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New Families in the Stable Homotopy of Spheres Revisited

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Abstract This paper constructs a new family in the stable homotopy of spheres $\pi_{t-6}S$ represented by $h_n g_0 \gamma_3 \in E_2^{6,t}$ in the Adams spectral sequence which revisits the $b_{n-1}g_0\gamma_3$-elements $\in \pi_{t-7}S$ constructed in [3], where $t = 2p^n(p-1) + 6(p^2+p+1)(p-1)$ and $p \geq 7$ is a prime, $n \geq 4$.

Keywords Stable homotopy of spheres, Adams spectral sequence, Toda-Smith spectrum, Adams resolution
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1 Introduction

Let A be the mod p Steenrod algebra and S the sphere spectrum localized at an odd prime p. To determine the stable homotopy groups of spheres π_*S is one of the central problems in homotopy theory. One of the main tools to reach it is the Adams spectral sequence (ASS) $E_2^{s,t} = \mathrm{Ext}_A^{s,t}(Z_p, Z_p) \Longrightarrow \pi_{t-s}S$, where the $E_2^{s,t}$-term is the cohomology of A.

From [1], $\mathrm{Ext}_A^{1,*}(Z_p, Z_p)$ has the Z_p-base consisting of $a_0 \in \mathrm{Ext}_A^{1,1}(Z_p, Z_p)$, $h_i \in \mathrm{Ext}_A^{1,p^iq}$ (Z_p, Z_p) for all $i \geq 0$ and $\mathrm{Ext}_A^{2,*}(Z_p, Z_p)$ has the Z_p-base consisting of $\alpha_2, a_0^2, a_0h_i (i > 0), g_i (i \geq 0), k_i (i \geq 0), b_i (i \geq 0)$ and $h_ih_j (j \geq i+2, i \geq 0)$ whose internal degrees are $2q+1, 2, p^iq+1, p^{i+1}q+2p^iq, 2p^{i+1}q+p^iq, p^{i+1}q$ and p^iq+p^jq, respectively, where $q = 2(p-1)$. From [2, p.110, Table 8.1], the Z_p-base of $\mathrm{Ext}_A^{3,*}(Z_p, Z_p)$ has been completely listed and there is a generator $\gamma_3 \in \mathrm{Ext}_A^{3,(3p^2+2p+1)q}(Z_p, Z_p)$ whose name in [2] is $h_{0,1,2,3}$.

In [3], a family in π_*S, which is represented by $b_{n-1}g_0\gamma_3 \in \mathrm{Ext}_A^{7,p^nq+3(p^2+p+1)q}$ (Z_p, Z_p) in the ASS, has been detected. The main purpose of this paper is to construct a new family in π_*S revisited [3]. Our result is the following theorem:

Theorem I *Let $p \geq 7, n \geq 4$. Then the product*

$$h_n g_0 \gamma_3 \neq 0 \in \mathrm{Ext}_A^{6,p^nq+3(p^2+p+1)q}(Z_p, Z_p)$$

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and it converges in the ASS to a nontrivial element in $\pi_{p^nq+3(p^2+p+1)q-6}S$ *of order* p.

The construction of the above $h_ng_0\gamma_3$-element is parallel to that of $b_{n-1}g_0\gamma_3$-element given in [3]. That is, Theorem I will also be proved on the basis of the following Theorem II revisited [3, Theorem II].

Let M be the Moore spectrum modulo a prime $p \geq 3$ given by the cofibration

$$S \xrightarrow{p} S \xrightarrow{i} M \xrightarrow{j} \Sigma S. \tag{1.1}$$

Let $\alpha : \Sigma^q M \longrightarrow M$ be the Adams map and K be its cofibre given by the cofibration

$$\Sigma^q M \xrightarrow{\alpha} M \xrightarrow{i'} K \xrightarrow{j'} \Sigma^{q+1} M, \tag{1.2}$$

where $q = 2(p-1)$. This spectrum, which we briefly write as K, is known as the Toda-Smith spectrum V(1). Theorem I will be proved on basis of the following theorem:

Theorem II *Let* $p \geq 5, n \geq 3$. *Then*

$$h_ng_0 \in \mathrm{Ext}_A^{3,p^nq+pq+2q}(H^*K, Z_p),$$

the reduction of $h_ng_0 \in \mathrm{Ext}_A^{3,p^nq+pq+2q}(Z_p, Z_p)$, *converges in the ASS to a nontrivial homotopy element in* $\pi_{p^nq+pq+2q-3}K$.

Parallel to the detection of the element $\zeta''_{n-1} \in [\Sigma^{p^nq+q-4}K, K]$ in [3], we will find an element $\eta''_n \in [\Sigma^{p^nq+q-3}K, K]$ (given in Prop. 3.4) so that $j'\eta''_n \in [\Sigma^{p^nq-4}K, M]$ is represented by $(jj')^*i_*(h_0h_n) \in \mathrm{Ext}_A^{2,p^nq-2}(H^*M, H^*K)$ in the ASS. Then $\eta''_n\beta i'i \in \pi_{p^nq+(p+2)q-3}K$ is our desired map in Theorem II and $jj'\bar{j}\gamma^3\bar{i}\eta''_n\beta i'i \in \pi_{p^nq+3(p^2+p+1)q-6}S$ is the $h_ng_0\gamma_3$-element, where $\beta \in [\Sigma^{(p+1)q}K, K]$ and $\gamma \in [\Sigma^{(p^2+p+1)q}V(2), V(2)]$ are the known v_2- and v_3-periodicity elements, respectively.

Note that the proof, in [3, Theorem II], of detecting ζ''_{n-1} relies on the fact that $a_0b_{n-1} \in \mathrm{Ext}_A^{3,p^nq+1}(Z_p, Z_p)$ is hit by a differential $d_2(h_n)$ and this no longer holds for $a_0h_n \in \mathrm{Ext}_A^{2,p^nq+1}(Z_p, Z_p)$. So, the arguments in [3] are not valid for proving the existence of η''_n here. However,we can say that the proof of the existence of η''_n given in this paper will be valid to prove the existence of ζ''_{n-1} in [3].

Some techniques on the derivation of maps between M-module spectra will play an important role in the proof of Theorem II and especially of Prop. 3.4. After giving some preliminaries on it and some low-dimensional Ext groups in Section 2, the proof of the main theorems will be given in Section 3.

2 Some Preliminaries on Derivations and Low-dimensional Ext Groups

In this section, we first recall some results on derivations of maps between M-module spectra developed in [4]. From [4, p. 204–206], the Moore spectrum M is a commutative ring spectrum with multiplication $m_M : M \wedge M \to M$ and there is $\overline{m}_M : \Sigma M \to M \wedge M$ such that

$$\begin{aligned} &m_M(i \wedge 1_M) = 1_M, \quad (j \wedge 1_M)\overline{m}_M = 1_M, \\ &m_M\overline{m}_M = 0, \quad \overline{m}_M(j \wedge 1_M) + (i \wedge 1_M)m_M = 1_{M\wedge M} \end{aligned} \tag{2.1}$$

and

$$m_M T = -m_M, \quad T\overline{m}_M = \overline{m}_M, \quad m_M(1_M \wedge i) = -1_M, \quad (1_M \wedge j)\overline{m}_M = 1_M, \tag{2.2}$$

where $T : M \wedge M \to M \wedge M$ is the switching map.

A spectrum X is called an M-module spectrum if $p \wedge 1_X = 0 \in [X, X]$, and consequently, the cofibration $X \xrightarrow{p\wedge 1_X} X \xrightarrow{i\wedge 1_X} M \wedge X \xrightarrow{j\wedge 1_X} \Sigma X$ splits, i.e. there is a homotopy equivalence $M \wedge X = X \vee \Sigma X$ and there are maps $m_X : M \wedge X \to X$, $\overline{m}_X : \Sigma X \to M \wedge X$ satisfying

$$\begin{aligned} &m_X(i \wedge 1_X) = 1_X, \quad (j \wedge 1_X)\overline{m}_X = 1_X, \\ &m_X\overline{m}_X = 0, \quad \overline{m}_X(j \wedge 1_X) + (i \wedge 1_X)m_X = 1_{M\wedge X}. \end{aligned}$$

The M-module actions $m_X, \overline{m}_X$ are called associative if there are commutativities $m_X(1_M \wedge m_X) = -m_X(m_M \wedge 1_X)$ and $(1_M \wedge \overline{m}_X)\overline{m}_X = (\overline{m}_M \wedge 1_X)\overline{m}_X$.

