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A Pull Back Theorem in the Adams Spectral Sequence

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Abstract This paper proves that, for any generator $x \in Ext_A^{s,tq}(Z_p, Z_p)$, if $(1_L \wedge i)_*\phi_*(x) \in Ext_A^{s+1,tq+2q}(H^*L \wedge M, Z_p)$ is a permanent cycle in the Adams spectral sequence (ASS), then $h_0x \in Ext_A^{s+1,tq+q}(Z_p, Z_p)$ also is a permenent cycle in the ASS. As an application, the paper obtains that $h_0h_nh_m \in Ext_A^{3,p^nq+p^mq+q}(Z_p, Z_p)$ is a permanent cycle in the ASS and it converges to elements of order p in the stable homotopy groups of spheres $\pi_{p^nq+p^mq+q-3}S$, where $p \geq 5$ is a prime, $s \leq 4, n \geq m+2 \geq 4$ and M is the Moore spectrum.

Keywords Adams spectral sequence, Toda spectrum, stable homotopy groups of spheres

MR(2000) Subject Classification 55Q45

1 Introduction

Let A be the mod p Steenrod algebra and S the sphere spectrum localized at an odd prime p. To determine the stable homotopy groups of spheres π_*S is one of the central problems in homotopy theory. One of the main tools to reach it is the Adams spectral sequence (ASS) $E_2^{s,t} = Ext_A^{s,t}(Z_p, Z_p) \Longrightarrow \pi_{t-s}S$, where the $E_2^{s,t}$-term is the cohomology of A.

Let M be the Moore spectrum given by the cofibration

$$S \xrightarrow{p} S \xrightarrow{i} M \xrightarrow{j} \Sigma S. \tag{1.1}$$

Let $\alpha : \Sigma^q M \to M$ be the Adams map and K be its cofibre given by the cofibration

$$\Sigma^q M \xrightarrow{\alpha} M \xrightarrow{i'} K \xrightarrow{j'} \Sigma^{q+1} M; \tag{1.2}$$

K is the Toda–Smith spectrum $V(1)$. Let L be the cofibre of $\alpha_1 = j\alpha i : \Sigma^{q-1}S \to S$ given by the cofibration

$$\Sigma^{q-1}S \xrightarrow{\alpha_1} S \xrightarrow{i''} L \xrightarrow{j''} \Sigma^q S. \tag{1.3}$$

Since $\alpha_1 \cdot \alpha_1 = j\alpha i j\alpha i = 0$, then there are $\phi \in [\Sigma^{2q-1}S, L]$ and $(\alpha_1)_L \in [\Sigma^{q-1}L, S]$ such that

$$j'' \cdot \phi \quad = \quad \alpha_1 \quad = \quad (\alpha_1)_L \cdot i''. \tag{1.4}$$

From [1], there are $a_0 \in Ext_A^{1,1}(Z_p, Z_p), h_0 \in Ext_A^{1,q}(Z_p, Z_p)$, $\widetilde{\alpha}_2 \in Ext_A^{2,2q+1}(Z_p, Z_p)$ which converge in the ASS to $p \in \pi_0 S, \alpha_1 = j\alpha i \in \pi_{q-1}S$, $\alpha_2 = j\alpha^2 i \in \pi_{2q-1}S$, respectively. Then, for any $\sigma \in Ext_A^{s,tq}(Z_p, Z_p)$, the products $h_0\sigma = j_*\alpha_*i_*(\sigma) \in Ext_A^{s+1,tq+q}(Z_p, Z_p)$ and $\widetilde{\alpha}_2\sigma =$

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$j_*\alpha_*\alpha_*i_*(\sigma) \in Ext_A^{s+1,tq+2q+1}(Z_p, Z_p)$, where α_*, is the connecting homomorphism induced by α and $j_*\alpha_*\alpha_*i_*$, is the following composition:

$$Ext_A^{s,tq}(Z_p, Z_p) \xrightarrow{i_*} Ext_A^{s,tq}(H^*M, Z_p) \xrightarrow{\alpha_*} Ext_A^{s+1,tq+q+1}(H^*M, Z_p) \xrightarrow{\alpha_*} Ext_A^{s+2,tq+2q+2}(H^*M, Z_p) \xrightarrow{j_*} Ext_A^{s+2,tq+2q+1}(Z_p, Z_p).$$

If a generator $x \in E_2^{s,u} = Ext_A^{s,u}(Z_p, Z_p)$ is a permanent cycle in the ASS, then it is well known that $i_*(x) \in Ext_A^{s,u}(H^*M, Z_p)$ also is a permanent cycle in the ASS. The reverse problem is in general not true. But it may be true in some case such as $x = h_0\sigma$, and we call it a pull back problem in the ASS. The main purpose of this paper is to prove the following pull back theorem with a rather stonger supposition:

Theorem A *Let $p \geq 5, s \leq 4$ and assume that:*

(I) (a) $Ext_A^{s,tq}(Z_p, Z_p) \cong Z_p\{\sigma\}, Ext_A^{s+1,tq+q}(Z_p, Z_p) \cong Z_p\{h_0\sigma\}$, $Ext_A^{s+2,tq+2q+1}(Z_p, Z_p) \cong Z_p\{\widetilde{\alpha}_2\sigma\}$;

(b) $Ext_A^{s+1,tq+u}(Z_p, Z_p) \cong Z_p\{a_0\sigma\}$ *for* $u = 1$, *is zero for* $u = 2, 3$ *and* $a_0^2\sigma \neq 0$, $Ext_A^{s+1,tq}(Z_p, Z_p) = 0$ *or has (one or two) generator* σ' *(both) satisfying* $a_0\sigma' \neq 0, h_0\sigma' \neq 0$, $Ext_A^{s+1,tq+rq+u}(Z_p, Z_p) = 0$ *for* $r = 1, u = -2, -1, 1, 2, 3$ *or* $r = -1, 2, 3, u = -2, -1, 0, 1, 2, 3$;

(c) $Ext_A^{s,tq+u}(Z_p, Z_p) = 0$ *for* $u = -1, 1, 2, 3$. $Ext_A^{s,tq+rq+u}(Z_p, Z_p) = 0$ *for* $r = -2, -1, 1, 2, 3, u = -2, -1, 0, 1, 2, 3$;

(II) $(1_L \wedge i)_*(\phi)_*(\sigma) \in Ext_A^{s+1,tq+2q}(H^*L \wedge M, Z_p)$ *is a permanent cycle in the ASS. Then,* $(\alpha i)_*(\sigma) \in Ext_A^{s+1,tq+q+1}(H^*M, Z_p)$ *also is a permanent cycle so that* $h_0\sigma = j_*(\alpha i)_*(\sigma) \in Ext_A^{s+1,tq+q}(Z_p, Z_p)$ *converges in the ASS to an element in* $\pi_{tq+q-s-1}S$ *of order* p.

As an application of Theorem A to $(s, tq, \sigma) = (2, p^nq + p^mq, h_nh_m)$, we obtain the following result in which the geometric input (II) comes from [2]:

Theorem B *Let $p \geq 5, m \geq n + 2 \geq 4$. Then*

$$h_0h_nh_m \in Ext_A^{3,p^nq+p^mq+q}(Z_p, Z_p)$$

is a permanent cycle in the ASS and it converges to an element in $\pi_{p^nq+p^mq+q-3}S$ *of order* p.

Remark By the result in [3], there is $\gamma_{p^{n-2}/p^{n-2}-p^{m-1},p^{m-1}-1} \in Ext_{BP_*BP}^{3,p^nq+p^mq+q}(BP_*, BP_*)$ whose image under the Thom map is $\Phi(\gamma_{p^{n-2}/p^{n-2}-p^{m-1},p^{m-1}-1}) = h_0h_nh_m \in Ext_A^{3,p^nq+p^mq+q}(Z_p, Z_p)$, then the element obtained in Theorem B is represented by

$$\gamma_{p^{n-2}/p^{n-2}-p^{m-1},p^{m-1}-1} + \text{ other terms } \in Ext_{BP_*BP}^{3,p^nq+p^mq+q}(BP_*, BP_*)$$

in the Adams–Novikov spectral sequence. The result of Theorem B is outside from [3] and it is still open until now.

Theorem A will be proved by some techniques processing in the Adams resolution of certain spectra related to S, which is equivalent to computing the differentials of the ASS. After giving some preminilaries on low-dimensional Ext groups in §2, the proofs of Theorems A, B are given in §3.

2 Some Preliminaries on Low-Dimensional Ext Groups and Others

A spectrum V is called an M-module spectrum if $p \wedge 1_V = 0 \in [V, V]$, and consequently, the cofibration $V \xrightarrow{p\wedge 1_V} V \xrightarrow{i\wedge 1_V} M \wedge V \xrightarrow{j\wedge 1_V} \Sigma V$ splits, i.e. there is a homotopy equivalence

$M \wedge V = V \vee \Sigma V$ and there are maps $m_V : M \wedge V \to V$, $\overline{m}_V : \Sigma V \to M \wedge V$ which are called the M-module actions of V, satisfying

$$m_V(i \wedge 1_V) = 1_V, \quad (j \wedge 1_V)\overline{m}_V = 1_V,$$
$$m_V\overline{m}_V = 0, \quad \overline{m}_V(j \wedge 1_V) + (i \wedge 1_V)m_V = 1_{M\wedge V}. \tag{2.1}$$

Let V and V' be M-module spectra. Then we define a homomorphism $d : [\Sigma^s V', V] \to [\Sigma^{s+1}V', V]$ by $d(f) = m_V(1_M \wedge f)\overline{m}_{V'}$ for $f \in [\Sigma^s V', V]$. This operation d is called a derivation (of maps between M-module spectra), which has the following properties:

Proposition 2.2 ([4, p. 272 Prop. 1.1] and [5, p. 210 Theorem 2.2(iii)]) (i) *d is derivative: $d(fg) = fd(g) + (-1)^{|g|} d(f)g$ for $f \in [\Sigma^s V', V]$, $g \in [\Sigma^t V'', V']$, where V, V', V'' are M-module spectra;*

(ii) *Let W', W be arbirtary spectra and $h \in [\Sigma^r W', W]$. Then $d(h \wedge f) = (-1)^{|h|} h \wedge d(f)$.*

From [4 pp. 271–277], K and M are M-module spectra, i.e. there are M-module actions $m_M : M \wedge M \to M$, $\overline{m}_M : \Sigma M \to M \wedge M$ and $m_K : K \wedge M \to K$, $\overline{m}_K : \Sigma M \to K \wedge M$ satisfying

$$m_M(i \wedge 1_M) = m_M(1_M \wedge i) = 1_M, \quad (j \wedge 1_M)\overline{m}_M = 1_M, (1_M \wedge j)\overline{m}_M = -1_M,$$
$$m_M\overline{m}_M = 0, \quad \overline{m}_M(j \wedge 1_M) + (i \wedge 1_M)m_M = 1_{M\wedge M},$$
$$m_K(1_K \wedge i) = 1_K, (1_K \wedge j)\overline{m}_K = 1_K, m_K\overline{m}_K = 0, (1_K \wedge i)m_K + (1_K \wedge j)\overline{m}_K = 1_{K\wedge M},$$
$$d(ij) = -1_M, \quad d(i') = 0, \quad d(j') = 0. \quad \text{(cf. [4, p. 277 and p. 272 line -5])}. \tag{2.3}$$

Proposition 2.4 ([6, Cor. 2.7]) *Let X, V,V' and V'' be arbitrary spectra and $g : V \to V'$, $g' : V' \to V''$ be maps. If $[V'' \wedge M, X \wedge M] \xrightarrow{(g'\wedge 1_M)^*} [V' \wedge M, X \wedge M] \xrightarrow{(g\wedge 1_M)^*} [V \wedge M, X \wedge M]$ is an exact sequence, then $\ker d \cap [V'' \wedge M, X \wedge M] \xrightarrow{(g'\wedge 1_M)^*} \ker d \cap [V' \wedge M, X \wedge M] \xrightarrow{(g\wedge 1_M)^*} \ker d \cap [V \wedge M, X \wedge M]$ is also exact, where d is the derivation defined on the corresponding group. Moreover, the result also holds in the dual form.*

Proposition 2.5 *Let $p \geq 5$ and V, V' be arbitrary spectra. Then there is a direct sum decomposition*

$$[\Sigma^* V \wedge M, V' \wedge K] = (\ker d) \cdot (1_V \wedge i') \oplus (\ker d) \cdot (1_V \wedge i'ij),$$

where $(\ker d) = (\ker d) \cap [\Sigma^* V \wedge K, V' \wedge K]$, the subgroup of $[\Sigma^* V \wedge K, V' \wedge K]$ consisting of the maps $f : \Sigma^* V \wedge K \to V' \wedge K$ such that $d(f) = 0$.

Proof For any $f \in [\Sigma^r V \wedge M, V' \wedge K]$, $f(1_V \wedge i) = (1_{V'} \wedge \mu(1_K \wedge i'i))f(1_V \wedge i) = (1_{V'} \wedge \mu)(f(1_V \wedge i) \wedge 1_K)(1_V \wedge i'i)$, where $\mu : K \wedge K \to K$ is the multiplication of the ring spectrum K satisfying $\mu(i'i \wedge 1_K) = 1_K = \mu(1_K \wedge i'i)$, then $f = (1_{V'} \wedge \mu)(f(1_V \wedge i) \wedge 1_K)(1_V \wedge i') + f_2(1_V \wedge j) = (1_{V'} \wedge \mu)(f(1_V \wedge i) \wedge 1_K)(1_V \wedge i') + (1_{V'} \wedge \mu)(f_2 \wedge 1_K)(1_V \wedge i'ij)$, and the result follows since $d(f_2 \wedge 1_K) = f_2 \wedge d(1_K) = 0$ and $d(1_{V'} \wedge \mu) = 1_{V'} \wedge d(\mu) = 0$ (cf. [8, p.437 Lemma 6.4(G)]).

Proposition 2.6 *Under the assumption I of Theorem A we have:*

(1) *$Ext_A^{s,tq}(H^*M, H^*M) \cong Z_p\{\tilde{\sigma}\}$ satisfying $i^*(\tilde{\sigma}) = i_*(\sigma) \in Ext_A^{s,tq}(H^*M, Z_p)$, $j_*(\tilde{\sigma}) = j^*(\sigma) \in Ext_A^{s,tq-1}(Z_p, H^*M)$ and $Ext_A^{s,tq+u}(H^*M, H^*M) = 0$ for $u = 1, 2$;*

(2) *$Ext_A^{s+1,tq}(H^*M, H^*M)$ is zero or has (one or two) generator $\tilde{\sigma}'$ such that $i^*(\tilde{\sigma}') = i_*(\sigma'), j^*(\tilde{\sigma}') = j_*(\sigma')$;*

(3) *$Ext_A^{s+1,tq+q+1}(H^*M, H^*M)$ has a unique generator $\alpha_*(\tilde{\sigma}) = (\alpha)^*(\tilde{\sigma})$, where α_* : $Ext_A^{s,tq}\ (H^*M, H^*M) \to Ext_A^{s+1,tq+q+1}(H^*M, H^*M)$ is the connecting homomorphism induced by $\alpha : \Sigma^q M \to M$;*

(4) *$Ext_A^{s+1,tq+q}(H^*M, H^*M) \cong Z_p\{(ij)_*\alpha_*(\tilde{\sigma}), (ij)^*\alpha_*(\tilde{\sigma})\}$.*

Proof (1) We first prove that $Ext_A^{s,tq+u}(H^*M, Z_p) = 0$ for $u = 1, 2, 3$. Consider the exact sequence ($u = 1, 2, 3$)

$$Ext_A^{s,tq+u}(Z_p, Z_p) \xrightarrow{i_*} Ext_A^{s,tq+u}(H^*M, Z_p) \xrightarrow{j_*} Ext_A^{s,tq+u-1}(Z_p, Z_p) \xrightarrow{p_*}$$

induced by (1.1). By the assumption I(c), the left group is zero for $u = 1, 2, 3$. The right group is zero for $u = 2, 3$ and has a unique generator σ for $u = 1$ which satisfies $p_*(\sigma) = a_0\sigma \neq 0$. Then im $j_* = 0$ and the middle group is zero for $u = 1, 2, 3$ and so $Ext_A^{s,tq+u}(H^*M, H^*M) = 0$ for $u = 1, 2$. Look at the exact sequence

$$\begin{aligned} 0 = {} & Ext_A^{s,tq+1}(H^*M, Z_p) \\ & \xrightarrow{j^*} Ext_A^{s,tq}(H^*M, H^*M) \xrightarrow{i^*} Ext_A^{s,tq}(H^*M, Z_p) \xrightarrow{p^*} \end{aligned}$$

induced by (1.1). The right group has a unique generator $i_*(\sigma)$ since $Ext_A^{s,tq-r}(Z_p, Z_p) = 0$ for $r = 1$, and has a unique generator σ for $r = 0$ by assumption I(c). Moreover, $p^*i_*(\sigma) = i_*p^*(\sigma) = i_*(a_0\sigma) = i_*p_*(\sigma) = 0$, then the middle group has a unique generator $\tilde{\sigma}$ such that $i^*(\tilde{\sigma}) = i_*(\sigma)$ as desired. The proof of the second relation is similar.

