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THE SEGAL CONJECTURE FOR CYCLIC GROUPS AND ITS CONSEQUENCES

By Douglas C. Ravenel*

with an appendix by Haynes R. Miller**

This paper was submitted before the announcement of Carlsson's proof of the Segal conjecture for all finite groups, so the result cited in the title is now obsolete. However the methods used here and some of their consequences (e.g. 1.11(b), which concerns complex projective space, 1.14 and 1.15, which concern certain Thom spaces) are not evident in Carlsson's work. Our main technical tool, the modified Adams spectral sequence of section 2 may be of some independent interest. It has recently been applied by Miller—Wilkerson [19] to periodic groups; their paper also provides some further insight into the proofs presented here.

The Segal conjecture (unpublished, c. 1970) concerns the stable homotopy type of BG, the classifying space for a finite group G. We will state it here in three forms. Recall that the Burnside ring $A(G)$ is the ring of isomorphism classes of virtual finite G-sets. There is a ring homomorphism $\epsilon : A(G) \to Z$ which assigns to each virtual G-set its cardinality. Let $I = \ker \epsilon$ and let $A(G)^\wedge$ denote the I-adic completion of G. Let $\pi_S^i(X) = \varinjlim\, [\Sigma^n X, S^{n+i}]$, the ith stable cohomotopy group of X, and let BG_+ denote BG union a disjoint base point. Then we have

Segal Conjecture I. For each finite group G,

$$\pi_S^i(BG_+) = \begin{cases} A(G)^\wedge & \text{if} \quad i = 0 \\ 0 & \text{if} \quad i > 0. \end{cases} \qquad \square$$

For the second form of the conjecture we consider a spectrum $DBG+$, the functional dual of the suspension spectrum for BG_+, such

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that $\pi_i(DBG_+) = \pi_S^{-i}(BG_+)$. For a spectrum X, let DX be the representing spectrum for the functor $[W \wedge X, S^0]$, regarded as a cohomology theory on W. The conjecture above says that DBG_+ is (-1)-connected with $\pi_0(DBG_+) = A(G).^\wedge$. The next conjecture describes DBG_+. Given a subgroup $H \subseteq G$, let N_H be its normalizer, i.e. the largest subgroup of G in which H is normal, and let $W_H = N_H/H$, the Weyl group of H. Then we have

Segal Conjecture II. For each finite group G. DBG_+ is a suitable completion of $\vee_{H \subseteq G} BW_{H+}$ where the wedge sum is over all conjugacy classes of subgroups of G. If G is a p-group, this completion is obtained by p-adically completing each BW_H for $H \neq G$. □

To see that this is consistent with I, recall that additively $A(G)$ is isomorphic to the free abelian group on the conjugacy classes of subgroups $H \subseteq G$. For G a p-group, $A(G)^\wedge$ is obtained from $A(G)$ by p-adically completing the direct summand I (see section 1 of [12]).

The third and strongest form of the conjecture asserts that a certain map $\vee_{H \subseteq G} BW_{H+} \xrightarrow{f} DBG_+$ becomes an equivalence after completing the source. The map f is obtained as follows. A map $BW_{H+} \to DBG_+$ by definition corresponds to a stable map $BW_{H+} \wedge BG_+ \to S^0$. Now $BW_{H+} \wedge BG_+$ is the suspension spectrum of the space $(BW_H \times BG)_+$, so we need a map from it to the space QS^0. W_H is the automorphism set of the G-set G/H, which is also acted on by G. If G/H has n elements we get a homomorphism $W_H \times G \to \Sigma_n$. Passing to classifying spaces and composing with the Barratt-Quillen map $B\Sigma_n \to QS^0$ gives us the desired map $BW_H \times BG \to QS^0$, and we have

Segal Conjecture III. The map $f : \vee_{H \subseteq G} BW_{H+} \to DBG_+$ described above factors through an equivalence from a suitable completion of the source. □

Our main result is 1.11, which implies Segal Conjecture II for cyclic groups (1.19) and certain metacyclic groups (1.21). 1.11 also gives a local description of DCP^∞ and of the duals of related Thom spectra. It seems quite likely that our methods would apply to the Thom spectra of more general vector bundles than those considered here; we leave the details to the interested reader.

Segal Conjecture III for cyclic groups is proved in the Appendix.

In 1.15 we identify certain inverse limits of Thom spectra, generalizing Lin's result [13] 1.2 that $\varprojlim RP^\infty_{-n}$ is the 2-adic completion of S^{-1}.

This result was conjectured by Mahowald and discussed by Adams in [2]. We use it to define an invariant due to Mahowald in the stable homotopy groups of spheres which has been the subject of much speculation by him and his coworkers.

Most of the results of section 1 are derived from 1.7 which states that certain maps induce isomorphisms of cohomotopy groups above certain dimensions. Lin [13] proved the special case of this result that he needed with the Adams spectral sequence (ASS). In section 2 we prove 1.7 with our main technical tool, the modified Adams spectral sequence (MASS), after explaining why the usual ASS will not work in general.

In section 3 we prove the requisite technical properties of the MASS including convergence (3.6) and identification of the E_2-term (3.4).

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1. Main results. In this section we prove Segal Conjecture II for cyclic groups (1.19) and certain metacyclic groups (1.21). These are corollaries of a more general result (1.11) which describes the duals of the Thom spectra of multiples of the canonical complex line bundle over $BZ/(p^n)$ and CP^∞. Inverse limits of these Thom spectra, which have cells in all negative dimensions, are shown (1.15) to be equivalent to certain connective spectra, thereby providing spectacular examples of the failure of homology to commute with inverse limits.

1.11 and its consequences are derived from 1.7 which says that certain maps between these Thom spectra induce isomorphisms in cohomotopy above a given dimension. The proof uses a new generalization of the Adams spectral sequence and is given in section 2.

We have to state our results in terms of functional duals and p-adic completions and cocompletions so we begin by defining these objects and giving their basic properties.

1.1 *Definition.* Given a spectrum X let $X/(p^i)$ be the cofibre of $X \xrightarrow{p^i} X$. Let $\hat{X}$, the *p-adic completion* of X, be $\varprojlim X/(p^i)$, and let $\tilde{X}$, the *p-adic cocompletion* of X be $\varinjlim \Sigma^{-1}X/(p^i)$.

1.2 Proposition.

(a) *If $\pi_*(X)$ is all p-torsion then $\tilde{X} = X$. If in addition each $\pi_i(X)$ has finite exponent the $\hat{X} = X$.*

(b) *$\tilde{S}^1$ is the Moore spectrum for $Z/(p^\infty) = \varinjlim Z/(p^i)$ and $\hat{S}^0$ is the Moore spectrum for $Z_p = \varprojlim Z/(p^i)$, the p-adic integers.*

(c) $\pi^*(\tilde{X}) = \varprojlim \pi^*(X) \otimes Z/(p^i)$. □

Before proceeding further we need some properties of the duality functor D defined in the introduction.

1.3 PROPOSITION.

(a) *D is a contravariant functor from the stable homotopy category to itself which preserves cofibre sequences.*

(b) *If X is a finite spectrum, DX is its Spanier-Whitehead dual.*

(c) $\pi^{-n}(X) = \pi_n(DX)$.

(d) *$D \varinjlim X_i = \varprojlim DX_i$ (It is not generally true that $D \varprojlim X_i = \varinjlim DX_i$.).*

(e) *$D(\tilde{X}) = \hat{D}X$, where $\hat{D}X$ denotes the p-adic completion of DX.* □

Before stating our main results we need some notation.

Let λ be the canonical complex line bundle over $BZ/(p^n)$ and let U_i denote the Thom spectrum of the i-fold Whitney sum of λ. It is easy to see that U_i is a CW-spectrum with one cell in each dimension $\geq 2i$. The cell in dimension $2i + 1$ is trivially attached to the bottom cell and the $(2i + 2)$-cell is attached to the $(2i + 1)$-cell by a map of degree p^n. These statements hold for any integer i.

Similarly let CP_i denote the Thom spectrum of the i-fold Whitney sum of the canonical complex line bundle over CP^∞. CP_i is a CW-spectrum with one cell in every even dimension $\geq 2i$.

1.4 LEMMA. *$U_i/U_i^{2i+1} = U_{i+1}$ and $CP_i/CP_i^{2i} = CP_{i+1}$, where X^j denotes the j-skeleton of X.*

Proof. Given a vector bundle ξ over a space X. Let X^ξ denote the Thom space of ξ and X_ξ the associated sphere bundle. It follows immediately from the definitions that given vector bundles α and β over a space B there is a cofibre sequence (up to homotopy equivalence)

$$(B_\alpha)^\beta \to B^\beta \to B^{\alpha\oplus\beta}. \tag{1.5}$$

To prove the lemma for U_i with $i > 0$ let $B = BZ/(p^n)$, $\alpha = \lambda$ and $\beta = i\lambda$. Then $B^\beta = U_i$, $B^{\alpha\oplus\beta} = U_{i+1}$, $B_\alpha = S_1$, and β is trivial over B_α so $(B_\alpha)^\beta = S^{2i} \vee S^{2i+1} = U_i^{2i+1}$.

For $i < 0$ we can prove the lemma through a range of dimensions by letting B be a suitable skeleton of $BZ/(p^n)$ and $\beta = (p^m + i)\lambda$ for m sufficiently large. Passing to spectra 1.5 gives the desired result through a range up to suspension. This range of dimensions can be made arbitrarily large, so the result follows.