Let X and X' be M-module spectra. Then we define a homomorphism $d : [\Sigma^s X', X] \to [\Sigma^{s+1} X', X]$ by $d(f) = m_X(1_M \wedge f)\overline{m}_{X'}$ for $f \in [\Sigma^s X', X]$. This operation d is called a derivation (of maps between M-module spectra) which has the following properties:

Proposition 2.3 [4, p. 210, Theorem 2.2] (i) *d is derivative*: $d(fg) = fd(g) + (-1)^{|g|}\, d(f)g$ *for $f \in [\Sigma^s X', X]$, $g \in [\Sigma^t X'', X']$, where X, X', X'' are M-module spectra.*

(ii) *Let W', W be arbitrary spectra and $h \in [\Sigma^r W', W]$. Then $d(h \wedge f) = (-1)^{|h|} h \wedge d(f)$ for $f \in [\Sigma^s X', X]$.*

(iii) $d^2 = 0 : [\Sigma^s X', X] \to [\Sigma^{s+2} X', X]$ *for associative spectra X', X.*

From [4, p. 217, (3.4)], K is an M-module spectrum, i.e. there are M-module actions $m_K : K \wedge M \to K$, $\overline{m}_K : \Sigma K \to K \wedge M$ satisfying

$$\begin{aligned} &m_K(1_K \wedge i) = 1_K, \quad (1_K \wedge j)\overline{m}_K = 1_K, \\ &m_K\overline{m}_K = 0, \quad (1_K \wedge i)m_K + (1_K \wedge j)\overline{m}_K = 1_{K\wedge M}. \end{aligned} \tag{2.4}$$

Moreover, from [4, p. 218, (3.7)] we have

$$d(ij) = -1_M, \quad d(\alpha) = 0, \quad d(i') = 0, \quad d(j') = 0. \tag{2.5}$$

The following proposition is a generalization of Theorem A(c) in [5]:

Proposition 2.6 *Let V, V' be arbitrary spectra. Then there is a direct sum decomposition*

$$[\Sigma^* V \wedge M, V' \wedge M] = (\ker d) \oplus (1_{V'} \wedge ij)(\ker d),$$

where $\mathrm{ker} d = [\Sigma^* V \wedge M, V' \wedge M] \cap (\ker d)$.

Proof The proof is a modification of the proof of Theorem A(c) in [5, p. 631]. Let $\delta_L(f) = (1_{V'} \wedge ij)f$ for $f \in [\Sigma^* V \wedge M, V' \wedge M]$. Then we have exact sequences

$$[\Sigma^s V \wedge M, V' \wedge M] \xrightarrow{d} [\Sigma^{s+1} V \wedge M, V' \wedge M] \xrightarrow{d} [\Sigma^{s+2} V \wedge M, V' \wedge M],$$
$$[\Sigma^s V \wedge M, V' \wedge M] \xleftarrow{\delta_L} [\Sigma^{s+1} V \wedge M, V' \wedge M] \xleftarrow{\delta_L} [\Sigma^{s+2} V \wedge M, V' \wedge M],$$

which split each other. To prove this, we claim that $V \wedge M, V' \wedge M$ are associative M-module spectra, then $d^2 = 0$ and $\delta_L^2 = 0$, since $ijij = 0$. On the other hand, by Prop. 2.3(i) and $d(1_{V'} \wedge ij) = -1_{V' \wedge M}$, we have $d((1_{V'} \wedge ij)f) = \pm f + (1_{V'} \wedge ij)d(f)$, then if $d(f) = 0$, $f = \pm d((1_{V'} \wedge ij)f)$ and if $\delta_L(f) = 0$, $f = \pm(1_{V'} \wedge ij)d(f)$, which shows the result.

To prove the claim, we need to show that $m_{V\wedge M}(1_M \wedge m_{V\wedge M}) = -m_{V\wedge M}(m_M \wedge 1_{V\wedge M})$ and $(1_M \wedge \overline{m}_{V\wedge M})\overline{m}_{V\wedge M} = (\overline{m}_M \wedge 1_{V\wedge M})\overline{m}_{V\wedge M}$, where $m_{V\wedge M} = (1_V \wedge m_M)(T_{M,V} \wedge 1_M) : M \wedge V \wedge M \xrightarrow{T_{M,V}\wedge 1_M} V \wedge M \wedge M \xrightarrow{1_V \wedge m_M} V \wedge M$ and $\overline{m}_{V\wedge M} = (T_{V,M} \wedge 1_M)(1_V \wedge \overline{m}_M) : \Sigma V \wedge M \xrightarrow{1_V \wedge \overline{m}_M} V \wedge M \wedge M \xrightarrow{T_{V,M}\wedge 1_M} M \wedge V \wedge M$ are the M-module action of $V \wedge M$ in which $T_{M,V} : M \wedge V \to V \wedge M$, $T_{V,M} : V \wedge M \to M \wedge V$ are the switching maps. In fact, we have

$$\begin{aligned}
m_{V\wedge M}(1_M \wedge m_{V\wedge M}) &= (1_V \wedge m_M)(T_{M,V} \wedge 1_M)(1_M \wedge 1_V \wedge m_M)(1_M \wedge T_{M,V} \wedge 1_M) \\
&= (1_V \wedge m_M)(1_V \wedge 1_M \wedge m_M)(T_{M\wedge M,V} \wedge 1_M) \text{ with } T_{M\wedge M,V} : (M \wedge M) \wedge V \to V \wedge (M \wedge M) \\
&= -(1_V \wedge m_M)(1_V \wedge m_M \wedge 1_M)(T_{M\wedge M,V} \wedge 1_M), \text{ by the associativity of } m_M \\
&= -(1_V \wedge m_M)(T_{M,V} \wedge 1_M)(m_M \wedge 1_V \wedge 1_M) \\
&= -m_{V\wedge M}(m_M \wedge 1_{V\wedge M}).
\end{aligned}$$

This shows the first associativity of the M-module spectrum $V \wedge M$, while the proof of the second one is similar. Q.E.D.

Corollary 2.7 *Let X, V,V' and V'' be arbitrary spectra and $g : V \to V'$, $g' : V' \to V''$ be maps. If $[V'' \wedge M, X \wedge M] \xrightarrow{(g'\wedge 1_M)^*} [V' \wedge M, X \wedge M] \xrightarrow{(g\wedge 1_M)^*} [V \wedge M, X \wedge M]$ is an exact sequence, then $\ker d \cap [V'' \wedge M, X \wedge M] \xrightarrow{(g'\wedge 1_M)^*} \ker d \cap [V' \wedge M, X \wedge M] \xrightarrow{(g\wedge 1_M)^*} \ker d \cap [V \wedge M, X \wedge M]$ is also exact, where d is the derivation defined on the corresponding group.*

Proof For any $f \in \ker d \cap [V' \wedge M, X \wedge M]$ such that $f \in \ker(g \wedge 1_M)^*$, there is $f' \in [V'' \wedge M, X \wedge M]$ so that $f'(g' \wedge 1_M) = f$. By Prop. 2.6, $f' = f_1' + (1_X \wedge ij)f_2'$ with $f_1' \in \ker d \cap [V'' \wedge M, X \wedge M]$ and $f_2' \in \ker d \cap [\Sigma V'' \wedge M, X \wedge M]$. Then, by applying d on the equation $f = f_1'(g' \wedge 1_M) + (1_X \wedge ij)f_2'(g' \wedge 1_M)$ we have $f_2'(g' \wedge 1_M) = 0$ and so $f = f_1'(g' \wedge 1_M)$ with $f_1' \in \ker d \cap [V'' \wedge M, X \wedge M]$ as desired. Q.E.D.

Now we turn to considering some results on low-dimensional Ext groups which will be used in the proof of the main theorems and especially of Prop. 3.4.

Proposition 2.8 *Let $p \geq 7, n \geq 4$. Then the product $h_n g_0 \gamma_3 \neq 0 \in \mathrm{Ext}_A^{6,p^n q+3(p^2+2p+1)q}(Z_p, Z_p)$, where $\gamma_3 = h_{0,1,2,3} \in \mathrm{Ext}_A^{3,(3p^2+2p+1)q}(Z_p, Z_p)$ as in* [2, *Table* 8.1].

Proof The proof is similar to that given in the proof of [3, Prop. 2.2] and is omitted here.

Proposition 2.9 *Let $p \geq 3, n \geq 2$. Then* $\mathrm{Ext}_A^{2,p^n q+q}(H^*M, H^*M) \cong Z_p\{(ij)_*\alpha_*(\tilde{h}_n), \alpha_*(ij)^*$

$(\tilde{h}_n)\}$ *and* $\mathrm{Ext}_A^{2,p^nq+q}(H^*K,H^*M)\cong Z_p\{i'_*(ij)_*\alpha_*(\tilde{h}_n)=i'_*(\alpha_1\wedge 1_M)_*(\tilde{h}_n)\}$, *where* $\alpha_1=j\alpha i:\Sigma^{q-1}S\to S$ *and* $\tilde{h}_n$ *is the unique generator of* $\mathrm{Ext}_A^{1,p^nq}(H^*M,H^*M)$ *stated in* [3, *Prop.* 2.4(2)].