(2) The proof is similar to that given in (1) by replacing σ with σ'.

(3) Consider the exact sequence

$$\begin{aligned} Ext_A^{s+1,tq+q+2}(H^*M, Z_p) & \xrightarrow{j^*} Ext_A^{s+1,tq+q+1}(H^*M, H^*M) \\ & \xrightarrow{i^*} Ext_A^{s+1,tq+q+1}(H^*M, Z_p) \xrightarrow{p^*} \end{aligned}$$

induced by (1.1). The left group is zero since $Ext_A^{s+1,tq+q+r}(Z_p, Z_p) = 0$ for $r = 1, 2$ by assumption I(b). The right group has a unique generator $(\alpha i)_*(\sigma) = i^*\alpha_*(\tilde{\sigma})$ since $Ext_A^{s+1,tq+q+r}(Z_p, Z_p) = 0$ for $r = 1$, and has a unique generator $h_0\sigma = j_*(\alpha i)_*(\sigma)$ for $r = 0$. Moreover $p^*(\alpha i)_*(\sigma) = (\alpha i)_*p^*(\sigma) = (\alpha i)_*p_*(\sigma) = 0$, then the middle group $\cong Z_p\{\alpha_*(\tilde{\sigma})\}$ and $\alpha^*(\tilde{\sigma}) = \alpha_*(\tilde{\sigma})$ since $i^*j_*\alpha_*(\tilde{\sigma}) = j_*\alpha_*i_*(\sigma) = h_0\sigma = (j\alpha i)^*(\sigma) = i^*j_*\alpha^*(\tilde{\sigma})$.

(4) Consider the following exact sequence:

$$\begin{aligned} Ext_A^{s+1,tq+q+1}(H^*M, Z_p) & \xrightarrow{j^*} Ext_A^{s+1,tq+q}(H^*M, H^*M) \\ & \xrightarrow{i^*} Ext_A^{s+1,tq+q}(H^*M, Z_p) \xrightarrow{p^*} \end{aligned}$$

induced by (1.1). The left group has a unique generator $\alpha_*i_*(\sigma) = i^*\alpha_*(\tilde{\sigma})$ as shown in (3). The right group has a unique generator $i_*(h_0\sigma) = i^*(ij)_*\alpha_*(\tilde{\sigma})$ by assumption I(a). Moreover, $p^*i_*(h_0\sigma) = i_*p_*(h_0\sigma) = 0$, then the result follows.

Proposition 2.7 *Under the assumption* I *of Theorem* A *we have*:

(1) *$Ext_A^{s+1,tq+2q+u}(H^*K, H^*M) = 0$ for $u = 1, 2$, $Ext_A^{s+1,tq+2q+1}(H^*K, H^*K) = 0$;*

(2) *$Ext_A^{s+1,tq+u}(H^*M, Z_p) = 0$ for $u = 1, 2, 3$, $Ext_A^{s+1,tq+q+u}(H^*K, H^*M) = 0$ for $u = 1, 2$, $Ext_A^{s+1,tq+u}(H^*K, H^*M) = 0$ for $u = 1, 2$;*

(3) *$Ext_A^{s+1,tq+rq+1}(H^*K, H^*K) = 0$ for $r = -1, 0, 1$.*

Proof (1) Look at the following exact sequence $(u = 1, 2)$:

$$Ext_A^{s+1,tq+2q+u}(H^*M, H^*M) \xrightarrow{i'_*} Ext_A^{s+1,tq+2q+u}(H^*K, H^*M)$$
$$\xrightarrow{j'_*} Ext_A^{s+1,tq+q+u-1}(H^*M, H^*M) \xrightarrow{\alpha_*}$$

induced by (1.2). The left group is zero since $Ext_A^{s+1,tq+2q+k}(Z_p, Z_p) = 0$ for $k = 0, 1, 2, 3$ by assumption I(b). The right group has two generators $(ij)_*\alpha_*(\tilde{\sigma})$ and $\alpha_*(ij)_*(\tilde{\sigma})$ for $u = 1$ and has a unique generator $\alpha_*(\tilde{\sigma})$ for $u = 2$ (cf. Prop. 2.6). We claim that (i) $\alpha_*[\lambda_1(ij)_*\alpha_*(\tilde{\sigma}) + \lambda_2\alpha_*(ij)_*(\tilde{\sigma})] \neq 0$; (ii) $\alpha_*\alpha_*(\tilde{\sigma}) \neq 0$. Then the above α_* is monic and so im $j'_* = 0$. This shows that $Ext_A^{s+1,tq+2q+u}(H^*K, H^*M) = 0$ for $u = 1, 2$ and consequently we have $Ext_A^{s+1,tq+2q+1}$ $(H^*K, H^*K) = 0$ since $Ext_A^{s+1,tq+3q+2}(H^*K, H^*M) = 0$ by $Ext_A^{s+1,tq+uq+k}(Z_p, Z_p) = 0$ for $u = 2, 3,\ k = 0, 1, 2, 3$ in assumption I(b).

To prove the claim, we recall from the assumpition I(a) that $\widetilde{\alpha}_2\sigma = j_*\alpha_*\alpha_*i_*(\sigma) \neq 0 \in Ext_A^{s+2,tq+2q+1}(Z_p, Z_p)$, then $i_*(\widetilde{\alpha}_2\sigma) \neq 0 \in Ext_A^{s+2,tq+2q+1}(H^*M, Z_p)$ since $Ext_A^{s+1,tq+2q}(Z_p, Z_p) = 0$ (cf. assumption I(b)) and then $j^*i_*(\widetilde{\alpha}_2\sigma) \neq 0 \in Ext_A^{s+2,tq+2q}(H^*M, H^*M)$ since $Ext_A^{s+1,tq+2q}$ $(H^*M, Z_p) = 0$ by assumption I(b). Hence, by $2\alpha ij\alpha = ij\alpha^2 + \alpha^2 ij$ (cf. [8, p. 428 line 20]) we have $\alpha_*[\lambda_1(ij)_*\alpha_*(\tilde{\sigma}) + \lambda_2\alpha_*(ij)_*(\tilde{\sigma})] = \frac{1}{2}\lambda_1(ij)_*\alpha_*\alpha_*(\tilde{\sigma}) + (\frac{1}{2}\lambda_1 + \lambda_2)\alpha_*\alpha_*(ij)_*(\tilde{\sigma}) \neq 0$ since the two terms are linearly independent by $(j)_*\alpha_*\alpha_*(ij)_*(\tilde{\sigma}) = j^*(\widetilde{\alpha}_2\sigma) \neq 0$ and $i^*(ij)_*\alpha_*\alpha_*(\tilde{\sigma}) = j^*i_*(\widetilde{\alpha}_2\sigma) \neq 0$. This shows the claim (i). The claim (ii) follows by $j_*\alpha_*\alpha_*i^*(\tilde{\sigma}) = \widetilde{\alpha}_2\sigma \neq 0$.

(2) Consider the exact sequence $(u = 1, 2, 3)$

$$Ext_A^{s+1,tq+u}(Z_p, Z_p) \xrightarrow{i_*} Ext_A^{s+1,tq+u}(H^*M, Z_p) \xrightarrow{j_*} Ext_A^{s+1,tq+u-1}(Z_p, Z_p) \xrightarrow{p_*}$$

induced by (1.1). The right group is zero for $u = 3$ and has a unique generator $a_0\sigma$ for $u = 2$ which satisfies $p_*(a_0\sigma) = a_0^2\sigma \neq 0$. For $u = 1$, the right group is zero or has one (or two) generator σ' which satisfies $p_*(\sigma') = a_0\sigma' \neq 0$, then im $j_* = 0$. The left group is zero for $u = 2, 3$ and has a unique generator $a_0\sigma = p_*(\sigma)$ for $u = 1$ so that im $i_* = 0$. Then the middle group is zero for $u = 1, 2, 3$ and so $Ext_A^{s+1,tq+u}(H^*M, H^*M) = 0$ for $u = 1, 2$.

For the second result, look at the following exact sequence $(u = 1, 2, 3)$:

$$Ext_A^{s+1,tq+q+u}(H^*M, Z_p) \xrightarrow{i'_*} Ext_A^{s+1,tq+q+u}(H^*K, Z_p) \xrightarrow{j'_*} Ext_A^{s+1,tq+(u-1)}(H^*M, Z_p) \xrightarrow{\alpha_*}$$

induced by (1.2). The left group is zero for $u = 2, 3$ since $Ext_A^{s+1,tq+q+k}(Z_p, Z_p) = 0$ for $k = 1, 2, 3$ by assumption I(b) and has a unique generator $\alpha_*i_*(\sigma)$ for $u = 1$ (cf. Prop. 2.6(3)) so that im $i'_* = 0$. The right group is zero for $u = 2, 3$, as shown above. For $u = 1$, the right group is zero or has one (or two) generator $i_*(\sigma')$ (cf. assumption I(b)) which satisfies $\alpha_*i_*(\sigma') \neq 0 \in Ext_A^{s+1,tq+q+1}(H^*M, Z_p)$ by the assumption I(b) on $j_*(\alpha i)_*(\sigma') = h_0\sigma' \neq 0$. Then, im $j'_* = 0$ and so the middle group is zero for $u = 1, 2, 3$ and the result follows.

The third result follows by the following exact sequence $(u = 1, 2)$:

$$0 = Ext_A^{s+1,tq+u}(H^*M, H^*M) \xrightarrow{(i')_*} Ext_A^{s+1,tq+u}(H^*K, H^*M)$$
$$\xrightarrow{(j')_*} Ext_A^{s+1,tq-q+u-1}(H^*M, H^*M) = 0$$

induced by (1.2), where the left group is zero as shown above and the right group also is zero since $Ext_A^{s+1,tq-q+k}(Z_p, Z_p) = 0$ for $k = -1, 0, 1, 2$ by the assumption I(b).

(3) Consider the exact sequence ($r=-1,0,1$)

$$Ext_A^{s+1,tq+(r+1)q+2}(H^*K,H^*M)\xrightarrow{(j')^*}Ext_A^{s+1,tq+rq+1}(H^*K,H^*K)$$
$$\xrightarrow{(i')^*}Ext_A^{s+1,tq+rq+1}(H^*K,H^*M)$$

induced by (1.2). The left group is zero for $r=-1,0,1$ by (1)(2) and the right group is zero for $r=0,1$ by (2). For $r=-1$, it is also zero by $Ext_A^{s+1,tq-rq+k}(Z_p,Z_p)=0$ for $r=1,2$, $k=-1,0,1,2$ in the assumption. Then the middle group is zero.

Let K' be the cofibre of $jj':\Sigma^{-1}K\to\Sigma^{q+1}S$ given by the cofibration

$$\Sigma^{-1}K\xrightarrow{jj'}\Sigma^{q+1}S\xrightarrow{z}K'\xrightarrow{x}K. \tag{2.8}$$

As that in [6, (2.14)], K' is also the cofibre of $\alpha i:\Sigma^qS\to M$ given by the cofibration

$$\Sigma^qS\xrightarrow{\alpha i}M\xrightarrow{v}K'\xrightarrow{y}\Sigma^{q+1}S. \tag{2.9}$$

Let Y be the cofibre of $i'i:S\to K$ given by the cofibration

$$S\xrightarrow{i'i}K\xrightarrow{r}Y\xrightarrow{\epsilon}\Sigma S. \tag{2.10}$$

Then Y also is the cofibre of $j\alpha:\Sigma^qM\to\Sigma S$ given by the cofibration

$$\Sigma^qM\xrightarrow{j\alpha}\Sigma S\xrightarrow{\overline{w}}Y\xrightarrow{\overline{u}}\Sigma^{q+1}M. \tag{2.11}$$

This can be seen by the following homotopy commutative (up to sign) diagram of 3×3-Lemma in the stable homotopy category (cf. [7, pp. 292–293]):

$$\begin{array}{ccccccc}
S & \xrightarrow{i'i} & K & \xrightarrow{j'} & \Sigma^{q+1}M & & \\
& \searrow i \quad \nearrow i' & & \searrow r \quad \nearrow \overline{u} & & & \\
& M & & Y & & & \\
& \nearrow \alpha \quad \searrow j & & \nearrow \overline{w} \quad \searrow \epsilon & & & \\
\Sigma^qM & \xrightarrow{j\alpha} & \Sigma S & \xrightarrow{p} & \Sigma S. & &
\end{array} \tag{2.12}$$

Note that $d((1_Y\wedge i)r)=d((r\wedge 1_M)(1_K\wedge i))=(r\wedge 1_M)d(1_K\wedge i)=(r\wedge 1_M)(1_K\wedge m_M)(T_{K,M}\wedge 1_M)(1_M\wedge 1_K\wedge i)\overline{m}_K=(r\wedge 1_M)(1_K\wedge m_M(1_M\wedge i))\overline{m}_K=(r\wedge 1_M)\overline{m}_K$. Moreover, $(\overline{u}\cdot r\wedge 1_M)\overline{m}_K=(j'\wedge 1_M)\overline{m}_K=\lambda\overline{m}_M\cdot j'$ and by composing $(1_M\wedge j)$, we have $-\lambda j'=-j'(1_K\wedge j)\overline{m}_K=-j'$ so that $\lambda=1$ and we have $(j\overline{u}\cdot r\wedge 1_M)\overline{m}_K=j'$, that is,

$$d((1_Y\wedge i)r)=(r\wedge 1_M)\overline{m}_K,\qquad (j\overline{u}\cdot r\wedge 1_M)\overline{m}_K=j'. \tag{2.13}$$

Moreover, the cofibre of $j\overline{u}:\Sigma^{-2}Y\to\Sigma^qS$ is L given by the cofibration

$$\Sigma^{-2}Y\xrightarrow{j\overline{u}}\Sigma^qS\xrightarrow{\pi}L\xrightarrow{\bar{h}}\Sigma^{-1}Y. \tag{2.14}$$

This can be seen by the following commutative diagram of 3×3-Lemma using (1.1), (1.3) :

$$\begin{array}{ccccc}
\Sigma^{-2}Y & \xrightarrow{j\overline{u}} & \Sigma^qS & \xrightarrow{p} & \Sigma^qS \\
\searrow \overline{u} \quad \nearrow j & & \searrow \pi \quad \nearrow j'' & & \searrow i \\
\Sigma^{q-1}M & & L & & \Sigma^qM \\
\nearrow i \quad \searrow j\alpha & & \nearrow i'' \quad \searrow \bar{h} & & \nearrow \overline{u} \\
\Sigma^{q-1}S & \xrightarrow{\alpha_1} & S & \xrightarrow{\overline{w}} & \Sigma^{-1}Y.
\end{array} \tag{2.15}$$

From the diagram (2.12), we have $\epsilon \cdot \overline{w} = p$ (up to sign), then one can easily prove that $\overline{w} \cdot \epsilon = (1_Y \wedge p)$, and by the following commutative diagram of 3×3-Lemma:

$$\begin{array}{ccccccc}
\Sigma^q M & \xrightarrow{\alpha' i'} & & \Sigma K & & \xrightarrow{r} & \Sigma Y \\
& \searrow j\alpha & \nearrow i'i & & \searrow^{(r\wedge 1_M)\overline{m}_K} & \nearrow_{1_Y \wedge j} & \\
& \Sigma S & & & Y \wedge M & & \\
& \nearrow \epsilon & \searrow \overline{w} & & \nearrow_{1_Y \wedge i} & \searrow^{m_M(\overline{u}\wedge 1_M)} & \\
Y & \xrightarrow{1_Y \wedge p} & & Y & & \xrightarrow{\overline{u}} & \Sigma^{q+1} M,
\end{array}$$

we know that the cofibre of $\alpha' i' : \Sigma^q M \to \Sigma K$ is $Y \wedge M$ given by the cofibration

$$\Sigma^q M \xrightarrow{\alpha' i'} \Sigma K \xrightarrow{(r\wedge 1_M)\overline{m}_K} Y \wedge M \xrightarrow{m_M(\overline{u}\wedge 1_M)} \Sigma^{q+1} M. \tag{2.16}$$

