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The convergence of $\tilde{\alpha}_s^{(n)} h_0 h_k$ *

WANG Xiangjun (王向军) and ZHENG Qibing (郑弃冰)

(Department of Mathematics, Nankai University, Tianjin 300071, China)

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Abstract It is proved that (i) for $p \geqslant 5$, $2 \leqslant s \leqslant p-1$, $k \geqslant 2$, $\tilde{\beta}_s h_0 h_{k+1}$ survives to E_∞; (ii) for $p \geqslant 7$, $3 \leqslant s \leqslant p-1$, $k \geqslant 3$, $\tilde{\gamma}_s h_0 h_{k+1}$ survives to E_∞.

Keywords: stable homotopy groups, Adams spectral sequence, convergence.

The stable homotopy ring has long been an important problem of algebraic topology. Low-dimensional computation made little progress and hardly gave an insight into $\pi_*^s(s^0)$. In recent years, however, infinite families of elements of $\pi_*^s(s^0)$ have been discovered[1—5]. In this paper, we will prove the convergence of some $\tilde{\alpha}_s^{(n)} h_0 h_k$, and then get infinite families of non-zero elements in the stable homotopy groups of spheres.

Reference [6] gave a proof of the convergence of $\tilde{\alpha}_s^{(2)} h_0 h_k$. But that proof was based on the convergence of $h_0 h_k$[7]. Recently the proof for the convergence of $h_0 h_k$ was demenstracted to be wrong[7] (i.e. the convergence of $h_0 h_k$ is uncertain), and so the proof for the convergence of $\tilde{\alpha}_s^{(n)} h_0 h_k$ is not completed. In this paper, using the convergence of $h_0 b_k$ in ref. [8] instead of the convergence of $h_0 h_k$ and the viewpoint of the Greek letter families in the classical Adams spectral sequence we will prove the convergence of some $\tilde{\alpha}_s^{(n)} h_0 h_k$.

1 Preliminaries

Let p be an odd prime, let A be the mod p Steenrod algebra, let $\varepsilon: A \longrightarrow Z_p$ be the augmentation of A, and let $\bar{A} = \ker \varepsilon$ be the augmentation idea of A. Let A^* be the dual of A. Then as a Hopf algebra,

$$A^* = P[\xi_1, \xi_2, \cdots] \otimes E[\tau_0, \tau_1, \tau_2, \cdots]$$

and the coproduct $\Delta: A^* \longrightarrow A^* \otimes A^*$ induced by the product φ: $A \otimes A \longrightarrow A$ in A is given by

$$\Delta(\xi_n) = \sum_{i=0}^{n} \xi_{n-i}^{p^i} \otimes \xi_i,$$

$$\Delta(\tau_n) = \tau_n \otimes 1 + \sum_{i=0}^{n} \xi_{n-i}^{p^i} \otimes \tau_i,$$

where $\xi_0 = 1$.

Let M, N be modules of the Steenrod algebra A, and let φ_M: $A \otimes M \longrightarrow M$, φ_N: $A \otimes N \longrightarrow N$ be the A module structures of M, N. Then their duals M^*, N^* have natural A^*

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comodule structure. Let

$$\cdots \longrightarrow B_{s+1}(M) \xrightarrow{d_s} B_s(M) \longrightarrow \cdots \longrightarrow B_0(M) \xrightarrow{\varepsilon} M \longrightarrow 0$$

be the bar-resolution of M, where $B_s(M) = A \otimes \underbrace{\bar{A} \otimes \cdots \otimes \bar{A}}_{s} \otimes M$, the elements in $B_s(M)$ is denoted by $a[a_1 \mid a_2 \mid \cdots \mid a_s \mid]m$, and its differential d_s is given by

$$d_s[a_1 \mid a_2 \mid \cdots \mid a_s \mid] = a_1[a_2 \mid \cdots \mid a_s] + \sum_{i=1}^{s-1}(-1)^{\lambda(i)}[a_1 \mid \cdots \mid a_i a_{i+1} \mid \cdots \mid a_s]m,$$
$$-(-1)^{\lambda(s-1)}[a_1 \mid \cdots \mid a_{s-1}]a_s m,$$

