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SOME PRODUCTS OF β-ELEMENTS IN THE NOVIKOV E_2 TERM OF MOORE SPECTRA

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ABSTRACT. In this note, we prove trivialities and nontrivialities of products of some higher-order $\beta'_{tp^n/s}$ elements in the E_2 terms of the Adams-Novikov spectral sequence of Moore spectra.

1. INTRODUCTION

Let S be the sphere spectrum and M the Moore spectrum modulo a prime $p \geq 5$ given by the cofibration

$$S \xrightarrow{p} S \xrightarrow{i} M \xrightarrow{j} \Sigma S.$$

Consider the Brown-Peterson spectrum BP at p and the Adams-Novikov spectral sequence (ANSS) $\mathrm{Ext}^{s,t} M = \mathrm{Ext}^{s,t}_{BP_*BP}(BP_*, BP_*M) \Rightarrow \pi_{t-s}M$. According to Miller and Wilson [1] and Miller, Ravenel, and Wilson [2], there are β-elements

$$\beta'_{tp^n/s} \in \mathrm{Ext}^{1,*} M \quad \text{for} \begin{cases} 1 \leq s \leq p^n + p^{n-1} - 1 & \text{if } p \nmid t \geq 2, \\ 1 \leq s \leq p^n & \text{if } t = 1, \end{cases}$$

such that their images under the boundary homomorphisms associated with the short exact sequence $0 \to BP_* \xrightarrow{p} BP_* \to BP_*/(p) \to 0$ are $\beta_{tp^n/s} \in \mathrm{Ext}^{2,*} S = \mathrm{Ext}^{2,*}_{BP_*BP}(BP_*, BP_*)$. We will write $\beta'_{tp^n/1}$ as β'_{tp^n} in $\mathrm{Ext}^1 M$ and $\beta_{tp^n/1}$ as β_{tp^n} in $\mathrm{Ext}^2 S$.

The present note gives some results on trivialities and nontrivialities of products of higher-order β-elements in $\mathrm{Ext}^{*,*} M$, and we will consider p as a prime ≥ 5 throughout this note.

Theorem 1.1. *The following relations on products of β-elements in the E_2 terms of the ANSS of M hold*:

(1) $\beta'_{mp^{n-1}/p^{n-1}}\beta_{tp^n/p^n} = 0$ *in* $\mathrm{Ext}^3 M$ *for* $p \nmid t \geq 2$, $n \geq 2$, *and* $m = k(tp-1)$, $k \not\equiv 0, -1 \pmod p$.

(2) $\beta'_{mp^{n-1}}\beta_{tp^n/a} = \beta'_{mp^{n-1}/p^{n-1}+1}\beta_{tp^n/p^{n-1}} = -\beta'_{mp^{n-1}/2}\beta'_{tp^n/a} \cdot h_0 \neq 0$ *in* $\mathrm{Ext}^3 M$ *for* $p \nmid t \geq 2$, $n \geq 2$, *and* $m = k(tp-1)$, $k \not\equiv 0, -1 \pmod p$, *where*

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$a = p^n + p^{n-1} - 1$ *and* $h_0 \in \mathrm{Ext}^1 M$ *is the* v_1*-torsion free generator stated in* [1, *Theorem* 1.1].

The triviality in (1) will be proved by using a result in [1], and the nontriviality in (2) will rely on the result on $\beta'_s \beta_{tp^n/a}$ in [7]. By using the results in Theorem 1.1, we have

Theorem 1.2. (1) *The product* $\beta'_{bp^{n-1}/p^{n-1}} \beta'_{tp^n/p^n} = 2t \cdot \beta'_{bp^{n-1}+(tp-1)p^{n-1}} \cdot h_0 \neq 0$ *in* $\mathrm{Ext}^2 M$ *for* $p \nmid t \geq 2$, $n \geq 2$, *and* $b \not\equiv 0, 1 \pmod p$.

(2) $\beta'_{bp^n/s} \cdot h_0 \neq 0$ *in* $\mathrm{Ext}^2 M$ *for* $2 \leq s \leq p^n + p^{n-1} - 1$, $n \geq 1$, $p \nmid b \geq 2$, *and* $b \not\equiv -1 \pmod p$ *or* $b = tp - 1$ *with* $p \nmid t \geq 2$.

(3) $\beta'_{bp^n} \cdot h_0 = 0$ *in* $\mathrm{Ext}^2 M$ *for* $n \geq 1$ *and* $b = tp - 1$ *with* $p \nmid t \geq 2$.