Let W be the cofibre of $\phi : \Sigma^{2q-1} S \to L$ (*cf.* (1.4)) given by the cofibration

$$\Sigma^{2q-1} S \xrightarrow{\phi} L \xrightarrow{w} W \xrightarrow{j''u} \Sigma^{2q} S. \tag{2.17}$$

Then, W also is the cofibre of $(\alpha_1)_L : \Sigma^{q-1} L \to S$ given by the cofibration

$$\Sigma^{q-1} L \xrightarrow{(\alpha_1)_L} S \xrightarrow{wi''} W \xrightarrow{u} \Sigma^q L. \tag{2.18}$$

This can be seen by the following homotopy commutative diagram of 3×3-Lemma:

$$\begin{array}{ccccccc}
\Sigma^{2q-1} S & \xrightarrow{\alpha_1} & & \Sigma^q S & & \xrightarrow{\alpha_1} & \Sigma S \\
& \searrow \phi & \nearrow j'' & & \searrow i'' & \nearrow (\alpha_1)_L & \\
& L & & & \Sigma^q L & & \\
& \nearrow i'' & \searrow w & & \nearrow u & \searrow j'' & \\
S & \xrightarrow{wi''} & & W & & \xrightarrow{j''u} & \Sigma^{2q} S.
\end{array}$$

Now we consider the ring spectrum properties of K. From [8, p. 433], there is a homotopy equivalence $K \wedge K = K \vee \Sigma L \wedge K \vee \Sigma^{q+2} K$ and there are maps

$$\begin{aligned}
&\mu : K \wedge K \to K, \\
&\mu_2 : K \wedge K \to \Sigma L \wedge K, \\
&jj' \wedge 1_K : K \wedge K \to \Sigma^{q+2} K, \\
&i'i \wedge 1_K : K \to K \wedge K, \\
&\nu_2 : \Sigma L \wedge K \to K \wedge K, \\
&\nu : \Sigma^{q+2} K \to K \wedge K,
\end{aligned} \tag{2.19}$$

such that $\mu(i'i \wedge 1_K) = 1_K = \mu(1_K \wedge i'i)$, $(jj' \wedge 1_K)\nu = 1_K$, $(i'i \wedge 1_K)\mu + \nu_2\mu_2 + (jj' \wedge 1_K)\nu = 1_{K\wedge K}$ and $\mu_2(i'i \wedge 1_K) = 0$. Then, by (2.10), there is $\overline{\mu}_2 \in [Y \wedge K, \Sigma L \wedge K]$ such that

$$\overline{\mu}_2(r \wedge 1_K) = \mu_2 \in [K \wedge K, \Sigma L \wedge K]. \tag{2.20}$$

By (1.1), (1.3), (2.9) and a commutative diagram of 3×3-Lemma, we know that the cofibre of $vi : S \to K'$ is ΣL given by the cofibration

$$S \xrightarrow{vi} K' \xrightarrow{k} \Sigma L \xrightarrow{\xi} \Sigma S \tag{2.21}$$

with the relation that $\xi \cdot i'' = p$ so that $\xi i'' \wedge 1_K = p \wedge 1_K = 0$ and so $\xi \wedge 1_K \in (j'' \wedge 1_K)^*[\Sigma^q K, K] = 0$. Hence, the cofibration (2.21) induces a split cofibration $K \xrightarrow{vi\wedge 1_K} K' \wedge K \xrightarrow{k \wedge 1_K} \Sigma L \wedge K$ and there is a reverse split cofibration : $\Sigma L \wedge K \xrightarrow{\nu_2'} K' \wedge K \xrightarrow{\mu(x\wedge 1_K)} K$. Moreover, $x(1_{K'}\wedge\epsilon) = (1_K\wedge\epsilon)(x\wedge 1_Y) = 0 \in [\Sigma^{-1}K'\wedge Y, K]$, then, by (2.8), $1_{K'}\wedge\epsilon = z\cdot\widetilde{\nu}$ with $\widetilde{\nu}\in[K'\wedge Y, \Sigma^{q+2}S]$. Then, by the following diagram of 3×3-Lemma:

$$\begin{array}{ccccccc}
K'\wedge Y & & \xrightarrow{1_{K'}\wedge\epsilon} & \Sigma K' & \xrightarrow{x} & & \Sigma K \\
 & \searrow\widetilde{\nu} & \nearrow z & & \searrow^{1_{K'}\wedge i'i} & \nearrow^{\mu(x\wedge 1_K)} & \\
 & \Sigma^{q+2}S & & & \Sigma K'\wedge K & & \\
 & \nearrow jj' & \searrow 0 & & \nearrow \nu_2' & \searrow^{1_{K'}\wedge r} & \\
K & & \xrightarrow{0} & \Sigma^2 L\wedge K & \xrightarrow{\tilde{\nu}_2} & & \Sigma K'\wedge Y.
\end{array}$$

We have a split cofibration $\Sigma L\wedge K \xrightarrow{\tilde{\nu}_2} K'\wedge Y \xrightarrow{\tilde{\nu}} \Sigma^{q+2}S$ and so there is $\widetilde{\tau}:\Sigma^{q+2}S\to K'\wedge Y, \widetilde{\mu}_2: K'\wedge Y\to\Sigma L\wedge K$ such that

$$\widetilde{\nu}\cdot\widetilde{\tau} = 1_S,\quad \widetilde{\mu}_2\widetilde{\nu}_2 = 1_{L\wedge K},\quad \widetilde{\tau}\cdot\widetilde{\nu}+\widetilde{\nu}_2\widetilde{\mu}_2 = 1_{K'\wedge Y}. \tag{2.22}$$

Let U be the cofibre of $\bar{h}\phi:\Sigma^{2q-1}S\to\Sigma^{-1}Y$ given by the cofibration

$$\Sigma^{2q-1}S \xrightarrow{\bar{h}\phi} \Sigma^{-1}Y \xrightarrow{w_2} U \xrightarrow{u_2} \Sigma^{2q}S. \tag{2.23}$$

Since $j''\phi\cdot p = \alpha_1\cdot p = 0$, then $\phi\cdot p = i''j\alpha^2 i$ up to scalar since $\pi_{2q-1}S\cong Z_{(p)}\{j\alpha^2 i\}$. Then we have $\bar{h}\phi\cdot p = 0$ and so there exist $\widetilde{w}\in[\Sigma^{2q}S, U]$ and $\alpha_{Y\wedge M}\in[\Sigma^{2q+1}M, Y\wedge M]$ such that $u_2\widetilde{w} = p, (j\wedge 1_Y)\alpha_{M\wedge Y}i = \bar{h}\phi$.

Let X be the cofibre of $\widetilde{w}:\Sigma^{2q}S\to U$ given by the cofibration

$$\Sigma^{2q}S \xrightarrow{\tilde{w}} U \xrightarrow{\tilde{u}} X \xrightarrow{j\tilde{\psi}} \Sigma^{2q+1}S. \tag{2.24}$$

Then the cofibre of $\widetilde{u}w_2:\Sigma^{-1}Y\to X$ is $\Sigma^{2q}M$ given by the cofibration

$$\Sigma^{-1}Y \xrightarrow{\tilde{u}w_2} X \xrightarrow{\tilde{\psi}} \Sigma^{2q}M \xrightarrow{(1_Y\wedge j)\alpha_{Y\wedge M}} Y. \tag{2.25}$$

This can be seen by the following homotopy commutative diagram of 3×3-Lemma:

$$\begin{array}{ccccccc}
\Sigma^{-1}Y & & \xrightarrow{\tilde{u}w_2} & X & \xrightarrow{j\tilde{\psi}} & & \Sigma^{2q+1}S \\
 & \searrow w_2 & \nearrow \widetilde{u} & & \searrow\widetilde{\psi} & \nearrow j & \\
 & U & & & \Sigma^{2q}M & & \\
 & \nearrow\widetilde{w} & \searrow u_2 & & \nearrow i & \searrow_{(1_Y\wedge j)\alpha_{Y\wedge M}} & \\
\Sigma^{2q}S & & \xrightarrow{p} & \Sigma^{2q}S & \phi\xrightarrow{\bar{h}} & & Y.
\end{array} \tag{2.26}$$

The cofibre of $w\pi:\Sigma^q S\to W$ is U given by the cofibration

$$\Sigma^q S \xrightarrow{w\pi} W \xrightarrow{w_3} U \xrightarrow{u_3} \Sigma^{q+1}S. \tag{2.27}$$

This can be seen by the following diagram of 3×3-Lemma using (2.14), (2.17), (2.23):

$$\begin{array}{ccccccc}
\Sigma^q S & & \xrightarrow{w\pi} & W & \xrightarrow{j''u} & & \Sigma^{2q}S \\
 & \searrow\pi & \nearrow w & & \searrow w_3 & \nearrow u_2 & \\
 & L & & & U & & \\
 & \nearrow\phi & \searrow\bar{h} & & \nearrow w_2 & \searrow u_3 & \\
\Sigma^{2q-1}S & & \xrightarrow{\bar{h}\phi} & \Sigma^{-1}Y & \xrightarrow{j\overline{u}} & & \Sigma^{q+1}S.
\end{array} \tag{2.28}$$

By the following homotopy commutative diagram of 3×3-Lemma using $u_3\cdot\widetilde{w}=\alpha_1$:

$$\begin{array}{ccccccc} W & \overset{\tilde{u}w_3}{\longrightarrow} & X & \overset{j\tilde{\psi}}{\longrightarrow} & \Sigma^{2q+1}S & & \\ & \searrow w_3 \quad \nearrow \widetilde{u} & & \searrow u'' \quad \nearrow j'' & & & \\ & U & & \Sigma^{q+1}L & & & \\ & \nearrow \widetilde{w} \quad \searrow u_3 & & \nearrow i'' \quad \searrow w'(\pi\wedge 1_L) & & & \\ \Sigma^{2q}S & \overset{\alpha_1}{\longrightarrow} & \Sigma^{q+1}S & \overset{w\pi}{\longrightarrow} & \Sigma W, & & \end{array} \tag{2.29}$$

we know that the cofibre of $\widetilde{u}w_3 : W\to X$ is $\Sigma^{q+1}L$ given by the cofibration

$$W\overset{\tilde{u}w_3}{\longrightarrow} X\overset{u''}{\longrightarrow}\Sigma^{q+1}L\overset{w'(\pi\wedge 1_L)}{\longrightarrow}\Sigma W \tag{2.30}$$

with $w'\in[L\wedge L, W]$ such that $w'(1_L\wedge i'')=w$.

By $m_M(\overline{u}\wedge 1_M)\alpha_{Y\wedge M}=\alpha$, (2.16), (1.2), (1.3) and a diagram of 3×3-Lemma we know that the cofibre of $\alpha_{Y\wedge M}:\Sigma^{2q+1}M\to Y\wedge M$ is $\Sigma L\wedge K$ given by the cofibration

$$\Sigma^{2q+1}M\overset{\alpha_{Y\wedge M}}{\longrightarrow} Y\wedge M\overset{\overline{\mu}_2(1_Y\wedge i')}{\longrightarrow}\Sigma L\wedge K\overset{j'(j''\wedge 1_K)}{\longrightarrow}\Sigma^{2q+2}M. \tag{2.31}$$

Since $((1_Y\wedge j)\alpha_{Y\wedge M}\wedge 1_M)\overline{m}_M=\alpha_{Y\wedge M}$, then the cofibre of $m_M(\widetilde{\psi}\wedge 1_M):X\wedge M\to\Sigma^{2q}M$ is $\Sigma L\wedge K$ given by the cofibration

$$X\wedge M\overset{m_M(\tilde{\psi}\wedge 1_M)}{\longrightarrow}\Sigma^{2q}M\overset{(\phi\wedge 1_K)i'}{\longrightarrow}\Sigma L\wedge K\overset{u'}{\longrightarrow}\Sigma X\wedge M. \tag{2.32}$$

This can be seen by the following commutative diagram of 3×3-Lemma using (2.31), (2.25):

$$\begin{array}{ccccc} X\wedge M & \overset{m_M(\tilde{\psi}\wedge 1_M)}{\longrightarrow} & \Sigma^{2q}M & \overset{0}{\longrightarrow} & \Sigma^{2q+2}M \\ & \searrow \widetilde{\psi}\wedge 1_M \nearrow m_M & & \searrow (\phi\wedge 1_K)i' \nearrow j'(j''\wedge 1_K) & \\ & \Sigma^{2q}M\wedge M & & \Sigma L\wedge K & \\ & \nearrow \overline{m}_M \quad \searrow^{(1_Y\wedge j)\alpha_{Y\wedge 1_M}\wedge 1_M} & & \nearrow^{\overline{\mu}_2(1_Y\wedge i')} \quad \searrow u' & \\ \Sigma^{2q+1}M & \overset{\alpha_{Y\wedge M}}{\longrightarrow} & Y\wedge M & \overset{\tilde{u}w_2\wedge 1_M}{\longrightarrow} & \Sigma X\wedge M. \end{array} \tag{2.33}$$

Since $(\phi\wedge 1_K)i'\alpha=0$, then, by (2.32), there is $\alpha_{X\wedge M}\in[\Sigma^{3q}M, X\wedge M]$ such that $m_M(\widetilde{\psi}\wedge 1_M)\alpha_{X\wedge M}=\alpha$. Moreover, $m_M(\widetilde{\psi}\wedge 1_M)\alpha_{X\wedge M}m_M(\widetilde{\psi}\wedge 1_M)=\alpha m_M(\widetilde{\psi}\wedge 1_M)=m_M(\widetilde{\psi}\wedge 1_M)(1_X\wedge\alpha)$ and so, by (2.32), we have $\alpha_{X\wedge M}m_M(\widetilde{\psi}\wedge 1_M)=1_X\wedge\alpha$ modulo $(u')_*[\Sigma^q X\wedge M, L\wedge K]=0$ since $[\Sigma^q L\wedge K, L\wedge K]=0$ and $[\Sigma^{3q}M, L\wedge K]=0$. In addition, $(1_L\wedge i')(\phi\wedge 1_M)m_M(\widetilde{\psi}\wedge 1_M)=0$, then up to nonzero scalar we have $(\phi\wedge 1_M)m_M(\widetilde{\psi}\wedge 1_M)=(1_L\wedge\alpha)(u''\wedge 1_M)$ since $[\Sigma^{-q-1}X\wedge M, L\wedge M]\cap(\ker d)\cong Z_p\{u''\wedge 1_M\}$. That is,

$$\alpha_{X\wedge M}m_M(\widetilde{\psi}\wedge 1_M)=1_X\wedge\alpha,\quad (\phi\wedge 1_M)m_M(\widetilde{\psi}\wedge 1_M)=(u''\wedge 1_M)(1_X\wedge\alpha). \tag{2.34}$$

We claim that the cofibre of $\alpha_{X\wedge M}:\Sigma^{3q}M\to X\wedge M$ is $W\wedge K$ given by the cofibration

$$\Sigma^{3q}M\overset{\alpha_{X\wedge M}}{\longrightarrow} X\wedge M\overset{\mu_{X\wedge M}}{\longrightarrow} W\wedge K\overset{j'(j''u\wedge 1_K)}{\longrightarrow}\Sigma^{3q+1}M. \tag{2.35}$$

This can be seen by the following homotopy commutative diagram of 3×3-Lemma:

$$\begin{array}{ccccc} \Sigma^{3q}M & \overset{\alpha}{\longrightarrow} & \Sigma^{2q}M & \overset{(\phi\wedge 1_K)i'}{\longrightarrow} & \Sigma L\wedge K \\ & \searrow \alpha_{X\wedge M} \nearrow_{m_M(\tilde{\psi}\wedge 1_M)} & & \searrow i' \quad \nearrow (\phi\wedge 1_K) & \\ & X\wedge M & & \Sigma^{2q}K & \\ & \nearrow u' \quad \searrow \mu_{X\wedge M} & & \nearrow j''u\wedge 1_K \searrow j' & \\ L\wedge K & \overset{w\wedge 1_K}{\longrightarrow} & K\wedge W & \overset{j'(j''u\wedge 1_K)}{\longrightarrow} & \Sigma^{3q+1}M. \end{array}$$

