Acta Mathematica Sinica, English Series
Jun., 2007, Vol. 23, No. 6, pp. 1025–1032
Published online: Dec. 14, 2006
DOI: 10.1007/s10114-005-0863-3
Http://www.ActaMath.com

Acta Mathematica Sinica, English Series

On the Convergence of Products $\tilde{\gamma}_s h_1 h_n$ in the Adams Spectral Sequence

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Abstract Let A be the mod p Steenrod algebra and S the sphere spectrum localized at p, where p is an odd prime. In 2001 Lin detected a new family in the stable homotopy of spheres which is represented by $(b_0 h_n - h_1 b_{n-1}) \in \mathrm{Ext}_A^{3,(p^n+p)q}(\mathbb{Z}_p, \mathbb{Z}_p)$ in the Adams spectral sequence. At the same time, he proved that $i_*(h_1 h_n) \in \mathrm{Ext}_A^{2,(p^n+p)q}(H^*M, \mathbb{Z}_p)$ is a permanent cycle in the Adams spectral sequence and converges to a nontrivial element $\xi_n \in \pi_{(p^n+p)q-2}M$. In this paper, with Lin's results, we make use of the Adams spectral sequence and the May spectral sequence to detect a new nontrivial family of homotopy elements $jj'\bar{j}\gamma^s\bar{i}i'\xi_n$ in the stable homotopy groups of spheres. The new one is of degree $p^n q + sp^2 q + spq + (s-2)q + s - 6$ and is represented up to a nonzero scalar by $h_1 h_n \tilde{\gamma}_s$ in the $E_2^{s+2,*}$-term of the Adams spectral sequence, where $p \geq 7$, $q = 2(p-1)$, $n \geq 4$ and $3 \leq s < p$.

Keywords stable homotopy groups of spheres, Adams spectral sequence, May spectral sequence
MR(2000) Subject Classification 55Q45

1 Introduction and the Main Results

Homotopy groups of spheres are among the most fundamental algebraic invariants of topological spaces. The higher homotopy groups of spheres are very difficult to compute and all known methods of computation apply only to limited classes. All the homotopy groups of spheres are not known. Since homotopy groups carry substantial amount of information, their computation often results in nontrivial interesting applications. For example, Adams's partial computation of certain homotopy groups of spheres implies that $\mathbb{R}^n$ is not a normed algebra for $n \neq 1, 2, 4, 8$.

Throughout this paper, p will denote an odd prime. Let A be the mod p Steenrod algebra and S be the sphere spectrum localized at p. So far, not so many families of homotopy elements in π_*S have been detected. Recently, Lin got a series of results and detected some new families in π_*S. Let $q = 2(p-1)$.

In [1], Lin and Zheng obtained the following theorem:

Theorem 1.1 *Let $p \geq 7$, $n \geq 4$. Then the product $b_{n-1} g_0 \gamma_3 \neq 0 \in \mathrm{Ext}_A^{7,p^n q+3(p^2+p+1)q}(\mathbb{Z}_p, \mathbb{Z}_p)$ and it converges in the Adams spectral sequence to a nontrivial element in $\pi_{p^n q+3(p^2+p+1)q-7}S$ of order p.*

Lin [2] detected a new family in π_*S and proved the following theorem:

Theorem 1.2 *Let $p \geq 7$, $n \geq 4$. Then the product $h_n g_0 \gamma_3 \neq 0 \in \mathrm{Ext}_A^{6,p^n q+3(p^2+p+1)q}(\mathbb{Z}_p, \mathbb{Z}_p)$ and it converges in the Adams spectral sequence to a nontrivial element in $\pi_{p^n q+3(p^2+p+1)q-6}S$ of order p.*

In 2001, Lin gave the following theorem in [3]:

Received May 21, 2004, Accepted January 11, 2005
Supported by the National Natural Science Foundation of China (No. 10501045, 10426028), the China Postdoctoral Science Foundation and the Fund of the Personnel Division of Nankai University

Theorem 1.3 *Let $p \geq 5$, $n \geq 3$. Then :*

(1) *$i_*(h_1h_n) \neq 0 \in \mathrm{Ext}_A^{2,p^nq+pq}(H^*M, \mathbb{Z}_p)$ is a permanent cycle in the Adams spectral sequence and converges to a nontrivial element $\xi_n \in \pi_{p^nq+pq-2}M$;*

(2) *For $\xi_n \in \pi_{p^nq+pq-2}M$ obtained in (1), $j\xi_n \in \pi_{p^nq+pq-3}S$ is a nontrivial element of order p which is represented (up to a nonzero scalar) by $(b_0h_n + h_1b_{n-1}) \in \mathrm{Ext}_A^{3,p^nq+pq}(\mathbb{Z}_p, \mathbb{Z}_p)$ in the Adams spectral sequence.*

In this paper, we make use of Lin's results in Theorem 1.3 to detect a new family in π_*S. Our result is the following theorem:

Theorem 1.4 *Let $p \geq 7$, $n \geq 4$. Then $\tilde{\gamma}_sh_1h_n \neq 0 \in \mathrm{Ext}_A^{s+2,p^nq+sp^2q+spq+(s-2)q+s-3}(\mathbb{Z}_p, \mathbb{Z}_p)$ is a permanent cycle in the Adams spectral sequence and converges to a nontrivial element $jj'\bar{j}\gamma^s\bar{i}i'\xi_n \in \pi_{p^nq+sp^2q+spq+(s-2)q-5}S$ of order p, where $3 \leq s < p$, $q = 2(p-1)$.*

Remark The element $\tilde{\gamma}_sh_1h_n$ obtained in Theorem 1.4 is an indecomposable element in the stable homotopy groups of spheres π_*S, i.e., it is not a composition of elements of lower filtration in π_*S, because h_n $(n > 0)$ is known to die in the Adams spectral sequence.