$\lambda(i) = i + \sum_{j-1}^{i} \deg a_j$. Let

$$C^s(M,N) = \mathrm{Hom}_A(B_s(M),N) \cong N \otimes \bar{A}^* \otimes \cdots \otimes \bar{A}^* \otimes M^*$$

and $\delta_s = d_s^*$ be the coboundary induced by d_s. Then the elements in $C^s(M,N)$ may be denoted by $n[\alpha_1 \mid \alpha_2 \mid \cdots \mid \alpha_s \mid]\mu$. Suppose that $\bar{A}$ has Z_p-basis $\{\theta_\lambda\}_{\lambda \in I}$ and $\bar{A}^*$ has dual basis $\{\bar{\theta}_\lambda\}_{\lambda \in I}$. Then δ_s is given by

$$\delta_s n[\alpha_1 \mid \cdots \mid \alpha_s]\mu = \sum_{\theta_\lambda} \theta_\lambda n[\bar{\theta}_\lambda \mid \alpha_1 \mid \cdots \mid \alpha_s \mid]\mu$$
$$- \sum_{i=1}^{s}(-1)^{\lambda(i)} n[\alpha_1 \mid \cdots \mid \alpha_i' \mid \alpha_i'' \mid \cdots \mid \alpha_s]\mu$$
$$-(-1)^{\lambda(s+1)} n[\alpha_1 \mid \cdots \mid \alpha_s \mid \alpha']\mu',$$

where

$$\lambda(i) = i + \deg\alpha_1 + \cdots + \deg\alpha_i',$$
$$\lambda(s+1) = s + 1 + \deg\alpha_1 + \cdots + \deg\alpha_s + \deg\alpha',$$

and $\Delta(\alpha_i) = \sum \alpha_i' \otimes \alpha_i''$, $\Delta(\mu) = \sum \alpha' \otimes \mu'$ are given by the coproduct in $\bar{A}^*$ and M^* respectively. So we get a cochain complex

$$0 \longrightarrow C^0(M,N) \xrightarrow{\delta_0} C^1(M,N) \longrightarrow \cdots \longrightarrow C^s(M,N) \xrightarrow{\delta_s} C^{s+1}(M,N) \longrightarrow \cdots$$

and $H^{s,t}(C(M,N),\delta) = \mathrm{Ext}_A^{s,t}(M,N)$.

For finite spectra X, Y, if $H^*(X, Z_p) = M$, $H^*(Y, Z_p) = N$, then we have the Adams spectral sequence $\{E_k^{s,t}, d_k\}$ with its E_2-term $E_2^{s,t} = \mathrm{Ext}_A^{s,t}(M,N)$ that converges to $[Y,X]_p$.

Let M, N, L be three A modules. Then we have product in cochain complexes

$$\bar{\Lambda}: C^s(N,L) \otimes C^t(M,N) \longrightarrow C^{s+t}(M,L)$$

given by

$$(l[\alpha_1 \mid \cdots \mid \alpha_s]\mu)\bar{\Lambda}(n[\beta_1 \mid \cdots \mid \beta_t]\omega) \mapsto \langle\mu, n\rangle l[\alpha_1 \mid \cdots \mid \alpha_s \mid \beta_1 \cdots \mid \beta_t]\omega,$$

where $\langle\mu, n\rangle$ is the value of n under the map $\mu: N \longrightarrow Z_p$. And then $\bar{\Lambda}$ induces

$$\mathrm{Ext}_A^s(N,L) \otimes \mathrm{Ext}_A^t(M,N) \longrightarrow \mathrm{Ext}_A^{s+t}(M.L).$$

For any A modules M, N, $\mathrm{Ext}_A^{s,t}(M,N)$ is computable in theory. But because $C^s(M,N)$ has too many generators, the computation of $\mathrm{Ext}_A^{s,t}(M,N)$ becomes impossible. Zhou introduced order to $C^{s,t}(Z_p, Z_p)$, and then he found an acyclic subcomplex $\bar{B}(\infty)$ in $C^{s,t}(Z_p, Z_p)$ such that[9]

$$C^{s,t}(Z_p, Z_p)/\bar{B}(\infty) = P[\tau_0, \tau_1, \cdots] \otimes P[b(i,j); i \geqslant 1, j \geqslant 0]$$

$$\otimes E[\xi_i^{p^j};\ i \geqslant 1,\ j \geqslant 0].$$

And then $\mathrm{Ext}_A^{s,t}(Z_p, Z_p) = H^{s,t}(C(Z_p, Z_p), \delta) \cong H^{s,t}(C(Z_p, Z_p)/\bar{B}(\infty), \bar{\delta})$ where $\bar{\delta}$ is the coboundary induced by δ.