2. Proof of the main theorems

Let $\alpha: \Sigma^q M \to M$ be the Adams map and K_r be the cofibre of α^r given by the cofibration

$$\Sigma^{rq} M \xrightarrow{\alpha^r} M \xrightarrow{i'_r} K_r \xrightarrow{j'_r} \Sigma^{rq+1} M. \tag{2.1}$$

The cofibration (2.1) induces a short exact sequence $0 \to BP_*/(p) \xrightarrow{v_1^r} BP_*/(p) \to BP_*/(p, v_1^r) \to 0$ and then induces the Ext exact sequence

$$\cdots \to \mathrm{Ext}^{s,t} M \xrightarrow{v_1^r} \mathrm{Ext}^{s,t+rq} M \xrightarrow{(i'_r)_*} \mathrm{Ext}^{s,t+rq} K_r \xrightarrow{(j'_r)_*} \mathrm{Ext}^{s+1,t} M \to \cdots$$

where we write $(j'_r)_*$ as the boundary homomorphism and $(i'_r)_*$ as the reduction.

From [3, p. 422], $i'_u j'_r: K_r \to \Sigma^{rq+1} K_u$ induces a cofibration

$$\Sigma^{rq} K_u \xrightarrow{\psi} K_{r+u} \xrightarrow{\rho} K_r \xrightarrow{i'_u j'_r} \Sigma^{rq+1} K_u, \tag{2.2}$$

which realizes the short exact sequence $0 \to BP_*/(p, v_1^u) \xrightarrow{v_1^r} BP_*/(p, v_1^{u+r}) \to BP_*(p, v_1^r) \to 0$ and induces the Ext exact sequence

$$\cdots \to \mathrm{Ext}^{s,t} K_u \xrightarrow{\psi_*} \mathrm{Ext}^{s,t+rq} K_{u+r} \xrightarrow{\rho_*} \mathrm{Ext}^{s,t+rq} K_r \xrightarrow{(i'_u j'_r)_*} \mathrm{Ext}^{s+1,t} K_u \to \cdots$$

where we write $(i'_u j'_r)_*$ as the boundary homomorphism and $\psi_* = v_1^r$. From the 3×3 lemma in the stable homotopy category, we can easily have

$$\psi i'_u = i'_{u+r} \alpha^r, \qquad j'_r \rho = \alpha^u j'_{u+r}. \tag{2.3}$$

Note that the behavior of ψ_*, ρ_*, and $(i'_u j'_r)_*$ in the above Ext exact sequence is compatible with that of ψ, ρ, and $i'_u j'_r$ in the cofibration, i.e., we also have

$$\psi_* (i'_u)_* = (i'_{u+r})_* v_1^r, \qquad (j'_r)_* \rho_* = v_1^u (j'_{u+r})_* \tag{2.4}$$

in the Ext stage.

If $r \equiv 0 \pmod p$, K_r is a split ring spectrum (cf. [5]), there exists $\overline{\delta} \in [\Sigma^{-1} K_r, K_r]$ such that

$$\overline{\delta} i'_r = i'_r \delta, \quad j'_r \overline{\delta} = -\delta j'_r, \qquad (\delta = ij) \tag{2.5}$$

and $\overline{\delta}$ is a derivation behaved on the products in $\pi_* K_r$, i.e.,

$$\overline{\delta}\,\overline{\delta} = 0, \qquad \overline{\delta} \mu = \mu(\overline{\delta} \wedge 1_{K_r}) + \mu(1_{K_r} \wedge \overline{\delta}),$$

where $\mu\colon K_r \wedge K_r \to K_r$ is the associative and commutative multiplication of K_r. Hence $\overline{\delta}_*\colon \mathrm{Ext}^{s,t} K_r \to \mathrm{Ext}^{s+1,t} K_r$ also is a derivation behaved on the products in $\mathrm{Ext}^{*,*} K_r$, i.e., $\overline{\delta}_*\overline{\delta}_* = 0$, and

$$(2.6) \qquad \overline{\delta}_*(xy) = (\overline{\delta}_* x)y + (-1)^{|x|} x(\overline{\delta}_* y), \qquad x, y \in \mathrm{Ext}^{*,*} K_r.$$

Moreover, from (2.5) we have

$$(2.7) \qquad \begin{aligned} \overline{\delta}_*(i'_r)_* &= (i'_r)_*\delta_*\colon \mathrm{Ext}^{s,*} M \to \mathrm{Ext}^{s+1,*} K_r, \\ (j'_r)_*\overline{\delta}_* &= -\delta_*(j'_r)_*\colon \mathrm{Ext}^{s,*} K_r \to \mathrm{Ext}^{s+2,*} M, \end{aligned}$$

where $\delta_*\colon \mathrm{Ext}^{s,t} M \to \mathrm{Ext}^{s+1,t} M$ is the boundary homomorphism induced by $\delta = ij \in [\Sigma^{-1}M, M]$, and it is similar that δ_* is a derivation behaved on the products in $\mathrm{Ext}^{*,*} M$

$$(2.8) \qquad \delta_*(xy) = (\delta_* x)y + (-1)^{|x|} x(\delta_* y), \qquad x, y \in \mathrm{Ext}^{*,*} M.$$