Proof Since $\mathrm{Ext}_A^{2,p^nq+q}(Z_p,Z_p)$ has the unique generator $h_0h_n=j_*\alpha_*i_*(h_n)=j_*\alpha_*i^*(\tilde{h}_n)$, then the first result follows from the following exact sequence:

$$\xrightarrow{p^*}\mathrm{Ext}_A^{2,p^nq+q+1}(H^*M,Z_p)\xrightarrow{j^*}\mathrm{Ext}_A^{2,p^nq+q}(H^*M,H^*M)\xrightarrow{i^*}\mathrm{Ext}_A^{2,p^nq+q}(H^*M,Z_p)\xrightarrow{p^*}$$

induced by (1.1), where the right group has the unique generator $i^*(ij)_*\alpha_*(\tilde{h}_n)=(ij)_*\alpha_*i_*(h_n)$ satisfying $p^*(ij)_*\alpha_*i_*(h_n)=(ij)_*\alpha_*i_*p_*(h_n)=0$ and the left group has the unique generator $\alpha_*i_*(h_n)=i^*\alpha_*(\tilde{h}_n)$ (cf. [3, Prop. 2.4(2)]).

Look at the following exact sequence:

$$\mathrm{Ext}_A^{2,p^nq+q}(H^*M,H^*M)\xrightarrow{i'_*}\mathrm{Ext}_A^{2,p^nq+q}(H^*K,H^*M)\xrightarrow{j'_*}\mathrm{Ext}_A^{2,p^nq-1}(H^*M,H^*M)\xrightarrow{\alpha_*}$$

induced by (1.2). Since $\mathrm{Ext}_A^{2,p^nq-r}(Z_p,Z_p)=0$ for $r=1,2$ and has the unique generator b_{n-1} for $r=0$, then the right group has the unique generator $(ij)^*(\tilde{b}_{n-1})$ satisfying $\alpha_*(ij)^*(\tilde{b}_{n-1})=j^*\alpha_*i_*(b_{n-1})\neq 0\in\mathrm{Ext}_A^{3,p^nq+q}(H^*M,H^*M)$ (cf. [3, Prop. 2.4(1)]). So $\mathrm{Ext}_A^{2,p^nq+q}(H^*K,H^*M)$ $=i'_*\mathrm{Ext}_A^{2,p^nq+q}(H^*M,H^*M)$ has the unique generator $(i')_*(ij)_*\alpha_*\tilde{h}_n=i'_*(\alpha_1\wedge 1_M)_*(\tilde{h}_n)$, since $(\alpha_1\wedge 1_M)_*(\tilde{h}_n)=(ij)_*\alpha_*(\tilde{h}_n)-\alpha_*(ij)_*(\tilde{h}_n)$ by the fact that $\alpha_1\wedge 1_M=ij\alpha-\alpha ij$ (cf. [6, p. 428, (5.1)]). Q.E.D.

Proposition 2.10 *Let* $p\geq 3,n\geq 2$. *Then:*

(1) $\mathrm{Ext}_A^{2,p^nq+2q+r}(H^*K,H^*M)=0$ *for* $r=0,1,2$, $\mathrm{Ext}_A^{2,p^nq+2q+1}(H^*K,Z_p)=0$;

(2) $\mathrm{Ext}_A^{2,p^nq+q+r}(H^*K,Z_p)=0$ *for* $r=1,2,3$, $\mathrm{Ext}_A^{2,p^nq+q+r}(H^*K,H^*M)=0$ *for* $r=1,2$;

(3) $\mathrm{Ext}_A^{2,p^nq+q}(H^*K,H^*K)\cong Z_p\{(h_0h_n)'\}$ *with* $(i')^*(h_0h_n)'=(i'ij\alpha)_*(\tilde{h}_n)$.

Proof (1) Look at the following exact sequence:

$$\mathrm{Ext}_A^{2,p^nq+2q+r}(H^*M,H^*M)\xrightarrow{i'_*}\mathrm{Ext}_A^{2,p^nq+2q+r}(H^*K,H^*M)$$
$$\xrightarrow{j'_*}\mathrm{Ext}_A^{2,p^nq+q+r-1}(H^*M,H^*M)\xrightarrow{\alpha_*}$$

induced by (1.2). The left group is zero since $\mathrm{Ext}_A^{2,p^nq+2q+t}(Z_p,Z_p)=0$ for $t=-1,0,1,2,3$ (cf. [1]). The right group has the unique generator $(ij)^*(ij)_*\alpha_*(\tilde{h}_n)$ for $r=0$, and has two generators $(ij)_*\alpha_*(\tilde{h}_n)$ and $(ij)^*\alpha_*(\tilde{h}_n)$ for $r=1$ and has the unique generator $\alpha_*(\tilde{h}_n)$ for $r=2$ (cf. [3, Prop. 2.4 (2)]). We claim that (i) $\alpha_*(ij)^*(ij)_*\alpha_*(\tilde{h}_n)\neq 0$; (ii) $\alpha_*[\lambda_1(ij)_*\alpha_*(\tilde{h}_n)+\lambda_2\alpha_*(ij)^*(\tilde{h}_n)]\neq 0$; (iii) $\alpha_*\alpha_*(\tilde{h}_n)\neq 0$. Then the above α_* is monic and so $imj'_*=0$. This shows that $\mathrm{Ext}_A^{2,p^nq+2q+r}(H^*K,H^*M)=0$ for $r=0,1,2$ and consequently we have $\mathrm{Ext}_A^{2,p^nq+2q+1}$ $(H^*K,Z_p)=0$.

To prove the claim, we recall from [2, Table 8.1] that $\alpha_2h_n=j_*\alpha_*\alpha_*i_*(h_n)\neq 0\in\mathrm{Ext}_A^{3,p^nq+2q+1}$ (Z_p,Z_p), then $i_*(\alpha_2h_n)\neq 0\in\mathrm{Ext}_A^{3,p^nq+2q+1}(H^*M,Z_p)$ since $\mathrm{Ext}_A^{2,p^nq+2q}(Z_p,Z_p)=0$ (cf. [1]). We also have $j^*i_*(\alpha_2h_n)\neq 0\in\mathrm{Ext}_A^{3,p^nq+2q}(H^*M,H^*M)$ since $\mathrm{Ext}_A^{2,p^nq+2q}$ $(H^*M,Z_p)=0$. Hence, by $2\alpha ij\alpha=ij\alpha^2+\alpha^2ij$ (cf. [6, p. 428 line 20]),

$$\alpha_*(ij)^*(ij)_*\alpha_*(\tilde{h}_n)=j^*\alpha_*(ij)_*\alpha_*i_*(h_n)=\frac{1}{2}j^*(ij)_*\alpha_*\alpha_*i_*(h_n)=\frac{1}{2}j^*i_*(\alpha_2h_n)\neq 0.\quad(2.11)$$

This shows (i). For the claim (ii),

$$\alpha_*[\lambda_1(ij)_*\alpha_*(\tilde{h}_n)+\lambda_2\alpha_*(ij)^*(\tilde{h}_n)]=\frac{1}{2}\lambda_1(ij)_*\alpha_*\alpha_*(\tilde{h}_n)+\left(\frac{1}{2}\lambda_1+\lambda_2\right)\alpha_*\alpha_*(ij)^*(\tilde{h}_n)\neq 0,$$

since the two terms are linearly independent by the fact that $(ij)_*\alpha_*\alpha_*(ij)^*(\tilde{h}_n)\neq 0$ (cf. (2.11)). The claim (iii) is self-evident since $i^*j_*\alpha_*\alpha_*(\tilde{h}_n)=j_*\alpha_*\alpha_*i_*(h_n)=\alpha_2 h_n\neq 0$.

(2) Consider the following exact sequence $(r=1,2,3)$:

$$\mathrm{Ext}_A^{2,p^nq+q+r}(H^*M,Z_p)\xrightarrow{i'_*}\mathrm{Ext}_A^{2,p^nq+q+r}(H^*K,Z_p)\xrightarrow{j'_*}\mathrm{Ext}_A^{2,p^nq+r-1}(H^*M,Z_p)\xrightarrow{\alpha_*}$$

induced by (1.2). The left group is zero for $r=2,3$ since $\mathrm{Ext}_A^{2,p^nq+q+t}(Z_p,Z_p)=0$ for $t=1,2,3$ (cf. [1]) and has the unique generator $\alpha_*i_*(h_n)$ for $r=1$ so that im $i'_*=0$. The right group is zero for $r=2,3$ (cf. [3, Prop. 2.3(1)]) and has the unique generator $i_*(b_{n-1})$ for $r=1$ satisfying $\alpha_*i_*(b_{n-1})\neq 0\in\mathrm{Ext}_A^{3,p^nq+q+1}(H^*M,Z_p)$ so that im $j'_*=0$ and so the result follows.