Our method is to use the Adams spectral sequence. We also use the May spectral sequence to determine $\mathrm{Ext}_A^{s,t}(\mathbb{Z}_p, \mathbb{Z}_p)$.

The paper is arranged as follows. After recalling some useful knowledge about our methods in Section 2, we will make use of the May spectral sequence and the Adams spectral sequence to prove the existence of a new nontrivial family in the stable homotopy groups of spheres in Section 3.

2 The Adams Spectral Sequence and the May Spectral Sequence

In this section, we first recall some knowledge on the Adams spectral sequence and the May spectral sequence. One of the main tools for determining the stable homotopy groups of spheres π_*S is the Adams spectral sequence.

Let p be a prime, X a spectrum of finite type and Y a finite-dimensional spectrum. Then there is a natural spectral sequence $\{E_r^{s,t}, d_r\}$, which is called the Adams spectral sequence

$$E_2^{s,t} = \mathrm{Ext}_A^{s,t}((H^*X; \mathbb{Z}_p), H^*(Y; \mathbb{Z}_p)) \Rightarrow ([Y, X]_{t-s})_p, \tag{2.1}$$

where $d_r : E_r^{s,t} \to E_r^{s+r,t+r-1}$.

If X and Y are sphere spectra S, then the Adams spectral sequence

$$E_2^{s,t} = \mathrm{Ext}_A^{s,t}(\mathbb{Z}_p, \mathbb{Z}_p) \Rightarrow (\pi_{t-s}S)_p. \tag{2.2}$$

If S is localized at p, then the Adams spectral sequence

$$E_2^{s,t} = \mathrm{Ext}_A^{s,t}(\mathbb{Z}_p, \mathbb{Z}_p) \Rightarrow \pi_{t-s}S. \tag{2.3}$$

There are three problems in using the Adams spectral sequence: calculation of the E_2-term, computation of the differentials and determination of the nontrivial extensions from E_∞ to π_*S. So, for computing the stable homotopy groups of spheres with the Adams spectral sequence, we must compute the E_2-term of the Adams spectral sequence, $\mathrm{Ext}_A^{*,*}(\mathbb{Z}_p, \mathbb{Z}_p)$. The most successful method for computing $\mathrm{Ext}_A^{*,*}(\mathbb{Z}_p, \mathbb{Z}_p)$ is the May spectral sequence.

From [4], there is a May spectral sequence $\{E_r^{s,t,*}, d_r\}$ which converges to $\mathrm{Ext}_A^{s,t}(\mathbb{Z}_p, \mathbb{Z}_p)$ with E_1-term

$$E_1^{*,*,*} = E(h_{m,i} | m > 0, i \geq 0) \bigotimes P(b_{m,i} | m > 0, i \geq 0) \bigotimes P(a_n | n \geq 0), \tag{2.4}$$

where E is the exterior algebra, P is the polynomial algebra, and $h_{m,i} \in E_1^{1,2(p^m-1)p^i,2m-1}$, $b_{m,i} \in E_1^{2,2(p^m-1)p^{i+1},p(2m-1)}$, $a_n \in E_1^{1,2p^n-1,2n+1}$. One has $d_r : E_r^{s,t,u} \to E_r^{s+1,t,u-r}$ and if $x \in E_r^{s,t,*}$ and $y \in E_r^{s',t',*}$, then $d_r(x \cdot y) = d_r(x) \cdot y + (-1)^s x \cdot d_r(y)$, $x \cdot y = (-1)^{ss'+tt'} y \cdot x$ for $x, y = h_{m,i}, b_{m,i}$ or a_n. The first May differential d_1 is given by

$$d_1(h_{i,j}) = \sum_{0<k<i} h_{i-k,k+j}h_{k,j}, \quad d_1(a_i) = \sum_{0 \leq k < i} h_{i-k,k}a_k, \quad d_1(b_{i,j}) = 0. \tag{2.5}$$

For any element $x \in E_1^{s,t,*}$, define dim $x = s$, deg $x = t$. Then we have

$$\begin{aligned}
&\dim h_{i,j} = \dim a_i = 1, \dim b_{i,j} = 2,\\
&\deg h_{i,j} = 2(p^i-1)p^j = 2(p-1)(p^{i+j-1}+\cdots+p^j),\\
&\deg b_{i,j} = 2(p^i-1)p^{j+1} = 2(p-1)(p^{i+j}+\cdots+p^{j+1}),\\
&\deg a_i = 2p^i-1 = 2(p-1)(p^{i-1}+\cdots+1)+1,\\
&\deg a_0 = 1,
\end{aligned} \tag{2.6}$$

where $i \geq 1$, $j \geq 0$.

Note By the knowledge of the p-adic expression, for any integral $t \geq 0$, t can be always expressed as $t = q(c_n p^n + c_{n-1}p^{n-1} + \cdots + c_1 p + c_0) + e$, where $0 \leq c_i < p$ $(0 \leq i < n)$, $c_n > 0$, $0 \leq e < q$, $q = 2(p-1)$.

3 The Products $\tilde{\gamma}_s h_1 h_n$

From [5], $\mathrm{Ext}_A^{1,*}(\mathbb{Z}_p, \mathbb{Z}_p)$ has $\mathbb{Z}_p$-bases consisting of $a_0 \in \mathrm{Ext}_A^{1,1}(\mathbb{Z}_p,\mathbb{Z}_p)$, $h_i \in \mathrm{Ext}_A^{1,p^iq}(\mathbb{Z}_p,\mathbb{Z}_p)$ for all $i \geq 0$ and $\mathrm{Ext}_A^{2,*}(\mathbb{Z}_p,\mathbb{Z}_p)$ has $\mathbb{Z}_p$-bases consisting of α_2, a_0^2, $a_0h_i (i>0)$, $g_i (i \geq 0)$, $k_i (i \geq 0)$, $b_i (i \geq 0)$, and $h_ih_j (j \geq i+2, i \geq 0)$ whose internal degrees are $2q+1$, 2, $p^iq+1, p^{i+1}q+2p^iq$, $2p^{i+1}q+p^iq$, $p^{i+1}q$ and p^iq+p^jq, respectively.