The argument for CP_i is similar. $\square$

1.6 LEMMA. *For* $j > i$, $DU_i^{2j-1} = \Sigma U_{-j}^{-2i-1}$ *and* $DCP_i^{2j-2} = \Sigma^2 CP_{-j}^{-2i-2}$.

Proof. First consider the case $i = 0$ for U_i^{2j-1}. U_0^{2j-1} is the suspension spectrum of the $(2j-1)$-skeleton of $BZ/(p^n)_+$, i.e. a lens space L_+^{2j-1} with disjoint base point. This lens space is a smooth manifold whose tangent bundle is stably equivalent to $j\lambda$. By Atiyah duality [4], L_+^{2j-1} is dual to the appropriate desuspension of the Thom spectrum of its stable normal bundle, i.e. $DU_0^{2j-1} = \Sigma U_{-j}^{-1}$ as claimed.

Now for $i > 0$ we have a cofibre sequence

$$U_0^{2i-1} \to U_0^{2j-1} \to U_i^{2j-1}$$

which dualizes to give $DU_i^{2j-1} = \Sigma U_{-j}^{-2i-1}$.

For $i < 0$ we use James periodicity, i.e. we know that $\lambda - 1$ has finite order in $K^*(L^{2j-1-2i})$ so $U_i^{2j-1} = \Sigma^{-2p^m} U_{i+p^m}^{2j++2p^m-1}$ for m sufficiently large. Hence the result for $i < 0$ follows from the result for $i \geq 0$.

The argument for CP_i^{2j-2} is similar. $\square$

The following result is crucial. Its proof involves a new generalization of the Adams spectral sequence and will be presented in section 2, while the requisite properties of the spectral sequence will be established in section 3. Let V_i be defined similarly to U_i using $BZ/(p^{n-1})$ instead of $BZ/(p^n)$. Note that U_0 and V_0 are the suspension spectra of $BZ/(p^n)_+$ and $BZ/(p^{n-1})_+$ respectively, so we have a reduction map $U_0 \to V_0$ which we denote by r. The pth power map on CP^∞ induces a map on CP_0 which we denote by $[p]$. From 1.4 we get inverse systems

$$U_0 \leftarrow U_{-1} \leftarrow U_{-2} \leftarrow \cdots$$

and

$$CP_0 \leftarrow CP_{-1} \leftarrow CP_{-2} \leftarrow \cdots.$$

1.7 THEOREM. *For $i > 0$ the composites*

$$\tilde{U}_{-i} \longrightarrow \tilde{U}_0 \xrightarrow{r} \tilde{V}_0$$

and

$$\widetilde{CP}_i \longrightarrow \widetilde{CP}_0 \xrightarrow{[p]} \widetilde{CP}_0$$

induce isomorphisms in π^k for $k > 1 - 2i$. □

The spectra above have been p-adically cocompleted to ensure that the mod (p) Adams spectral sequences for their cohomotopies each converge.

Now we are ready to analyze DU_{-k} and DCP_{-k}. From 1.4 we get cofibre sequences

$$U_{-k-i} \to U_{-k} \to \Sigma U_{-k-i}^{-2k-1}$$

and

$$CP_{-k-i} \to CP_{-k} \to \Sigma CP_{-k-i}^{-2k-2} \quad \text{for} \quad i > 0.$$

Using 1.6 the right hand maps dualize to

$$U_k^{2i+2k-1} \to DU_{-k} \quad \text{and} \quad \Sigma CP_k^{2i+2k-2} \to DCP_{-k}.$$

Letting i go to ∞ we get maps

1.8 $$U_k \xrightarrow{f} DU_{-k} \quad \text{and} \quad \Sigma CP_k \xrightarrow{g} DCP_{-k}$$

for all integers k.

For $k > 0$ we have composite maps

$$DV_0 \xrightarrow{Dr} DU_0 \longrightarrow DU_{-k} \quad \text{and} \quad DCP_0 \xrightarrow{D[p]} DCP_0 \longrightarrow DCP_{-k}$$

which we denote simply by Dr and $D[p]$ respectively. Composing iterates of Dr with f and summing we get a map

1.9 $$U_k \vee \bigvee_{i=0}^{n-1} BZ/(p^i)_+ \xrightarrow{h} DU_{-k}$$

for $k \leq 0$. Similarly, using $D[p]$ and g we get

1.10 $$\Sigma CP_k \vee \bigvee_{i=1}^{\infty} \Sigma CP_0 \xrightarrow{e} DCP_{-k}$$

for $k \leq 0$. We will deal with the case $k < 0$ below.

Now we come to our main result.

1.11 THEOREM. (a) *The map h of 1.9 is an equivalence after p-adic completion, i.e. the p-adic completion of the source is equivalent to $D\tilde{U}_{-k}$ for $k \geq 0$.*

(b) *There is a map*

$$e' : D\tilde{C}P_{-k} \to \Sigma C\hat{P}_k \vee \prod_{i=1}^{\infty} \Sigma C\hat{P}_0$$

for $k \geq 0$ such that

(i) *$e'e$ is the composite of p-adic completion and the inclusion of a sum into a product,*

(ii) *e' has fibre $\hat{S}^0$ and*

(iii) *e' is a retraction so $DCP_{-k} \cong \hat{S}^0 \vee \Sigma C\hat{P}_k \vee \Pi_{i=1}^{\infty} \Sigma C\hat{P}_0$.* □

Note that 1.11(a) for $k = 0$ is equivalent to Segal Conjecture II for $G = Z/(p^n)$. The only difference between U_0 and $\tilde{U}_0$ is that the bottom cell (which is a retract) of the former is cocompleted in the latter. Thus in $D\tilde{U}_0$ all $n + 1$ bottom cells are completed whereas in DU_0 one of them is not, as the conjecture prescribes.

For $k > 0$, U_{-k} differs from $\tilde{U}_{-k}$ only in that the bottom cell is cocompleted in the latter and it follows that

1.12 $$DU_{-k} = U_k \vee \bigvee_{i=0}^{n-1} \hat{B}Z/(p^i)_+ .$$

The relation of DCP_{-k} to $DC\tilde{P}_{-k}$ is still unclear.

Proof. (a) We will argue by induction on n. For $n = 0$ the statement is simply that $D\tilde{S}^0 = \hat{S}^0$. For the inductive step we need to show that $\hat{U}_k$ is the cofibre of Dr and that f gives a splitting.

Using 1.3(c) and dualizing 1.7 we see that the composite

$$D\tilde{V}_0 \xrightarrow{Dr} D\tilde{U}_0 \longrightarrow D\tilde{U}_{-i}$$

is a $(2i-2)$-equivalence. Taking the direct limit as i approaches ∞ we find that $D\tilde{V}_0 = \lim_{\rightarrow} D\tilde{U}_{-i}$ and that this spectrum is a retract of $D\tilde{U}_0$.

From the definition of f (1.8) we have a cofibre sequence

$$\varinjlim D\tilde{U}_{-i} \longleftarrow D\tilde{U}_{-k} \xleftarrow{f} \hat{U}_k$$

for $k \leq 0$. The left hand spectrum has just been identified with $D\tilde{V}_0$ and the left hand map has been shown to be a retraction, so the result follows.

(b) Arguing as above we get an equivalence of $\varinjlim D C\tilde{P}_{-i}$ with $DC\tilde{P}_0$ and a cofibre sequence

$$\varinjlim DC\tilde{P}_{-1} \leftarrow DC\tilde{P}_{-k} \leftarrow \Sigma C\hat{P}_k$$

for $k \geq 0$ in which the left hand map is a retraction onto $DC\tilde{P}_0$ split by $D[p]$ and in which the right hand map is a splitting for g. In particular we have $DC\tilde{P}_{-k} = C\hat{P}_k \vee DC\tilde{P}_0$ for $k \geq 0$.

For $k = 0$ the cofibre of $D[p]$ is $\Sigma C\hat{P}_0$ so we have a diagram

$$\begin{tikzcd}
DC\tilde{P}_0 \arrow[r, "D[p^n]"] & DC\tilde{P}_0 \arrow[r] & \bigvee_{i=1}^{n} \Sigma C\hat{P}_0 \\
DC\tilde{P}_0 \arrow[u, "D[p]"] \arrow[r, "D[p^{n+1}]"] & DC\tilde{P}_0 \arrow[u, equal] \arrow[r] & \bigvee_{i=1}^{n+1} \Sigma C\hat{P}_0 \arrow[u]
\end{tikzcd}$$

where the rows are cofibre sequences and the right hand vertical map collapses the $(n+1)$th wedge summand. Letting n go to ∞ we get a cofibre sequence

1.3 $$\varprojlim_{D[p]} DC\tilde{P}_0 \longrightarrow DC\tilde{P}_0 \xrightarrow{e'} \prod_{i=1}^{\infty} \Sigma C\hat{P}_0.$$

The relation of e to e' (i) is straightforward. For (ii) we have

$$\varprojlim_{D[p]} DC\tilde{P}_0 = D \varinjlim_{[p]} C\tilde{P}_0$$

by 1.3(d). Now $C\tilde{P}_1$ is the suspension spectrum of $B\mathbf{Q}/Z_{(p)}$, $[p]$ is induced by multiplication by p in $L/Z_{(p)}$, and $\varinjlim_{p} \mathbf{Q}/Z_{(p)} = 0$. It follows that $\varinjlim_{[p]} C\tilde{P}_0 = \tilde{S}^0$ and e' has fibre $\hat{S}^0$. This $\hat{S}^0$ is dual to the bottom of $C\tilde{P}_0$ which is a retract, so e' must be a retraction. □

We now analyze DU_k and DCP_k for $k > 0$. Our results here are less precise. From the cofibre sequences (1.4)

$$U_{k-i}^{2k-1} \to U_{k-i} \to U_k \quad \text{and} \quad CP_{k-i}^{2k-2} \to CP_{k-i} \to CP_k$$

we dualize (using 1.6) and get

$$\Sigma U_{-k}^{2i-2k-1} \leftarrow DU_{k-i} \leftarrow DU_k \quad \text{and} \quad \Sigma^2 CP_{-k}^{2i-2k-1} \leftarrow DCP_{k-i} \leftarrow DCP_k.$$

Completing, letting i go to ∞, and identifying middle terms as above we get

1.14 $$\Sigma \hat{U}_{-k} \leftarrow D\tilde{V}_0 \leftarrow D\tilde{U}_k \quad \text{and} \quad \Sigma^2 C\hat{P}_{-k} \leftarrow D\tilde{CP}_0 \leftarrow D\tilde{CP}_k.$$

We saw above that when $k \leq 0$ the left hand maps are trivial, but that is apparently not the case when $k > 0$.