By (2.26), $ijm_M(\widetilde{\psi}\cdot\widetilde{u}\wedge 1_M) = ij(u_2\wedge 1_M) = (u_2\wedge 1_M)(1_U\wedge ij) = m_M(\widetilde{\psi}\cdot\widetilde{u}\wedge 1_M)(1_U\wedge ij) = m_M(\widetilde{\psi}\wedge 1_M)(1_X\wedge ij)(\widetilde{u}\wedge 1_M)$, then by (2.24), we have $ijm_M(\widetilde{\psi}\wedge 1_M) = m_M(\widetilde{\psi}\wedge 1_M)(1_X\wedge ij) + \lambda(j\widetilde{\psi}\wedge 1_M)$ for some $\lambda\in Z_p$. It follows that $\lambda j(j\widetilde{\psi}\wedge 1_M) = -jm_M(\widetilde{\psi}\wedge 1_M)(1_X\wedge ij) = -j\widetilde{\psi}(1_X\wedge j) = j(j\widetilde{\psi}\wedge 1_M)$, so that $\lambda = 1$. In addition, $i'(\alpha_1\wedge 1_M)m_M(\widetilde{\psi}\wedge 1_M) = (j''\wedge 1_K)(1_L\wedge i')(\phi\wedge 1_M)m_M(\widetilde{\psi}\wedge 1_M) = 0$, then by (2.16), $m_M(\widetilde{\psi}\wedge 1_M) = m_M(\overline{u}\wedge 1_M)\psi_{X\wedge M}$ with $\psi_{X\wedge M}\in[\Sigma^{-q+1}X\wedge M, Y\wedge M]$. Moreover, $[\Sigma^{-q+1}X\wedge M, Y\wedge M]\cong Z_p\{\psi_{X\wedge M}\}$ by $[\Sigma^{-2q}X\wedge M, M]\cong Z_p\{m_M(\widetilde{\psi}\wedge 1_M)\}$, (2.16) and $[\Sigma^{-q}X\wedge M, K] = 0$. Then, by $j'(j''u\wedge 1_K)\cdot\mu_{X\wedge M} = 0$ and (2.31) we have $(u\wedge 1_K)\mu_{X\wedge M} = \overline{\mu}_2(1_Y\wedge i')\psi_{X\wedge M}$ up to nonzero scalar. In conclusions, we have

$$
\begin{aligned}
&j\widetilde{\psi}\wedge 1_M = ijm_M(\widetilde{\psi}\wedge 1_M) - m_M(\widetilde{\psi}\wedge 1_M)(1_X\wedge ij),\\
&(u\wedge 1_K)\mu_{X\wedge M} = \overline{\mu}_2(1_Y\wedge i')\psi_{X\wedge M} \text{ up to nonzero scalar},\\
&[\Sigma^{-q+1}X\wedge M, Y\wedge M]\cong Z_p\{\psi_{X\wedge M}\},\quad m_M(\overline{u}\wedge 1_M)\psi_{X\wedge M} = m_M(\widetilde{\psi}\wedge 1_M).
\end{aligned}
\tag{2.36}
$$

By the following homotopy commutative diagram of 3×3-Lemma:

$$
\begin{array}{ccccccc}
L\wedge K & \xrightarrow{(1_X\wedge j)u'} & \Sigma X & \xrightarrow{1_X\wedge p} & \Sigma X & & \\
& \searrow u' & & \nearrow 1_X\wedge j & & \searrow \omega & & \nearrow \widetilde{u}w_2 \\
& & X\wedge M & & & & Y & \\
& \nearrow 1_X\wedge i & & \searrow^{m_M(\tilde{\psi}\wedge 1_M)} & & \nearrow_{(1_Y\wedge j)\alpha_{Y\wedge M}} & & \searrow^{\overline{\mu}_2(1_Y\wedge i'i)} \\
X & \xrightarrow{\tilde{\psi}} & \Sigma^{2q}M & \xrightarrow{(\phi\wedge 1_K)i'} & \Sigma L\wedge K, & &
\end{array}
$$

we know that the cofibre of $(1_X\wedge j)u' : L\wedge K\to\Sigma X$ is Y given by the cofibration

$$
L\wedge K \xrightarrow{(1_X\wedge j)u'} \Sigma X \xrightarrow{\omega} Y \xrightarrow{\overline{\mu}_2(1_Y\wedge i'i)} \Sigma L\wedge K. \tag{2.37}
$$

Moreover, by the commutativity in the above rectangle, we have

$$
\omega\wedge 1_M = \alpha_{Y\wedge M}m_M(\widetilde{\psi}\wedge 1_M). \tag{2.38}
$$

Proposition 2.39 *Under the assumption* I *of Theorem* A *we have*:

(1) $Ext_A^{s+1,tq+q+1}(H^*W\wedge K, H^*X\wedge M) = 0$;

(2) $Ext_A^{s+1,tq+2q+1}(H^*Y, H^*M)\cong Z_p\{((1_Y\wedge j)\alpha_{Y\wedge M})_*(\tilde{\sigma})\}$, $Ext_A^{s+1,tq+q}(H^*Y, H^*Y)\cong Z_p\{(\overline{u})^*((1_Y\wedge j)\alpha_{Y\wedge M})_*(\tilde{\sigma})\}$;

(3) $Ext_A^{s+1,tq+3q}(H^*X, H^*M)\cong Z_p\{((1_X\wedge j)\alpha_{X\wedge M})_*(\tilde{\sigma})\}$.

Proof (1) Consider the exact sequence

$$
\begin{aligned}
0 = Ext_A^{s+1,tq+3q+1}(H^*W\wedge K, H^*M) &\xrightarrow{m_M(\tilde{\psi}\wedge 1_M)^*} Ext_A^{s+1,tq+q+1}(H^*W\wedge K, H^*X\wedge M)\\
&\xrightarrow{(u')^*} Ext_A^{s+1,tq+q+1}(H^*W\wedge K, H^*L\wedge K)
\end{aligned}
$$

induced by (2.32). The right group is zero by $Ext_A^{s+1,tq+rq+1}(H^*K, H^*K) = 0$ for $r = -1, 0, 1, 2$ (cf. Prop. 2.7) and (2.17), (1.3). The left group also is zero by $Ext_A^{s+1,tq+rq+1}(H^*K, H^*M) = 0$ for $r = 1, 2, 3$ (cf. Prop. 2.7), then the middle group is zero, as desired.

(2) Since $\overline{u}(1_Y\wedge j)\alpha_{Y\wedge M}\in[\Sigma^{q-1}M, M]\cong Z_p\{ij\alpha, \alpha ij\}$, then $\overline{u}(1_Y\wedge j)\alpha_{Y\wedge M} = \lambda_1 ij\alpha + \lambda_1\alpha ij$ with the scalar $\lambda_1, \lambda_2\in Z_p$ so that $\lambda_1 j\alpha ij\alpha + \lambda_2 j\alpha^2 ij = 0$. Consider the exact sequence

$$
Ext_A^{s+1,tq+2q}(Z_p, H^*M) \xrightarrow{(\overline{w})_*} Ext_A^{s+1,tq+2q+1}(H^*Y, H^*M)
$$

$$\xrightarrow{(\overline{u})_*} Ext_A^{s+1,tq+q}(H^*M, H^*M) \xrightarrow{(j\alpha)_*}$$

induced by (2.11). The left group is zero since $Ext_A^{s+1,tq+2q+k}(Z_p, Z_p) = 0$ for $k = 0, 1$ in the assumption I(b). The right group has two generators $(ij)_*\alpha_*(\tilde{\sigma})$ and $\alpha_*(ij)_*(\tilde{\sigma})$ by Prop. 2.6(4). Then $(\overline{u})_* Ext_A^{s+1,tq+2q+1}(H^*Y \wedge H^*M)$ has a unique generator $(\overline{u})_*((1_Y \wedge j)\alpha_{Y\wedge M})_*(\tilde{\sigma})$ and the first result follows. For the second result, look at the exact sequence

$$Ext_A^{s+1,tq+q}(Z_p, Z_p) \xrightarrow{(\overline{w})_*} Ext_A^{s+1,tq+q+1}(H^*Y, Z_p) \xrightarrow{(\overline{u})_*} Ext_A^{s+1,tq}(H^*M, Z_p) \xrightarrow{(j\alpha)_*}$$

induced by (2.11). The left group has a unique generator $h_0\sigma = (j\alpha i)_*(\sigma)$ so that im $(\overline{w})_* = 0$. The right group is zero or has one (or two) generator $i_*(\sigma')$ which satisfies $(j\alpha)_* i_*(\sigma') = h_0\sigma' \neq 0$. Then the middle group is zero and the second result follows.

(3) Since $\widetilde{\psi}(1_X \wedge j)\alpha_{X\wedge M} \in [\Sigma^{q-1}M, M] \cong Z_p\{ij\alpha, \alpha ij\}$, then $\widetilde{\psi}(1_X \wedge j)\alpha_{X\wedge M} = \lambda_3 ij\alpha + \lambda_4 \alpha ij$ with the scalar $\lambda_3, \lambda_4 \in Z_p$ so that $\lambda_3(1_Y \wedge j)\alpha_{Y\wedge M} ij\alpha + \lambda_4(1_Y \wedge j)\alpha_{Y\wedge M}\alpha ij = 0$. Hence, similarly to that in (2), $(\widetilde{\psi})_* Ext_A^{s+1,tq+3q}(H^*X, H^*M)$ has a unique generator $(\widetilde{\psi})_*((1_X \wedge j)\alpha_{X\wedge M})_*(\tilde{\sigma})$ so that $Ext_A^{s+1,tq+3q}(H^*X, H^*M)$ has a unique generator $((1_X \wedge j)\alpha_{X\wedge M})_*(\tilde{\sigma})$ since $Ext_A^{s+1,tq+3q+1}(H^*Y, H^*M) = 0$ by $Ext_A^{s+1,tq+rq+k}(Z_p, Z_p) = 0$ for $r = 1, 2, k = -1, 0, 1, 2$ in the assumption I(b).

Proposition 2.40 *Under the assumption I of Theorem* A, *we have*:

(1) $Ext_A^{s,tq-2q}(H^*M, H^*X \wedge M) \cong Z_p\{m_M(\widetilde{\psi} \wedge 1_M)^*(\tilde{\sigma})\}$;

(2) $Ext_A^{s,tq+rq+u}(H^*K, H^*M) = 0$ *for* $r = -1, 1, 2, 3, u = 0, 1, 2$, $Ext_A^{s,tq}(H^*K, H^*K) \cong Z_p\{\sigma_K\}$ *satisfying* $(i')^*(\sigma_K) = (i')_*(\tilde{\sigma})$;

(3) $Ext_A^{s,tq}(H^*L\wedge K, H^*L\wedge K) \cong Z_p\{\sigma_{L\wedge K}\}$ *satisfying* $(j'' \wedge 1_K)_*(\sigma_{L\wedge K}) = (j'' \wedge 1_K)^*(\sigma_K)$ $Ext_A^{s,tq+rq+u}(H^*L \wedge K, H^*M) = 0$ *for* $r = 1, 2, 3, u = 0, 1, 2$;

(4) $Ext_A^{s,tq+rq+u}(H^*W \wedge K, H^*M) = 0$ *for* $r = 1, 2, 3, u = 0, 1, 2$, $Ext_A^{s,tq+q}(H^*W \wedge K, H^*X \wedge M) = 0$.

Proof (1) Consider the exact sequence

$$Ext_A^{s,tq}(H^*M, H^*M \wedge M) \xrightarrow{(\tilde{\psi}\wedge 1_M)^*} Ext_A^{s,tq-2q}(H^*M, H^*X \wedge M)$$
$$\xrightarrow{(\tilde{u}w_2\wedge 1_M)^*} Ext_A^{s,tq-2q}(H^*M, H^*Y \wedge M)$$

induced by (2.25). Since $Ext_A^{s,tq-rq+u}(Z_p, Z_p) = 0$ for $r = 1, 2$ and $u = 0, 1, 2$ by the supposition and the degree of the top cell of $Y \wedge M$ being $q + 3$, then the right group is zero. Since $(\overline{m}_M)^* Ext_A^{s,tq}(H^*M, H^*M \wedge M) \subset Ext_A^{s,tq+1}(H^*M, H^*M) = 0$ (cf. Prop. 2.6(1)), then the left group has a unique generator $(m_M)^*(\tilde{\sigma})$ and the result follows.

(2) Consider the exact sequence ($r = -1, 1, 2, 3, u = 0, 1, 2$)

$$Ext_A^{s,tq+rq+u}(H^*M, H^*M) \xrightarrow{(i')_*} Ext_A^{s,tq+rq+u}(H^*K, H^*M)$$
$$\xrightarrow{(j')_*} Ext_A^{s,tq+(r-1)q+u-1}(H^*M, H^*M) \xrightarrow{\alpha_*}$$

induced by (1.2). The left group is zero for $r = -1, 1, 2, 3, u = 0, 1, 2$ since $Ext_A^{s,tq+rq+k}(Z_p, Z_p) = 0$ for $r = -1, 1, 2, 3, k = -1, 0, 1, 2, 3$ in the supposition I(c). The right group is also zero by the supposition and Prop. 2.6(1) unless it has a unique generator $(ij)_*(\tilde{\sigma})$ and $\tilde{\sigma}$ for $r = 1, u = 0, 1$, respectively. However, it satisfies $\alpha_*(ij)_*(\tilde{\sigma}) \neq 0$, $\alpha_*(\tilde{\sigma}) \neq 0$ by Prop. 2.6, then the middle group is zero as desired. Look at the exact sequence

$$0 = Ext_A^{s,tq+q+1}(H^*K, H^*M) \xrightarrow{(j')^*} Ext_A^{s,tq}(H^*K, H^*K)$$

$$\xrightarrow{(i')^*} Ext_A^{s,tq}(H^*K, H^*M) \xrightarrow{\alpha^*}$$

induced by (1.2). The left group is zero as shown above. The right group has a unique generator $(i')_*(\tilde{\sigma})$ since $(j')_* Ext_A^{s,tq}(H^*K, H^*M) \subset Ext_A^{s,tq-q-1}(H^*M, H^*M) = 0$ by the supposition and $Ext_A^{s,tq}(H^*M, H^*M) \cong Z_p\{\tilde{\sigma}\}$. Then the middle group has a unique σ_K as desired.

(3) Consider the exact sequence $(r = -1, 0)$

$$Ext_A^{s,tq+(r+1)q}(H^*K, H^*K) \xrightarrow{(j''\wedge 1_K)^*} Ext_A^{s,tq+rq}(H^*K, H^*L\wedge K)$$
$$\xrightarrow{(i''\wedge 1_K)^*} Ext_A^{s,tq+rq}(H^*K, H^*K) \xrightarrow{(\alpha_1\wedge 1_K)^*}$$

induced by (1.3). The left group is zero for $r = 0$ since $(i')^* Ext_A^{s,tq+q}(H^*K, H^*K) \subset Ext_A^{s,tq+q}(H^*K, H^*M) = 0$ and $Ext_A^{s,tq+2q+1}(H^*K, H^*M) = 0$ by (2). The left group has a unique generator σ_K for $r = -1$ by (2). The right group is zero for $r = -1$ since $(i')^* Ext_A^{s,tq-q}(H^*K, H^*K) \subset Ext_A^{s,tq-q}(H^*K, H^*M) = 0$ by (2) and $Ext_A^{s,tq+1}(H^*K, H^*M) = 0$ by Prop. 2.6(1) and the supposition. The right group has a unique generator σ_K for $r = 0$ which satisfies $(\alpha_1 \wedge 1_K)^*(\sigma_K) \neq 0 \in Ext_A^{s+1,tq+q}(H^*K, H^*K)$ since $(i')^*(\alpha_1 \wedge 1_K)^*(\sigma_K) = (\alpha_1\wedge 1_M)^*(i')^*(\sigma_K) = (\alpha_1\wedge 1_M)^*(i')_*(\tilde{\sigma}) = (i')_*(\alpha_1\wedge 1_M)_*(\tilde{\sigma}) \neq 0 \in Ext_A^{s+1,tq+q}(H^*K, H^*M)$. Then the middle group is zero for $r = 0$, has a unique generator $(j''\wedge 1_K)^*(\sigma_K)$ for $r = -1$ and the first result follows by the exact sequence:

$$0 = Ext_A^{s,tq}(H^*K, H^*L\wedge K) \xrightarrow{(i''\wedge 1_K)_*} Ext_A^{s,tq}(H^*L\wedge K, H^*L\wedge K)$$
$$\xrightarrow{(j''\wedge 1_K)_*} Ext_A^{s,tq-q}(H^*K, H^*L\wedge K) \xrightarrow{(\alpha_1\wedge 1_K)_*}$$

induced by (1.3). For the second result, consider the exact sequence $(r = 1, 2, 3, u = 0, 1, 2)$

$$0 = Ext_A^{s,tq+rq+u}(H^*K, H^*M) \xrightarrow{(i''\wedge 1_K)_*} Ext_A^{s,tq+rq+u}(H^*L\wedge K, H^*M)$$
$$\xrightarrow{(j''\wedge 1_K)_*} Ext_A^{s,tq+(r-1)q+u}(H^*K, H^*M) \xrightarrow{(\alpha_1\wedge 1_K)_*}$$

induced by (1.3). The left group is zero for $r = 1, 2, 3, u = 0, 1, 2$ by (2) and the right group is also zero for $r = 2, 3, u = 0, 1, 2$ by (2). The right group is also zero for $r = 1, u = 1, 2$ by Prop. 2.6(1) and the supposition. It has a unique generator $(i')_*(\tilde{\sigma})$ for $r = 1, u = 0$ which satisfies $(\alpha_1 \wedge 1_K)_*(i')_*(\tilde{\sigma}) \neq 0$. Then the middle group is zero for $r = 1, 2, 3, u = 0, 1, 2$.