Let M be the Moore spectrum modulo a prime $p \geq 5$ given by the cofibration

$$S \xrightarrow{p} S \xrightarrow{i} M \xrightarrow{j} \Sigma S. \tag{3.1}$$

Let $\alpha: \Sigma^q M \to M$ be the Adams map and K be its cofibre given by the cofibration

$$\Sigma^q M \xrightarrow{\alpha} M \xrightarrow{i'} K \xrightarrow{j'} \Sigma^{q+1} M, \tag{3.2}$$

where $q = 2(p-1)$. This spectrum, which we write for short as K, is known to be the Toda–Smith spectrum $V(1)$. Let $V(2)$ be the cofibre of $\beta: \Sigma^{(p+1)q}K \to K$ given by the cofibration

$$\Sigma^{(p+1)q} K \xrightarrow{\beta} K \xrightarrow{\bar{i}} V(2) \xrightarrow{\bar{j}} \Sigma^{(p+1)q+1} K.. \tag{3.3}$$

Let $\gamma: \Sigma^{q(p^2+p+1)}V(2) \to V(2)$ be the v_3-map. As we know, in the Adams spectral sequence, for $p \geq 7$ the γ-element $\gamma_t = jj'\bar{j}\gamma^t\bar{i}i'i$ is a nontrivial element of order p in $\pi_{tq(p^2+p+1)-q(p+2)-3}S$ (see [6, Theorem 2.12]).

In [7], the following theorem was given:

Theorem 3.1 *Let $p \geq 7$, $0 \leq s < p-3$. Then the permanent cycle $a_3^s h_{3,0}h_{2,1}h_{1,2} \in E_r^{s+3,t,*}$ converges to the third Greek letter family element $\tilde{\gamma}_{s+3} \in \mathrm{Ext}_A^{s+3,t}(\mathbb{Z}_p,\mathbb{Z}_p)$ in the May spectral sequence, where $r \geq 1$, $t = (s+3)p^2q+(s+2)pq+(s+1)q+s$ and $\tilde{\gamma}_{s+3}$ converges to the γ-element $\gamma_{s+3} \in \pi_{(s+3)p^2q+(s+2)pq+(s+1)q-3}S$ in the Adams spectral sequence, where $\gamma_{s+3} = jj'\bar{j}\gamma^{s+3}\bar{i}i'i \in \pi_{t-s-3}S$, $\tilde{\gamma}_{s+3}$ is given in* [8].

Lemma 3.1 *Let $p \geq 7$, $n \geq 4$. Then in the May spectral sequence we have*

(1) *If $0 \leq s < p-4$, $E_1^{s+4,p^nq+(s+3)p^2q+(s+3)pq+(s+1)q+s,*} = 0$;*

(2) $E_1^{p,p^nq+(p-1)p^2q+(p-1)pq+(p-3)q+(p-4),*} = \begin{cases} 0 & n=4,\\ \mathbb{Z}_p\{a_n^{p-4}h_{n,0}h_{n-1,1}h_{n-3,3}h_{3,1}\} & n>4. \end{cases}$

Proof (1) Let $t = p^nq+(s+3)p^2q+(s+3)pq+(s+1)q+s$. Suppose that $h = x_1x_2\cdots x_m$ is a generator of $E_1^{s+4,t,*}$, where $m \leq s+4$, x_i is one of a_k, $h_{l,j}$ or $b_{u,z}$, $0 \leq k \leq n+1$, $0 \leq l+j \leq n+1$, $0 \leq u+z \leq n$, $l>0$, $j \geq 0$, $u>0$, $z \geq 0$. Assume that $\deg x_i = q(c_{i,n}p^n + c_{i,n-1}p^{n-1}+\cdots+c_{i,0})+e_i$, where $c_{i,j} = 0$ or 1, $e_i = 1$ if $x_i = a_{k_i}$, or $e_i = 0$. Then

$$\begin{aligned}
\deg h = \sum_{i=1}^m \deg x_i &= q\left(\left(\sum_{i=1}^m c_{i,n}\right)p^n+\cdots+\left(\sum_{i=1}^m c_{i,2}\right)p^2+\left(\sum_{i=1}^m c_{i,1}\right)p+\left(\sum_{i=1}^m c_{i,0}\right)\right)+\left(\sum_{i=1}^m e_i\right)\\
&= q(p^n+(s+3)p^2+(s+3)p+(s+1))+s.
\end{aligned} \tag{3.4}$$

Note that $0 \le s, s+1, s+3 < p-1$. So from (3.4), we have

$$\begin{cases} \sum_{i=1}^{m} e_i = s + \lambda_{-1} q, & \lambda_{-1} \ge 0, \\ \sum_{i=1}^{m} c_{i,0} + \lambda_{-1} = s + 1 + \lambda_0 p, & \lambda_0 \ge 0, \\ \sum_{i=1}^{m} c_{i,1} + \lambda_0 = s + 3 + \lambda_1 p, & \lambda_1 \ge 0, \\ \sum_{i=1}^{m} c_{i,2} + \lambda_1 = s + 3 + \lambda_2 p, & \lambda_2 \ge 0, \\ \sum_{i=1}^{m} c_{i,3} + \lambda_2 = 0 + \lambda_3 p, & \lambda_3 \ge 0, \\ \cdots & \cdots, \\ \sum_{i=1}^{m} c_{i,n-1} + \lambda_{n-2} = 0 + \lambda_{n-1} p, & \lambda_{n-1} \ge 0, \\ \sum_{i=1}^{m} c_{i,n} + \lambda_{n-1} = 1. \end{cases} \tag{3.5}$$