(3) Consider the following exact sequence:

$$\mathrm{Ext}_A^{2,p^nq+2q+1}(H^*K,H^*M)\xrightarrow{(j')^*}\mathrm{Ext}_A^{2,p^nq+q}(H^*K,H^*K)\xrightarrow{(i')^*}\mathrm{Ext}_A^{2,p^nq+q}(H^*K,H^*M)$$

induced by (1.2). The left group is zero by (1) and the right group has the unique generator $i'_*(ij)_*\alpha_*(\tilde{h}_n)$ by Prop. 2.9 which satisfies $\alpha^*i'_*(ij)_*\alpha_*(\tilde{h}_n)=i'_*(ij)_*\alpha_*\alpha^*(\tilde{h}_n)=i'_*(ij)_*\alpha_*\alpha_*(\tilde{h}_n)=0$ since $i'ij\alpha^2=2i'\alpha ij\alpha-i'\alpha^2ij=0\in[\Sigma^{2q-1}M,K]$. Then the result follows. Q.E.D.

Proposition 2.12 *Let* $p\geq 3, n\geq 2$. *Then* $\mathrm{Ext}_A^{2,p^nq+q-1}(H^*K,H^*K)\cong Z_p\{(h_0h_n)''\}$ *with* $(i')^*(h_0h_n)''=i'_*(ij)_*(\alpha_1\wedge 1_M)_*(\tilde{h}_n)$.

Proof Look at the following exact sequence:

$$\mathrm{Ext}_A^{2,p^nq+2q}(H^*K,H^*M)\xrightarrow{(j')^*}\mathrm{Ext}_A^{2,p^nq+q-1}(H^*K,H^*K)\xrightarrow{(i')^*}\mathrm{Ext}_A^{2,p^nq+q-1}(H^*K,H^*M)$$

induced by (1.2). The left group is zero by Prop. 2.10(1) and similarly to Prop. 2.9, the right group has the unique generator $(ij)^*i'_*(ij)_*\alpha_*(\tilde{h}_n)=i'_*(ij)_*(\alpha_1\wedge 1_M)_*(\tilde{h}_n)$ satisfying $\alpha^*i'_*(ij)_*(\alpha_1\wedge 1_M)_*(\tilde{h}_n)=i'_*(ij)_*(\alpha_1\wedge 1_M)_*\alpha_*(\tilde{h}_n)=0\in\mathrm{Ext}_A^{3,p^nq+2q}(H^*K,\ H^*M)$ since $i'ij(\alpha_1\wedge 1_M)\alpha=0\in[\Sigma^{2q-2}M,K]$. Then the result follows. Q.E.D.

Let K' be the cofibre of $jj':\Sigma^{-1}K\to\Sigma^{q+1}S$ given by the cofibration

$$\Sigma^{-1}K\xrightarrow{jj'}\Sigma^{q+1}S\xrightarrow{z}K'\xrightarrow{x}K. \tag{2.13}$$

As stated in [3, pp. 191–192], K' is also the cofibre of $\alpha i:\Sigma^qS\to M$ given by the cofibration

$$\Sigma^qS\xrightarrow{\alpha i}M\xrightarrow{v}K'\xrightarrow{y}\Sigma^{q+1}S, \tag{2.14}$$

and we also have another cofibration

$$\Sigma^{-1}K\xrightarrow{\alpha ijj'}\Sigma M\xrightarrow{\psi}K'\wedge M\xrightarrow{\rho}K \tag{2.15}$$

with the relation that $(1_{K'}\wedge j)\psi=v$, $\rho(1_{K'}\wedge i)=x$. (cf. [3, (2.9), (2.10)]).

Since $(1_{K'} \wedge j)(v \wedge 1_M)\overline{m}_M = v(1_M \wedge j)\overline{m}_M = v = (1_{K'} \wedge j)\psi$, then $(v \wedge 1_M)\overline{m}_M = \psi$ since $[\Sigma M, K'] = 0$ by the fact that $[\Sigma M, M] = 0$, $[\Sigma M, \Sigma^{q+1}S] = 0$. So we have $d(\psi) = 0 \in [\Sigma^2 M, K' \wedge M]$ since $d(v \wedge 1_M) = v \wedge d(1_M) = 0$ and $d(\overline{m}_M) \in [\Sigma^2 M, M \wedge M] \cong [\Sigma^2 M, M] + [\Sigma M, M] = 0$. Since $m_K(x \wedge 1_M)(1_{K'} \wedge i) = m_K(1_K \wedge i)x = x = \rho(1_{K'} \wedge i)$, then $\rho = m_K(x \wedge 1_M)$ since $[\Sigma K', K] = 0$ by the fact that $[\Sigma M, K] = 0$ and $[\Sigma^{q+2}S, K] = 0$ (cf. [7, Theorem 5.2]). So we have $d(\rho) = 0$ since $d(x \wedge 1_M) = x \wedge d(1_M) = 0$ and $d(m_K) \in [\Sigma K \wedge M, K] \cong [\Sigma K, K] + [\Sigma^2 K, K] = 0$ (cf. [4, Theorem 3.6]). That is, up to a sign we have

$$\rho = m_K(x \wedge 1_M), \quad \psi = (v \wedge 1_M)\overline{m}_M, \quad d(\rho) = 0, \quad d(\psi) = 0. \tag{2.16}$$

Let $\alpha' = \alpha_1 \wedge 1_K \in [\Sigma^{q-1}K, K]$, where $\alpha_1 = j\alpha i \in \pi_{q-1}S$. Then $j'\alpha'\alpha' = 0$ and so by (2.15) there is $\alpha'_{K' \wedge M} \in [\Sigma^{q-1}K, K' \wedge M]$ such that $\rho\alpha'_{K' \wedge M} = \alpha'$. Moreover, $d(\alpha'_{K' \wedge M}) \in [\Sigma^q K, K' \wedge M] = 0$ since $[\Sigma^q K, K] = 0$ (cf. [4]) and $[\Sigma^{q-1}K, M] = 0$ by the following exact sequence:

$$[\Sigma^{2q}M, M] \xrightarrow{(j')^*} [\Sigma^{q-1}K, M] \xrightarrow{(i')^*} [\Sigma^{q-1}M, M] \xrightarrow{(\alpha)^*}, \tag{2.17}$$

where the left group has the unique generator α^2 so that im $(j')^* = 0$ and the right group has two generators $ij\alpha$ and αij so that the above $(\alpha)^*$ is monic. Then $\rho\alpha'_{K' \wedge M}i' = \alpha' i' = i'(\alpha_1 \wedge 1_M) = \rho(vi \wedge 1_M)(\alpha_1 \wedge 1_M)$ and we have $\alpha'_{K' \wedge M}i' = (vi \wedge 1_M)(\alpha_1 \wedge 1_M) + \lambda\psi(ij\alpha ij)$ for some $\lambda \in Z_p$ since $[\Sigma^{q-2}M, M] \cong Z_p\{ij\alpha ij\}$. Since $d(\alpha'_{K' \wedge M}) = 0, d(i') = 0, d(vi \wedge 1_M) = 0, d(\alpha_1 \wedge 1_M) = 0, d(\psi) = 0$ and $d(ij\alpha ij) = -\alpha_1 \wedge 1_M$, then by applying d to the above equation we have $\lambda\psi(\alpha_1 \wedge 1_M) = 0$ and the scalar $\lambda = 0$. So $\alpha_{K' \wedge M}i' = (vi \wedge 1_M)(\alpha_1 \wedge 1_M)$. Moreover, $\rho(1_{K'} \wedge ij)\alpha_{K' \wedge M} \neq 0 \in [\Sigma^{q-2}K, K] \cong Z_p\{\alpha''\}$ (cf. [6, p. 431, Lemma 5.6 (ii) and (5.12)]) since $d(\rho(1_{K'} \wedge ij)\alpha_{K' \wedge M}) = \rho\alpha_{K' \wedge M} = \alpha' \neq 0$. Then, in conclusion we have a map $\alpha'_{K' \wedge M} \in [\Sigma^{q-1}K, K' \wedge M]$ satisfying

$$\begin{aligned} &\rho\alpha'_{K' \wedge M} = \alpha', \quad \alpha'_{K' \wedge M}i' = (vi \wedge 1_M)(\alpha_1 \wedge 1_M), \\ &d(\alpha'_{K' \wedge M}) = 0, \quad \rho(1_{K'} \wedge ij)\alpha'_{K' \wedge M} = -\alpha'', \end{aligned} \tag{2.18}$$

since $d(\alpha'') = -\alpha'$ (cf. [6, p. 430, (5.10)]).

Proposition 2.19 *Let $p \geq 5$ and $f : \Sigma^t K' \to K$ be any map. Then $f \cdot z = 0 \in [\Sigma^{t+q+1}S, K]$.*

Proof From [6, p. 433], there is a commutative multiplication $\mu : K \wedge K \to K$ such that $\mu(i'i \wedge 1_K) = 1_K = \mu(1_K \wedge i'i)$ and there is an injection $\nu : \Sigma^{q+2}K \to K \wedge K$ such that $(jj' \wedge 1_K)\nu = 1_K$. Then by (2.13) we have $z \wedge 1_K = (z \wedge 1_K)(jj' \wedge 1_K)\nu = 0$ and so $f \cdot z = \mu(1_K \wedge i'i)f \cdot z = \mu(f \cdot z \wedge 1_K)i'i = 0$. Q.E.D.