2 Greek letter families in the classical Adams spectral sequence

After giving some basic knowledge about the homological algebras, we will introduce the Greek letter families in the classical Adams spectral sequence and their relationship to that in the Adams-Novikov spectral sequence.

Let $n \geqslant 1$ be a positive integer, and let $E(n-1)$ be the torsion part of the Steenrod A generated by $Q_0, Q_1, \cdots, Q_{n-1}$, i.e. $E(n-1) = E[Q_0, Q_1, \cdots, Q_{n-1}]$. Let $\{Q_0^{\varepsilon_0} Q_1^{\varepsilon_1} \cdots Q_{n-1}^{\varepsilon_{n-1}} \mid$, where $\varepsilon_i = 0$ or $1\}$ be the Z_p-basis of $E(n-1)$, and let $\{\tau_0^{\varepsilon_0}\tau_1^{\varepsilon_1}\cdots\tau_{n-1}^{\varepsilon_{n-1}} \mid \varepsilon_i = 0$ or $1\}$ be the dual basis of $E(n-1)^*$. Then we have a short exact sequence

$$0 \longrightarrow \Sigma^{|Q_n|} E(n-1) \xrightarrow{Q_n} E(n) \xrightarrow{\pi} E(n-1) \longrightarrow 0,$$

where Q_n is product Q_n on the right side, and π is the natural projection. And that exact sequence represents an element in $\mathrm{Ext}_A^{1,*}(E(n-1), E(n-1))$.

Consider the bar-resolution of $E(n-1)$ and the following commutative digram:

$$\begin{array}{ccccccc}
\cdots \longrightarrow A \otimes \bar{A} \otimes E(n-1) & \xrightarrow{d_1} & A \otimes E(n-1) & \xrightarrow{\varphi_{n-1}} & E(n-1) & \longrightarrow 0 \\
\downarrow \bar{v}_n & & \downarrow \varphi_n(1 \otimes i) & & \| & \\
0 \longrightarrow \quad \Sigma^{|Q_n|} E(n-1) & \longrightarrow & E(n) & \xrightarrow{\pi} & E(n-1) & \longrightarrow 0
\end{array}$$

where φ_{n-1}: $A \otimes E(n-1) \longrightarrow E(n-1)$ is the A module structure of $E(n-1)$.

Denoting $\varphi_m: A \otimes E(n) \longrightarrow E(n)$ the A module structure of $E(n)$, we get a Z_p module map i: $E(n-1) \longrightarrow E(n)$ such that $\pi \cdot i = 1_{E(n-1)}$, and i induces A module map

$$A \otimes E(n-1) \xrightarrow{1 \otimes i} A \otimes E(n) \xrightarrow{\varphi_n} E(n).$$

It is easy to show that

$$\begin{array}{ccc}
A \otimes E(n-1) & \xrightarrow{\varphi_{n-1}} & E(n-1) \longrightarrow 0 \\
\downarrow \varphi_n(1 \otimes i) & & \| \\
E(n) & \xrightarrow{\pi} & E(n-1) \longrightarrow 0
\end{array}$$

commute with each other.

From the definition of d_1, we know that

$$\varphi_n(1 \otimes i) d_1 = \varphi_n(1 \otimes i) = i\varphi_{n-1}: \bar{A} \otimes E(n-1) \longrightarrow E(n)$$

$\mathrm{im}[\varphi_n(1 \otimes i) d_1] = \mathrm{im} Q_n \cong E(n-1)$, and $\{Q_0^{\varepsilon_0} Q_1^{\varepsilon_1} \cdots Q_{n-1}^{\varepsilon_{n-1}} Q_n \mid \varepsilon_i = 0$ or $1\}$ is a Z_p basis of $\mathrm{im} Q_n$, while $\{\tau_0^{\varepsilon_0}\cdots\tau_{n-1}^{\varepsilon_{n-1}} \mid \varepsilon_i = 0$ or $1\}$ is the dual basis of $\mathrm{im} Q_n^*$. Meanwhile, we note that

$$\sum_{(\varepsilon_0, \cdots, \varepsilon_{n-1})} Q_0^{\varepsilon_0} \cdots Q_{n-1}^{\varepsilon_{n-1}} Q_n [\tau_0^{\varepsilon_0} \cdots \tau_{n-1}^{\varepsilon_{n-1}} \tau_n] = 1: \mathrm{im} Q_n \longrightarrow \mathrm{im} Q_n.$$

