of ∞-categories) of the tower

$$\cdots \xrightarrow{\Omega} \mathcal{S}_* \xrightarrow{\Omega} \mathcal{S}_* \xrightarrow{\Omega} \mathcal{S}_*,$$

where Ω is the loop functor $X \mapsto \mathcal{S}_*(S^1, X)$.

To unpack this definition, note that a vertex in this homotopy limit (meaning an object in the ∞-category Sp) consists of a sequence of vertices (i.e., pointed spaces) $X_0, X_1, X_2, \ldots$, along with weak equivalences $X_i \to \Omega X_{i+1}$ in $\mathcal{S}_*$. *This coincides with the original definition of an Ω-spectrum.*

This ∞-category is stable as in Definition 9. Its objects and morphisms are Ω-spectra and maps between them as classically defined. Unlike the classical case, it comes equipped with higher homotopies similar to those we described for the ∞-category $\mathcal{S}$ of spaces above.

Classically we know that the homotopy type of an Ω-spectrum E is determined by the homology theory E_* associated with it. The latter can be viewed as a functor with certain properties (spelled out in Brown's representability theorem, the subject of [Lur17, §1.4.1]) on the category of finite pointed CW-complexes. It is possible to define the ∞-category of spectra as the ∞-category of such functors. The following is [Bar22, Def. 4.1.1], and is indicated in [Lur17, Remark 1.4.3.3].

Definition 11. Spectra as functors. *A* spectrum *is a functor $E : \mathcal{S}_* \to \mathcal{S}_*$ which is*

(i) reduced, *meaning that it sends final objects to final objects,*

(ii) excisive, *meaning that it converts pushout diagrams to pullback diagrams, and*

(iii) of finite presentation, *meaning that it preserves filtered colimits.*

Applying a reduced functor F to a pushout square in $\mathcal{S}_*$

$$\begin{array}{ccc} X & \longrightarrow & * \\ \downarrow & & \downarrow \\ * & \longrightarrow & \Sigma X \end{array}$$

leads to a map $\sigma_X : FX \to \Omega F \Sigma X$. In [Bar22, Lemma 4.1.3] Barwick shows that such an F is excisive iff σ_X is an equivalence for each X.

The finiteness condition (iii) means that E is determined by its restriction to the subcategory $\mathcal{S}_*^{\text{fin}}$ of finite pointed CW-complexes. This category is generated by S^0 under finite colimits. Similarly the unpointed analog $\mathcal{S}^{\text{fin}}$ is generated by $*$ under finite colimits.

For a spectrum E as in Definition 11, one has pointed spaces $E_n := E(S^n)$ with equivalences $E_n \to \Omega E_{n+1}$ as in Definition 10.

The following is [Lur17, Definition 1.4.2.8].

Definition 12. Spectrum objects in a category. Let $\mathcal{C}$ be an ∞-category that admits finite limits. A *spectrum object* of $\mathcal{C}$ is a reduced, excisive functor $E : \mathcal{S}_*^{\text{fin}} \to \mathcal{C}$, and $\text{Sp}(\mathcal{C})$ is the full subcategory of $\text{Fun}(\mathcal{S}_*^{\text{fin}}, \mathcal{C})$ spanned by such functors.

In particular, the category of spectra Sp is $\text{Sp}(\mathcal{S}_*)$.

We define the *0-object functor* $\Omega_{\mathcal{C}}^\infty : \text{Sp}(\mathcal{C}) \to \mathcal{C}$ by $\Omega_{\mathcal{C}}^\infty E := E(S^0)$. $\Omega_{\mathcal{C}}^\infty$ has a left adjoint $\Sigma_{\mathcal{C}}^\infty{}_+ : \mathcal{C} \to \text{Sp}(\mathcal{C})$, the *suspension spectrum functor*, given (when $\mathcal{C}$ has small colimits) by

$$(\Sigma_{\mathcal{C}}^\infty{}_+ X)K := \operatorname*{colim}_n \Omega_{\mathcal{C}}^n(\Sigma^n K \wedge X_+) \qquad \text{for } X \text{ in } \mathcal{C} \text{ and } K \text{ in } \mathcal{S}_*^{\text{fin}},$$

where the smash product with the finite pointed CW-complex $\Sigma^n K$ is the appropriate finite colimit in $\mathcal{C}$. When $\mathcal{C}$ is pointed and has small colimits, we define $\Sigma_{\mathcal{C}}^\infty : \mathcal{C} \to \text{Sp}(\mathcal{C})$ by

$$(\Sigma_{\mathcal{C}}^\infty X)K := \operatorname*{colim}_n \Omega_{\mathcal{C}}^n(\Sigma^n K \wedge X).$$

The *sphere spectrum* $\mathbb{S}$ in Sp is defined by

$$\mathbb{S} := \Sigma_+^\infty(*) = \Sigma^\infty(S^0). \tag{10.13}$$

Lurie shows that $\text{Sp}(\mathcal{C})$ is stable as in Definition 9 [Lur17, Corollary 1.4.2.17], it has a description similar to that of Definition 10 [Lur17, Proposition 1.4.2.24], and it is presentable when $\mathcal{C}$ is presentable [Lur17, Proposition 1.4.4.4].

For presentable ∞-categories $\mathcal{C}$ and $\mathcal{D}$, let $\text{LFun}(\mathcal{C}, \mathcal{D})$ denote the full subcategory of the functor category $\text{Fun}(\mathcal{C}, \mathcal{D})$ spanned by those functors which preserve small colimits. The following is [Lur17, Corollary 1.4.4.6].

Proposition 14. *Let $\mathcal{D}$ be a presentable stable ∞-category. Then evaluation on the sphere spectrum $\mathbb{S}$ of (10.13) induces an equivalence of ∞-categories*

$$\theta : \text{LFun}(\text{Sp}, \mathcal{D}) \to \mathcal{D}.$$

This means, as Lurie puts it in [Lur17, page 9], that

> Sp is universal among stable ∞-categories: it is freely generated (as a stable ∞-category which admits small colimits) by a single object.

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