Noting that $0 \le \sum_{i=1}^{m} e_i, \sum_{i=1}^{m} c_{i,j} \le m \le s+4 < p$, it is easy to know that the sequence $(\lambda_{-1}, \lambda_0, \lambda_1, \lambda_2, \lambda_3, \dots, \lambda_{n-2}, \lambda_{n-1})$ must equal the sequence $(0,0,0,0,0,\dots,0,0)$. Thus, from (3.5), we get that

$$\begin{aligned} &\sum_{i=1}^{m} e_i = s, \quad \sum_{i=1}^{m} c_{i,0} = s+1, \quad \sum_{i=1}^{m} c_{i,1} = s+3, \\ &\sum_{i=1}^{m} c_{i,2} = s+3, \quad \sum_{i=1}^{m} c_{i,3} = \cdots = \sum_{i=1}^{m} c_{i,n-1} = 0, \quad \sum_{i=1}^{m} c_{i,n} = 1. \end{aligned} \tag{3.6}$$

From (3.6), it is easy to know that there exists a factor $h_{1,n}$ or $b_{1,n-1}$ among x_i's. By the graded commutativity of $E_1^{*,*,*}$, we can denote the factor $h_{1,n}$ or $b_{1,n-1}$ by x_m, then $h = x_1 x_2 \cdots x_{m-1} h_{1,n}$ or $h = x_1 x_2 \cdots x_{m-1} b_{1,n-1}$.

If $h = x_1 x_2 \cdots x_{m-1} b_{1,n-1}$, $h'' = x_1 x_2 \cdots x_{m-1} \in E_1^{s+2, t-p^n q, *}$ we have $\sum_{i=1}^{m-1} e_i = s$, $\sum_{i=1}^{m-1} c_{i,0} = s+1$, $\sum_{i=1}^{m-1} c_{i,1} = s+3$, $\sum_{i=1}^{m-1} c_{i,2} = s+3$. From $\sum_{i=1}^{m-1} c_{i,2} = s+3$, we would have that $m-1 \ge s+3$. Noting that $\dim x_i = 1$ or 2, we would have that $\dim h'' = \sum_{i=1}^{m-1} \dim x_i \ge m-1 \ge s+3$. This contradicts $\dim h'' = s+2$. So we know that it is impossible for the generator h'' to exist, and then it is impossible for the generator $h = x_1 x_2 \cdots x_{m-1} b_{1,n-1}$ to exist.

If $h = x_1 x_2 \cdots x_{m-1} h_{1,n}$, $h' = x_1 x_2 \cdots x_{m-1} \in E_1^{s+3, t-p^n q, *}$ we have $\sum_{i=1}^{m-1} e_i = s$, $\sum_{i=1}^{m-1} c_{i,0} = s+1$, $\sum_{i=1}^{m-1} c_{i,1} = s+3$, $\sum_{i=1}^{m-1} c_{i,2} = s+3$. By an argument similar to that used in the proof of Theorem 3.1 (see [7]), it is easy to show that $E_1^{s+3, t-p^n q, *} = 0$, so it is impossible for the generator $h = x_1 x_2 \cdots x_{m-1} h_{1,n}$ to exist.

From the above discussion, we get that, for $s+4 < p$, $E_1^{s+4,t,*} = 0$. This completes the proof of the first part of Lemma 3.1.

(2) Let $t' = p^n q + (p-1)p^2 q + (p-1)pq + (p-3)q + (p-4)$. Suppose that $h = x_1 x_2 \cdots x_m$ is a generator of $E_1^{p,t',*}$, where $m \le p$, x_i is one of a_k, $h_{l,j}$ or $b_{u,z}$, $0 \le k \le n+1$, $0 \le l+j \le n+1$, $0 \le u+z \le n$, $l > 0$, $j \ge 0$, $u > 0$, $z \ge 0$. Assume that $\deg x_i = q(c_{i,n} p^n + c_{i,n-1} p^{n-1} + \cdots + c_{i,0}) + e_i$, where $c_{i,j} = 0$ or 1, $e_i = 1$ if $x_i = a_{k_i}$, or $e_i = 0$. Then

$$\deg h = \sum_{i=1}^{m} \deg x_i = q\Big(\Big(\sum_{i=1}^{m} c_{i,n}\Big)p^n + \cdots + \Big(\sum_{i=1}^{m} c_{i,2}\Big)p^2 + \Big(\sum_{i=1}^{m} c_{i,1}\Big)p + \Big(\sum_{i=1}^{m} c_{i,0}\Big)\Big) + \Big(\sum_{i=1}^{m} e_i\Big)$$

$$= q(p^n + (p-1)p^2 + (p-1)p + (p-3)) + p - 4. \tag{3.7}$$

Note that $0 \le p-4, p-3, p-1 < p$, so, from (3.7), we have

$$\begin{cases} \sum_{i=1}^{m} e_i = p - 4 + \lambda_{-1} q, & \lambda_{-1} \ge 0, \\ \sum_{i=1}^{m} c_{i,0} + \lambda_{-1} = p - 3 + \lambda_0 p, & \lambda_0 \ge 0, \\ \sum_{i=1}^{m} c_{i,1} + \lambda_0 = p - 1 + \lambda_1 p, & \lambda_1 \ge 0, \\ \sum_{i=1}^{m} c_{i,2} + \lambda_1 = p - 1 + \lambda_2 p, & \lambda_2 \ge 0, \\ \sum_{i=1}^{m} c_{i,3} + \lambda_2 = 0 + \lambda_3 p, & \lambda_3 \ge 0, \\ \cdots & \cdots \\ \sum_{i=1}^{m} c_{i,n-1} + \lambda_{n-2} = 0 + \lambda_{n-1} p, & \lambda_{n-1} \ge 0, \\ \sum_{i=1}^{m} c_{i,n} + \lambda_{n-1} = 1. \end{cases} \tag{3.8}$$

Noting that $0 \le \sum_{i=1}^{m} e_i, \sum_{i=1}^{m} c_{i,j} \le m \le p$, it is easy to know that the sequence $(\lambda_{-1}, \lambda_0, \lambda_1, \lambda_2,)$ must equal the sequence $(0,0,0,0)$. From (3.8) we have that $\sum_{i=1}^{m} c_{i,3} = \lambda_3 p$. Note that $0 \le \sum_{i=1}^{m} c_{i,3} \le m \le p$. Thus we have that λ_3 may equal 0 or 1.