So

$$\begin{aligned}
\varphi_n(1 \otimes i) d_1 &= \sum_{(\varepsilon_0, \cdots, \varepsilon_{n-1})} Q_0^{\varepsilon_0} \cdots Q_{n-1}^{\varepsilon_{n-1}} Q_n [\tau_0^{\varepsilon_0} \cdots \tau_{n-1}^{\varepsilon_{n-1}} \tau_n](\varphi_n(1 \otimes i) - i\varphi_{n-1}) \\
&= \sum Q_0^{\varepsilon_0} \cdots Q_{n-1}^{\varepsilon_{n-1}} Q_n [\tau_0^{\varepsilon_0} \cdots \tau_{n-1}^{\varepsilon_{n-1}} \tau_n] \varphi_n(1 \otimes i).
\end{aligned}$$

We can choose $\tilde{v}_n$ as

$$\tilde{v}_n = \sum Q_0^{\varepsilon_0}\cdots Q_{n-1}^{\varepsilon_{n-1}} Q_n[\tau_0^{\varepsilon_0}\cdots\tau_{n-1}^{\varepsilon_{n-1}}\tau_n]\varphi_n(1\otimes i)$$
$$= \sum Q_0^{\varepsilon_0}\cdots Q_{n-1}^{\varepsilon_{n-1}}(1\otimes i^*)\varphi_n^*(\tau_0^{\varepsilon_0}\cdots\tau_{n-1}^{\varepsilon_{n-1}}\tau_n).$$

i^* is given by $i^*(\tau_n)=0$.

Consider $\tilde{j} = 1[\,]1 \in \mathrm{Ext}_A^0(E(n-1), Z_p)$, $\tilde{\phi} = Q_0Q_1\cdots Q_{n-1}[\,]1 \in \mathrm{Ext}_A^0(Z_p, E(n-1))$ and define $\tilde{j}\bar{\Lambda}\tilde{v}_n\bar{\Lambda}\tilde{v}_n\bar{\Lambda}\cdots\bar{\Lambda}\tilde{v}_n\bar{\Lambda}\tilde{\phi} = \tilde{\alpha}_s^{(n)} \in \mathrm{Ext}_A^{s,*}(Z_p, Z_p)$.

Theorem 1. *For $n<p$, $s\neq 0, 1, \cdots, n-1 \bmod p$, $\tilde{\alpha}_s^{(n)}\neq 0$ is called the n-th Greek letter family element in* Ext_A^s. *$n=1,2,3$, $\tilde{\alpha}_s^{(n)}$ are denoted by $\tilde{\alpha}_s$, $\tilde{\beta}_s$, $\tilde{\gamma}_s$ respectively.*

Proof. As we know, $\tilde{j} = 1[\,]1$, and for $(\varepsilon_0, \varepsilon_1, \cdots, \varepsilon_{n-1} \neq (0, 0, \cdots, 0)$, $\langle 1, Q_0^{\varepsilon_0}\cdots Q_{n-1}^{\varepsilon_{n-1}}\rangle = 0$. If we denote

$$E(n)^* \xrightarrow{\varphi_n^*} \bar{A}^*\otimes E(n-1)^* \xrightarrow{1\otimes i^*} \bar{A}^*\otimes E(n-1)^*,$$

by $\tilde{\Delta}$ then

$$\tilde{j}\bar{\Lambda}\tilde{v}_n = 1[\tau_n]1 + \sum_{i=0}^{n-1} 1[\xi_{n-1}^{p^i}]\tau_i$$
$$= 1[\,]\tilde{\Delta}(\tau_n)$$

and

$$\tilde{j}\bar{\Lambda}\tilde{v}_n\bar{\Lambda}\tilde{v}_n = 1[\tau_n]\tilde{\Delta}(\tau_n) + \sum_{i=0}^{n+1} 1[\xi_{n-1}^{p^i}]\tilde{\Delta}(\tau_i\tau_n).$$