Proposition 2.20 *Let $p \geq 3, n \geq 2$. Then*

$$\mathrm{Ext}_A^{2,p^n q+q+1}(H^*K' \wedge M, H^*M) \cong Z_p\{\psi_*(ij)_*\alpha_*(\tilde{h}_n), \psi_*(ij)^*\alpha_*(\tilde{h}_n)\}.$$

Proof Look at the following exact sequence:

$$\begin{aligned} &\mathrm{Ext}_A^{2,p^n q+q}(H^*M, H^*M) \xrightarrow{\psi_*} \mathrm{Ext}_A^{2,p^n q+q+1}(H^*K' \wedge M, H^*M) \\ &\quad \xrightarrow{\rho_*} \mathrm{Ext}_A^{2,p^n q+q+1}(H^*K, H^*M) = 0 \end{aligned}$$

induced by (2.15). The result follows from Prop. 2.9 and 2.10(2). Q.E.D.

Proposition 2.21 *Let* $p \geq 3, n \geq 2$. *Then* $\mathrm{Ext}_A^{1,p^n q}(H^*K, H^*K) \cong Z_p\{(h_n)'\}$ *with* $(i')^*(h_n)' = (i')_*(\tilde{h}_n)$.

Proof Consider the following exact sequence:

$$\mathrm{Ext}_A^{1,p^n q+q+1}(H^*K, H^*M) \xrightarrow{(j')^*} \mathrm{Ext}_A^{1,p^n q}(H^*K, H^*K) \xrightarrow{(i')^*} \mathrm{Ext}_A^{1,p^n q}(H^*K, H^*M)$$

induced by (1.2). Since $j'_*\mathrm{Ext}_A^{1,p^n q+q+1}(H^*K, H^*M) \subset \mathrm{Ext}_A^{1,p^n q}(H^*M, H^*M) \cong Z_p\{\tilde{h}_n\}$ (cf. [3, Prop. 2.4(2)]) and $\alpha_*(\tilde{h}_n) \neq 0 \in \mathrm{Ext}_A^{2,p^n q+q+1}(H^*M, H^*M)$, then $\mathrm{Ext}_A^{1,p^n q+q+1}(H^*K, H^*M) = i'_*\mathrm{Ext}_A^{1,p^n q+q+1}(H^*M, H^*M) = 0$ by the fact that $\mathrm{Ext}_A^{1,p^n q+q+r}(Z_p, Z_p) = 0$ for $r = 0, 1, 2$ (cf. [2]). Moreover, it is clear that $\mathrm{Ext}_A^{1,p^n q}(H^*K, H^*M)$ has the unique generator $i'_*(\tilde{h}_n)$ and it satisfies $\alpha^* i'_*(\tilde{h}_n) = i'_*\alpha_*(\tilde{h}_n) = 0$. Then the result follows. Q.E.D.

From [6, p. 430], there is $\alpha'' \in [\Sigma^{q-2}K, K]$ satisfying $\alpha'' i' = i' i j \alpha i j$. Let X be the cofibre of $\alpha'' : \Sigma^{q-2}K \to K$ given by the cofibration

$$\Sigma^{q-2}K \xrightarrow{\alpha''} K \xrightarrow{w} X \xrightarrow{u} \Sigma^{q-1}K. \tag{2.22}$$

Then α'' induces a boundary homomorphism $(\alpha'')^* : \mathrm{Ext}_A^{1,p^n q}(H^*K, H^*K) \to \mathrm{Ext}_A^{2,p^n q+q-1}(H^*K, H^*K)$. Since $\alpha'' i' = i' i j \alpha i j = i' i j(\alpha_1 \wedge 1_M)$, then $(i')^*(\alpha'')^*(h_n)' = (\alpha'' i')^*(h_n)' = (i' i j(\alpha_1 \wedge 1_M))^*(h_n)' = (\alpha_1 \wedge 1_M)^*(ij)^*(i')^*(h_n)' = (i' i j)_*(\alpha_1 \wedge 1_M)_*(\tilde{h}_n) = (i')^*(h_0 h_n)''$ (cf. Prop. 2.21 and 2.12). So, we have

$$(h_0 h_n)'' = (\alpha'')^*(h_n)' \in \mathrm{Ext}_A^{2,p^n q+q-1}(H^*K, H^*K), \tag{2.23}$$

since the above $(i')^*$ is monic by $\mathrm{Ext}_A^{2,p^n q+2q}(H^*K, H^*M) = 0$ (cf. Prop. 2.10(1)).

3 Proof of the Main Theorems

We will first prove Theorem II by an argument presented in the Adams resolution of certain spectra related to K. Recall from [3, p. 193] that

$$\begin{array}{ccccccc} \cdots \xrightarrow{\bar{a}_2} & \Sigma^{-2}E_2 & \xrightarrow{\bar{a}_1} & \Sigma^{-1}E_1 & \xrightarrow{\bar{a}_0} & E_0 = S \\ & \downarrow \bar{b}_2 & & \downarrow \bar{b}_1 & & \downarrow \bar{b}_0 \\ & \Sigma^{-2}KG_2 & & \Sigma^{-1}KG_1 & & KG_0 \end{array} \tag{3.1}$$

is the minimal Adams resolution of S satisfying the conditions (1)(2)(3) stated in [3, p. 194]. An Adams resolution of arbitrary spectrum V can be obtained by smashing V on (3.1). We first prove the following lemmas:

Lemma 3.2 *Let* $p \geq 5$ *and* $n \geq 2$. *Then there exist* $\tilde{\eta}_{n,2} \in [\Sigma^{p^n q+q}M, E_2 \wedge M]$ *and* $\eta'_{n,2} \in [\Sigma^{p^n q+q}K, E_2 \wedge K]$ *such that* $(\bar{b}_2 \wedge 1_M)\tilde{\eta}_{n,2} = h_0 h_n \wedge 1_M$ *and* $(\bar{b}_2 \wedge 1_K)\eta'_{n,2} = h_0 h_n \wedge 1_K$, *where* $h_0 h_n \in \pi_{p^n q+q} KG_2 \cong \mathrm{Ext}_A^{2,p^n q+q}(Z_p, Z_p)$.

Proof From [8, Theorem IV (b)(c)], a map $\overline{\zeta}_{n-1} \in [\Sigma^{p^nq+q-3}M, S]$ was constructed and shown to satisfy:

(i) The composition $\zeta_{n-1} = \overline{\zeta}_{n-1} i : \Sigma^{p^nq+q-3}S \xrightarrow{i} \Sigma^{p^nq+q-3}M \xrightarrow{\overline{\zeta}_{n-1}} S$ is represented by $h_0 b_{n-1} \in \mathrm{Ext}_A^{3,p^nq+q}(Z_p, Z_p)$ in the ASS;

(ii) $\overline{\zeta}_{n-1} : \Sigma^{p^nq+q-3}M \to S$ is represented by $j^*(h_0h_n) \in \mathrm{Ext}_A^{2,p^nq+q-1}(Z_p, H^*M)$ with $h_0h_n \in \mathrm{Ext}_A^{2,p^nq+q}(Z_p, Z_p)$.

So, in the Adams resolution, there is $\overline{\zeta}_{n-1,2} \in [\Sigma^{p^nq+q-1}M, E_2]$ such that $\bar{a}_0\bar{a}_1\overline{\zeta}_{n-1,2} = \overline{\zeta}_{n-1}$ and $\bar{b}_2\overline{\zeta}_{n-1,2} = h_0h_n \cdot j \in [\Sigma^{p^nq+q-1}M, KG_2]$, where $h_0h_n \in \pi_{p^nq+q}KG_2 \cong \mathrm{Ext}_A^{2,p^nq+q}(Z_p, Z_p)$. It follows that $\bar{c}_2(h_0h_n)j = 0$ and we have $\bar{c}_2(h_0h_n) = f_0 \cdot p$ for some $f_0 \in \pi_{p^nq+q}E_3$. So, $(\bar{c}_2 \wedge 1_M)(h_0h_n \wedge 1_M) = 0$ and $(\bar{c}_2 \wedge 1_K)(h_0h_n \wedge 1_K) = 0$ and the result follows. Q.E.D.