Case 1 If $\lambda_3 = 0$, then $\sum_{i=1}^{m} c_{i,3} = 0$.

When $n = 4$, we have that $\sum_{i=1}^{m} c_{i,4} = 1$. From the above results, it follows that there exists a factor $h_{1,4}$ or $b_{1,3}$ among x_i's.

When $n > 4$, we can similarly discuss and obtain that λ_4 may equal 0 or 1. We claim that $\lambda_4 = 0$, for otherwise, we would have that $\lambda_4 = 1$ and $\sum_{i=1}^{m} c_{i,4} = p$, then $m = p$. For any $1 \le i \le m$, deg x_i = higher terms $+p^4 q+$lower terms. Since $\sum_{i=1}^{p} e_i = p - 4 \equiv p - 4 (\bmod q)$, deg $a_i \equiv 1 (\bmod q)$ $(i \ge 0)$ and deg $h_{i,j} \equiv 0 (\bmod q)$ $(i > 0, j \ge 0)$, then there exists a factor $a_{j_1} a_{j_2} \cdots a_{j_{p-4}}$ $(0 \le j_1 \le j_2 \le \cdots \le j_{p-4} \le n+1)$ among x_i's with, for any $1 \le i \le p-4$, $j_i \ge 5$ and deg a_{j_i} =higher terms $+p^4 q + p^3 q + p^2 q + pq + q + 1$. It is obvious that $\sum_{i=1}^{m} c_{i,3} \ge p - 4$, which contradicts $\sum_{i=1}^{m} c_{i,3} = 0$, thus the claim is proved. By induction on j, we can get that $\lambda_j = 0$ $(4 \le j \le n-1)$. Thus we have that $\sum_{i=1}^{m} c_{i,j} = 0$ $(4 \le j \le n-1)$ and $\sum_{i=1}^{m} c_{i,n} = 1$. It follows that there is a factor $h_{1,n}$ or $b_{1,n-1}$ among x_i's .

All in all, at this time, for $n \ge 4$, there is a factor $h_{1,n}$ or $b_{1,n-1}$ among x_i's. We can denote the factor $h_{1,n}$ or $b_{1,n-1}$ by x_m, then $h = x_1 x_2 \cdots x_{m-1} h_{1,n}$ or $h = x_1 x_2 \cdots x_{m-1} b_{1,n-1}$. By the same argument as the proof of (1), we can show that at this time it is impossible for the generator h to exist.

Case 2 If $\lambda_3 = 1$, then $\sum_{i=1}^{m} c_{i,3} = p$.

Note that $c_{i,3} = 0$ or 1 and $m \le p$. It is easy to get that $m = p$. Noting that dim $h = p$, we can easily see that, for any i, dim $x_i = 1$ and $h = x_1 x_2 \cdots x_p \in E(h_{m,i} | m > 0, i \ge 0) \bigotimes P(a_n | n \ge 0)$.

For $n = 4$, we can easily get that $\sum_{i=1}^{p} c_{i,3} = p$, $\sum_{i=1}^{p} c_{i,4} = \cdots = \sum_{i=1}^{p} c_{i,n} = 0$.

For $n > 4$, from (3.8), we have that $\sum_{i=1}^{p} c_{i,4} + 1 = 0 + \lambda_4 p$; by the fact that $c_{1,4} = 0$ or 1 we have that $\lambda_4 = 1$. By induction on j, we have that $\lambda_j = 1$, $4 \le j \le n-1$. And then we have that $\sum_{i=1}^{p} c_{i,3} = p$, $\sum_{i=1}^{p} c_{i,4} = \cdots = \sum_{i=1}^{p} c_{i,n-1} = p - 1$ and $\sum_{i=1}^{p} c_{i,n} = 0$.

When $n = 4$, by the fact that $\sum_{i=1}^{p} e_i = p - 4$, $\sum_{i=1}^{p} c_{i,0} = p - 3$, $\sum_{i=1}^{p} c_{i,1} = p - 1$, $\sum_{i=1}^{p} c_{i,2} = p - 1$ and $\sum_{i=1}^{p} c_{i,3} = p$, we can prove that it is impossible for $h = x_1 x_2 \cdots x_p$ to exist by an argument similar to that used in the proof of Theorem 3.1.