If we let k go to infinity and take the inverse limit in 1.14, the fibres become contractible by 1.3(d) so we get

1.15 Corollary. *$\varprojlim \hat{U}_{-k} = \Sigma^{-1} D\tilde{V}_0$ and $\varprojlim C\hat{P}_{-k} \cong \Sigma^{-2} D\tilde{CP}_0$, where $D\tilde{V}_0$ and $D\tilde{CP}_0$ are identified in* 1.11. *In particular $\varprojlim U_{-k} \cong \hat{S}^{-1}$ for $n = 1$.* □

1.15 provides a spectacular example of the failure of homology to commute with inverse limits. If $n = 1$ and $p = 2$ then $\varinjlim H^*(U_{-i}; Z/(2)) = Z/2[x, x^{-1}] \neq H^*(\hat{S}^{-1}; Z/(2))$ with $x \in H^1$. The action of the Steenrod algebra A is given by $\mathrm{Sq}^k(x^i) = \binom{i}{k} x^{k+i}$, where the binomial coefficient $\binom{i}{k} = (\Pi_{j=0}^{k=1} (i - j))/k!$ is defined for any integer i. This A-module figures in the proof of 1.7 (to be given in the next section) and in an old conjecture of Mahowald which is discussed by Adams in [2] and of which 1.15 is a generalization.

For $n = 1$, 1.15 says $\hat{S}^{-1} \simeq \varprojlim U_{-i}$. Define X_i for $i \in Z$ by $X_{2i} = U_i$ and $X_{2i+1} = U_i/U_i^{2i}$. For $p = 2$ X_i is the stunted real projective space RP_i^∞ with bottom cell in dimension i. 1.15 implies $\varprojlim \hat{X}_{-i} = \hat{S}^{-1}$ and there are compatible maps $f_i : \hat{S}^{-1} \to \hat{X}_{-i}$ for $i > 0$. Similarly we have compatible maps $g_i : S^{-2} \to C\hat{P}_{-i}$ for $i > 0$ and 1.15 implies that $\hat{S}^{-2}$ is a retract of $\varprojlim C\hat{P}_{-i}$.

1.16 *Definition.* Let $\alpha \in \pi_k \hat{S}^0$ and let i be the smallest integer such that $f_i(\alpha) \neq 0$. Since $f_{i-1}(\alpha) = 0$, $f_i(\alpha)$ lifts to the bottom cell (p-adically completed) of $\hat{X}_i$. This lifting gives a coset $M(\alpha)$ of im $\pi_k(X_{1-i}) \subset \pi_{k-1}(\hat{S}^{-i})$ which we call the *Mahowald invariant* of α. Similarly let j be the smallest integer such that $g_j(\alpha) \neq 0$. Then we get a coset $CM(\alpha)$ of im $\pi_{k-1}(S^{-2j}) \subset \pi_{k-2}(\hat{S}^{-2j})$ which we call the *complex Mahowald invariant* of α.

Hence for $\alpha \in \pi_k(S^0)$ we have $M(\alpha) \subset \pi_{k+i-1}(S^0)$ and $CM(\alpha) \subset \pi_{k+2j-2}(\hat{S}^0)$ where i and j depend on α as well as on k. 1.15 guarantees that $0 \notin M(\alpha)$ and $0 \notin CM(\alpha)$. This invariant appears to be an interesting way of constructing new elements (modulo indeterminacy) in stable homotopy. For example, low dimensional calculations give

1.17 PROPOSITION. (a) *For $p = 2$, let $\eta \in \pi_1(\hat{S}^0)$, $\nu \in \pi_3(\hat{S}^0)$ and $\sigma \in \pi_7(\hat{S}^0)$ be the three Hopf fibrations and let $\iota \in \pi_0(\hat{S}^0)$ be the standard generator. Then $\eta \in M(2\iota)$, $\nu \in M(\eta)$, $\sigma \in M(\nu)$, $\sigma^2 \in M(\sigma)$, $\nu \in CM(\eta)$, and $\sigma \in CM(\nu)$.*

(b) For $p > 2$ let $\alpha_1 \in \pi_{2p-3}(\hat{S}^0)$ and $\beta_1 \in \pi_{2p^2-2p-2}(\hat{S}^0)$ be the standard generators. Then $\alpha_1 \in M(p\iota)$ and $\beta_1 \in M(\alpha_1)$. □

We will discuss further examples of this sort elsewhere. In general we have

1.18 LEMMA. *$M(\alpha) \ni \alpha$ (i.e. $i = 1$) only if α is a generator of $\pi_0\hat{S}^0$.*

Proof. Consider the cofibre sequence

$$\hat{S}^{-1} \xrightarrow{a} \hat{X}_{-1} \xrightarrow{b} \hat{X}_0 \xrightarrow{c} \hat{S}^0.$$

The lemma asserts that im $\pi_*(a)$ is $Z/(p)$ in dimension -1 and trivial in higher dimensions, which is equivalent to a similar description of coker $\pi_*(c)$. One knows that $\hat{X}_0 = \hat{X}_1 \vee \hat{S}^0$ and that the restriction of c to $\hat{S}^0$ has degree p. The Kahn-Priddy Theorem as formulated by Adams [3] asserts that any map $\hat{X}_1 \to \hat{S}^0$ satisfying a certain condition induces a surjection in homotopy in positive dimensions. The condition is that the map be surjective on $\pi_{2p-3} = Z/(p)$. The generator of this group is detected by the Steenrod operation P^1 (or Sq^2 if $p = 2$), which is easily seen to be nontrivial on the bottom class of $H^*(\hat{X}_{-1}; Z/(p))$. □

The analogous question for the complex Mahowald invariant is harder to answer. We have $CM(\alpha) \ni \alpha$ iff α is in the cokernel of the map induced by the S^1-transfer $\Sigma CP_0^\infty \to S^0$ (see Becker-Schultz [5]). It is clear that if dim $\alpha = 0$ then $M(\alpha) \ni \alpha$. In [5] it is shown that the 2-primary elements $\mu_k \in \pi_{8k+1}S^0$ the element detected by its image in π_*bo, and $\sigma_k \in \pi_{8k-1}S^0$, the generator of im J, are not in the image of this map. At odd primes it is known that im J is in the image, but K. Knapp [11] has shown that the map is not surjective for $p \geq 5$. The first positive dimensional element not in its image is β_{1+p}, which occurs in $\pi_{278}(S^0)$ for $p = 5$.

We close this section by showing that Segal Conjecture II can be extended to general cyclic and certain metacyclic groups with little additional effort. If $G = Z/(m)$ then stably BG is the wedge of the BG_p for all primes p dividing the order of G, where $G_p = Z/(p^i)$ is the p-Sylow subgroup of G. Hence from 1.9 we get

1.19 COROLLARY. *Let $G = Z/(m)$ with $m = \Pi_t\, p_t^{i_t}$ where the p_t are distinct primes. Then*

$$DBG = \bigvee_{0<j\leq i_t} \hat{B}Z/(p_t^j)_+$$

where $\hat{B}Z/(p^j)_+$ denotes the p-adic completion of $BZ/(p^j)_+$. $\square$

Now suppose G is a metacyclic group in which the subgroup and quotient group have coprime order, i.e. an extension of the form

$$Z/(m) \to G \to Z/(n)$$

with m and n relatively prime. Again BG is stably equivalent to the wedge of its p-adic completions for all primes p dividing the order of G. If p divides n then this completion is $BZ/(p^i)$ where p^i is the largest power of p dividing n.

If p divides m, the situation is more complicated. We illustrate it first with the simplest nontrivial example, i.e. $p = m = 3$, $n = 2$ and $G = \Sigma_3$, the symmetric group on three letters. The 3-adic completion of $B\Sigma_3$ is not $BZ/(3)$, but a retract of it having cells only in dimensions congruent to 0 and -1 mod (4). This splitting, originally due to Holzager [10], is obtained as follows. $Z/(2)$ acts on $Z/(3)$ and hence on $BZ/(3)$. Let τ denote the self-map of $BZ/(3)$ corresponding to the generator of $Z/(2)$. Then $(1 + \tau)/2$ is defined on $\Sigma BZ/(3)$ since it is 3-adically complete. It corresponds to an idempotent element in $Z_3[Z/(2)]$, the 3-adic group ring of $Z/(2)$, and is therefore an idempotent map which gives the desired splitting. When dualized this map gives a splitting of the 3-adic completion of $DB\Sigma_3$, and we get

$$DB\Sigma_3 = \hat{B}\Sigma_{3+} \vee \hat{B}Z/(2)_+$$

where the right hand summands are p-adically completed for $p = 3$ and 2 respectively.