Proof of Theorem 1.1. (1) Briefly write $a = p^n + p^{n-1} - 1$. According to [1], $\beta'_{tp^n/a} = c_1(tp^n) \in \mathrm{Ext}^1 M$ and $c_1(tp^n)$ is the v_1-torsion generator of $\mathrm{Ext}^1 M$ stated in [1, Theorem 1.1, p. 132]. Moreover, $\beta'_{tp^n/s} = v_1^{a-s} c_1(tp^n) = v_1^{a-s}\beta'_{tp^n/a}$, and $v_1^u \beta'_{tp^n/s+u} = \beta'_{tp^n/s}$ for $u + s \le a$. Briefly write $d = p^{n-1}$. Then

$$(2.9) \qquad \begin{aligned} (i'_d)_*(\beta'_{tp^n/p^n}) &= (i'_d)_* v_1^{d-1}\beta'_{tp^n/a} = \psi_* i'_*(\beta'_{tp^n/a}) \\ &= 2t \cdot \psi_*(v_2^{(tp-1)d} h_0) = 2t \cdot v_2^{(tp-1)d}\psi_*(h_0), \end{aligned}$$

where $\psi\colon \Sigma^{(d-1)q} K_1 \to K_d$ is the map in (2.2) and we use the relation $i'_* c_1(tp^n) = 2t \cdot v_2^{(tp-1)d} h_0$ in [1, Theorem 1.1(b)(iii)].

Since $h_0 \in \mathrm{Ext}^{1,q} K_1$ converges to $i'ij\alpha i \in \pi_{q-1} K_1$ in the ANSS and $\overline{\delta}\psi i'ij\alpha i = \overline{\delta} i'_d \alpha^{d-1} ij\alpha i = i'_d ij \alpha^{d-1} ij\alpha i = 0 \in \pi_* K_d$, it follows that $\overline{\delta}_*\psi_*(h_0) = 0 \in \mathrm{Ext}^2 K_d$. Hence, by applying the derivation $\overline{\delta}_*$ on (2.9), we have

$$(2.10) \qquad \begin{aligned} (i'_d)_*\delta_*(\beta'_{tp^n/n}) &= \overline{\delta}_*(i'_d)_*(\beta'_{tp^n/p^n}) \\ &= 2t \cdot \overline{\delta}_*(v_2^{(tp-1)d}\psi_*(h_0)) = 2t \cdot \overline{\delta}_*(v_2^{(tp-1)d}) \cdot \psi_*(h_0). \end{aligned}$$

Since $v_2^{(tp-1)d} \in \mathrm{Ext}^0 K_d$ by [1, Proposition 6.3] and $\overline{\delta}_*$ is a derivation (cf. (2.6)), then $\overline{\delta}_*(v_2^{(k+1)(tp-1)d}) = (k+1) \cdot v_2^{k(tp-1)d}\overline{\delta}_* v_2^{(tp-1)d}$, so by (2.10) we have

$$(2.11) \qquad \begin{aligned} v_2^{k(tp-1)d}(i'_d)_*\delta_*(\beta'_{tp^n/p^n}) &= 2t \cdot v_2^{k(tp-1)d}\overline{\delta}_* v_2^{(tp-1)d} \cdot \psi_* h_0 \\ &= \frac{2t}{k+1}\overline{\delta}_*(v_2^{(k+1)(tp-1)d}) \cdot \psi_*(h_0) \\ &= \frac{2t}{k+1}\overline{\delta}_*(v_2^{(k+1)(tp-1)d}\psi_*(h_0)) \quad [\text{by (2.6) and } \overline{\delta}_*\psi_* h_0 = 0] \\ &= \frac{2t}{k+1}\overline{\delta}_*\psi_*(v_2^{(k+1)(tp-1)d} h_0) \quad [\text{since } \psi_* = v_1^{d-1}]. \end{aligned}$$