When $n > 4$, by the fact that $\sum_{i=1}^{p} c_{i,3} = p$, $\sum_{i=1}^{p} c_{i,4} = \cdots = \sum_{i=1}^{p} c_{i,n-1} = p-1$, $\deg h_{k,j} = q(p^{k+j-1} + \cdots + p^j)$ $(k \geq 1, j \geq 0)$ and $\deg a_i = q(p^{i-1} + \cdots + p + 1) + 1$ $(i > 0)$, we can divide the p x_i's into two disjoint sets S_1 and S_2. The two disjoint sets are given by

$$S_1 = \{x | \deg x = q(p^{n-1} + p^{n-2} + \cdots + p^3) + \text{lower terms}\},$$
$$S_2 = \{x | \deg x = qp^3 + \text{lower terms}\}.$$

For a set S, define the number of elements in S by $N(S)$, then we can get $N(S_1) = p-1$ and $N(S_2) = 1$. Similarly, by the fact that $\sum_{i=1}^{p} e_i = p-4$, $\sum_{i=1}^{p} c_{i,0} = p-3$, $\sum_{i=1}^{p} c_{i,1} = p-1$, $\sum_{i=1}^{p} c_{i,2} = p-1$, $\sum_{i=1}^{p} c_{i,3} = p$, $\deg h_{k,j} = q(p^{k+j-1} + \cdots + p^j)$ $(k \geq 1, j \geq 0)$ and $\deg a_i = q(p^{i-1} + \cdots + p + 1) + 1$ $(i > 0)$, we can also divide the p x_i's into four disjoint sets. The four sets are given by

$$S_3 = \{x | \deg x = q(\text{higher terms} + p^3 + p^2 + p + 1) + 1\}, N(S_3) = p-4,$$
$$S_4 = \{x | \deg x = q(\text{higher terms} + p^3 + p^2 + p + 1)\}, N(S_4) = 1,$$
$$S_5 = \{x | \deg x = q(\text{higher terms} + p^3 + p^2 + p)\}, N(S_5) = 2,$$
$$S_6 = \{x | \deg x = q(\text{higher terms} + p^3)\}, N(S_6) = 1.$$

If $S_5 \subset S_1$, then there will be two $h_{n-1,1}$'s with $\deg h_{n-1,1} = q(p^{n-1} + \cdots + p^3 + p^2 + p)$. This is impossible since $h_{n-1,1}^2 = 0$, so one of the two elements in S_5 must be in S_2. This one is $h_{3,1}$ with $\deg h_{3,1} = q(p^3 + p^2 + p)$. Since $S_1 \bigcup S_2 = S_3 \bigcup S_4 \bigcup S_5 \bigcup S_6$, then we have that $S_3 \subset S_1$, $S_4 \subset S_1$, $S_6 \subset S_1$ and the other element of S_5 is in S_1. By these results, we can easily get that the set S_3 is made up of $p-4$ a_n's with $\deg a_n = q(p^{n-1} + \cdots + p^3 + p^2 + p + 1) + 1$, the set S_4 is made up of a $h_{n,0}$ with $\deg h_{n,0} = q(p^{n-1} + \cdots + p^3 + p^2 + p + 1)$, the set S_5 is made up of a $h_{3,1}$ and a $h_{n-1,1}$ with $\deg h_{3,1} = q(p^3 + p^2 + p)$ and $\deg h_{n-1,1} = q(p^{n-1} + \cdots + p^3 + p^2 + p)$ and the set S_6 is made up of an $h_{n-3,3}$ with $\deg h_{n-3,3} = q(p^{n-1} + \cdots + p^3)$. Therefore we have that, for $n > 4$, the generator h will exist and h can equal $a_n^{p-4} h_{n,0} h_{n-1,1} h_{n-3,3} h_{3,1}$ up to a sign.

From the above discussion, we get that

$$E_1^{p,t',*} = \begin{cases} \mathbb{Z}_p\{a_n^{p-4} h_{n,0} h_{n-1,1} h_{n-3,3} h_{3,1}\} & n > 4, \\ 0 & n = 4. \end{cases}$$

This completes the proof of (2).

Lemma 3.2 *Let $p \geq 7$, $n \geq 4$, $0 \leq s < p-3$. Then the product*

$$h_1 h_n \tilde{\gamma}_{s+3} \neq 0 \in \mathrm{Ext}_A^{s+5, p^n q + (s+3)p^2 q + (s+3)pq + (s+1)q + s}(\mathbb{Z}_p, \mathbb{Z}_p).$$

Proof It is known that $h_{1,n}$, $a_3^s h_{3,0} h_{2,1} h_{1,2} \in E_1^{*,*,*}$ are permanent cycles in the May spectral sequence and converge nontrivially to h_n, $\tilde{\gamma}_{s+3} \in \mathrm{Ext}_A^{*,*}(\mathbb{Z}_p, \mathbb{Z}_p)$ for $n \geq 0$, respectively (cf. Theorem 3.1). Thus $a_3^s h_{3,0} h_{2,1} h_{1,2} h_{1,1} h_{1,n} \in E_1^{s+5,t,*}$ is a permanent cycle in the May spectral sequence and converges to $\tilde{\gamma}_{s+3} h_1 h_n \in \mathrm{Ext}_A^{s+5,t}(\mathbb{Z}_p, \mathbb{Z}_p)$, where $t = p^n q + (s+3)p^2 q + (s+3)pq + (s+1)q + s$.

Case 1 When $\underline{0 \leq s < p-4 \text{ and } n \geq 4}$ or $\underline{s = p-4 \text{ and } n = 4}$, from Lemma 3.1, we know that, in the May spectral sequence, $E_1^{s+4,t,*} = 0$. Then we have $E_r^{s+4,t,*} = 0$ $(r \geq 1)$. It follows that the permanent cycle $a_3^s h_{3,0} h_{2,1} h_{1,2} h_{1,1} h_{1,n} \in E_r^{s+5,t,*}$ is not bounded and converges nontrivially to $\tilde{\gamma}_{s+3} h_1 h_n \in \mathrm{Ext}_A^{s+5,t}(\mathbb{Z}_p, \mathbb{Z}_p)$ in the May spectral sequence, then $\tilde{\gamma}_{s+3} h_1 h_n \neq 0 \in \mathrm{Ext}_A^{s+5,t}(\mathbb{Z}_p, \mathbb{Z}_p)$.