Before proceeding to the general case we need

1.20 LEMMA. *Let G be an extension of $Z/(m)$ by $Z/(n)$ with m and n relatively prime and let p a prime dividing m. Let $Z/(k) \subset Z/(p-1)$ be the image of the composite homomorphism $Z/(n) \to \mathrm{Aut}_{Z/(m)} \to \mathrm{Aut}_{Z/(p^i)} \to \mathrm{Aut}_{Z/(p)}$ where the first map is given by conjugation in G, the second by the projection of $Z/(m)$ onto its p-Sylow subgroup, and the third by* mod (p) *reduction. Then $\Sigma\hat{B}G$, the p-adic completion of ΣBG, is a retract of $\Sigma BZ/(p^i)$ with*

$$\tilde{H}^j(\hat{B}G; Z) = \begin{cases} Z/(p^i) & \text{if } \ 2k/j. \\ 0 & \text{otherwise.} \end{cases}$$

Proof. It follows from the Serre spectral sequence for the fibration

$$\hat{B}Z/(m) = BZ/(p^i) \to \hat{B}G \to BZ/(n)$$

that $H^*\hat{B}G$ is the subring of $H^*BZ/(p^i)$ left fixed by $Z/(n)$, i.e. by $Z/(k)$, so $\tilde{H}^*\hat{B}G$ is as indicated. For the retraction observe the $\Sigma BZ/(p^i)$ is a co-H-space on which $Z/(k)$ acts and this action induces a projection onto the indicated factor using the methods of Cooke-Smith [7]. The inclusion of this factor in $BZ/(p^i)$ followed by the map to $\hat{B}G$ induces a homology isomorphism and is therefore an equivalence. □

1.21 COROLLARY. *Let G be an extension of $Z/(m)$ by $Z/(n)$ with m and n relatively prime. For each prime power p^i dividing mn let $G_{p,i}$ be the induced extension of $Z/(p^i)$ by $Z/(n)$, if p divides m, and $Z/(p^i)$ if p divides n. Then*

$$DBG = \bigvee_{p,i} \hat{B}G_{p,i+}$$

where $\hat{B}G_{p,i}$ denotes the p-adic completion of $BG_{p,i}$ and prime powers $p^i > 1$ which divide mn. (Note that we are describing DBG, not DBG_+.)

Proof. We have $BG = \vee_p \hat{B}G_{p,j}$ where the sum is over all primes p dividing mn and j is maximal. If $p \mid n$, $G_{p,j} = Z/(p^j)$ and the case is 1.9. If $p \mid m$ then $\hat{B}G_{p,j}$ is stably a retract of $BZ/(p^j)$ by 1.20. The splitting of $BZ/(P^j)$ induces one of $DBZ/(p^j)$ which yields the result. □

It is clear that our methods would work for any group having cyclic p-

Sylow subgroups for all p, but it is known ([9] p. 146) that any such group is metacyclic as in 1.21.

2. The modified Adams spectral sequence. In this section we introduce the modified Adams spectral sequence (MASS) and use it to prove 1.7. To construct the usual Adams spectral sequence (ASS) for $[X, Y]$ one uses an Adams resolution of Y, which is a diagram

$$Y = Y_0 \leftarrow Y_1 \leftarrow Y_2 \leftarrow \cdots$$

having certain properties. In the MASS we use extra data, i.e. a diagram

$$X = X_0 \rightarrow X_1 \rightarrow X_2 \rightarrow \cdots.$$

For X finite and $Y = S^0$ our MASS is essentially the same as the spectral sequence constructed by Milgram in 5.3 of [15].

1.7 says that certain maps of Thom spectra induce isomorphisms of cohomotopy groups. Lin [13] proves the special case of 1.7 relevant to the group $Z/(2)$ by showing that the map induces an isomorphism of ASS E_2-terms. In the general case we calculate the target E_2-term (2.5) and observe that it cannot possibly be isomorphic to the source E_2-term. We surmount this obstacle by replacing the latter with a MASS E_2-term (2.12) which maps isomorphically (2.13, 2.16 and 2.17) to the target E_2-term. Convergence and other technical properties of MASS are proved in section 3.

The first statement of 1.7 for $n = 1$ and $p = 2$ was proved as Theorem 1.3(iii) of Lin [13], and for $p > 2$ by Gunawardena [8]. In this case $V_0 = S^0$ and the map from U_0 is the retraction onto the bottom cell. The argument involves the Adams spectral sequence (ASS) in the following way. One has the ASS for $\pi^*(\tilde{U}_{-i})$ with $E_2^{s,t} = \mathrm{Ext}_A^{s,t}(H^*(S^0), H^*(\tilde{U}_{-i}))$. Here A denotes the mod (p) Steenrod algebra and all cohomology groups have coefficients in $Z/(p)$. The standard convergence theorems for the ASS (see Adams [1] III.15) require the source spectrum to be finite and hence do not apply here. This problem is dealt with by Lin [13] and we will address it in section 3. For now we will assume that all spectral sequences in sight converge in the appropriate sense.

The above E_2-term cannot be readily computed. Instead, let $P = \lim_{\overrightarrow{i}} H^*(\tilde{U}_{-i})$; for $p = 2$ we have $P = Z/(2)[x, x^{-1}]$ with $\dim x = 1$ and $\mathrm{Sq}^k x^i = \binom{i}{k} x^{i+k}$. For an A-module M we abbreviate $\mathrm{Ext}_A(Z/(p), M)$ by

$\operatorname{Ext}(M)$; this group is the E_2-term for the *co*homotopy of a spectrum with cohomology M. The following is elementary.

2.1 PROPOSITION. *If* $M = \varinjlim M_i$ *and if each* M_i *is isomorphic to* M *in dimensions* $\geq -i$ *then*

$$\operatorname{Ext}(M) = \varinjlim \operatorname{Ext}(M_i). \qquad \square$$

It follows that there is a map $\operatorname{Ext}^{s,t}(H^*(\tilde{U}_i)) \to \operatorname{Ext}^{s,t}(P)$ which is an isomorphism for $t - s > 2i$ since $H^*(U_{-i})$ and P are isomorphic in dimensions $\geq -2i$. Therefore it suffices (modulo convergence) to show that the induced map

$$\operatorname{Ext}(Z/(p)) \to \operatorname{Ext}(P)$$

is an isomorphism, which is done for $p = 2$ by Lin et al [14] and for $p > 2$ by Gunawardena [8]. A simpler proof has recently been divised by Haynes Miller, Adams and Gunawardena [16]. Their argument leads to a proof of Segal Conjecture III for elementary abelian groups $(Z/(p))^n$.

Now we consider what happens when we try to mimic this proof for $n \geq 2$, i.e. for cyclic p-groups of order $>p$. We let $R = \varinjlim_i H^*(\tilde{U}_{-i})$; it is easily seen that as an A-module $H^*(\tilde{U}_{-i})$ and hence R are independent of n as long as $n > 1$. Then we will want to show that

2.2 $$\operatorname{Ext}(H^*(\tilde{V}_0)) \to \operatorname{Ext}(R)$$

is an isomorphism. (Here the alert reader may already suspect trouble; $H^*(\tilde{V}_0)$ as an A-module depends on whether $n = 2$ or $n \geq 3$.) Let $C \subset P$ denote the submodule generated by even dimensional elements. Then we have short exact sequences of A-modules

2.3 $$0 \to C \to P \to \Sigma C \to 0$$

and

2.4 $$0 \to C \to R \to \Sigma C \to 0.$$

The latter is split but the former is not since the Bockstein sends an odd dimensional generator to an even dimensional one. In view of 2.4, to compute $\mathrm{Ext}(R)$ it suffices to compute $\mathrm{Ext}(C)$ which we do with 2.3 and our knowledge of $\mathrm{Ext}(P)$. Note that $C = \lim_{\vec{i}} H^*(CP_{-i})$.

2.5 LEMMA. (a) *With notation as above,* $\mathrm{Ext}(C)$ *is a free module over* $\mathrm{Ext}(Z/(p))$ *on generators* $c_{2k} \in \mathrm{Ext}^{k,-k}(C)$ *for all* $k \geq 0$, *and* $\mathrm{Ext}(R)$ *is a free module on generators* $r_k \in \mathrm{Ext}^{[k/2],[k/2]-k}$ *for all* $k \geq 0$, *where* $[k/2]$ *is the integer part* of $k/2$.

(b) *The image of* c_{2k} *in* $\mathrm{Ext}(H^*C^{2k})$ *is* $h_0^k x_{2k}$, *where* $C^{2k} = \varinjlim H^*CP_{-i}^{2k}$, h_0 *is the generator of* $\mathrm{Ext}^{1,1}(Z/(p))$ *and* $x_{2k} \in \mathrm{Ext}^{0,-2k}(H^*CP_{-i}^{2k})$ *(for any* $i \geq 0$*) is the generator corresponding to the top cell, and the image of* r_k *is* $\mathrm{Ext}(\varinjlim \mathrm{H}^*(\tilde{U}_{-i}^k))$ *is* $h_0^{[k/2]} y_k$ *where* $y_k \in \mathrm{Ext}^0(H^*(\tilde{U}_{-i}^k))$ *(for any* $i \geq 0$*) is the generator corresponding to the top cell. Moreover this property characterizes* c_{2k}.

Proof. It is evident that the statements about $\mathrm{Ext}(R)$ follow immediately from those about $\mathrm{Ext}(C)$ and the splitting of 2.4, so we will only prove the results on $\mathrm{Ext}(C)$.

For (a), 2.3 gives a long exact sequence of Ext groups which form an exact couple. The associated spectral sequence converges to $\mathrm{Ext}(C)$ and has an E_2-term (using the isomorphism $\mathrm{Ext}(P) = \mathrm{Ext}(Z/(p))$ such that the result will follow if it collapses, i.e. the E_2-term is a free $\mathrm{Ext}(Z/(p))$-module on appropriate generators. The spectral sequence is one of $\mathrm{Ext}(Z/(p))$-modules so it suffices to show that these generators are all permanent cycles, i.e. to construct the c_{2k} in $\mathrm{Ext}(C)$.

The c_{2k} can be manufactured as follows. The short exact sequence 2.3 gives an element in $\mathrm{Ext}_A^{1,-1}(C, C)$ and its kth power lies in $\mathrm{Ext}_A^{k,-k}(C, C)$. c_{2k} is the image of this element under the map to $\mathrm{Ext}(C)$ induced by $Z/(p) \to C$.