By applying the boundary homomorphism $(j'_d)_*\colon \mathrm{Ext}^2 K_d \to \mathrm{Ext}^3 M$, the left-hand side of (2.11) becomes $\beta'_{k(tp-1)d/d}\beta_{tp^n/p^n}$ by the Yoneda product and the right-hand side of (2.11) becomes

$$\begin{aligned}\frac{2t}{k+1}\cdot (j'_d)_*\overline{\delta}_*\psi_*(v_2^{(k+1)(tp-1)d}h_0)\\ &= \frac{-2t}{k+1}\delta_* j'_*(v_2^{(k+1)(tp-1)d}h_0) \quad [\text{cf. } (2.5), (2.4)]\\ &= 0 \in \mathrm{Ext}^3 M,\end{aligned}$$

since $j_* j'_*(v_2^{(k+1)(tp-1)d}h_0) = \beta_{(k+1)(tp-1)d}\alpha_1 \in \mathrm{Ext}^3 S$ is divisible by p (cf. [2, Theorem 2.8(c), p. 477]). So we have the desired triviality.

(2) Let $m'\colon M\wedge S \to M$ be the restriction of the multiplication $m\colon M\wedge M \to M$. Since $m' = m(1_M \wedge i)$, the following diagram commutes:

$$\begin{array}{c} \beta'_{mp^{n-1}} \otimes \beta_{tp^n/a} \in \mathrm{Ext}^1(BP_*, BP_*M) \otimes \mathrm{Ext}^2(BP_*, BP_*) \\ \big\downarrow 1\otimes i_* \\ \beta'_{mp^{n-1}} \otimes \delta_*\beta'_{tp^n/a} \in \mathrm{Ext}^1(BP_*, BP_*M) \otimes \mathrm{Ext}^2(BP_*, BP_*M) \\ \longrightarrow \mathrm{Ext}^3(BP_*, BP_*M\wedge S) \\ \big\downarrow (1_M\wedge i)_* \\ \longrightarrow \mathrm{Ext}^3(BP_*, BP_*M\wedge M) \\ \xrightarrow{m'_*} \mathrm{Ext}^3(BP_*, BP_*M) \ni \beta'_{mp^{n-1}}\beta_{tp^n/a} \\ \Vert \qquad\qquad \Vert \\ \xrightarrow{m_*} \mathrm{Ext}^3(BP_*, BP_*M) \ni \beta'_{mp^{n-1}}\delta_*\beta'_{tp^n/a} \end{array}$$

where the top and bottom rows are products in the ANSS. Hence, we have $\beta'_{mp^{n-1}}\beta_{tp^n/a} = \beta'_{mp^{n-1}}\delta_*\beta'_{tp^n/a}$ and by (2.8)

$$\begin{aligned}\beta'_{mp^{n-1}}\beta_{tp^n/a} &= \beta'_{mp^{n-1}}\delta_*\beta'_{tp^n/a} = \beta'_{mp^{n-1}/p^{n-1}+1}v_1^{p^{n-1}}\delta_*\beta'_{tp^n/a}\\ &= \beta'_{mp^{n-1}/p^{n-1}+1}\delta_*(v_1^{p^{n-1}}\beta'_{tp^n/a}) = \beta'_{mp^{n-1}/p^{n-1}+1}\beta_{tp^n/p^n-1}\end{aligned}$$

as desired. Moreover,

$$\begin{aligned}\beta'_{mp^{n-1}}\beta_{tp^n/a} &= \beta'_{mp^{n-1}/p^{n-1}+1}\delta_*(v_1\beta'_{tp^n/p^n})\\ &= \beta'_{mp^{n-1}/p^{n-1}+1}v_1\delta_*\beta'_{tp^n/p^n} + \beta'_{mp^{n-1}/p^{n-1}+1}(\delta_*v_1)\beta'_{tp^n/p^n}\\ &= \beta'_{mp^{n-1}/p^{n-1}}\beta_{tp^n/p^n} - \beta'_{mp^{n-1}/p^{n-1}+1}\beta'_{tp^n/p^n}\cdot h_0\\ &= -\beta'_{mp^{n-1}/2}\beta'_{tp^n/a}\cdot h_0 \quad [\text{the 1st term is zero from (1)}].\end{aligned}$$

From [7, (4.1.3), p. 132] $\beta'_r\beta_{tp^n/a} \neq 0$ in $\mathrm{Ext}^3 M$ if and only if $r \neq (up^e - p^{e-1}) - (tp^n - p^{n-1})$ for any $p\nmid u \geq 2$ and $e \geq 1$. Now if $mp^{n-1} = k(tp-1)p^{n-1} = (up^e - p^{e-1}) - (tp^n - p^{n-1})$ for some $p\nmid u \geq 2$ and $e \geq 1$, then $(k+1)(tp-1)p^{n-1} = (up-1)p^{e-1}$, and it is impossible since $k \not\equiv 0, -1 \pmod p$. Hence, $\beta'_{mp^{n-1}}\beta_{tp^n/a} \neq 0$. Q.E.D.