Lemma 3.3 *Let $p \geq 3, n \geq 2$ and $(h_0h_n)'' \in [\Sigma^{p^nq+q-1}K, KG_2 \wedge K]$ be the d_1-cycle which represents the element $(h_0h_n)'' = (\alpha'')^*(h_n)' \in \mathrm{Ext}_A^{2,p^nq+q-1}(H^*K, H^*K)$ stated in Prop.* 2.12 *and* (2.23). *Then $(\bar{c}_2 \wedge 1_K)(h_0h_n)'' = (1_{E_3} \wedge \alpha'')(\kappa \wedge 1_K)$, where κ is an element in $\pi_{p^nq+1}E_3$ satisfying $\bar{a}_2\kappa = \bar{c}_1h_n$ with $h_n \in \pi_{p^nq}KG_1 \cong \mathrm{Ext}_A^{1,p^nq}(Z_p, Z_p)$.*

Proof Recall that X is the cofibre of $\alpha'' : \Sigma^{q-2}K \to K$ given by the cofibration (2.22). Since $(h_0h_n)'' \in [\Sigma^{p^nq+q-1}K, KG_2 \wedge K]$ represents $(h_0h_n)'' = (\alpha'')^*(h_n)' \in \mathrm{Ext}_A^{2,p^nq+q-1}$ (H^*K, H^*K), then $(h_0h_n)''u \in [\Sigma^{p^nq}X, KG_2 \wedge K]$ is a d_1-boundary and so $(\bar{c}_2 \wedge 1_K)(h_0h_n)''u = 0$ and $(\bar{c}_2 \wedge 1_K)(h_0h_n)'' = f'\alpha''$ with $f' \in [\Sigma^{p^nq+1}K, E_3 \wedge K]$. It follows that $(\bar{a}_2 \wedge 1_K)f'\alpha'' = 0$ and $(\bar{a}_2 \wedge 1_K)f' = f_2'w$ with $f_2' \in [\Sigma^{p^nq}X, E_2 \wedge K]$. Hence, $(\bar{b}_2 \wedge 1_K)f_2'w = 0$ and $(\bar{b}_2 \wedge 1_K)f_2' = g' \cdot u$ with $g' \in [\Sigma^{p^nq+q-1}K, KG_2 \wedge K]$. This g' is a d_1-cycle since $(\bar{b}_3\bar{c}_2 \wedge 1_K)g' = g_2'\alpha''$ (with $g_2' \in [\Sigma^{p^nq+1}K, KG_3 \wedge K]) = 0$ by the fact that α'' induces zero homomorphism in Z_p-cohomology. So, by Prop. 2.12 and (2.23), g' represents $(h_0h_n)'' = (\alpha'')^*(h_n)' \in \mathrm{Ext}_A^{2,p^nq+q-1}(H^*K, H^*K)$ and so $g' \cdot u$ is a d_1-boundary, i.e. $g' \cdot u = (\bar{b}_2\bar{c}_1 \wedge 1_K)g_3'$ with $g_3' \in [\Sigma^{p^nq}X, KG_1 \wedge K]$. It follows from $(\bar{b}_2 \wedge 1_K)f_2' = (\bar{b}_2\bar{c}_1 \wedge 1_K)g_3'$ that $f_2' = (\bar{c}_1 \wedge 1_K)g_3' + (\bar{a}_2 \wedge 1_K)f_3'$ with $f_3' \in [\Sigma^{p^nq+1}X, E_3 \wedge K]$ and we have $(\bar{a}_2 \wedge 1_K)f' = f_2'w = (\bar{c}_1 \wedge 1_K)g_3'w + (\bar{a}_2 \wedge 1_K)f_3'w$. Clearly, $g_3'w \in [\Sigma^{p^nq}K, KG_1 \wedge K]$ is a d_1-cycle which represents an element in $\mathrm{Ext}_A^{1,p^nq}(H^*K, H^*K) \cong Z_p\{(h_n)'\}$ (cf. Prop. 2.21). Then $g_3'w = h_n \wedge 1_K$ up to a scalar with $h_n \in \pi_{p^nq}KG_1 \cong \mathrm{Ext}_A^{1,p^nq}(Z_p, Z_p)$. So we have $(\bar{a}_2 \wedge 1_K)f' = (\bar{c}_1 \wedge 1_K)(h_n \wedge 1_K) + (\bar{a}_2 \wedge 1_K)f_3'w = (\bar{a}_2 \wedge 1_K)(\kappa \wedge 1_K) + (\bar{a}_2 \wedge 1_K)f_3'w$, where $\kappa \in \pi_{p^nq+1}E_3$ satisfies $\bar{a}_2\kappa = \bar{c}_1h_n$. It follows that $f' = \kappa \wedge 1_K + f_3'w + (\bar{c}_2 \wedge 1_K)g_4'$ for some $g_4' \in [\Sigma^{p^nq+1}K, KG_2 \wedge K]$ and we have $(\bar{c}_2 \wedge 1_K)(h_0h_n)'' = f'\alpha'' = (\kappa \wedge 1_K)\alpha'' = (1_{E_3} \wedge \alpha'')(\kappa \wedge 1_K)$. Q.E.D.

Proposition 3.4 *Let $p \geq 5, n \geq 2$ and $(h_0h_n)'' \in [\Sigma^{p^nq+q-1}K, KG_2 \wedge K]$ be the d_1-cycle as in Lemma* 3.3. *Then $(\bar{c}_2 \wedge 1_K)(h_0h_n)'' = 0$.*

Proof By Lemma 3.3, it suffices to prove that $(1_{E_3} \wedge \alpha'')(\kappa \wedge 1_K) = 0$. Note that, by $\bar{a}_2\kappa = \bar{c}_1h_n$, we have $\bar{a}_2(1_{E_3} \wedge \alpha_1)\kappa = \bar{c}_1(1_{KG_1} \wedge \alpha_1)h_n = 0$ and $(1_{E_3} \wedge \alpha_1)\kappa = \bar{c}_2(h_0h_n)$ (up to a scalar) since $\pi_{p^nq+q}KG_2 \cong \mathrm{Ext}_A^{2,p^nq+q}(Z_p, Z_p) \cong Z_p\{h_0h_n\}$. Hence, by Lemma 3.2 we have

$$(1_{E_3} \wedge \alpha_1 \wedge 1_M)(\kappa \wedge 1_M) = 0, \quad (1_{E_3} \wedge \alpha_1 \wedge 1_K)(\kappa \wedge 1_K) = 0. \tag{3.5}$$

Moreover, from (2.18) we have

$$\begin{aligned}
&(1_{E_3}\wedge\alpha'_{K'\wedge M})(\kappa\wedge 1_K)\rho(v\wedge 1_M)\\
&\quad=(1_{E_3}\wedge\alpha'_{K'\wedge M})(\kappa\wedge 1_K)\rho(v\wedge 1_M)(i\wedge 1_M)m_M\\
&\qquad+(1_{E_3}\wedge\alpha'_{K'\wedge M})(\kappa\wedge 1_K)\rho(v\wedge 1_M)\overline{m}_M(j\wedge 1_M)\\
&\quad=(\kappa\wedge 1_{K'\wedge M})\alpha'_{K'\wedge M}i'm_M\ \ (\text{since}\,\rho(v\wedge 1_M)\overline{m}_M=0,\ \rho(vi\wedge 1_M)=i')\\
&\quad=(\kappa\wedge 1_{K'\wedge M})(vi\wedge 1_M)(\alpha_1\wedge 1_M)m_M\qquad\text{by (2.18)}\\
&\quad=(1_{E_3}\wedge vi\wedge 1_M)(\kappa\wedge 1_M)(\alpha_1\wedge 1_M)m_M=0\quad\text{by (3.5)},
\end{aligned}$$

and so by (2.14) and Cor. 2.7, $(1_{E_3}\wedge\alpha'_{K'\wedge M})(\kappa\wedge 1_K)\rho=f(y\wedge 1_M)$ for some $f\in[\Sigma^{p^nq+2q+1}M,E_3\wedge K'\wedge M]\cap\ker d$.

It follows that $(\bar{a}_2\wedge 1_{K'\wedge M})f(y\wedge 1_M)=(\bar{a}_2\wedge 1_{K'\wedge M})(1_{E_3}\wedge\alpha'_{K'\wedge M})\ (\kappa\wedge 1_K)\rho=(\bar{c}_1\wedge 1_{K'\wedge M})(1_{KG_1}\wedge\alpha'_{K'\wedge M})(h_n\wedge 1_K)\rho=0$, then by (2.14) and Cor. 2.7 we have

$$(\bar{a}_2\wedge 1_{K'\wedge M})f=f_2(\alpha i\wedge 1_M)\tag{3.6}$$

for some $f_2\in[\Sigma^{p^nq+q}M\wedge M,E_2\wedge K'\wedge M]\cap\ker d$.

Observe that $(\bar{b}_2\wedge 1_{K'\wedge M})f_2=(\bar{b}_2\wedge 1_{K'\wedge M})f_2(i\wedge 1_M)m_M+(\bar{b}_2\wedge 1_{K'\wedge M})f_2\overline{m}_M(j\wedge 1_M)$ and we claim that $(\bar{b}_2\wedge 1_{K'\wedge M})f_2(i\wedge 1_M)=\lambda_1(1_{KG_2}\wedge vi\wedge 1_M)(h_0h_n\wedge 1_M)$ and $(\bar{b}_2\wedge 1_{K'\wedge M})f_2\overline{m}_M=\lambda_2(1_{KG_2}\wedge(v\wedge 1_M)\overline{m}_M)(h_0h_n\wedge 1_M)$ modulo d_1-boundary with $\lambda_1,\lambda_2\in Z_p$.