We prove (b) by induction on k, the statement being obvious for $k = 0$. Consider the commutative diagram

2.6
$$\begin{tikzcd}
0 \arrow[r] & \Sigma^{2k}Z/(p) \arrow[r] \arrow[d] & \Sigma^{2k-1}B \arrow[r] \arrow[d] & \Sigma^{2k-1}Z/(p) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C^{2k} \arrow[r] & P^{2k} \arrow[r] & \Sigma C^{2k-2} \arrow[r] & 0 \\
0 \arrow[r] & C \arrow[r] \arrow[u] & P \arrow[r] \arrow[u] & \Sigma C \arrow[r] \arrow[u] & 0
\end{tikzcd}$$

where the rows are exact, B is the cohomology of the mod (p) Moore spectrum, M^{2k} denotes the $2k$-skeleton of the A-module M, the upper vertical maps are inclusions, and the lower vertical maps are surjections. Each row gives a long exact sequence of Ext groups with connecting homomorphism δ. The bottom row gives $\delta\Sigma c_{2k-2} = c_{2k}$ and our inductive assumption is that the image of Σc_{2k-2} is $h_0^{k-1}\Sigma x_{2k-2}$, so it suffices to show that $\delta\Sigma x_{2k-1} = h_0 x_{2k}$. But x_{2k} and Σx_{2k-2} are the images of the generators a and b respectively coming from the top, which is the short exact sequence representing h_0, so $\delta b = h_0 a$. The characterization of c_{2k} follows because the map $\mathrm{Ext}^{k,-k}(C) \to \mathrm{Ext}^{k,-k}(C^{2k})$ is a monomorphism. □

Now we have computed the target of the map 2.2. We cannot compute the source explicitly, but it can be shown that the ASS for $\pi^*(\tilde{V}_0)$ has an E_1-term isomorphic to $H^*(\tilde{V}_0) \otimes \mathrm{Ext}(Z/(p))$ where the d_1 reflects the action of A on $H^*(\tilde{V}_0)$. From this it is clear that 2.2 cannot be an isomorphism and our attempt to extend Lin's proof to the case $n > 1$ is doomed.

Our way out of this difficulty is to modify the ASS for $\pi^*(\tilde{V}_0)$ such that the resulting E_2-term is isomorphic to $\mathrm{Ext}(R)$. We will refer to this spectral sequence as the MASS and denote its E_2-term by $\mathrm{Mext}(H^*(V_0))$. Our aim is to prove

2.7 THEOREM. *There is a* MASS *converging to* $\pi^*(\tilde{V}_0)$ *with* E_2*-term* Mext $(H^*(\tilde{V}_0)) \simeq \mathrm{Ext}(R)$ *which is compatible with the* ASS *for* $\pi^*(\tilde{U}_{-i})$ *in the sense that the maps* $\tilde{U}_{-i} \to \tilde{V}_0$ *of* 1.7 *induce the above isomorphism.* □

The first part of 1.7 will follow easily from this result; we replace R by $H^*(\tilde{U}_{-i})$ and get an isomorphism of E_2-terms and hence of cohomotopy groups in the appropriate range. In a similar way the statement in 1.7 about CP^∞ will follow from

2.8 THEOREM. *There is a* MASS *converging to* $\pi^*(\tilde{C}P_0)$ *with* E_2*-term* $\mathrm{Mext}(H^*\tilde{C}P_0)) \simeq \mathrm{Ext}(C)$ *which is compatible with the* ASS *for* $\pi^*(\tilde{C}P_{-i})$ *in the sense that the maps* $\tilde{C}P_{-i} \to \tilde{C}P_0$ *of* 1.7 *induce the above isomorphism.* □

We will see below the class $r_k \in \mathrm{Ext}^{[k/2],[k/2]-k}(R)$ corresponds to the k-cell in V_0. In the E_1-term of the ASS for $\pi^*(\tilde{V}_0)$ mentioned above, the class corresponding to the k-cell has Adams filtration 0, i.e. it lies in $E_1^{0,-k}$. In our MASS for $\pi^*(\tilde{V}_0)$ the corresponding class has filtration $[k/2]$ and survives to E_2 for all $k \geq 0$.

Before constructing our MASS we need a little general nonsense.

2.9 *Definition.* Given spectra X and Y, the *function spectrum*

$F(X, Y)$ is representing object for $[W \wedge X, Y]$ regarded as a cohomology theory on W.

Note that $F(X, Y)$ is defined only up to homotopy.

2.10 PROPOSITION.

(i) $\pi_*(F(X, Y)) = [Z, Y]_*$

(ii) *$F(X, Y)$ is functorial on both variables (contravariant in X and covariant in Y) and cofibre sequences in either variable induce cofibre sequences of function spectra.*

(iii) $DX = F(X, S^0)$. □

Next we recall the construction of the ASS for $[X, Y]$. An *Adams resolution* for Y is a diagram

$$Y = Y_0 \leftarrow Y_1 \leftarrow Y_2 \leftarrow \cdots$$

such that $H^*(\varprojlim Y_i) = 0$, each map is trivial in cohomology and the cofibre of each map is a generalized mod (p) Eilenberg-MacLane spectrum. Then we have a diagram

$$F(X, Y) = F(X, Y_0) \leftarrow F(X, Y_1) \leftarrow F(X, Y_2) \leftarrow \cdots.$$

The cofibre sequences given by these maps induce long exact sequences of homotopy groups and these form an exact couple which give the ASS for $[X, Y]_*$. In general this ASS is not the same as the spectral sequence obtained from an Adams resolution for $F(X, Y)$.

Now suppose that we also have a diagram

$$X = X_0 \to X_1 \to X_2 \to \cdots$$

with $H^*(\varinjlim X_i) = 0$. (This vanishing condition is not necessary to set up the MASS, but it is needed to prove convergence.) Then we have a homotopy commutative diagram

2.11
$$\begin{tikzcd}
F(X, Y) = F(X_0, Y_0) & F(X_0, Y_1) \arrow[l] & \cdots \arrow[l] \\
F(X_1, Y_0) \arrow[u] & F(X_1, Y_1) \arrow[l] \arrow[u] & \cdots \arrow[l] \\
\vdots \arrow[u] & \vdots \arrow[u] &
\end{tikzcd}$$

We will show below (3.1) that any diagram of this form is equivalent to a strictly commutative one in which all maps are inclusions. This allows us to define $W_s = \bigcup_{i+j=s} F(X_i, Y_j)$.

2.12 *Definition.* The MASS for $[X, Y]_*$ is the spectral sequence associated with the homotopy exact couple given by the diagram

$$F(X, Y) = W_0 \leftarrow W_1 \leftarrow W_2 \leftarrow \cdots .$$

We will see that the MASS is independent (from E_2 onward) of the choice of Y_j, but it depends very much on the choice of X_i. In particular the choice $X_i =$ pt. for $i > 0$ gives the usual ASS. We ask the reader's forgiveness for not making this dependence explicit in our notation. Now we will prove 2.7 and 2.8, and hence 1.7, modulo various properties of the MASS to be proved in section 3. To compute the E_2-term we have

2.13 LEMMA. *With notations as above suppose each map $X_i \to X_{i+1}$ is injective in* mod (p) *cohomology. Then the E_2-term for the* MASS *for $[X, Y]$ is given by*

$$E_2^{s,t} = \bigoplus_{i\geq 0} \operatorname{Ext}_A^{s-i,t+1-i}(H^*(Y), H^*(X_{i+1}, X_i)). \qquad \square$$

This will be proved in section 3.

For 2.7 and 2.8 we use $X_i = \tilde{V}_i$ and $X_i = C\tilde{P}_i$ respectively with $Y = S^0$ in both cases. Then 2.13 applies and shows that the E_2-terms are as specified in 2.7 and 2.8, but we still have to show that these isomorphisms are induced by the indicated maps. We will do this only for 2.7, leaving the similar argument for 2.8 to the reader.

To prove 2.7 we must show that the above MASS for $\pi^*(\tilde{V}_0)$ is compatible with the standard ASS for $\pi^*(\tilde{U}_i)$. The latter involves the diagram

2.14 $$\tilde{U}_{-i} \to * \to * \to \cdots$$

which does *not* map to

2.15 $$\tilde{V}_0 \to \tilde{V}_1 \to \tilde{V}_2 \to \cdots,$$

the diagram used in the former MASS. We will replace 2.14 by a diagram which does map to 2.15 and which gives the same standard ASS for $\pi^*(\tilde{U}_{-i})$. The following result complements 2.13 and will be proved in section 3.

2.16 Lemma. *With notation as in 2.12, suppose each map $X_i \to X_{i+1}$ is trivial in* mod (p) *cohomology. Then the resulting* MASS *for* $[X, Y]$ *is isomorphic (from E_2 onward) to the standard* ASS *for* $[X, Y]$. □

Now recall U_{-i} is the Thom spectrum of $-i\lambda$ over $BZ/(p^n)$, where λ is the canonical complex line bundle. Let $U_{-i,j}$ denote the Thom spectrum of $-i\lambda + j\lambda^p$, where λ^p is the p-fold tensor product of λ.