Because $\tilde{\phi} = Q_0\cdots Q_{n-1}[\,]1$, in $\tilde{j}\bar{\Lambda}\underbrace{\tilde{v}_n\bar{\Lambda}\cdots\bar{\Lambda}\tilde{v}_n}_{s}$, only the elements of the form $1[\theta_1 \mid \cdots \mid \theta_s]\tau_0\cdots\tau_{n-1}$ have non-zero product with $\tilde{\phi}$. And then, although $\tilde{\alpha}_s^{(n)} = \tilde{j}\bar{\Lambda}\tilde{v}_n\bar{\Lambda}\cdots\bar{\Lambda}\tilde{v}_n\bar{\Lambda}\tilde{\phi}$ is very complecated modular $\bar{B}(\infty)$, the largest word[9] in $\tilde{\alpha}_s^{(n)}$ is

$$[\tau_n \mid \cdots \mid \tau_n \mid \xi_n \mid \xi_{n-1}^p \mid \cdots \mid \xi_1^{p^{n-1}}].$$

Considering where ξ_n, ξ_{n-1}^p, $\cdots$, $\xi_1^{p^{n-1}}$ come from, we will find that

$$\tilde{\alpha}_s^{(n)} = \frac{s!}{(s-n)!}[\tau_n \mid \cdots \mid \tau_n \mid \xi_n \mid \xi_{n-1}^p \mid \cdots \mid \xi_1^{p^{n-1}}] + \cdots \bmod \bar{B}(\infty).$$

For $n<p$, $s\neq 0, 1, \cdots, n-1 \bmod p$, if the coboundary of $[x_1 \mid x_2 \mid \cdots \mid x_{s-1}] \in C^{s-1}(Z_p, Z_p)/\bar{B}(\infty)$ contains $[\tau_n \mid \cdots \mid \tau_n \mid \xi_n \mid \cdots \mid \xi_1^{p^{n-1}}]$, then there should be $s-n$ τ_{k_i}'s in x_1, $x_2, \cdots, x_{s-1}$ and each k_i should be no less than n. Meanwhile, each k_i should not be greater than n, otherwise the degree of τ_{k_i} will be greater than $\deg[\tau_n \mid \xi_n \mid \cdots \mid \xi_1^{p^{n-1}}]$. So $[x_1 \mid \cdots \mid x_{s-1}] = [\tau_n \mid \cdots \mid \tau_n \mid x_1 \mid \cdots \mid x_{n-1}]$, and x_1, $x_2, \cdots, x_{n-1}$ are elements in $P[b(i,j); i\geqslant 1, j\geqslant 0]\otimes E[\xi_i^{p^j}; i\geqslant 1, j\geqslant 0]$.

Suppose that $\deg x_i = 2(p-1)(a_{ki}p^k + \cdots + a_{1i}p + a_{0i})$. Then $a_{ji} = 0$ or 1, and

$$\deg[x_1 \mid \cdots \mid x_{n-1}] = 2(p-1)\left[\left(\sum_{i=1}^{n-1} a_{ki}\right)p^k + \cdots + \left(\sum_{i=1}^{n-1} a_{1i}\right)p + \left(\sum_{i=1}^{n-1} a_{0i}\right)\right]$$
$$= \deg[\xi_n \mid \cdots \mid \xi_1^{p^{n-1}}] = 2(p-1)[np^{n-1} + (n-1)p^{n-2} + \cdots + 2p + 1].$$

That is impossible, because $n<p$, and $\sum_{i=1}^{n-1} a_{(n-1)i} \leqslant n-1$ should never be n. That is to say,

$[\tau_n \mid \cdots \mid \tau_n \mid \xi_n \mid \cdots B\xi_1^{p^{n-1}}]$ is not contained in any coboundary of the elements in $C^{s-1}(Z_p, Z_p)/\bar{B}(\infty)$.

After giving the Greek letter families in the E_2-term of the classical Adams spectral sequence, we will discuss their relationship to the stable homotopy elements α_s, β_s, γ_s.

As we know, for $n\leqslant 3$, $p\geqslant 7$, the cohomological module $E(n)$ has geometric realization $V(n)$[10,11] i.e. $H^*(V(n);\ Z_p) = E(n)$, and we have the following cofibration sequence:

$$\Sigma^{2(p^n-1)}V(n-1) \xrightarrow{v_n} V(n-1) \longrightarrow V(n) \longrightarrow \Sigma^{2(p^n-1)+1}V(n-1) \longrightarrow \cdots$$

and this colibration sequence induces a short exact sequence

$$\begin{array}{ccccccccc} 0 \longrightarrow & H^*(\Sigma^{2(p^n-1)-1}V(n-1);\ Z_p) & \longrightarrow & H^*(V(n);\ Z_p) & \longrightarrow & H^*(V(n-1);\ Z_p) & \longrightarrow 0 \\ & \| & & \| & & \| & \\ 0 \longrightarrow & \Sigma^{|Q_n|}E(n-1) & \longrightarrow & E(n) & \longrightarrow & E(n-1) & \longrightarrow 0 \end{array}$$