To prove this, note that the d_1-cycle $(\bar{b}_2\wedge 1_{K'\wedge M})f_2(i\wedge 1_M)$ represents an element $[(\bar{b}_2\wedge 1_{K'\wedge M})f_2(i\wedge 1_M)]\in\mathrm{Ext}_A^{2,p^nq+q}(H^*K'\wedge M,H^*M)$ and $[(\bar{b}_2\wedge 1_K)(1_{E_2}\wedge\rho)f_2(i\wedge 1_M)]\in\mathrm{Ext}_A^{2,p^nq+q}(H^*K,H^*M)\cong Z_p\{[(1_{KG_2}\wedge i')(h_0h_n\wedge 1_M)]\}$ (cf. Prop. 2.9). Then $(\bar{b}_2\wedge 1_K)(1_{E_2}\wedge\rho)f_2(i\wedge 1_M)=\lambda_1(1_{KG_2}\wedge\rho(vi\wedge 1_M))(h_0h_n\wedge 1_M)+(\bar{b}_2\bar{c}_1\wedge 1_K)g$ for some $g\in[\Sigma^{p^nq+q}M,KG_1\wedge K]$. Since $(1_{KG_1}\wedge j'\alpha')g=0$, then $g=(1_{KG_1}\wedge\rho)g_2$ with $g_2\in[\Sigma^{p^nq+q}M,KG_1\wedge K'\wedge M]$. It follows that $(\bar{b}_2\wedge 1_{K'\wedge M})f_2(i\wedge 1_M)=\lambda_1(1_{KG_2}\wedge vi\wedge 1_M)(h_0h_n\wedge 1_M)+(\bar{b}_2\bar{c}_1\wedge 1_{K'\wedge M})g_2+(1_{KG_2}\wedge\psi)g_3$ for some $g_3\in[\Sigma^{p^nq+q-1}M,KG_2\wedge M]\cong Z_p\{(h_0h_n\wedge 1_M)ij\}$, then $g_3=\lambda'(h_0h_n\wedge 1_M)ij$ for some $\lambda'\in Z_p$. However, $d(i\wedge 1_M)=0$ and $d(f_2)=0$ implies that $d(f_2(i\wedge 1_M))=0$, then by applying d to the above equation we have $(1_{KG_2}\wedge\psi)d(g_3)+(\bar{b}_2\bar{c}_1\wedge 1_{K'\wedge M})d(g_2)=0$, i.e. $\lambda'(1_{KG_2}\wedge\psi)(h_0h_n\wedge 1_M)=(\bar{b}_2\bar{c}_1\wedge 1_{K'\wedge M})d(g_2)$ and this means that the scalar $\lambda'=0$ since $\psi_*[h_0h_n\wedge 1_M]\neq 0\in\mathrm{Ext}_A^{2,p^nq+q+1}(H^*K'\wedge M,H^*M)$. This shows that $(\bar{b}_2\wedge 1_{K'\wedge M})f_2(i\wedge 1_M)=\lambda_1(1_{KG_2}\wedge vi\wedge 1_M)(h_0h_n\wedge 1_M)$ modulo d_1-boundary. In addition, since $d(\overline{m}_M)\in[\Sigma^2M,M\wedge M]\cong[\Sigma^2M,M]+[\Sigma M,M]=0$, then, similarly, by Prop. 2.20 and $d(f_2\overline{m}_M)=0$ we have $(\bar{b}_2\wedge 1_{K'\wedge M})f_2\overline{m}_M=\lambda_2(1_{KG_2}\wedge\psi)(h_0h_n\wedge 1_M)$ modulo d_1-boundary. This shows the claim.

Hence we have

$$\begin{aligned}
&(\bar{b}_2\wedge 1_{K'\wedge M})f_2=(\bar{b}_2\wedge 1_{K'\wedge M})f_2(i\wedge 1_M)m_M+(\bar{b}_2\wedge 1_{K'\wedge M})f_2\overline{m}_M(j\wedge 1_M)\\
&\quad=\lambda_1(1_{KG_2}\wedge vi\wedge 1_M)(h_0h_n\wedge 1_M)m_M+\lambda_2(1_{KG_2}\wedge\psi)(h_0h_n\wedge 1_M)(j\wedge 1_M)\\
&\quad=\lambda_1(h_0h_n\wedge 1_{K'\wedge M})(v\wedge 1_M)(i\wedge 1_M)m_M+\lambda_2(h_0h_n\wedge 1_{K'\wedge M})(v\wedge 1_M)\overline{m}_M(j\wedge 1_M)
\end{aligned}$$

modulo d_1-boundary. Moreover, $(1_{KG_2}\wedge\rho(1_{K'}\wedge ij))(h_0h_n\wedge 1_{K'\wedge M})(v\wedge 1_M)=(h_0h_n\wedge 1_K)\rho(1_{K'}\wedge ij)(v\wedge 1_M)=(h_0h_n\wedge 1_K)\rho(1_{K'}\wedge i)v(1_M\wedge j)=(h_0h_n\wedge 1_K)i'(1_M\wedge j)$ (Note:

$\rho(1_{K'}\wedge i)v = xv = i'$, cf. (2.15)). Then modulo a d_1-boundary $(\bar b_2\bar c_1\wedge 1_K)g_4$ we have

$$\begin{aligned}&(\bar b_2\wedge 1_K)(1_{E_2}\wedge\rho(1_{K'}\wedge ij))f_2\\&\quad=\lambda_1(h_0h_n\wedge 1_K)i'(1_M\wedge j)(i\wedge 1_M)m_M+\lambda_2(h_0h_n\wedge 1_K)i'(1_M\wedge j)\overline{m}_M(j\wedge 1_M)\\&\quad=\lambda_1(\bar b_2\wedge 1_K)\eta'_{n,2}i'(1_M\wedge j)(i\wedge 1_M)m_M+\lambda_2(\bar b_2\wedge 1_K)\eta'_{n,2}i'(1_M\wedge j)\overline{m}_M(j\wedge 1_M)\end{aligned}$$

by Lemma 3.2. It follows that $(1_{E_2}\wedge\rho(1_{K'}\wedge ij))f_2=(\bar a_2\wedge 1_K)f_3+\lambda_1\eta'_{n,2}i'(1_M\wedge j)(i\wedge 1_M)m_M+\lambda_2\eta'_{n,2}i'(1_M\wedge j)\overline{m}_M(j\wedge 1_M)+(\bar c_1\wedge 1_K)g_4$ for some $f_3\in[\Sigma^{p^nq+q}M\wedge M,E_3\wedge K]$ and we have $(\bar a_2\wedge 1_K)(1_{E_3}\wedge\rho(1_{K'}\wedge ij))f=(1_{E_2}\wedge\rho(1_{K'}\wedge ij))f_2(\alpha i\wedge 1_M)=(\bar a_2\wedge 1_K)f_3(\alpha i\wedge 1_M)+\lambda_1\eta'_{n,2}i'(1_M\wedge j)(i\wedge 1_M)m_M(\alpha i\wedge 1_M)+\lambda_2\eta'_{n,2}i'(1_M\wedge j)\overline{m}_M(j\wedge 1_M)(\alpha i\wedge 1_M)$.

By (2.2), $m_M(\alpha\wedge 1_M)(1_M\wedge i)=m_M(1_M\wedge i)\alpha=-\alpha=\alpha m_M(1_M\wedge i)$, then $m_M(\alpha\wedge 1_M)=\alpha m_M$ since $[\Sigma^{q+1}M,M]=0$. So $m_M(\alpha i\wedge 1_M)=\alpha m_M(i\wedge 1_M)=\alpha$ and we have

$$\begin{aligned}\sigma_1&=\eta'_{n,2}i'(1_M\wedge j)(i\wedge 1_M)m_M(\alpha i\wedge 1_M)=\eta'_{n,2}i'ij\alpha=\eta'_{n,2}\alpha' i',\\ \sigma_2&=\eta'_{n,2}i'(1_M\wedge j)\overline{m}_M(j\alpha i\wedge 1_M)=\eta'_{n,2}i'(\alpha_1\wedge 1_M)=\eta'_{n,2}\alpha' i'.\end{aligned}$$

So, $\lambda_1\sigma_1+\lambda_2\sigma_2=(\lambda_1+\lambda_2)\eta'_{n,2}\alpha' i'$. On the other hand, $\lambda_1\sigma_1+\lambda_2\sigma_2=(\lambda_1-\lambda_2)\sigma_1+\lambda_2\eta'_{n,2}i'(1_M\wedge j)((i\wedge 1_M)m_M+\overline{m}_M(j\wedge 1_M))(\alpha i\wedge 1_M)=(\lambda_1-\lambda_2)\eta'_{n,2}\alpha' i'+\lambda_2\eta'_{n,2}i'\alpha ij=(\lambda_1-\lambda_2)\eta'_{n,2}\alpha' i'$ and similarly $\lambda_1\sigma_1+\lambda_2\sigma_2=(\lambda_2-\lambda_1)\sigma_2=(\lambda_2-\lambda_1)\eta'_{n,2}\alpha' i'$. This shows that $\lambda_1\sigma_1+\lambda_2\sigma_2=(\lambda_1+\lambda_2)\eta'_{n,2}\alpha' i'=(\lambda_1-\lambda_2)\eta'_{n,2}\alpha' i'=(\lambda_2-\lambda_1)\eta'_{n,2}\alpha' i'=0$, so we have

$$(\bar a_2\wedge 1_K)(1_{E_3}\wedge\rho(1_{K'}\wedge ij))f=(\bar a_2\wedge 1_K)f_3(\alpha i\wedge 1_M).$$