To proceed further we need

2.17 Lemma. *Let λ be a complex line bundle and α a vector bundle over a space X. Then there is a map of Thom spectra*

$$X^{\alpha\oplus\lambda} \to X^{\alpha\oplus\lambda^p}$$

which has degree p on the bottom cell and which is trivial in mod (p) *cohomology.*

Proof. Let X_ξ denote the total space of the sphere bundle associated with a vector bundle ξ over X. Then the group $Z/(p)$ acts freely on X_λ by multiplication by $e^{2\pi i/p}$ on each fibre, and the orbit space is X_{λ^p}. This action can be extended linearly over $X_{\alpha\oplus\lambda}$ by letting $Z/(p)$ act trivially on $S(\alpha)$ and the orbit space is $X_{\alpha\oplus\lambda^p}$. In both cases the projection onto orbit space covers the identity on X and has degree p on each fibre. The induced map of Thom spectra therefore has the desired properties. □

Using 2.17 we get a diagram

2.18 $$\tilde{U}_{-i} = \tilde{U}_{-i,0} \to \tilde{U}_{-1-i,1} \to \tilde{U}_{-2-i,2} \to \cdots$$

to which 2.16 applies and which maps to 2.15. To complete the proof of 2.7 (modulo convergence, 2.13 and 2.16) we must show that this map induces the isomorphism of E_2-terms. Let $\mathrm{v}_k \in \mathrm{Mext}^{[k/2],[k/2]-k}(H^*(\tilde{V}_0))$ be the generator given by 2.13 (note that $V_{i+1}/V_i = S^{2i+1} \vee S^{2i+2}$); we want to show that it maps to the class $r_k \in \mathrm{Ext}(R)$ of 2.5. The cases $k = 2i$ and $k = 2i + 1$ are similar so we treat only the former. Consider the diagram

$$\begin{tikzcd}
\mathrm{Ext}^{i,-i}(H^*(S^{2i})) \arrow[r, equal] & \mathrm{Ext}^{i,-i}(H^*(S^{2i})) & \mathrm{Mext}^{i,-i}(H^*(S^{2i})) \arrow[l] \\
\mathrm{Ext}^{i,-i}(R^{2i}) \arrow[u] & \mathrm{Ext}^{i,-i}(H^*(U_0^{2i})) \arrow[l] \arrow[u] & \mathrm{Mext}^{i,-i}(H^*(V_0^{2i})) \arrow[l] \arrow[u] \\
\mathrm{Ext}^{i,-i}(R) \arrow[u] & \mathrm{Ext}^{i,-i}(H^*(U_0)) \arrow[l] \arrow[u] & \mathrm{Mext}^{i,-i}(H^*(V_0)) \arrow[l] \arrow[u]
\end{tikzcd}$$

where $R^{2i} = \varinjlim H^*(U_{-j}^{-2i})$ and $\operatorname{Mext}(H^*(S^{2i}))$ is associated with the diagram

$$S^{2i} \to S^{2i} \to \cdots\cdot S^{2i} \to * \to * \to \cdots$$

where the first i maps are the identity. It follows from 2.13 that $\operatorname{Mext}^{s,t}(H^*(S^{2i})) = \operatorname{Ext}^{s-i,t-i}(H^*(S^{2i}))$. Then $v_{2i} \in \operatorname{Mext}^{i,-i}(H^*(V_0))$ is characterized by its image (the generator x) in $\operatorname{Mext}^{i,-i}(H^*(S^{2i}))$ while r_{2i} is characterized by its image in $\operatorname{Ext}^{i,-i}(H^*(S^{2i}))$, which is h_0^i times the generator y of $\operatorname{Ext}^{0,-2i}(H^*(S^{2i}))$. A simple calculation in integral cohomology shows that the map $U_0 \to V_0$ has degree p^i in dimension $2i$, so x maps to $h_0^i y$ and 2.7 follows.

3. Convergence and other properties of the MASS. In this section we verify the technical properties of the MASS which are needed to complete the proof of 1.7. With an eye to future applications we prove some of our results (3.4 and 3.6) in greater generality than required above. We begin with a technical lemma (3.1) about homotopy commutative diagrams needed in 2.12 to construct the MASS. Then we define Mext (3.3), a homological functor with which the MASS E_2-term will be identified in 3.4. It is a generalization of Ext in which one of the variables is replaced by a chain complex. Next we show (3.5) the MASS is independent from E_2 onward of all the choices made in its construction. Finally we prove our convergence theorem (3.6).

To construct the MASS we need to know that the homotopy commutative diagram 2.11 is equivalent to one which is strictly commutative and in which every map is an inclusion. This result for diagrams of spectra follows from the analogous result for diagrams of spaces. The proof is a generalized mapping cylinder construction; recall that a map $f: X \to Y$ is equivalent to the inclusion of X in the mapping cylinder of f. More precisely we have

3.1 LEMMA. *Suppose we have a homotopy commutative diagram of spectra $X_{i,j}(i, j \in Z)$ and maps $f: X_{i,j} \to X_{i-1,j}$ and $g: X_{i,j} \to X_{i,j-1}$ (subscripts on f and g can be omitted without ambiguity). Then there is a homotopy equivalent diagram $(X'_{i,j}, f', g')$ (i.e. there are homotopy equivalences $k: X_{i,j} \to X_{i,j}$ with $f'k \sim kf$ and $g'k \sim kg$) which is strictly commutative and in which all the maps are inclusions.*

The construction in the proof will depend on the choice of homotopy (for each i and j) between fg and gf.

Proof. It suffices to prove the analogous statement for diagrams of pointed spaces in such a way that everything in sight commutes with reduced suspension. Let I denote the closed unit interval $[0, 1]$. Let $h : X_{i,j} \times I \to X_{i-1,j-1}$ be a base point preserving homotopy between fg and gf with $h(x, 0) = gf(x)$ and $h(x, 1) = fg(x)$. Let X' be the quotient of $\bigcup_{i,j} X_{i,j} \times I^2$ obtained by the following identifications for $x \in I_{i,j}$ and $s, t \in I$:

(a) $(x, t, 0) = (f(x), 0, (t + 1)/2)$
(b) $(x, t, 1) = (g(x), 0, (1 - t)/2)$
(c) $(x, 1, t) = (h(x, t), 0, ½)$
(d) $(x_0, s, t) = (x_0, 0, 0)$ where

$x_0 \in X_{i,j}$ is the base point. We must verify that this makes sense by showing that the two identifications of $(x, 1, 0)$ and $(x, 1, 1)$ are the same. In the former case (the latter is similar), (a) gives

$$(x, 1, 0) = (f(x), 0, 1) = (gf(x), 0, ½) \qquad \text{by (b)}$$

while (c) gives

$$(x, 1, 0) = (h(x, 0), 0, ½) = (gf(x), 0, ½)$$

by our assumption on the homotopy h.

Let $X'_{i,j} \subset X'$ be the image of

$$X_{i,j} \times \{(0, ½)\} \cup \bigcup_{s\geq 1} (X_{i,j+s} \times \{0\} \times [½, 1] \cup X_{i+s,j} \times \{0\} \times [0, ½])$$

$$\cup \bigcup_{s,t\geq 1} X_{i+s,j+t} \times I^2.$$

The map $k : X_{i,j} \to X'_{i,j}$ is defined by $k(x) = (x, 0, ½)$. f' and g' are the inclusions. Homotopies $m : X_{i+1,j} \times I \to X'_{i,j}$ between $f'k$ and kf, and $n : X_{i,j+1} \times I \to X'_{i,j}$ between $g'k$ and kg are given by

$$m(x, t) = (x, 0, (1 - t)/2) \quad \text{and} \quad n(x, t) = (x, 0, (1 + t)/2).$$

To show k is a homotopy equivalence it suffices to construct a map $p : X'_{i,j} \times [0, \infty) \to X'_{i,j}$ satisfying

(e) $p(x', 0) = x'$
(f) $p(k(x), t) = k(x)$ and
(g) $p(x, t) \in \operatorname{Im} k$ for large t.

We define p by (f) and the following:

(h) $p(x', u_1 + u_2) = p(p(x', u_1), u_2)$,
(i) for $x \in X_{i,j+s}$ $(s > 0)$, $p((x, 0, t), u) = (x, 0, t + u/2)$ for $0 \leq u \leq 2(1 - t)$,
(j) for $x \in X_{i+s,j}$ $(s > 0)$, $p((x, 0, t), u) = (x, 0, t - u/2)$ for $0 \leq u \leq 2t$ and
(k) for $x \in X_{i+s,j+t}$ $(s, t > 0)$, $p((x, a, b), u) = (x, a + u, b)$ for $0 \leq u \leq 1 - a$.

To verify that p has the desired properties note that in (i), (j) and (k) when u has its prescribed maximum value, $p(x', u)$ can be identified using rules (b), (a) and (c) respectively. Then the value of $p(x', u)$ for larger values of u can be determined by (h). For example when $x \in X_{i+s,j+t}$ $(s, t \geq 2)$ we get

$$\begin{aligned} p((x, a, b), 1) &= p((x, 1, b), a)) \\ &= p((h(x, b)0, \tfrac{1}{2}), a) \\ &= (h(x, b), a, \tfrac{1}{2}) \end{aligned}$$

and when $x \in X_{i+s,j}$ $(s \geq 2)$

$$\begin{aligned} p((x, 0, t), 1) &= p((x, 0, 0), 1 - 2t) \\ &= p((f(x), 0, \tfrac{1}{2}), 1 - 2t) \\ &= (f(x), 0, t). \end{aligned}$$

It follows that when $x \in X_{i+s,j+t}$ $(s, t \geq 0)$, $p((x, a, b), u) \in \operatorname{Im} k$ if $u \geq \max(s, t)$.

The verification that p is well defined and continuous is straightforward and left to the reader. □

Now we turn to identifying the E_2-term of the MASS. We will prove a general result of which 2.13 and 2.16 are special cases. Recall that the spectral sequence for $[X, Y]$ is constructed from a diagram

$$X = X_0 \to X_1 \to X_2 \to \cdots$$

For $i \leq j$ let X_i^j be the fibre of the map $X_i \to X_{j+1}$. Then we get a diagram

3.2
$$X_0^0 \leftarrow \Sigma^{-1} X_1^1 \leftarrow \Sigma^{-2} X_2^2 \leftarrow \cdots$$

which in cohomology gives a cochain complex of A-modules. We need to define a functor generalizing Ext which accepts such a cochain complex as its second variable.

3.3 *Definition.* Let M be an A-module and N^* a cochain complex of A-modules. Let P^*M be a projective A-resolution of M. Let

$$C^n = \bigoplus_{i+j=n} \operatorname{Hom}_A(P^i M, N^j);$$

using appropriate sign conventions to combine the dual of the boundary operator in P^*M and the coboundary operator in N^* we can make C^* into a cochain complex. $\operatorname{Mext}_A^*(M, N^*)$ is the cohomology of C^*. For $M = Z/(p)$ we abbreviate to $\operatorname{Mext}(N^*)$. In the case associated with the MASS for $[X, Y]$, we denote this group abusively by $\operatorname{Mext}_A(H^*(Y), H^*(X))$.