(4) Consider the exact sequence $(r = 1, 2, 3, u = 0, 1, 2)$

$$0 = Ext_A^{s,tq+rq+u}(H^*L\wedge K, H^*M) \xrightarrow{(w\wedge 1_K)_*} Ext_A^{s,tq+rq+u}(H^*W\wedge K, H^*M)$$
$$\xrightarrow{(j''u\wedge 1_K)_*} Ext_A^{s,tq+(r-2)q+u}(H^*K, H^*M) \xrightarrow{(\phi\wedge 1_K)_*}$$

induced by (2.17). The left group is zero for $r = 1, 2, 3, u = 0, 1, 2$ by (3). The right group is also zero for $r = 1, 3, u = 0, 1, 2$ by (2) and it is zero for $r = 2, u = 1$ by Prop. 2.6(1) and the supposition and it has a unique generator $(i')_*(\tilde{\sigma})$ for $r = 2, u = 0$. However, it satisfies $(\phi \wedge 1_K)_*(i')_*(\tilde{\sigma}) \neq 0 \in Ext_A^{s+1,tq+2q}(H^*L \wedge K, H^*M)$. Then the middle group is zero for $r = 1, 2, 3, u = 0, 1, 2$ as desired.

Since $(\widetilde{u}w_2\overline{w} \wedge 1_M)^* Ext_A^{s,tq+q}(H^*W \wedge K, H^*X \wedge M) \subset Ext_A^{s,tq+q}(H^*W \wedge K, H^*M) = 0$ as shown above, then, by (2.11), $(\widetilde{u}w_2 \wedge 1_M)^* Ext_A^{s,tq+q}(H^*W \wedge K, H^*X \wedge M) = (\overline{u} \wedge 1_M)^* Ext_A^{s,tq+2q+1}(H^*W \wedge K, H^*M \wedge M) = 0$. It follows by (2.25) that $Ext_A^{s,tq+q}(H^*W \wedge K, H^*X \wedge M) = (\widetilde{\psi} \wedge 1_M)^* Ext_A^{tq+3q}(H^*W \wedge K, H^*M \wedge M) = 0$.

3 Proof of Main Theorems

Theorem A will be proved by an argument processing in the Adams resolution of certain spectra related to K, which is equivalent to computing the differentials of the ASS. Let

$$\begin{array}{ccccccc} \cdots \xrightarrow{\bar{a}_2} & \Sigma^{-2}E_2 & \xrightarrow{\bar{a}_1} & \Sigma^{-1}E_1 & \xrightarrow{\bar{a}_0} & E_0 = S \\ & \downarrow \bar{b}_2 & & \downarrow \bar{b}_1 & & \downarrow \bar{b}_0 \\ & \Sigma^{-2}KG_2 & & \Sigma^{-1}KG_1 & & KG_0 \end{array} \tag{3.1}$$

be the minimal Adams resolution of S satisfying:

(1) $E_s \xrightarrow{\bar{b}_s} KG_s \xrightarrow{\bar{c}_s} E_{s+1} \xrightarrow{\bar{a}_s} \Sigma E_s$ are cofibrations for all $s \geq 0$ which induce short exact sequences in Z_p-cohomology;

(2) KG_s is a wedge sum of suspensions of Eilenberg–Maclane spectra of type KZ_p;

(3) $\pi_t KG_s$ are the $E_1^{s,t}$-terms, $(\bar{b}_s\bar{c}_{s-1})_* : \pi_t KG_{s-1} \to \pi_t KG_s$ are the $d_1^{s-1,t}$-differentials of the ASS and $\pi_t KG_s \cong Ext_A^{s,t}(Z_p, Z_p)$ (cf. [9] p. 180).

Then, an Adams resolution of an arbitrary spectrum V can be obtained by smashing V on (3.1). We first prove the following Lemmas:

Lemma 3.2 *Under the assumption of Theorem* A*, we have*:

(1) *Let* $\widetilde{h_0\sigma} \in [\Sigma^{tq+q+1}M, KG_{s+1} \wedge M]$ *be the* d_1*-cycle which represents* $\alpha_*(\tilde{\sigma}) \in Ext_A^{s+1,tq+q+1}(H^*M, H^*M)$*. Then* $(\bar{c}_{s+1} \wedge 1_M)\widetilde{h_0\sigma} = (1_{E_{s+2}} \wedge \alpha)(\kappa \wedge 1_M)$ *up to scalar, where* $\kappa \in \pi_{tq+1}E_{s+2}$ *such that* $\bar{a}_{s+1} \cdot \kappa = \bar{c}_s \cdot \sigma$ *with* $\sigma \in \pi_{tq}KG_s \cong Ext_A^{s,tq}(Z_p, Z_p)$;

(2) $(1_{E_{s+2}} \wedge \phi \wedge 1_M)(\kappa \wedge 1_M) = 0$, $(1_{E_{s+2}} \wedge \alpha_1 \wedge 1_M)(\kappa \wedge 1_M) = 0$.

Proof (1) Since $(1_{KG_{s+1}} \wedge i')\widetilde{h_0\sigma}$ is a d_1-boundary, then $(\bar{c}_{s+1} \wedge 1_K)(1_{KG_{s+1}} \wedge i')(\widetilde{h_0\sigma}) = 0$ and so $(\bar{c}_{s+1} \wedge 1_M)\widetilde{h_0\sigma} = (1_{E_{s+2}} \wedge \alpha)f'$ with $f' \in [\Sigma^{tq+1}M, E_{s+2} \wedge M]$. It follows that $(\bar{a}_{s+1} \wedge 1_M)(1_{E_{s+2}} \wedge \alpha)f' = 0$ and so $(\bar{a}_{s+1} \wedge 1_M)f' = (1_{E_{s+1}} \wedge j')f_2'$ for some $f_2' \in [\Sigma^{tq+q+1}M, E_{s+1} \wedge K]$. The d_1-cycle $(\bar{b}_{s+1} \wedge 1_K)f_2'$ represents an element in $Ext_A^{s+1,tq+q+1}(H^*K, H^*M) = 0$ by Prop. 2.7(2), then $(\bar{b}_{s+1} \wedge 1_K)f_2' = (\bar{b}_{s+1}\bar{c}_s \wedge 1_K)g_0'$ for some $g_0' \in [\Sigma^{tq+q+1}M, KG_s \wedge K]$. Hence, $f_2' = (\bar{c}_s \wedge 1_K)g_0' + (\bar{a}_{s+1} \wedge 1_K)f_3'$ for some $f_3' \in [\Sigma^{tq+q+2}M, E_{s+2} \wedge K]$ and so $(\bar{a}_{s+1} \wedge 1_M)f' = (\bar{a}_{s+1} \wedge 1_M)(1_{E_{s+2}} \wedge j')f_3' + (\bar{c}_s \wedge 1_M)(1_{KG_s} \wedge j')g_0' = (\bar{a}_2 \wedge 1_M)(1_{E_{s+2}} \wedge j')f_3' + (\bar{c}_s \wedge 1_M)(\sigma \wedge 1_M) = (\bar{a}_{s+1} \wedge 1_M)(1_{E_{s+2}} \wedge j')f_3' + (\bar{a}_{s+1} \wedge 1_M)(\kappa \wedge 1_M)$, where the d_1-cycle $(1_{KG_s} \wedge j')g_0' \in [\Sigma^{tq}M, KG_s \wedge M]$ represents a unique generator $\widetilde{\sigma}$ of $Ext_A^{s,tq}(H^*M, H^*M)$ so that it is equal to $\sigma \wedge 1_M$ modulo d_1-boundary. Consequently we have $f' = (1_{E_{s+1}} \wedge j')f_3' + (\kappa \wedge 1_M) + (\bar{c}_{s+1} \wedge 1_M)\tilde{g}_1$ for some $\tilde{g}_1 \in [\Sigma^{tq+1}M, KG_{s+1} \wedge M]$. It follows that $(\bar{c}_{s+1} \wedge 1_M)\widetilde{h_0\sigma} = (1_{E_{s+2}} \wedge \alpha)f' = (1_{E_{s+1}} \wedge \alpha)(\kappa \wedge 1_M)$ as desired.

(2) Since $Ext_A^{s+1,tq+rq}(Z_p, Z_p) = 0$ for $r = 2$ and has a unique generator $h_0\sigma = (j'')_*(\phi)_*(\sigma)$ for $r = 1$, then $Ext_A^{s+1,tq+2q}(H^*L, Z_p) \cong Z_p\{(\phi)_*(\sigma)\}$ and $Ext_A^{s+1,tq+2q}(H^*W, Z_p) = 0$. By this fact and an argument similar to that given in (1) we have $(1_{E_{s+2}} \wedge \phi)\kappa = (\bar{c}_{s+1} \wedge 1_L)\sigma\phi$ (up to scalar), where $\sigma\phi \in \pi_{tq+2q}(KG_{s+1} \wedge L)$ is the d_1-cycle which represents $(\phi)_*(\sigma) \in Ext_A^{s+1,tq+2q}(H^*L, Z_p)$. By the supposition (II) of Theorem A we have $(1_{E_{s+2}} \wedge \phi \wedge 1_M)(\kappa \wedge 1_M) = (\bar{c}_{s+1} \wedge 1_{L \wedge M})(\sigma\phi \wedge 1_M) = 0$ and the result follows.

Lemma 3.3 *Under the supposition* I *of Theorem* A*, we have*:

(1) $Ext_A^{s,tq}(H^*X \wedge M, H^*X \wedge M) \cong Z_p\{[\sigma \wedge 1_{X \wedge M}]\}$;

(2) *For any d_1-cycle $g_0 \in [\Sigma^{tq+q}X, KG_{s+1}\wedge X]$, $g_0 = \lambda'(h_0\sigma\wedge 1_X)$ modulo d_1-boundary with $\lambda'\in Z_p$ and $(\psi_{X\wedge M})_*[h_0\sigma\wedge 1_{X\wedge M}]\neq 0\in Ext_A^{s+1,tq+1}(H^*Y\wedge M, H^*X\wedge M)$.*

Proof (1) Consider the exact sequence

$$Ext_A^{s,tq+2q}(H^*L\wedge K, H^*M) \stackrel{m_M(\tilde\psi\wedge 1_M)^*}{\longrightarrow} Ext_A^{s,tq}(H^*L\wedge K, H^*X\wedge M)$$
$$\stackrel{(u')^*}{\longrightarrow} Ext_A^{s,tq}(H^*L\wedge K, H^*L\wedge K) \stackrel{((1_L\wedge i')(\phi\wedge 1_M))^*}{\longrightarrow}$$

induced by (2.32). The left group is zero by Prop. 2.40(3). The right group has a unique generator $\sigma_{L\wedge K}$ by Prop. 2.40(3), which satisfies $((1_L\wedge i')(\phi\wedge 1_M))^*(\sigma_{L\wedge K})\neq 0\in Ext_A^{s+1,tq+2q}(H^*L\wedge K, H^*M)$ since $(j''\wedge 1_K)_*((1_L\wedge i')(\phi\wedge 1_M))^*(\sigma_{L\wedge K}) = ((1_L\wedge i')(\phi\wedge 1_M))^*(j''\wedge 1_K)_*(\sigma_{L\wedge K}) = ((1_L\wedge i')(\phi\wedge 1_M)^*(j''\wedge 1_K)^*(\sigma_K) = ((\alpha_1\wedge 1_K)i')^*(\sigma_K) = (\alpha_1\wedge 1_M)^*(i')_*(\tilde\sigma) = (i'(\alpha_1\wedge 1_M))_*(\tilde\sigma)\neq 0\in Ext_A^{s+1,tq+q}(H^*K, H^*M)$ by Prop. 2.6(4). Then the middle group is zero. Look at the exact sequence

$$Ext_A^{s,tq}(H^*L\wedge K, H^*X\wedge M) \stackrel{(u')_*}{\longrightarrow} Ext_A^{s,tq}(H^*X\wedge M, H^*X\wedge M)$$
$$\stackrel{m_M(\tilde\psi\wedge 1_M)_*}{\longrightarrow} Ext_A^{s,tq-2q}(H^*M, H^*X\wedge M) \stackrel{((1_L\wedge i')(\phi\wedge 1_M))_*}{\longrightarrow}$$

induced by (2.32). The left group is zero as shown above. The right group has a unique generator $m_M(\widetilde\psi\wedge 1_M)^*(\tilde\sigma) = m_M(\widetilde\psi\wedge 1_M)^*[\sigma\wedge 1_M] = [(\sigma\wedge 1_M)m_M(\widetilde\psi\wedge 1_M)] = [(1_{KG_s}\wedge m_M(\widetilde\psi\wedge 1_M))(\sigma\wedge 1_{X\wedge M})] = m_M(\widetilde\psi\wedge 1_M)_*[\sigma\wedge 1_{X\wedge M}]$ by Prop. 2.40(1) which satisfies $((1_L\wedge i')(\phi\wedge 1_M))_* m_M(\widetilde\psi\wedge 1_M)^*(\tilde\sigma) = ((1_L\wedge i')(\phi\wedge 1_M))_* m_M(\widetilde\psi\wedge 1_M)_*[\sigma\wedge 1_{X\wedge M}] = 0$. Then the middle group has a unique generator $[\sigma\wedge 1_{X\wedge M}]$ as desired.