Proof of Theorem 1.2. (1) From (2.9) we have

$$\begin{aligned} v_2^{bd}(i'_d)_*(\beta'_{tp^n/p^n}) &= 2t \cdot v_2^{bd} \cdot v_2^{(tp-1)d} \psi_*(h_0) \\ &= 2t \cdot \psi_*(v_2^{(b+tp-1)d} h_0), \qquad d = p^{n-1}. \end{aligned}$$

Hence, by applying the boundary homomorphism $(j'_d)_*\colon \mathrm{Ext}^1 K_r \to \mathrm{Ext}^2 M$, we have $\beta'_{bd/d}\beta'_{tp^n/p^n} = 2t \cdot j'_*(v_2^{(b+tp-1)d} h_0) = 2t\beta'_{(b+tp-1)d} h_0$. Moreover, if $b + tp - 1 \not\equiv 0,\ -1 \pmod p$, it follows from

$$0 \neq \beta_{(b+tp-1)d}\alpha_1 = j_*(\beta'_{(b+tp-1)d} h_0)$$

(cf. [2, Theorem 2.8(b)(i), p. 477]) that $\beta'_{(b+tp-1)d} h_0 \neq 0$.

(2) If $b \not\equiv -1 \pmod p$ and $\beta'_{bp^n/s} h_0 = 0$, then $\beta'_{bp^n} \cdot h_0 = 0$ in $\mathrm{Ext}^2 M$, and so in $\mathrm{Ext}^3 S$ we have

$$0 \neq \beta_{bp^n}\alpha_1 = j_*(\beta'_{bp^n} \cdot h_0) = 0,$$

which is a contradiction, where we use the result in [2, Theorem 2.8(b)(i), p. 477] on $\beta_{bp^n}\alpha_1 \neq 0$ in $\mathrm{Ext}^3 S$ if $b \not\equiv -1 \pmod p$.

If $b = tp - 1$ with $p \nmid t \geq 2$, from Theorem 1.1(1) we have

$$\beta'_{bp^n/s}\beta'_{tp^{n+1}/p^{n+1}+p^n+1-s} \cdot h_0 \neq 0,$$

and so $\beta'_{bp^n/s} \cdot h_0 \neq 0$ in $\mathrm{Ext}^2 M$.

(3) Let $b = tp - 1$ with $p \nmid t \geq 2$. It is known that there exists $f \in [\Sigma^* K_1, K_1]$ such that the induced BP_* homomorphism $f_* = v_2^{bp^n}$, and so $2t \cdot fi'ij\alpha i \in \pi_* K_1$ is detected by $2t \cdot v_2^{bp^n} h_0 = i'_* c_1(tp^{n+1}) \in \mathrm{Ext}^1 K_1$ (cf. [1, Theorem 1.1(b)(iii)]). Hence $j'fi'ij\alpha i \in \pi_* M$ has BP filtration > 2 since $j'_* i'_* c_1(tp^{n+1}) = 0$. This means that $\beta'_{bp^n} \cdot h_0 = 0$ in $\mathrm{Ext}^2 M$. Q.E.D.

References

1. H. R. Miller and W. S. Wilson, *On Novikov's* Ext¹ *modulo an invariant prime ideal*, Topology **15** (1976), 131–141.
2. H. R. Miller, D. C. Ravenel, and W. S. Wilson, *Periodic phenomena in the Adams-Novikov spectral sequence*, Ann. of Math. (2) **106** (1977), 469–516.
3. S. Oka, *Multiplicative structure of finite ring spectra and stable homotopy of spheres*, Algebraic Topology (Aarhus 1982), Lecture Notes in Math., vol. 1051, Springer-Verlag, New York, 1984, pp. 418–441.
4. S. Oka, *Realizing some cyclic BP_* modules and applications to stable homotopy of spheres*, Hiroshima Math. J. 7 (1977), 427–447.
5. S. Oka, *Small ring spectra and p-rank o f the stable homotopy of spheres*, Contemp. Math., vol. 19, Amer. Math. Soc., Providence, RI, 1983, pp. 267–308.
6. D. C. Ravenel, *The nonexistence of odd primary Arf invariant elements in the stable homotopy*, Math. Proc. Cambridge Philos. Soc. **83** (1978), 429–443.
7. K. Shimomura and H. Tamura, *Nontriviality of some compositions of β-elements in the stable homotopy of Moore spaces*, Hiroshima Math. J. **16** (1986), 121–133.

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