Case 2 When $\underline{s = p-4 \text{ and } n > 4}$, from Lemma 3.5, we have that $E_1^{p,t,*} = \mathbb{Z}_p\{a_n^{p-4} h_{n,0} h_{n-1,1} h_{n-3,3} h_{3,1}\}$. By the use of the first May differential, we can get that

$$d_1(a_n^{p-4} h_{n,0} h_{n-1,1} h_{n-3,3} h_{3,1}) = \begin{cases} a_n^{p-4} h_{n,0} h_{n-1,1} h_{1,3} h_{1,4} h_{3,1} + \cdots \neq 0 & \text{if } n = 5, \\ -a_n^{p-4} h_{n,0} h_{n-1,1} h_{n-3,3} h_{1,1} h_{2,2} + \cdots \neq 0 & \text{if } n \geq 6. \end{cases}$$

Thus $E_r^{p,t,*} = 0$ $(r \geq 2)$. Meanwhile, it is easy to see that the first May differential of $a_n^{p-4} h_{n,0} h_{n-1,1} h_{n-3,3} h_{3,1}$ does not equal $a_3^{p-4} h_{3,0} h_{2,1} h_{1,2} h_{1,1} h_{1,n}$ up to a sign. From the above results we know that the permanent cycle $a_3^{p-4} h_{3,0} h_{2,1} h_{1,2} h_1 h_n \in E_r^{p+1,t,*}$ is not bounded and converges nontrivially to $\tilde{\gamma}_{p-1} h_1 h_n \in \mathrm{Ext}_A^{p+1,t}(\mathbb{Z}_p, \mathbb{Z}_p)$. That is to say, $\tilde{\gamma}_{p-1} h_1 h_n \neq 0 \in \mathrm{Ext}_A^{p+1,t}(\mathbb{Z}_p, \mathbb{Z}_p)$.

From Case 1 and Case 2, the lemma follows.

Lemma 3.3 *Let $p \geq 7$, $n \geq 4$, $0 \leq s < p-3$ and $r \geq 2$. Then we have that the groups*

$$\mathrm{Ext}_A^{s+5-r, p^n q + (s+3)p^2 q + (s+3)pq + (s+1)q + s - r + 1}(\mathbb{Z}_p, \mathbb{Z}_p) = 0.$$

Proof If $r > s+5$, it is obvious.

Now we assume that $2 \leq r \leq s+5$. Let $t'' = q(p^n + (s+3)p^2 + (s+3)p + (s+1)) + (s-r+1)$. To prove $\mathrm{Ext}_A^{s+5-r,t''}(\mathbb{Z}_p, \mathbb{Z}_p) = 0$, it suffices to show that in the May spectral sequence $E_1^{s+5-r,t'',*} = 0$. Suppose that $h = x_1 x_2 \cdots x_m$ is a generator of $E_1^{s+5-r,t'',*}$, where $m \leq s+5-r$, x_i is one of a_k, $h_{l,j}$ or $b_{u,z}$, $0 \leq k \leq n+1$, $0 \leq l+j \leq n+1$, $0 \leq u+z \leq n$, $l > 0$, $j \geq 0$, $u > 0$, $z \geq 0$. Assume that $\deg x_i = q(c_{i,n} p^n + c_{i,n-1} p^{n-1} + \cdots + c_{i,0}) + e_i$, where $c_{i,j} = 0$ or 1, $e_i = 1$ if $x_i = a_{k_i}$, or $e_i = 0$. Then

$$\begin{aligned}\deg h &= \sum_{i=1}^m \deg x_i \\ &= q\left(\left(\sum_{i=1}^m c_{i,n}\right)p^n + \cdots + \left(\sum_{i=1}^m c_{i,2}\right)p^2 + \left(\sum_{i=1}^m c_{i,1}\right)p + \left(\sum_{i=1}^m c_{i,0}\right)\right) + \left(\sum_{i=1}^m e_i\right) \\ &= q(p^n + (s+3)p^2 + (s+3)p + (s+1)) + s - r + 1.\end{aligned}$$

We claim that $s-r+1 \geq 0$; otherwise, we would have $p > \sum_{i=1}^m e_i = q + (s-r+1) \geq q-4 > p$. That is impossible. The claim follows.

Note the suppositions that $c_{i,j} = 0$ or 1, $e_i = 0$ or 1 and $m \leq s+5-r \leq s+5-2 = s+3 < p$. By the same argument as in the proof of (1) in Lemma 3.1, we can get

$$\sum_{i=1}^m e_i = s-r+1, \quad \sum_{i=1}^m c_{i,0} = s+1, \quad \sum_{i=1}^m c_{i,1} = s+3,$$
$$\sum_{i=1}^m c_{i,2} = s+3, \quad \sum_{i=1}^m c_{i,3} = \cdots = \sum_{i=1}^m c_{i,n-1} = 0, \quad \sum_{i=1}^m c_{i,n} = 1.$$

It is easy to see that there exists a factor $h_{1,n}$ or $b_{1,n-1}$ among x_i's. By the graded commutativity of $E_1^{*,*,*}$, we can denote the factor $h_{1,n}$ or $b_{1,n-1}$ by x_m, then $h = x_1 x_2 \cdots x_{m-1} h_{1,n}$ or $h = x_1 x_2 \cdots x_{m-1} b_{1,n-1}$.