(2) Since $(\widetilde\psi)_*(\widetilde u w_2)^* Ext_A^{s+1,tq+q}(H^*X, H^*X)\subset Ext_A^{s+1,tq-q-1}(H^*M, H^*Y)$ which is zero or has one (or two) generator $(\overline u)^*(\tilde\sigma')$ by Prop. 2.6(2) and it satisfies $((1_Y\wedge j)\alpha_{Y\wedge M})_*(\overline u)^*(\tilde\sigma') = ((1_Y\wedge j)\alpha_{Y\wedge M})_*(\overline u)_*[\sigma'\wedge 1_Y] = (1_Y\wedge\alpha_1)_*[\sigma'\wedge 1_Y] = [h_0\sigma'\wedge 1_Y]\neq 0$, then $(\widetilde\psi)_*(\widetilde u w_2)^*$ $Ext_A^{s+1,tq+q}$ $(H^*X, H^*X) = 0$ and so $(\widetilde u w_2)^* Ext_A^{s+1,tq+q}(H^*X, H^*X) = (\widetilde u w_2)_*$ $Ext_A^{s+1,tq+q}(H^*Y, H^*Y) = 0$ since $Ext_A^{s+1,tq+q}(H^*Y, H^*Y)\cong Z_p\{((1_Y\wedge j)\alpha_{Y\wedge M})_*\ (\overline u)^*(\tilde\sigma)\}$ by Prop. 2.39(2). Then $Ext_A^{s+1,tq+q}(H^*X, H^*X) = (\widetilde\psi)^* Ext_A^{s+1,tq+3q}(H^*X,\ H^*M)$, which has a unique generator $(\widetilde\psi)^*((1_X\wedge j)\alpha_{X\wedge M})_*(\tilde\sigma) = ((1_X\wedge j)\alpha_{X\wedge M})_*[(\sigma\wedge 1_M)\widetilde\psi] = ((1_X\wedge j)\alpha_{X\wedge M})_*[(1_{KG_{s+1}}\wedge\widetilde\psi)(\sigma\wedge 1_X)] = ((1_X\wedge j)\alpha_{X\wedge M})_* m_M(\widetilde\psi\wedge 1_M)_*(1_X\wedge i)_*[\sigma\wedge 1_X] = (1_X\wedge j\alpha i)_*[\sigma\wedge 1_X] = [h_0\sigma\wedge 1_X]$ by Prop. 2.39(3). Then the first result follows. Moreover, by (2.36), the d_1-cycle $(1_{KG_{s+1}}\wedge m_M(\overline u\wedge 1_M)\psi_{X\wedge M})(h_0\sigma\wedge 1_{X\wedge M}) = (1_{KG_{s+1}}\wedge m_M(\widetilde\psi\wedge 1_M))(h_0\sigma\wedge 1_{X\wedge M}) = (h_0\sigma\wedge 1_M)m_M(\widetilde\psi\wedge 1_M)$ represents $m_M(\widetilde\psi\wedge 1_M)^*[h_0\sigma\wedge 1_M] = m_M(\widetilde\psi\wedge 1_M)^*(\alpha_1\wedge 1_M)_*(\tilde\sigma)\neq 0$ and so the second result follows.

Proof of Theorem A By Lemma 3.2(1), it is suffices to prove that $(\bar c_{s+1}\wedge 1_M)\widetilde{h_0\sigma} = (1_{E_{s+1}}\wedge\alpha)(\kappa\wedge 1_M) = 0$. The proof is divided into the following two steps:

Step 1 To prove $(\kappa\wedge 1_{X\wedge M})(1_X\wedge\alpha) = 0$.

By (2.34), $(\phi\wedge 1_M)m_M(\widetilde\psi\wedge 1_M) = (u''\wedge 1_M)(1_X\wedge\alpha)$, then we have $(1_{E_{s+2}}\wedge u''\wedge 1_M)(1_{E_{s+2}}\wedge 1_X\wedge\alpha)(\kappa\wedge 1_{X\wedge M}) = (1_{E_{s+2}}\wedge\phi\wedge 1_M)(\kappa\wedge 1_M)m_M(\widetilde\psi\wedge 1_M) = 0$ by Lemma 3.2(2). It follows by (2.30) that $(1_{E_{s+2}}\wedge 1_X\wedge\alpha)(\kappa\wedge 1_{X\wedge M}) = (1_{E_{s+2}}\wedge\widetilde u w_3\wedge 1_M)f$, for some $f\in[\Sigma^{tq+q+1}X\wedge M, E_{s+2}\wedge W\wedge M]\cap(\ker d)$ (cf. Prop. 2.4). By composing $(1_{E_{s+2}}\wedge 1_X\wedge i'i\wedge 1_M)$, we have $(1_{E_{s+2}}\wedge\widetilde u w_3\wedge 1_{K\wedge M})(1_{E_{s+2}}\wedge 1_W\wedge i'i\wedge 1_M)f = (1_{E_{s+2}}\wedge(1_X\wedge(i'i\wedge 1_M)\alpha)(\kappa\wedge 1_{X\wedge M}) = (1_{E_{s+2}}\wedge 1_X\wedge\overline m_K i'(\alpha_1\wedge 1_M)))(\kappa\wedge 1_{X\wedge M}) = 0$ by Lemma 3.2(2) on $(1_{E_{s+2}}\wedge\alpha_1\wedge 1_M)(\kappa\wedge 1_M) = 0$.

It follows by (2.30) that $(1_{E_{s+2}} \wedge 1_W \wedge i'i \wedge 1_M)f = (1_{E_{s+2}} \wedge w'(\pi \wedge 1_L) \wedge 1_{K\wedge M})f_2 = 0$ (with $f_2 \in [\Sigma^{tq+1}X \wedge M, E_{s+2} \wedge L \wedge K \wedge M]$) since $\pi \wedge 1_K = 0$. Hence, by (2.10), we have $f = (1_{E_{s+2}} \wedge 1_W \wedge \epsilon \wedge 1_M)f_3 = (1_{E_{s+2}} \wedge 1_W \wedge \alpha m_M(\overline{u} \wedge 1_M)f_3$ for some $f_3 \in [\Sigma^{tq+q+2}X \wedge M, E_{s+2} \wedge W \wedge Y \wedge M] \cap (\ker d)$ (cf. Prop. 2.4) and we have

$$\begin{aligned}(1_{E_{s+2}} \wedge 1_X \wedge \alpha)(\kappa \wedge 1_{X\wedge M}) &= (1_{E_{s+2}} \wedge \widetilde{u}w_3 \wedge 1_M)(1_{E_{s+2}} \wedge 1_W \wedge \alpha m_M(1_M \wedge \overline{u}))f_3 \\ &= (1_{E_{s+2}} \wedge \alpha_{X\wedge M}(j''u \wedge 1_M))(1_{E_{s+2}} \wedge 1_W \wedge m_M(1_M \wedge \overline{u}))f_3, \end{aligned} \tag{3.4}$$

by (2.34), (2.26), (2.28).

It follows from (3.4) that $(\bar{a}_{s+1} \wedge 1_{X\wedge M})(1_{E_{s+2}} \wedge \widetilde{u}w_3 \wedge 1_M)(1_{E_{s+2}} \wedge (1_W \wedge \alpha m_M(\overline{u} \wedge 1_M))f_3 = (\bar{a}_{s+1} \wedge 1_{X\wedge M})(1_{E_{s+2}} \wedge 1_X \wedge \alpha)(\kappa \wedge 1_{X\wedge M}) = (\bar{c}_s \wedge 1_{X\wedge M})(1_{KG_s} \wedge 1_X \wedge \alpha)(\sigma \wedge 1_{X\wedge M}) = 0$ since α induces zero homomorphism in Z_p-cohomology. Then, by (2.30) and $w'(\pi \wedge 1_L) \wedge 1_M = (w \wedge 1_M)(1_L \wedge \alpha)$, we have

$$(\bar{a}_{s+1} \wedge 1_{W\wedge M})(1_{E_{s+2}} \wedge 1_W \wedge \alpha m_M(\overline{u} \wedge 1_M))f_3 = (1_{E_{s+1}} \wedge (1_W \wedge \alpha)(w \wedge 1_M))f_5, \tag{3.5}$$

for some $f_5 \in [\Sigma^{tq}X \wedge M, E_{s+1} \wedge L \wedge M] \cap (\ker d)$ (cf. Prop. 2.4).

It follows from (3.5), (1.2) that $(\bar{a}_{s+1} \wedge 1_{W\wedge M})(1_{E_{s+2}} \wedge 1_W \wedge m_M(\overline{u} \wedge 1_M))f_3 = (1_{E_{s+1}} \wedge w \wedge 1_M)f_5 + (1_{E_{s+1}} \wedge 1_W \wedge j')f_6$ with $f_6 \in [\Sigma^{tq+q+1}X \wedge M, E_{s+1} \wedge W \wedge K] \cap (\ker d)$ (cf. Prop. 2.5). Since $(1_W \wedge \alpha_1)w = w(1_L \wedge \alpha_1) = w \cdot \phi j'' = 0$, then $w = (1_W \wedge j'')\psi_W$ with $\psi_W \in [\Sigma^q L, W \wedge L]$ and so $w \wedge 1_M = (1_W \wedge j'')\psi_W \wedge 1_M = (1_W \wedge m_M(\overline{u} \wedge 1_M))((1_W \wedge \bar{h})\psi_W \wedge 1_M)$. Then we have $-(\bar{a}_{s+1} \wedge 1_{W\wedge Y\wedge M})f_3 = (1_{E_{s+2}} \wedge (1_W \wedge h)\psi_W \wedge 1_M)f_5 + (1_{E_{s+1}} \wedge 1_W \wedge (1_Y \wedge i)r)f_6 + (1_{E_{s+1}} \wedge 1_W \wedge (r \wedge 1_M)\overline{m}_K)f_7$ and by Prop. 2.5, $f_7 = f_8(1_X \wedge i') + f_9(1_X \wedge i'ij)$ with $f_8 \in [\Sigma^{tq+q}X \wedge K, E_{s+1} \wedge W \wedge K] \cap (\ker d)$ and $f_9 \in [\Sigma^{tq+q+1}X \wedge K, E_{s+1} \wedge W \wedge K] \cap (\ker d)$. By applying d using Prop. 2.2(i) and $d((1_Y \wedge i)r) = (r \wedge 1_M)\overline{m}_K$ in (2.13), we have $-(1_{E_{s+1}} \wedge 1_W \wedge (r \wedge 1_M)\overline{m}_K)f_6 - (1_{E_{s+1}} \wedge 1_W \wedge (r \wedge 1_M)\overline{m}_K)f_9(1_X \wedge i') = 0$ (Note : f_6 has odd degree) and so we have

$$\begin{aligned}-(\bar{a}_{s+1} \wedge 1_{W\wedge Y\wedge M})f_3 = {} & (1_{E_{s+1}} \wedge (1_W \wedge \bar{h})\psi_W \wedge 1_M)f_5 \\ & + (1_{E_{s+1}} \wedge 1_W \wedge (1_Y \wedge i)r)f_6 \\ & + (1_{E_{s+1}} \wedge 1_W \wedge (r \wedge 1_M)\overline{m}_K)f_8(1_X \wedge i') \\ & - (1_{E_{s+1}} \wedge 1_W \wedge (r \wedge 1_M)\overline{m}_K)f_6(1_X \wedge ij). \end{aligned} \tag{3.6}$$

Moreover, the d_1-cycle $(\bar{b}_{s+1} \wedge 1_{W\wedge K})f_6 \in [\Sigma^{tq+q+1}X \wedge M, KG_{s+1} \wedge W \wedge K] \cap (\ker d)$ represents an element in $Ext_A^{s+1,tq+q+1}(H^*W \wedge K, H^*X \wedge M) = 0$ by Prop. 2.39(1), then $(\bar{b}_{s+1} \wedge 1_{W\wedge K})f_6 = (\bar{b}_{s+1}\bar{c}_s \wedge 1_{W\wedge K})g$ for some $g \in [\Sigma^{tq+q+1}X \wedge M, KG_s \wedge W \wedge K] \cap (\ker d)$ (cf. Prop. 2.5) and so $f_6 = (\bar{c}_s \wedge 1_{W\wedge K})g + (\bar{a}_{s+1} \wedge 1_{W\wedge K})f'$ with $f' \in [\Sigma^{tq+q+2}X \wedge M, E_{s+2} \wedge W \wedge K] \cap (\ker d)$ (cf. Prop. 2.5) and we have

$$\begin{aligned}-(\bar{a}_{s+1} \wedge 1_{W\wedge Y\wedge M})f_3 = {} & (1_{E_{s+1}} \wedge (1_W \wedge \bar{h})\psi_W \wedge 1_M)f_5 \\ & + (\bar{a}_{s+1} \wedge 1_{W\wedge Y\wedge M})(1_{E_{s+2}} \wedge 1_W \wedge (1_Y \wedge i)r)f' \\ & + (\bar{c}_s \wedge 1_{W\wedge Y\wedge M})(1_{KG_s} \wedge 1_W \wedge (1_Y \wedge i)r)g \\ & + (\bar{a}_{s+1} \wedge 1_{W\wedge Y\wedge M})(1_{E_{s+2}} \wedge 1_W \wedge (r \wedge 1_M)\overline{m}_K)f'(1_X \wedge ij) \\ & - (\bar{c}_s \wedge 1_{W\wedge Y\wedge M})(1_{KG_s} \wedge 1_W \wedge (r \wedge 1_M)\overline{m}_K)g(1_X \wedge ij) \end{aligned}$$

$$+ (1_{E_{s+1}} \wedge 1_W \wedge (r \wedge 1_M)\overline{m}_K) f_8 (1_X \wedge i'). \tag{3.7}$$

Let P be the cofibre of $(1_W \wedge \bar{h})\psi_W : \Sigma^{q+1}L \to W \wedge Y$ given by the cofibration

$$\Sigma^{q+1}L \xrightarrow{(1_W\wedge\bar{h})\psi_W} W \wedge Y \xrightarrow{w_5} P \xrightarrow{u_5} \Sigma^{q+2}L. \tag{3.8}$$

Then, the cofibre of $w_5(1_W \wedge r) : W \wedge K \to P$ is ΣX given by the cofibration

$$W \wedge K \xrightarrow{w_5(1_W\wedge r)} V \xrightarrow{w_6} \Sigma X \xrightarrow{u_6} \Sigma W \wedge K. \tag{3.9}$$

This can be seen by the following homotopy commutative diagram of 3×3-Lemma:

$$\begin{array}{ccccccc} W\wedge K & & \xrightarrow{w_5(1_W\wedge r)} & P & \xrightarrow{u_5} & & \Sigma^{q+2}L \\ & \searrow 1_W\wedge r & \nearrow w_5 & & \searrow w_6 & \nearrow u'' & \\ & W\wedge Y & & & & \Sigma X & \\ & \nearrow^{(1_W\wedge\bar{h})\psi_W} & \searrow 1_W\wedge\epsilon & & \nearrow \widetilde{u}w_3 & \searrow u_6 & \\ \Sigma^{q+1}L & & \xrightarrow{w'(\pi\wedge 1_L)} & \Sigma W & \xrightarrow{1_W\wedge i'i} & & \Sigma W\wedge K. \end{array}$$

Note that $u_6 = \mu_{X\wedge M}(1_X \wedge i)$, then by composing $(\bar{b}_{s+1} \wedge 1_P)(1_{E_{s+1}} \wedge w_5 \wedge j)$ on the left and composing $(1_X \wedge i)$ on the right of (3.7), we have $(\bar{b}_{s+1} \wedge 1_P)(1_{E_{s+1}} \wedge w_5(1_W \wedge r)) f_8(1_X \wedge i'i) = 0$ and so $(\bar{b}_{s+1} \wedge 1_{W\wedge K}) f_8(1_X \wedge i'i) = (1_{KG_{s+1}} \wedge u_6) g_0 = (1_{KG_{s+1}} \wedge \mu_{X\wedge M}(1_X \wedge i)) g_0 = (1_{KG_{s+1}} \wedge \mu_{X\wedge M})(g_0 \wedge 1_M)(1_X \wedge i)$ with d_1-cycle $g_0 \in [\Sigma^{tq+q}X, KG_{s+1} \wedge X]$ so that $g_0 = \lambda_1(h_0\sigma \wedge 1_X)$ modulo d_1-boundary with $\lambda_1 \in Z_p$ by Lemma 3.3(2). Moreover, by applying d to $(\bar{b}_{s+1} \wedge 1_{W\wedge K}) f_8(1_X \wedge i'ij) = (1_{KG_{s+1}} \wedge \mu_{X\wedge M})(g_0 \wedge 1_M)(1_X \wedge ij)$, we have

$$(\bar{b}_{s+1} \wedge 1_{W\wedge K}) f_8(1_X \wedge i') == (1_{KG_{s+1}} \wedge \mu_{X\wedge M})(g_0 \wedge 1_M) \tag{3.10}$$

and $g_0 = \lambda_1(h_0\sigma \wedge 1_X) \in [\Sigma^{tq+q}X, KG_{s+1} \wedge X]$ modulo d_1-boundary.