Now consider how this definition applies to the special cases addressed by 2.13 and 2.16. In the former we assume each map $X_i \to X_{i+1}$ is injective in mod (p) cohomology. It follows that the cochain complex arising from 3.2 has trivial coboundary, so C^* is a direct sum of resolutions and $\operatorname{Mext}^s(H^*(Y), H^*(X)) = \bigoplus_i \operatorname{Ext}^{s-i}(H^*(Y), H^*(X_i^i))$ as specified in 2.13.

In 2.16 we assume each map $X_i \to X_{i+1}$ is trivial in mod (p) cohomology. This means $H^*(X_i^i) = H^*(X_i) \oplus \Sigma^{-1} H^*(X_{i+1})$ (as vector spaces but not necessarily as A-modules) so the cochain complex of 3.2 is acyclic. In this case the cohomology of C^* can be computed by the spectral sequence obtained by filtering by the resolution degree. We have $E_2^{i,*} = H(\operatorname{Hom}_A(P^i M, N^*))$. Since N^* is acyclic, so is $\operatorname{Hom}_A(P^i M, N^*)$ and we have

$$E_2^{i,j} = \begin{cases} \operatorname{Ext}_A^i(H^*(Y), H^*(X)) & \text{if} \quad j = 0 \\ 0 & \text{if} \quad j > 0 \end{cases}$$

and the spectral sequence collapses. Hence we get

$$\mathrm{Mext}_A(H^*(Y), H^*(X)) = \mathrm{Ext}_A(H^*(Y), H^*(X))$$

as claimed in 2.16.

Hence 2.13 and 2.16 are both corollaries of the following.

3.4 THEOREM. *With notation as in* 2.12 *and* 3.3, *the* E_2*-term of the* MASS *for* $[X, Y]$ *is* $\mathrm{Mext}_A(H^*(Y), H^*(X))$.

Proof. Let L_i be the cofibre of the map $W_i \leftarrow W_{i+1}$ of 2.12. It follows from the definitions that $E_1^{s,t} = \pi_{t-s}(L_s)$ and $L_s = \vee_i F(X_i^i, K_{s-i})$ where K_i is the cofibre of $Y_i \leftarrow Y_{i+1}$. We have

$$\pi_*(F(X_j^j, K_i)) = \mathrm{Hom}_A(H^*(K_i), H^*(X_j^j)).$$

The $H^*(K_i)$ form a projective A-resolution of $H^*(Y)$ so we can identify the MASS E_1-term with the complex C^* of 3.3. □

Next we need to know that the MASS is independent of the choices made in its construction, i.e. the choices of Adams resolution of Y and of homotopies in 2.11. It is not independent of the choice of maps $X_i \to X_{i+1}$; indeed its dependence on this choice is our reason for studying it.

3.5 LEMMA. *The* MASS *of* 2.12 *depends from* E_2 *onward only on* Y *and the maps* $X_i \to X_{i+1}$.

Proof. We need to show that given two diagrams $\{W_s\}$ and $\{W_s'\}$ used to construct the MASS there are compatible maps $W_s \to W_s'$ which induce an isomorphism of E_2-terms. Let K_s' be the cofibres in the second Adams resolution for Y. Standard arguments give suitable maps $Y_s \to Y_s'$ and $K_s \to K_s'$. Since $L_s' = \vee_i F(X_i^i, K_{s-i}')$ we get maps $L_s \to L_s'$ which allow us to construct compatible maps $W_s \to W_s'$ by induction on s and standard algebraic arguments show that it induces an isomorphism of E_2-terms as desired. □

We now turn to the convergence question. Let X^i be the fibre of $X \to X_{i+1}$. Then our main result is

3.6 THEOREM. *With notation as above suppose*

(i) *either* Y *is p-adically complete or* X^i *is p-adically cocomplete* (1.1) *for all* i,

(ii) X^i *(for all i) and Y are connective and of finite type (or if Y is complete, each* $\pi_k Y$ *is finitely generated as a* Z_p*-module) and*

(iii) *either* X^i *is finite for all i or* $\pi_*(Y)$*/torsion has finite rank.*

Then the MASS of 2.12 converges to $[X, Y]$, i.e. $E_\infty^{s,t}$ is the associated bigraded group to a convergent filtration of $[X, Y]_*$. □

Note that the hypotheses above are adequate for our applications. In each case X^i is p-adically cocomplete and finite.

The proof requires the following technical lemmas which will be proved below.

3.7 LEMMA. *The* MASS *converges to* $[X, Y]$ *if* $\varprojlim W_s = pt$, *i.e. in that case* $E_\infty^{s,t}$ *is the subquotient* $\operatorname{im} \pi_{t-s}(W_s)/\operatorname{im} \pi_{t-s}(W_{s+1})$ *of* $\pi_{t-s}(W)$ *and* $\bigcap \operatorname{im} \pi_*(W_s) = 0$. □

3.8 LEMMA. *If* X^i *and Y are as in* 3.6 *then* $\lim_{\overleftarrow{s}} F(X^i, Y_s) = pt.$ □

3.9 LEMMA. *Suppose we have spectra* $A_{i,j}$ *for* $i, j \geq 0$ *with maps* $f: A_{i+1,j} \to A_{i,j}$ *and* $g: A_{i,j+1} \to A_{i,j}$ *commuting up to homotopy. Then the two limits*

$$\varprojlim_i \varprojlim_j A_{i,j} \quad \text{and} \quad \varprojlim_j \varprojlim_i A_{i,j}$$

are well defined and equivalent. □

Our $\varprojlim$ here is the holim of Bousfield-Kan [6], which is not a categorical inverse limit, hence the need to prove 3.9.

Proof of 3.6. Let $W_s^n = \bigcup_i F(X_i^n, Y_{s-i})$ and for $m \leq n$ let $W_{s,m}^n$ be the cofibre of $W_s^n \to W_s^{m-1}$. Since $X_i = \lim_{\overrightarrow{n}} X_i^n$, $F(X_i, Y_j) = \lim_{\overleftarrow{n}} F(X_i^n, Y_j)$ (2.10) and $W_s = \lim_{\overleftarrow{n}} W_s^n$, so by 3.7 and 3.9 it suffices to show $\lim_{\overleftarrow{s}} W_s^n = pt.$ for each n. One sees easily that

$$W_{s,m}^n = \bigcup_{m \leq i \leq n} F(X_i^n, Y_{s-i}),$$

e.g. $W_{s,n}^n = F(X_n^n, Y_s)$. Hence it follows from 3.8 that $\lim_{\overleftarrow{s}} W_s^n = \lim_{\overleftarrow{s}} W_{s,0}^n = pt.$ and the result follows. □

Proof of 3.7. For the triviality of the intersection we have $\varprojlim \pi_*(W_0) = 0$ since $\varprojlim W_s = pt.$ Let $G_s = \pi_*(W_0)$ and

$$G_s^t = \begin{cases} G_s & \text{if } s \geq t \\ \operatorname{Im} G_t \subset G_s & \text{if } t \geq s \end{cases}$$

We have injections $G_s^t \to G_s^{t-1}$, and surjections $G_s^t \to G_{s-1}^t$, so $\lim_{\overleftarrow{t}} G_s^t = \bigcap_t G_s^t$ and $\lim_{\overleftarrow{s}} G_s^t = G_t$. We are trying to show $\lim_{\overleftarrow{t}} G_0^t = 0$. $\lim_{\overleftarrow{t}} G_s^t$ maps onto $\lim_{\overleftarrow{t}} G_{s-1}^t$, so $\lim_{\overleftarrow{s}} \lim_{\overleftarrow{t}} G_s^t$ maps onto $\lim_{\overleftarrow{t}} G_0^t$. But $\lim_{\overleftarrow{s}} \lim_{\overleftarrow{t}} G_s^t = \lim_{\overleftarrow{t}} \lim_{\overleftarrow{s}} G_s^t = \lim_{\overleftarrow{t}} G_t = 0$.

For the identification of E_∞, let K_s be the cofibre of $W_{s+1} \to W_s$ and let

$$\cdots \longrightarrow \pi_n(W_{s+1}) \xrightarrow{\alpha_s} \pi_n(W_s) \xrightarrow{\beta_s} \pi_n(K_s) \xrightarrow{\partial_s} \pi_{n-1}(W_{s+1}) \longrightarrow \cdots$$

be the associated long exact sequence. A nontrivial class $[x] \in E_\infty^{s,t}$ is represented by an element $x \in \pi_n K_s$ where $u = t - s$. Since $d_r[x] = 0$, $\partial_s(x)$ can be lifted to $\pi_{n-1}(W_{s+r+1})$ for each $r > 0$. It follows that $\partial_s(x) \in \operatorname{im} \lim_{\overleftarrow{r}} \pi_{n-1}(W_{s+r}) = 0$ so $\partial_s(x) = 0$.