It follows that $(1_{E_3}\wedge\rho(1_{K'}\wedge ij))f=f_3(\alpha i\wedge 1_M)+(\bar c_2\wedge 1_K)g_5$ for some $g_5\in[\Sigma^{p^nq+2q}M,KG_2\wedge K]$, then we have

$$\begin{aligned}-(1_{E_3}\wedge\alpha'')(\kappa\wedge 1_K)\rho&=((1_{E_3}\wedge\rho(1_{K'}\wedge ij))\alpha'_{K'\wedge M})(\kappa\wedge 1_K)\rho\,(\text{cf. }(2.18))\\&=(1_{E_3}\wedge\rho(1_{K'}\wedge ij))f(y\wedge 1_M)=(\bar c_2\wedge 1_K)g_5(y\wedge 1_M).\end{aligned}$$

This g_5 is a d_1-cycle since $(\bar b_3\bar c_2\wedge 1_K)g_5(y\wedge 1_M)=0$ and so $(\bar b_3\bar c_2\wedge 1_K)g_5=g_6(\alpha i\wedge 1_M)=0$ (with $g_6\in[\Sigma^{p^nq+q}M\wedge M,KG_3\wedge K]$). Then g_5 represents an element in $\mathrm{Ext}_A^{2,p^nq+2q}(H^*K,H^*M)=0$ (cf. Prop. 2.10(1)). That is, g_5 is a d_1-boundary and we have $(1_{E_3}\wedge\alpha'')(\kappa\wedge 1_K)\rho=(\bar c_2\wedge 1_K)g_5(y\wedge 1_M)=0$.

It follows that $(1_{E_3}\wedge\alpha'')(\kappa\wedge 1_K)=f_4\alpha ijj'$ with $f_4\in[\Sigma^{p^nq+q+1}M,E_3\wedge K]$ and $(\bar a_2\wedge 1_K)f_4\alpha ijj'=(\bar a_2\wedge 1_K)(1_{E_3}\wedge\alpha'')(\kappa\wedge 1_K)=(\bar c_1\wedge 1_K)(1_{KG_1}\wedge\alpha'')(h_n\wedge 1_K)=0$. Then , by (2.13), we have $(\bar a_2\wedge 1_K)f_4\alpha i=f_5z$ with $f_5\in[\Sigma^{p^nq+q-1}K',E_2\wedge K]$. From Prop. 2.19, $(\bar a_0\bar a_1\wedge 1_K)f_5z=0$, then $f_5z=(\bar c_1\wedge 1_K)g_7=0$ since the d_1-cycle $g_7\in[\Sigma^{p^nq+2q}S,KG_1\wedge K]$ represents an element in $\mathrm{Ext}_A^{1,p^nq+2q}(H^*K,Z_p)=0$. Hence $(\bar a_2\wedge 1_K)f_4\alpha i=0$, $f_4\alpha i=(\bar c_2\wedge 1_K)g_8$ for some $g_8\in[\Sigma^{p^nq+2q+1}S,KG_2\wedge K]$ and we have $(1_{E_3}\wedge\alpha'')(\kappa\wedge 1_K)=f_4\alpha ijj'=(\bar c_2\wedge 1_K)g_8jj'$. This g_8 is a d_1-cycle since $(\bar b_3\bar c_2\wedge 1_K)g_8jj'=0$, $(\bar b_3\bar c_2\wedge 1_K)g_8=g_9z=0$ (with $g_9\in[\Sigma^{p^nq+q}K',KG_3\wedge K]$), then g_8 represents an element in $\mathrm{Ext}_A^{2,p^nq+2q+1}(H^*K,Z_p)=0$ (cf. Prop. 2.10(1)). That is, g_8 is a d_1-boundary and so $(1_{E_3}\wedge\alpha'')(\kappa\wedge 1_K)=(\bar c_2\wedge 1_K)g_8jj'=0$. This shows the lemma. Q.E.D.

Proof of Theorem II From Prop. 3.4, we have $(\bar c_2\wedge 1_K)(h_0h_n)''=0$, then there is $\eta''_{n,2}\in[\Sigma^{p^nq+q-1}K,E_2\wedge K]$ such that $(\bar b_2\wedge 1_K)\eta''_{n,2}=(h_0h_n)''\in[\Sigma^{p^nq+q-1}K,KG_2\wedge K]$. Let

$\eta_n'' = (\bar{a}_0\bar{a}_1 \wedge 1_K)\eta_{n,2}'' \in [\Sigma^{p^nq+q-3}K, K]$ and consider the map $\eta_n''\beta i'i \in \pi_{p^nq+pq+2q-3}K$, where $\beta \in [\Sigma^{(p+1)q}K, K]$ is the known v_2-map (cf. [6, p. 426]) which has filtration 1 in the ASS. Since η_n'' is represented by $(h_0h_n)'' \in \mathrm{Ext}_A^{2,p^nq+q-1}(H^*K, H^*K)$ in the ASS, then similarly to that is given at the bottom of [3, p. 202], $\eta_n''\beta i'i$ is represented by $(\beta i'i)^*(h_0h_n)'' = (\beta i'i)^*\alpha_*''(h_n)' = (\alpha'')_*(\beta i'i)^*(h_n)' = (\alpha'')_*(\beta i'i)_*(h_n) = (i'i)_*(h_ng_0) \neq 0 \in \mathrm{Ext}_A^{3,p^nq+pq+2q}(H^*K, Z_p)$. Moreover, $(i'i)_*(g_0h_n) \in \mathrm{Ext}_A^{3,p^nq+pq+2q}(H^*K, Z_p)$ cannot be hit by a differential since $\mathrm{Ext}_A^{3-r,p^nq+pq+2q-r+1}(H^*K, Z_p) = 0$ for $r \geq 2$ by several steps of exact sequences induced by (1.2) (1.1) and using [3, Prop. 2.1 (3)]. This finishes the proof of the theorem. Q.E.D.

Proof of Theorem I Let $V(2)$ be the cofibre of $\beta : \Sigma^{(p+1)q}K \to K$ given by the cofibration

$$\Sigma^{(p+1)q}K \xrightarrow{\beta} K \xrightarrow{\bar{i}} V(2) \xrightarrow{\bar{j}} \Sigma^{(p+1)q+1}K.$$

From Theorem II, there is $\eta_n''\beta i'i \in \pi_{p^nq+pq+2q-3}K$, which is represented by $(i'i)_*(h_ng_0) \in \mathrm{Ext}_A^{3,p^nq+pq+2q}(H^*K, Z_p)$. Let $\gamma : \Sigma^{(p^2+p+1)q}V(2) \to V(2)$ be the v_3-map for $p \geq 7$ (cf. [6, p. 426]) and consider the following composition ($t = p^nq + pq + 2q - 3$):

$$\tilde{f} : \Sigma^tS \xrightarrow{\bar{i}\eta_n''\beta i'i} V(2) \xrightarrow{\gamma^3} \Sigma^{-3(p^2+p+1)q}V(2) \xrightarrow{jj'\bar{j}} \Sigma^{-3(p^2+p+1)q+(p+2)q+3}S.$$

Since $\eta_n''\beta i'i$ is represented by $(i'i)_*(h_ng_0) \in \mathrm{Ext}_A^{3,p^nq+pq+2q}(H^*K, Z_p)$, then the above $\tilde{f}$ is represented by

$$c = (jj'\bar{j})_*(\gamma_*)^3(\bar{i}i'i)_*(h_ng_0) \in \mathrm{Ext}_A^{6,p^nq+3(p^2+p+1)q}(Z_p, Z_p).$$

Similarly to what is given in [1, p. 203], $\tilde{f} \in \pi_*S$ is represented by $c = h_ng_0\gamma_3 \neq 0 \in \mathrm{Ext}_A^{6,p^nq+3(p^2+p+1)q}(Z_p, Z_p)$ (up to a nonzero scalar) in the ASS. Moreover, from [1, Prop. 2.1(3)], $\mathrm{Ext}_A^{6-r,p^nq+3(p^2+p+1)q-r+1}(Z_p, Z_p) = 0$ for $r \geq 2$, then $h_ng_0\gamma_3$ cannot be hit by differentials in the ASS and so $\tilde{f} \in \pi_*S$ is nontrivial and of order p. Q.E.D.

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