Look at the following diagram of cofibrations (2.17), (3.8):

$$\begin{array}{ccccccc} \Sigma^{q+1}L\wedge M & \xrightarrow{w\wedge 1_M} & \Sigma^{q+1}W\wedge M & \xrightarrow{j''u\wedge 1_M} & \Sigma^{3q+1}M & \xrightarrow{\phi\wedge 1_M} & \Sigma^{q+2}L\wedge M \\ \uparrow 1_{L\wedge M} & & \uparrow 1_W\wedge m_M(\overline{u}\wedge 1_M) & & \uparrow u_7 & & \uparrow 1_{L\wedge M} \\ \Sigma^{q+1}L\wedge M & \xrightarrow{(1_W\wedge\bar{h})\psi_W\wedge 1_M} & W\wedge Y\wedge M & \xrightarrow{w_5\wedge 1_M} & P\wedge M & \xrightarrow{u_5\wedge 1_M} & \Sigma^{q+2}L\wedge M. \end{array}$$

Since the left rectangle of the above diagram commutes up to homotopy, then there exists $u_7 \in [\Sigma^{-3q-1}P \wedge M, M]$ such that all the rectangles commute. That is, we have

$$u_7(w_5 \wedge 1_M) = (j''u \wedge 1_M)(1_W \wedge m_M(\overline{u} \wedge 1_M)), \quad (\phi \wedge 1_M)u_7 = \pm \quad u_5 \wedge 1_M \tag{3.11}$$

with $u_7 \in [\Sigma^{-3q-1}P \wedge M, M]$. By the second equation we have the following homotopy commutative diagram of 3×3-Lemma using (2.17), (3.8), (2.16):

$$\begin{array}{ccccccc} P\wedge M & \xrightarrow{u_5\wedge 1_M} & \Sigma^{q+2}L\wedge M & \xrightarrow{w\wedge 1_M} & \Sigma^{q+2}W\wedge M & & \\ \searrow u_7 & \nearrow \phi\wedge 1_M & \searrow^{((1_W\wedge\bar{h})\psi_W\wedge 1_M} & \nearrow_{1_W\wedge m_M(\overline{u}\wedge 1_M)} & \searrow j''u\wedge 1_M & & \\ \Sigma^{3q+1}M & & \Sigma W\wedge Y\wedge M & & \Sigma^{3q+2}M & & \\ \nearrow j''u\wedge 1_M & \searrow^{(\phi_W\wedge 1_K)i'} & \nearrow_{1_W\wedge(r\wedge 1_M)\overline{m}_K} & \searrow w_5\wedge 1_M & \nearrow u_7 & & \\ \Sigma^{q+1}W\wedge M & \xrightarrow{\tilde{\lambda}(1_W\wedge\alpha' i')} & \Sigma^2 W\wedge K & \longrightarrow & \Sigma P\wedge M. & & \end{array} \tag{3.12}$$

That is, we have the following cofibration:

$$\Sigma^{3q-1}M \xrightarrow{(\phi_W\wedge 1_K)i'} W \wedge K \xrightarrow{(w_5\wedge 1_M)(1_W\wedge(r\wedge 1_M)\overline{m}_K)} \Sigma^{-1}P \wedge M \xrightarrow{u_7} \Sigma^{3q}M, \tag{3.13}$$

where $\phi_W \in [\Sigma^{3q-1}S, W]$ such that $u \cdot \phi_W = \phi \in [\Sigma^{2q-1}S, L]$. Since $(\phi \wedge 1_K)i' \cdot u_7 = (u \cdot \phi_W \wedge 1_K)i' \cdot u_7 = 0$, then by (2.32), we have

$$u_7 = m_M(\widetilde{\psi} \wedge 1_M)u_8, \tag{3.14}$$

with $u_8 \in [\Sigma^{-q-1}P \wedge M, X \wedge M]$. Moreover, by (2.38), $(\omega \wedge 1_M)u_8(w_5(1_W \wedge r) \wedge 1_M) = \alpha_{Y\wedge M} m_M(\widetilde{\psi}\wedge 1_M)u_8(w_5(1_W \wedge r)\wedge 1_M) = \alpha_{Y\wedge M} u_7(w_5(1_W \wedge r)\wedge 1_M) = \alpha_{Y\wedge M} j'(j''u\wedge 1_K)(1_W \wedge m_K) = 0$ (cf. (2.31)). Then, by (2.37), $u_8(w_5(1_W \wedge r) \wedge 1_M) = ((1_X \wedge j)u' \wedge 1_M)\Delta_1$ with $\Delta_1 \in [\Sigma^{-q}W \wedge K \wedge M, L \wedge K \wedge M] \cap (\ker d)$. By composing $\mu_{X\wedge M}(1_X \wedge i) \wedge 1_M$ using (3.9), we have $((1_X \wedge j)u' \wedge 1_M)\Delta_1(\mu_{X\wedge M}(1_X \wedge i) \wedge 1_M) = 0$ and so by (2.37), (2.36) we have $\Delta_1(\mu_{X\wedge M}(1_X \wedge i)\wedge 1_M) = (\overline{\mu}_2(1_Y \wedge i'i)\wedge 1_M)\psi_{X\wedge M}$. It follows that $(j''\wedge 1_{K\wedge M})\Delta_1(\mu_{X\wedge M}(1_X \wedge i)\wedge 1_M) = ((j''\wedge 1_K)\overline{\mu}_2(1_Y \wedge i'i)\wedge 1_M)\psi_{X\wedge M} = (i'\overline{u}\wedge 1_M)\psi_{X\wedge M} = (i'i\wedge 1_M)m_M(\overline{u}\wedge 1_M)\psi_{X\wedge M} + (i'\wedge 1_M)\overline{m}_M(j\overline{u}\wedge 1_M)\psi_{X\wedge M}$ and so $(j''\wedge 1_{K\wedge M})\Delta_1(\mu_{X\wedge M}(1_X \wedge i)\cdot\widetilde{u}w_2\wedge 1_M) = 0$. Consequently, $(j''\wedge 1_{K\wedge M})\Delta_1(\mu_{X\wedge M}(\widetilde{u}w_2\wedge 1_M)\wedge 1_M) \in (1_Y \wedge j\wedge 1_M)^*[\Sigma^{-2q}Y\wedge M, K\wedge M] = 0$ since the degree of the top cell of $Y\wedge M$ is $q+3$. It follows that $(j''\wedge 1_{K\wedge M})\Delta_1(\mu_{X\wedge M}\wedge 1_M) \in (\widetilde{\psi}\wedge 1_{M\wedge M})^*[M\wedge M\wedge M, K\wedge M]$, then we have $(j'(j''\wedge 1_K)\wedge 1_M)\Delta_1(\mu_{X\wedge M}\wedge 1_M) = 0$ and so by (2.35), we have $(j'(j''\wedge 1_K)\wedge 1_M)\Delta_1 = \Delta_2(j'(j''u\wedge 1_K)\wedge 1_M) = \lambda(j'(j''u\wedge 1_K)\wedge 1_M)$ for some $\lambda \in Z_p$ since $\Delta_2 \in [M\wedge M, M\wedge M]\cap(\ker d) \cong Z_p\{1_{M\wedge M}\}$. Hence we have $(\widetilde{\psi}\wedge 1_M)u_8(w_5(1_W \wedge r)\wedge 1_M) = (\widetilde{\psi}(1_X \wedge j)u' \wedge 1_M)\Delta_1 = (j'(j''\wedge 1_K)\wedge 1_M)\Delta_1 = \lambda(j'(j''u\wedge 1_K)\wedge 1_M)$ and by (3.11), (3.14) we know that $\lambda = 1$ and we have

$$
\begin{aligned}
&m_M(\widetilde{\psi}\wedge 1_M)u_8(w_5\wedge 1_M)(1_W \wedge (1_Y \wedge i)r) = j'(j''u\wedge 1_K)\\
&\quad = (j\widetilde{\psi}\wedge 1_M)u_8(w_5\wedge 1_M)(1_W \wedge (r\wedge 1_M)\overline{m}_K), \qquad \text{(cf. (2.13))}\\
&(j\widetilde{\psi}\wedge 1_M)u_8(w_5\wedge 1_M)(1_W \wedge (1_Y \wedge i)r) = ijj'(j''u\wedge 1_K).
\end{aligned} \tag{3.15}
$$

Now by composing $(1_{E_{s+1}} \wedge u_8(w_5\wedge 1_M))$ (which has odd degree) on (3.7), we have

$$
\begin{aligned}
&(\bar{a}_{s+1}\wedge 1_{X\wedge M})(1_{E_{s+2}} \wedge u_8(w_5\wedge 1_M))f_3\\
&\quad = -(\bar{a}_{s+1}\wedge 1_{X\wedge M})(1_{E_{s+2}} \wedge u_8(w_5\wedge 1_M)(1_W \wedge (1_Y \wedge i)r))f'\\
&\qquad - \overline{\lambda}(\bar{a}_{s+1}\wedge 1_{X\wedge M})(1_{E_{s+2}} \wedge u'(u\wedge 1_K))f'(1_X \wedge ij)\\
&\qquad + (\bar{c}_s\wedge 1_{X\wedge M})(1_{KG_s} \wedge u_8(w_5\wedge 1_M)(1_W \wedge (1_Y \wedge i)r))g\\
&\qquad - (\bar{c}_s\wedge 1_{X\wedge M})(1_{KG_s} \wedge u_8(w_5\wedge 1_M)(1_W \wedge (r\wedge 1_M)\overline{m}_K))g(1_X \wedge ij)\\
&\qquad + \overline{\lambda}(1_{E_{s+1}} \wedge u'(u\wedge 1_K))f_8(1_X \wedge i'),
\end{aligned} \tag{3.16}
$$

where we use $u_8(w_5\wedge 1_M)(1_W \wedge (r\wedge 1_M)\overline{m}_K) = \overline{\lambda}u'(u\wedge 1_K)$ with nonzero $\overline{\lambda} \in Z_p$. Moreover, by (3.10), (2.36), $(\bar{b}_{s+1}\wedge 1_{L\wedge K})(1_{E_{s+1}} \wedge u\wedge 1_K)f_8(1_X \wedge i') = (1_{KG_{s+1}} \wedge (u\wedge 1_K)\mu_{X\wedge M})(g_0 \wedge 1_M) = (1_{KG_{s+1}} \wedge \overline{\mu}_2(1_Y \wedge i')\psi_{X\wedge M})(g_0\wedge 1_M) = \lambda_1(1_{KG_{s+1}} \wedge \overline{\mu}_2(1_Y \wedge i')\psi_{X\wedge M})(h_0\sigma\wedge 1_{X\wedge M}) = \lambda_1(h_0\sigma\wedge 1_{L\wedge K})\overline{\mu}_2(1_Y \wedge i')\psi_{X\wedge M}$ modulo d_1-boundary. Then

$$
\begin{aligned}
&[(\bar{b}_{s+1}\wedge 1_{L\wedge K})(1_{E_{s+1}} \wedge u\wedge 1_K)f_8(1_X \wedge i')]\\
&\qquad = \lambda_1(\phi\wedge 1_K)_*(j''\wedge 1_K)_*[(\sigma\wedge 1_{L\wedge K})\overline{\mu}_2(1_Y \wedge i')\psi_{X\wedge M}]\\
&\qquad = \lambda_1(\phi\wedge 1_K)_*(j''\wedge 1_K)_*(\overline{\mu}_2(1_Y \wedge i'))_*(\psi_{X\wedge M})_*[\sigma\wedge 1_{X\wedge M}]\\
&\qquad = \lambda_1(\phi\wedge 1_K)_*(i')_*(m_M(\overline{u}\wedge 1_M))_*(\psi_{X\wedge M})_*[\sigma\wedge 1_{X\wedge M}]\\
&\qquad = \lambda_1((1_L \wedge i')(\phi\wedge 1_M))_*(m_M(\widetilde{\psi}\wedge 1_M)_*[\sigma\wedge 1_{X\wedge M}]\\
&\qquad = 0 \in Ext_A^{s+1,tq}(H^*L\wedge K, H^*X\wedge M).
\end{aligned}
$$

That is, we have $(\bar{b}_{s+1}\wedge 1_{L\wedge K})(1_{E_{s+1}} \wedge u\wedge 1_K)f_8(1_X \wedge i') = (\bar{b}_{s+1}\bar{c}_s\wedge 1_{L\wedge K})g_3$ with $g_3 \in [\Sigma^{tq}X\wedge M, KG_s\wedge L\wedge K]\cap(\ker d)$ (cf. Prop. 2.5) and so $(1_{E_{s+1}} \wedge u\wedge 1_K)f_8(1_X \wedge i') =$

$(\bar{c}_s \wedge 1_{L\wedge K})g_3 + (\bar{a}_{s+1} \wedge 1_{L\wedge K})f_2'$ for some $f_2' \in [\Sigma^{tq+1}X \wedge M, E_{s+2} \wedge L \wedge K] \cap (\ker d)$ (cf. Prop. 2.5). Hence, (3.16) becomes

$$\begin{aligned}
&(\bar{a}_{s+1} \wedge 1_{X\wedge M})(1_{E_{s+2}} \wedge u_8(w_5 \wedge 1_M))f_3 \\
&\quad = -(\bar{a}_{s+1} \wedge 1_{X\wedge M})(1_{E_{s+2}} \wedge u_8(w_5 \wedge 1_M)(1_W \wedge (1_Y \wedge i)r))f' \\
&\qquad - \overline{\lambda}(\bar{a}_{s+1} \wedge 1_{X\wedge M})(1_{E_{s+2}} \wedge u'(u \wedge 1_K))f'(1_X \wedge ij) \\
&\qquad + (\bar{c}_s \wedge 1_{X\wedge M})(1_{KG_s} \wedge u_8(w_5 \wedge 1_M)(1_W \wedge (1_Y \wedge i)r))g \\
&\qquad - (\bar{c}_s \wedge 1_{X\wedge M})(1_{KG_s} \wedge u_8(w_5 \wedge 1_M)(1_W \wedge (r \wedge 1_M)\overline{m}_K))g(1_X \wedge ij) \\
&\qquad + \overline{\lambda}(\bar{c}_s \wedge 1_{X\wedge M})(1_{KG_s} \wedge u')g_3 + \overline{\lambda}(\bar{a}_{s+1} \wedge 1_{X\wedge M})(1_{E_{s+2}} \wedge u')f_2'. \qquad (3.17)
\end{aligned}$$

By (3.17), $(1_{KG_s} \wedge u_8(w_5 \wedge 1_M)(1_W \wedge (1_Y \wedge i)r))g - (1_{KG_s} \wedge u_8(w_5 \wedge 1_M)(1_W \wedge (r \wedge 1_M)\overline{m}_K))g(1_X \wedge ij) + \overline{\lambda}(1_{KG_s} \wedge u')g_3 \in [\Sigma^{tq}X \wedge M, KG_s \wedge X \wedge M]$ is a d_1-cycle which represents an element in $Ext_A^{s,tq}(H^*X \wedge M, H^*X \wedge M) \cong Z_p\{[\sigma \wedge 1_{X\wedge M}]\}$ by Lemma 3.3(1). Then we have

$$\begin{aligned}
&(1_{KG_s} \wedge u_8(w_5 \wedge 1_M)(1_W \wedge (1_Y \wedge i)r))g + \overline{\lambda}(1_{KG_s} \wedge u')g_3 \\
&\quad - (1_{KG_s} \wedge u_8(w_5 \wedge 1_M)(1_W \wedge (r \wedge 1_M)\overline{m}_K))g(1_X \wedge ij) = \bar{\lambda}_0(\sigma \wedge 1_{X\wedge M}) \qquad (3.18)
\end{aligned}$$

modulo d_1-boundary. Now we consider the cases of $\bar{\lambda}_0 \neq 1$ or $\bar{\lambda}_0 = 1$, respectively.

If $\bar{\lambda}_0 \neq 1$, then by (3.17) and $\bar{c}_s \cdot \sigma = \bar{a}_{s+1} \cdot \kappa$, we have

$$\begin{aligned}
(1_{E_{s+2}} \wedge u_8(w_5 \wedge 1_M))f_3 = &-(1_{E_{s+2}} \wedge u_8(w_5 \wedge 1_M)(1_W \wedge (1_Y \wedge i)r))f' \\
&- \bar{\lambda}(1_{E_{s+2}} \wedge u'(u \wedge 1_K))f'(1_X \wedge ij) + \bar{\lambda}(1_{E_{s+2}} \wedge u')f_2' \\
&+ \bar{\lambda}_0(\kappa \wedge 1_{X\wedge M}) + (\bar{c}_{s+1} \wedge 1_{X\wedge M})g_4
\end{aligned}$$

with $g_4 \in [\Sigma^{tq+1}X \wedge M, KG_{s+1} \wedge X \wedge M]$. By composing $(1_{E_{s+2}} \wedge 1_X \wedge \alpha) = (1_{E_{s+2}} \wedge \alpha_{X\wedge M} m_M(\widetilde{\psi} \wedge 1_M))$, we have $(1_{E_{s+2}} \wedge 1_X \wedge \alpha)(\kappa \wedge 1_{X\wedge M}) = (1_{E_{s+2}} \wedge \alpha_{X\wedge M}(j''u \wedge 1_M)(1_W \wedge m_M(\overline{u} \wedge 1_M)))f_3 = (1_{E_{s+2}} \wedge \alpha_{X\wedge M} \cdot m_M(\widetilde{\psi} \wedge 1_M)u_8(w_5 \wedge 1_M))f_3 = \bar{\lambda}_0(1_{E_{s+2}} \wedge 1_X \wedge \alpha)(\kappa \wedge 1_{X\wedge M})$ and the result follows.