Hence $x = \beta_s(y)$ for some $y \in \pi_n(W_s)$. It suffices to show that y has a nontrivial image in $\pi_n(W)$. Suppose it does not and let r be the largest integer such that y has a nontrivial image $z \in \pi_n(W_{s+1+r})$. Then $z = \partial\pi_{s-r}(w)$ for some $w \in \pi_n(K_{s-r})$ and we have a differential $d_r[w] = [x]$, contradicting the nontriviality of $[x]$. □

Proof of 3.8. If X^i is finite this is the standard convergence result for the ASS, see e.g. [1] section III.15. If the other condition of 3.6(iii) is satisfied, suppose for simplicity that each X^i is p-adically complete and that Y has finite type. Since $\pi_*Y/\text{torsion}$ has finite rank we can form a fibre sequence

$$T \to Y \to E$$

where E is a finite wedge of integral Eilenberg-MacLane spectra and T has finite homotopy groups. Using standard properties of Adams resolutions one can make similar fibrations compatibly for each Y_s. Since inverse limits preserve fibrations, it suffices to show that $\lim_{\overleftarrow{s}} F(X^i, T_s)$ and $\lim_{\overrightarrow{s}} F(X^i, E_s)$ are contractible. Let $X^{i,n}$ denote the n-skeleton of X^i. According to Milnor [17] there is a short exact sequence

$$0 \to \lim_{\overleftarrow{n}}{}^1 [X^{i,n}, T_s] \to [X^i, T_s] \to \lim_{\overleftarrow{n}} [X^{i,n}, T_s] \to 0.$$

Since each group $[X^{i,n}, T_s]$ is locally finite the $\varprojlim^1$ vanishes so

$$[X^i, T_s] = \lim_{\overleftarrow{n}} [X^{i,n}, T_s] \quad \text{and} \quad \lim_{\overleftarrow{s}} [X^i, T_s] = \lim_{\overleftarrow{s}} \lim_{\overleftarrow{n}} [X^{i,n}, T_s]$$

$$= \varprojlim_n \varprojlim_s [X^{i,n}, T_s]$$

$$= 0 \text{ by the standard convergence result cited above.}$$

Replacing T_s by E_s we use a similar argument and we need to show $\varprojlim_n [X^{i,n}, E_s] = 0$.

Now E_s is a finite wedge of integral Eilenberg-MacLane spectra, so $[X^{i,n}, E_s]$ is a finite sum of integral cohomology groups of $X^{i,n}$. The $X^{i,n}$ are skeleta of X^i; so for n sufficiently large we have

$$[X^{i,n}, E_s] = [X^i, E_s]$$

and the $\varprojlim^1$ vanishes. □

Proof of 3.9. By 3.1 the homotopy commutative diagram of spectra $A_{i,j}$ can be replaced by one which is strictly commutative. Then the result is standard; see Bousfield-Kan [6] XI 4.3. □

Appendix by H. R. Miller

In this appendix we show that the form of Segal Conjecture II for cyclic groups proved here actually yields the more precise Segal Conjecture III. Thus we must show that the dual of the map

$$h : U_0 \to \Sigma U_{-i}^{-1}$$

dealt with above is homotopic to the composite

$$U_0^{2i-1} \longrightarrow U_0 \xrightarrow{\varphi} DU_0,$$

where φ is the map prescribed by Segal. We shall actually indicate a much more general identification.

For h we may take the following map. Let $W = Z/(p^n)$ act in the standard way on S^{2i-1} and on S^∞, and form the fiber bundle

$$S^{2i-1} \longrightarrow W\backslash(S^\infty \times S^{2i-1}) \xrightarrow{q_1} BW.$$

Let $q_2 : W\backslash(S^\infty \times S^{2i-1}) \to W\backslash S^{2i-1}$ be the natural projection to the lens space, and let τ be the tangent bundle of $W\backslash S^{2i-1}$. Then $q_2^*\tau$ (which we write, abusively, as τ) is the bundle of tangents along the fiber of q_1, and there is a transfer map (see below)

$$t(q_1): U_0 \to (W\backslash S^\infty \times S^{2i-1}))^{-\tau}.$$

Moreover, q_2 induces a homotopy-equivalence q_2 from the target here to the Thom space $(W\backslash S^{2i-1})^{-\tau} = \Sigma U_{-i}^{-1}$. Then we may take $h = q_2 \circ t(q_1)$ so the required identification follows as a special case from Propositions A and B below.

We begin by recalling Segal's map. Thus let G be a finite group, H a subgroup, N the normalizer of H in G, and $W = N/H$ the associated "Weyl group." Then N embeds "diagonally" in $G \times W$ as a subgroup, and the resulting map $p: BN \to BG \times BW$ is the finite cover formed from the universal $G \times W$-bundle by mixing in as fiber the $G \times W$-set G/H. Stably one has a transfer map $t(p)$, and the composite

$$BG_+ \wedge BW_+ \xrightarrow{t(p)} BN_+ \xrightarrow{c} S,$$

with c indicating the collapse, determines a stable map

$$\varphi_H: BW_+ \to DBG_+.$$

This is the map figuring in Segal Conjecture III.

Let M be a free smooth closed W-manifold, with orbit space $W\backslash M = B$ and classifying map $f: B \to BW$. We construct a map

$$\partial_{H,M}: BG_+ \to B^{-\tau},$$

τ the tangent bundle of B (or, below, any pull-back thereof), such that the following holds.

PROPOSITION A. *$D\partial_{H,M}$ is homotopic to the composite*

$$B_+ \xrightarrow{f} BW_+ \xrightarrow{\varphi_H} DBG_+,$$

where $DB^{-\tau} \cong B_+$ via Atiyah duality.

Recall [18] that if $\pi: E \to K$ is a smooth map with (virtual) normal bundle ν over K, and ξ is a (virtual) vector bundle over K, then the Pontrjagin-Thom construction gives a stable map

$$t(\pi): K^{\xi\oplus\nu} \to E^{\pi^*\xi}.$$

The following "pull-back" property [1] of this construction is the key element of this appendix. Let $f: K' \to K$ be a smooth map transverse to π, and form the pull-back

$$\begin{array}{ccc} E' & \xrightarrow{\bar{f}} & E \\ \downarrow{\scriptstyle \pi'} & & \downarrow{\scriptstyle \pi} \\ K' & \xrightarrow{f} & K \end{array}$$

Then π' has normal bundle $f^*\nu$, and the following stable diagram is commutative.

$$\begin{array}{ccc} K'^{f^*(\xi\oplus\nu)} & \xrightarrow{f} & K^{\xi\oplus\nu} \\ \downarrow{\scriptstyle t(\pi')} & & \downarrow{\scriptstyle t(\pi)} \\ E'^{(f\pi')^*\xi} & \xrightarrow{\bar{f}} & E^{\pi^*\xi} \end{array}$$

By suitable approximation, $t(\pi)$ may be defined in the contexts below, where π is a fibration with smooth fiber; and the pull-back property continues to hold.

We return to the construction of the map $\partial_{H,M}$. Let p_1 be the composite

$$p_1: N\backslash(EG \times M) \xrightarrow{\overline{pr}_1} N\backslash EG \longrightarrow G\backslash EG = BG;$$

this is a fibration with fiber $N\backslash(G \times M)$. Let

$$p_2: N\backslash(EG \times M) \xrightarrow{\overline{pr}_2} N\backslash M = B.$$

This map pulls τ back to the bundle of tangents along the fiber of p_1; so we have the indicated transfer in the following definition of $\partial_{H,M}$:

$$\partial_{H,M}: BG_+ \xrightarrow{t(p_1)} N\backslash(EG \times M)^{-\tau} \xrightarrow{p_2} B^{-\tau}.$$

We have the pull-back diagram

$$\begin{tikzcd}
N\backslash(EG \times M) \arrow[r, "p_2"] \arrow[d, "r"] & B \arrow[d, "\Delta"] \\
N\backslash(EG \times M) \times B \arrow[r, "p_2 \times 1"] & B \times B,
\end{tikzcd}$$

where $r(x) = (x, p_2(x))$.

With $\xi = -\tau \times 0$, the pull-back property of the transfer shows that the upper right square in the following stable diagram commutes.

$$\begin{tikzcd}
BG_+ \wedge B_+ \arrow[r, "t(p_1)\wedge 1"] \arrow[dr, "t(p)"] \arrow[dd, "1\wedge f"] & N\backslash(EG \times M)^{-\tau} \wedge B_+ \arrow[r, "p_2\wedge 1"] \arrow[d, "t(r)"] & B^{-\tau} \wedge B_+ \arrow[d, "t(\Delta)"] \arrow[dd, bend left=60, "\alpha"] \\
 & N\backslash(EG \times M)_+ \arrow[r, "p_2"] \arrow[d] & B_+ \arrow[d, "c"] \\
BG_+ \wedge BW_+ \arrow[r, "t(p)"] & BN_+ \arrow[r, "c"] & S^0
\end{tikzcd}$$

Here α is the Atiyah duality map. The other portions of the diagram commute trivially; for example, the upper left triangle commutes by virtue of the evident composition property of the transfer. This diagram verifies Proposition A.

In case H is normal in G, so $N = G$ and W is a quotient group of G, we may simplify the description of $\partial_{H,M}$ as follows.

Proposition B. *If $H \trianglelefteq G$, then $\partial_{H,M}$ is homotopic to the composite*

$$BG_+ \xrightarrow{\pi} BW_+ \underset{\simeq}{\xrightarrow{t(q_1)}} W\backslash(EW \times M)^{-\tau} \underset{\simeq}{\xrightarrow{q_2}} B^{-\tau},$$

where $q_1 : W\backslash(EW \times M) \to BW$ and $q_2 : W\backslash(EW \times M) \to W\backslash M = B$ are the evident projections.

Proof. We have a pull-back diagram

$$\begin{tikzcd}
G\backslash(EG \times M) \arrow[r, "\bar{\pi}"] \arrow[d, "p_1"] & W\backslash(EW \times M) \arrow[d, "q_1"] \\
BG \arrow[r, "\pi"] & BW.
\end{tikzcd}$$

With $\xi = 0$, the pull-back property of the transfer yields the commutativity of the following stable diagram.

$$\begin{tikzcd} BG_+ \arrow[r, "\pi"] \arrow[d, "t(p_1)"] & BW_+ \arrow[d, "t(q_1)"] \\ G\backslash(EG \times M)^{-\tau} \arrow[r, "\bar{\pi}"] \arrow[dr, "p_2"] & W\backslash(EW \times M)^{-\tau} \arrow[d, "q_2"] \\ & B^{-\tau} \end{tikzcd}$$

This verifies Proposition B.

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