Case 1 If $h = x_1 x_2 \cdots x_{m-1} h_{1,n}$, then $h' = x_1 x_2 \cdots x_{m-1} \in E_1^{s+4-r, t'' - p^n q, *}$ and we have

$$\sum_{i=1}^{m-1} e_i = s-r+1, \quad \sum_{i=1}^{m-1} c_{i,0} = s+1, \quad \sum_{i=1}^{m-1} c_{i,1} = s+3,$$
$$\sum_{i=1}^{m-1} c_{i,2} = s+3, \quad \sum_{i=1}^{m-1} c_{i,3} = \cdots = \sum_{i=1}^{m-1} c_{i,n-1} = 0, \quad \sum_{i=1}^{m-1} c_{i,n} = 0.$$

From $\sum_{i=1}^{m-1} c_{i,2} = s+3$, we can get that $m-1 \geq s+3$. Then $\dim h' \geq s+3$. On the other hand, we also have that $\dim h' = s+4-r \leq s+2$. There is a contradiction. Thus it is impossible for a generator of the form $h = x_1 x_2 \cdots x_{m-1} h_{1,n}$ to exist.

Case 2 If $h = x_1 x_2 \cdots x_{m-1} b_{1,n-1}$, then $h'' = x_1 x_2 \cdots x_{m-1} \in E_1^{s+3-r, t'' - p^n q, *}$ and we have $\sum_{i=1}^{m-1} e_i = s-r+1$, $\sum_{i=1}^{m-1} c_{i,0} = s+1$, $\sum_{i=1}^{m-1} c_{i,1} = s+3$, $\sum_{i=1}^{m-1} c_{i,2} = s+3$, $\sum_{i=1}^{m-1} c_{i,3} = \cdots = \sum_{i=1}^{m-1} c_{i,n-1} = 0$, $\sum_{i=1}^{m-1} c_{i,n} = 0$. By the same argument as in Case 1, we can show that it is impossible for a generator of the form $h = x_1 x_2 \cdots x_{m-1}\, b_{1,n-1}$ to exist either.

From Case 1 and Case 2, we see that $E_1^{s+5-r,t'',*} = 0$, so $\mathrm{Ext}_A^{s+5-r,t''}(\mathbb{Z}_p, \mathbb{Z}_p) = 0$. This finishes the proof of the lemma.

Now we give the proof of Theorem 1.4.

To prove Theorem 1.4, it is equivalent to proving the following:

Theorem 3.2 *Let $p \geq 7$, $n \geq 4$ and $0 \leq s < p-3$. Then the product $\tilde{\gamma}_{s+3}h_1h_n \neq 0 \in \mathrm{Ext}_A^{s+5,p^nq+(s+3)p^2q+(s+3)pq+(s+1)q+s}(\mathbb{Z}_p, \mathbb{Z}_p)$ is a permanent cycle in the Adams spectral sequence and converges to a nontrivial element $jj'\bar{j}\gamma^{s+3}\bar{i}i'\xi_n \in \pi_{p^nq+(s+3)p^2q+(s+3)pq+(s+1)q-5}S$ of order p, where $q = 2(p-1)$.*

Proof From Theorem 1.3, we see that $i_*(h_1h_n) \in \mathrm{Ext}_A^{2,p^nq+pq}(H^*M, \mathbb{Z}_p)$ is a permanent cycle in the Adams spectral sequence and converges to a nontrivial element $\xi_n \in \pi_{p^nq+pq-2}M$. Let $\gamma : \Sigma^{q(p^2+p+1)}V(2) \to V(2)$ be the v_3-map and consider the following composition:

$$\bar{f} = jj'\bar{j}\gamma^{s+3}\bar{i}i'\xi_n : \Sigma^{p^nq+pq-2}S \xrightarrow{\xi_n} M \xrightarrow{i'} K \xrightarrow{\bar{i}} V(2)$$
$$\xrightarrow{\gamma^{s+3}} \Sigma^{-(s+3)(p^2+p+1)q}V(2) \xrightarrow{jj'\bar{j}} \Sigma^{-(s+3)(p^2+p+1)q+(p+2)q+3}S.$$

Since ξ_n is represented by $i_*(h_1h_n) \in \mathrm{Ext}_A^{2,p^nq+pq}(H^*M, \mathbb{Z}_p)$ in the Adams spectral sequence, then the above $\bar{f}$ is represented by $\bar{c} = (jj'\bar{j})_*(\gamma_*)^{s+3}(\bar{i})_*(i')_*i_*(h_1h_n) = (jj'\bar{j}\gamma^{s+3}\bar{i}i'i)_*(h_1h_n)$ in the Adams spectral sequence.

From Theorem 3.1 and the knowledge of Yoneda products we know that the composition

$$\mathrm{Ext}_A^{0,0}(\mathbb{Z}_p, \mathbb{Z}_p) \xrightarrow{(\bar{i}i'i)_*} \mathrm{Ext}_A^{0,0}(H^*V(2), \mathbb{Z}_p)$$
$$\xrightarrow{(jj'\bar{j})_*(\gamma_*)^{s+3}} \mathrm{Ext}_A^{s+3,(s+3)p^2q+(s+2)pq+(s+1)q+s}(\mathbb{Z}_p, \mathbb{Z}_p)$$

is a multiplication (up to a nonzero scalar) by $\tilde{\gamma}_{s+3} \in \mathrm{Ext}_A^{s+3,(s+3)p^2q+(s+2)pq+(s+1)q+s}(\mathbb{Z}_p, \mathbb{Z}_p)$. Hence, $\bar{f}$ is represented (up to a nonzero scalar) by

$$\bar{c} = \tilde{\gamma}_{s+3}h_1h_n \neq 0 \in \mathrm{Ext}_A^{s+5,p^nq+(s+3)p^2q+(s+3)pq+(s+1)q+s}(\mathbb{Z}_p, \mathbb{Z}_p)$$

in the Adams spectral sequence (cf. Lemma 3.2).

Moreover, from Lemma 3.3, we know that $\tilde{\gamma}_{s+3}h_1h_n$ cannot be hit by the differentials in the Adams spectral sequence and so the corresponding homotopy element $\bar{f} = jj'\bar{j}\gamma^{s+3}\bar{i}i'\xi_n \in \pi_*S$ is nontrivial and of order p. This finishes the proof of the theorem.

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