If $\bar{\lambda}_0 = 1$, then by composing $(1_{KG_s} \wedge m_M(\widetilde{\psi} \wedge 1_M))$ on (3.18) using (3.15), we have $(1_{KG_s} \wedge j'(j''u \wedge 1_K))g = (1_{KG_s} \wedge m_M(\widetilde{\psi} \wedge 1_M)u_8(w_5 \wedge 1_M)(1_W \wedge (1_Y \wedge i)r))g = (\sigma \wedge 1_M)m_M(\widetilde{\psi} \wedge 1_M)$ modulo d_1-boundary. Moreover, by composing $(1_{KG_s} \wedge j\widetilde{\psi} \wedge 1_M)$ on (3.18) using (3.15) we have

$$\begin{aligned}
&(1_{KG_s} \wedge j\widetilde{\psi} \wedge 1_M)(\sigma \wedge 1_{X\wedge M}) \\
&\quad = (1_{KG_s} \wedge (j\widetilde{\psi} \wedge 1_M)u_8(w_5 \wedge 1_M)(1_W \wedge (1_Y \wedge i)r))g \\
&\qquad -(1_{KG_s} \wedge (j\widetilde{\psi} \wedge 1_M)u_8(w_5 \wedge 1_M)(1_W \wedge (r \wedge 1_M)\overline{m}_K))g(1_X \wedge ij) + \overline{\lambda}(1_{KG_s} \wedge (j\widetilde{\psi} \wedge 1_M)u')g_3 \\
&\quad = (1_{KG_s} \wedge ij(j'(j'' \wedge 1_K)(u \wedge 1_K)))g \\
&\qquad -(1_{KG_s} \wedge j'(j''u \wedge 1_K))g(1_X \wedge ij) + \bar{\lambda}(1_{KG_s} \wedge j'(j'' \wedge 1_K))g_3 \qquad \text{by (3.15)} \\
&\quad = (1_{KG_s} \wedge ij)(\sigma \wedge 1_M)m_M(\widetilde{\psi} \wedge 1_M) - (\sigma \wedge 1_M)m_M(\widetilde{\psi} \wedge 1_M)(1_X \wedge ij) \\
&\qquad +\overline{\lambda}(1_{KG_s} \wedge j'(j'' \wedge 1_K))g_3 \\
&\quad = (1_{KG_s} \wedge j\widetilde{\psi} \wedge 1_M)(\sigma \wedge 1_{X\wedge M}) + \overline{\lambda}(1_{KG_s} \wedge j'(j'' \wedge 1_K))g_3 \qquad \text{by (2.36)}
\end{aligned}$$

(modulo d_1-boundary) so that $(1_{KG_s} \wedge j'(j'' \wedge 1_K))g_3 = 0$ and we have $g_3 = (1_{KG_s} \wedge \overline{\mu}_2(1_Y \wedge i'))g_5$ (modulo d_1-boundary) for some $g_5 \in [\Sigma^{tq+q+1}X \wedge M, KG_s \wedge Y \wedge M]$. Hence, by (2.36), (3.10), we have

$$(1_{KG_{s+1}} \wedge \overline{\mu}_2(1_Y \wedge i')\psi_{X\wedge M})(g_0 \wedge 1_M) = (1_{KG_{s+1}} \wedge (u \wedge 1_K)\mu_{X\wedge M})(g_0 \wedge 1_M)$$

$$
\begin{aligned}
&= (\bar{b}_{s+1} \wedge 1_{L\wedge K})(1_{E_{s+1}} \wedge u \wedge 1_K) f_8 (1_X \wedge i') \\
&= (\bar{b}_{s+1}\bar{c}_s \wedge 1_{L\wedge K}) g_3 \\
&= (\bar{b}_{s+1}\bar{c}_s \wedge 1_{L\wedge K})(1_{KG_s} \wedge \overline{\mu}_2 (1_Y \wedge i')) g_5,
\end{aligned}
$$

and so $(1_{KG_{s+1}} \wedge \psi_{X\wedge M})(g_0 \wedge 1_M) = (\bar{b}_{s+1}\bar{c}_s \wedge 1_{Y\wedge M}) g_5$, which shows that $\lambda_1 (\psi_{X\wedge M})_* [h_0\sigma \wedge 1_{X\wedge M}] = (\psi_{X\wedge M})_* [g_0 \wedge 1_M] = 0 \in Ext_A^{s+1,tq+1}(H^*Y \wedge M, .H^*X \wedge M)$, and this implies $\lambda_1 = 0$ by Lemma 3.3(2). That is, $[g_0 \wedge 1_M] = 0$ and so $(\bar{b}_{s+1} \wedge 1_{W\wedge K}) f_8 (1_X \wedge i') = (\bar{b}_{s+1}\bar{c}_s \wedge 1_{W\wedge K}) g_6$ with $g_6 \in [\Sigma^{tq+q} X \wedge M, KG_s \wedge W \wedge K]$ and we have $f_8(1_X \wedge i') = (\bar{c}_s \wedge 1_{W\wedge K}) g_6 + (\bar{a}_{s+1} \wedge 1_{W\wedge K}) f_3'$ for some $f_3' \in [\Sigma^{tq+q+1} X \wedge M, E_{s+2} \wedge W \wedge K]$. Hence, by composing $(1_{E_{s+1}} \wedge w_5 \wedge 1_M)$ on (3.7), we have

$$
\begin{aligned}
&-(\bar{a}_{s+1} \wedge 1_{P\wedge M})(1_{E_{s+2}} \wedge w_5 \wedge 1_M) f_3 \\
&\quad = (\bar{a}_{s+1} \wedge 1_{P\wedge M})(1_{E_{s+2}} \wedge (w_5 \wedge 1_M)(1_W \wedge (1_Y \wedge i) r)) f' \\
&\qquad + (\bar{a}_{s+1} \wedge 1_{P\wedge M})(1_{E_{s+2}} \wedge (w_5 \wedge 1_M)(1_W \wedge (r \wedge 1_M)\overline{m}_K)) f' (1_X \wedge ij) \\
&\qquad + (\bar{a}_{s+1} \wedge 1_{P\wedge M})(1_{E_{s+2}} \wedge (w_5 \wedge 1_M)(1_W \wedge (r \wedge 1_M)\overline{m}_K)) f_3' + (\bar{c}_s \wedge 1_{P\wedge M}) g_7
\end{aligned}
$$

with d_1-cycle $g_7 = (1_{KG_s} \wedge (w_5 \wedge 1_M)(1_W \wedge (1_Y \wedge i) r)) g - (1_{KG_s} \wedge (w_5 \wedge 1_M)(1_W \wedge (r \wedge 1_M)\overline{m}_K)) g (1_X \wedge ij) + (1_{KG_s} \wedge (w_5 \wedge 1_M)(1_W \wedge (r \wedge 1_M)\overline{m}_K)) g_6 \in [\Sigma^{tq+q+1} X \wedge M, KG_s \wedge P \wedge M]$, which represents an element in $Ext_A^{s,tq+q+1}(H^*P \wedge M, H^*X \wedge M)$. However, this Ext group is zero which follows from the following exact sequence:

$$
\begin{aligned}
&0 = Ext_A^{s,tq+q}(H^*W \wedge K, H^*X \wedge M) \xrightarrow{((w_5\wedge 1_M)(1_W \wedge (r\wedge 1_M)\overline{m}_K)_*} \\
&\qquad Ext_A^{s,tq+q+1}(H^*P \wedge M, H^*X \wedge M) \xrightarrow{(u_7)_*} Ext_A^{s,tq-2q}(H^*M, H^*X \wedge M) \xrightarrow{((1_W\wedge i')(\phi_W \wedge M))_*}
\end{aligned}
$$

induced by (3.13), where the left group is zero by Prop. 2.40(4) and the right group has a unique generator $m_M(\widetilde{\psi} \wedge 1_M)^*(\tilde{\sigma})$ (cf. Prop. 2.40(1)), which satisfies $((1_W \wedge i')(\phi_W \wedge 1_M))_* m_M(\widetilde{\psi} \wedge 1_M)^*(\tilde{\sigma}) \neq 0 \in Ext_A^{s+1,tq+q}(H^*W \wedge K, H^*X \wedge M)$.

Hence, $(\bar{c}_s \wedge 1_{P\wedge M}) g_7 = 0$ and we have $-(1_{E_{s+2}} \wedge w_5 \wedge 1_M) f_3 = (1_{E_{s+2}} \wedge (w_5 \wedge 1_M) u_8 (1_W \wedge (1_Y \wedge i) r)) f' - (1_{E_{s+2}} \wedge (w_5 \wedge 1_M)(1_W \wedge (r \wedge 1_M)\overline{m}_K)) f' (1_X \wedge ij) + (1_{E_{s+2}} \wedge (w_5 \wedge 1_M)(1_W \wedge (r \wedge 1_M)\overline{m}_K)) f_3' + (\bar{c}_{s+1} \wedge 1_{P\wedge M}) g_8$ for some $g_8 \in [\Sigma^{tq+q+2} X \wedge M, KG_{s+1} \wedge P \wedge M]$. By composing $(1_{E_{s+2}} \wedge \alpha_{X\wedge M} \cdot u_7)$, we have $(1_{E_{s+2}} \wedge 1_X \wedge \alpha)(\kappa \wedge 1_{X\wedge M}) = (1_{E_{s+2}} \wedge \alpha_{X\wedge M}(j'' u \wedge 1_M)(1_W \wedge m_M(\overline{u} \wedge 1_M))) f_3 = (1_{E_{s+2}} \wedge \alpha_{X\wedge M} \cdot u_7 (w_5 \wedge 1_M)) f_3 = 0$, which shows the result.

Step 2 To prove $(\bar{c}_{s+1} \wedge 1_M)\widetilde{h_0\sigma} = (\kappa \wedge 1_M)\alpha = 0$.

By (2.34), (2.35), $\mu_{X\wedge M}(1_X \wedge \alpha i) = \mu_{X\wedge M}\alpha_{X\wedge M}\widetilde{\psi} = 0$, so that $\mu_{X\wedge M} = \mu_{X\wedge K'}(1_X \wedge v)$ with $\mu_{X\wedge K'} \in [X \wedge K', W \wedge K]$. We claim that $X \wedge K'$ splits into $W \wedge K \vee \Sigma^q Y$, that is, there is a split cofibration $\Sigma^q Y \to X \wedge K' \to W \wedge K$. This can be seen by the following homotopy commutative diagram of 3×3-Lemma using (2.9), (2.25), (2.35) and $(1_Y \wedge j)\alpha_{Y\wedge M} j' = r(1_K \wedge \alpha_1)$:

$$
\begin{array}{ccccc}
X \wedge M & \xrightarrow{\mu_{X\wedge M}} & W \wedge K & \xrightarrow{0} & \Sigma^{q+1} Y \\
\searrow 1_X \wedge v & \nearrow \mu_{X\wedge K'} & \searrow^{j'(j''u\wedge 1_K)} & \nearrow_{(1_Y\wedge j)\alpha_{Y\wedge M}} & \\
& X \wedge K' & & \Sigma^{3q+1} M & \\
\nearrow \widetilde{\tau}_{X\wedge K'} & \searrow 1_X \wedge y & \nearrow \widetilde{\psi} & \searrow \alpha_{X\wedge M} & \\
\Sigma^q Y & \xrightarrow{\tilde{u} w_2} & \Sigma^{q+1} X & \xrightarrow{1_X \wedge \alpha i} & \Sigma X \wedge M.
\end{array}
$$

That is, we have a split cofibration $\Sigma^q Y \xrightarrow{\tau_{X\wedge K'}} X \wedge K' \xrightarrow{\mu_{X\wedge K'}} W \wedge K$ and so there is $\nu_{X\wedge K'} : X \wedge K' \to \Sigma^q Y$ and $\widetilde{\nu}_{X\wedge K'} : W \wedge K \to X \wedge K'$ such that $\nu_{X\wedge K'} \cdot \tau_{X\wedge K'} = 1_Y$, $\mu_{X\wedge K'} \cdot \widetilde{\nu}_{X\wedge K'} = 1_{W\wedge K}$, $\widetilde{\tau}_{X\wedge K'} \cdot \nu_{X\wedge K'} + \widetilde{\nu}_{X\wedge K'} \cdot \mu_{X\wedge K'} = 1_{X\wedge K'}$.

It follows from Step 1 that $(\kappa\wedge 1_{M\wedge X\wedge K'})(\alpha\wedge 1_{X\wedge K'})=0$, then $(\kappa\wedge 1_{M\wedge Y})(\alpha\wedge 1_Y)=(1_{E_{s+2}}\wedge 1_M\wedge \nu_{X\wedge K'})(\kappa\wedge 1_{M\wedge X\wedge K'})(\alpha\wedge 1_{X\wedge K'})(1_M\wedge\tau_{X\wedge K'})=0$. By using the splitness in (2.22) again we have $(\bar{c}_{s+1}\wedge 1_M)\widetilde{h_0\sigma}=(\kappa\wedge 1_M)\alpha=(1_{E_{s+2}}\wedge 1_M\wedge\widetilde{\nu})(\kappa\wedge 1_{M\wedge Y\wedge K'})(\alpha\wedge 1_{Y\wedge K'})(1_M\wedge\widetilde{\tau})=0$, which shows the theorem.

Proof of Theorem B In the case $(s,tq,\sigma)=(2,p^nq+p^mq,h_nh_m)$, by [2, Prop. 2.1(1)] and the knowledge of Z_p-base of $Ext_A^{s,*}(Z_p,Z_p)$ for $s=1,2,3$ in [1] and [10, Table 8.1], all the supposition I of Theorem A holds. Moreover, from [2, (3.17)] we have $(1_{E_4}\wedge i)\kappa\cdot(\alpha_1)_L=0$, where $\kappa\in\pi_{p^nq+p^mq+1}E_4$ satisfying $\bar{a}_3\cdot\kappa=\bar{c}_2\cdot h_nh_m$. Then, by smashing 1_L and composing $\widetilde{i''}\in[\Sigma^qS,L\wedge L]$ we have $(1_{E_4}\wedge(i\wedge 1_L)\phi)\kappa=(1_{E_4}\wedge i\wedge 1_L)(\kappa\wedge 1_L)((\alpha_1)_L\wedge 1_L)\widetilde{i''}=0$, where $\widetilde{i''}\in[\Sigma^qS,L\wedge L]$ such that $(1_L\wedge j'')\widetilde{i''}=i''$ and satisfying $((\alpha_1)_L\wedge 1_L)\widetilde{i''}=\phi\in[\Sigma^{2q-1}S,L]$. As in the proof of Lemma 3.2(2), $(\bar{c}_{s+1}\wedge 1_L)\sigma\phi=(1_{E_{s+1}}\wedge\phi)\kappa$ (up to scalar), so the supposition II of Theorem A also holds for $(s,tq,\sigma)=(2,p^nq+p^mq,h_nh_m)$ and Theorem B follows.

Remark In the proof of Theorem A, we use only supposition (II) to input $(1_{E_{s+2}}\wedge\phi\wedge 1_M)(\kappa\wedge 1_M)m_M(\widetilde{\psi}\wedge 1_M)=0$. Then, the geometric supposition (II) of Theorem A can be weakened to suppose that $m_M(\widetilde{\psi}\wedge 1_M)^*(\phi\wedge 1_M)_*(\tilde{\sigma})\in Ext_A^{s+1,tq}(H^*L\wedge M,H^*X\wedge M)$ is a permanent cycle in the ASS. It is expected that the geometric supposition (II) of the pull back Theorem A also can be weakened to suppose that $i_*(h_0\sigma)\in Ext_A^{s+1,tq+q}(H^*M,Z_p)$ is a permanent cycle in the ASS, but it needs a large amount of work to prove it.

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