So in the classical Adams spectral sequence, $v_n: \Sigma^{2(p^n-1)}V(n-1) \longrightarrow V(n-1)$ is represented by $\tilde{v}_n \in \mathrm{Ext}_A^{1,*}(E(n-1),\ E(n-1))$ (i.e. $\tilde{v}_n$ survives to stable map v_n).

It is easy to show that $j:\ S^0 \to V(n-1)$ is represented by $\tilde{j} \in \mathrm{Ext}_A^0(E(n-1),\ Z_p)$, and $\phi: V(n-1) \longrightarrow S^{|Q_0 \cdots Q_{n-1}|}$ is represented by $\tilde{\phi} \in \mathrm{Ext}_A^0(Z_p,\ E(n-1))$. So we have the following result.

Theorem 2. *For* $s \not\equiv 0,\ 1,\ \cdots,\ n-1 \bmod p$, *the stable homotopy elements* α_s, β_s, γ_s *are respectively represented by* $\tilde{\alpha}_s$, $\tilde{\beta}_s$, $\tilde{\gamma}_s$ *in the classical Adams spectral sequence.*

Proof. From the definition of α_s, β_s, γ_s and the properties of the classical Adams spectral sequence.

3 The convergence of $\tilde{\alpha}_s^{(n)} h_0 h_k$

In the classical Adams spectral sequence, for $k\geqslant 2$, $h_0 b_k = b(1,\ k)\otimes \xi_1 \in Ext^3(Z_p,\ Z_p)$ survives to non-zero homotopy element ζ_k of order p[8]. Considering the following lefting diagram:

$$\begin{array}{ccccc} S^0 & \xrightarrow{p} & S^0 & \longrightarrow & M \\ & & \downarrow \xi_k & \swarrow \mu_k & \\ & & \Sigma^{-q}S^0 & & \end{array}$$

and the homeomorphism $\pi^*:\ \mathrm{Ext}_A^{*,*}(Z_p,\ Z_p) \longrightarrow \mathrm{Ext}_A^{*,*}(Z_p,\ H^*(M,\ Z_p))$ induced by $\pi: M \longrightarrow S^1$, we have the following lemma.

Lemma[8]. *In the classical Adams spectral sequence, the lefting* μ_k *is represented by* $\pi^*(h_0 h_k)$ *in* $\mathrm{Ext}_A^{2,*}(Z_p,\ H^*(M, Z_p))$.

As we know, $h_0 h_k = [\xi_1^{p^{k+1}} \mid \xi_1]$, and $\pi:\ M \longrightarrow S^1$ induces homeomorphism $\pi^*: H^*(S^1) \longrightarrow H^*(M)$, $\pi^*(1) = Q_0$. Then $\pi^*(h_0 h_{k+1}) = Q_0[\xi_1^{p^{k+1}} \mid \xi_1]$.

Considering the following composition of maps:

$$S^{2s(p^n-1)} \longrightarrow \Sigma^{2s(p^n-1)}V(n-1) \xrightarrow{v_n^s} V(n-1) \xrightarrow{\phi_1} \Sigma^{|Q_1 \cdots Q_{n-1}|}M \xrightarrow{\mu_k} \Sigma^{-*}S^0,$$

we know that ϕ_1 induces non-zero map $\phi_1^*: \Sigma^{|Q_1 \cdots Q_{n-1}|}E(0) \longrightarrow E(n-1)$, and then ϕ_1 is represented by

$$\tilde{\phi}_1 \in Ext_A^{0,*}(E(0), E(n-1)),$$

where $\tilde{\phi}_1 = Q_1\cdots Q_{n-1}[\,]1 + Q_0Q_1\cdots Q_{n-1}[\,]\tau_0$. So $\mu_k\phi_1$ is represented by

$$\tilde{\phi}_1\bar{\Lambda}\pi^*(h_0h_{k+1}) = \tilde{\phi}_1h_0h_{k+1} = Q_0Q_1\cdots Q_{n-1}[\xi_1^{p^{k+1}} \mid \xi_1].$$

Theorem 3. (1) *For* $p\geqslant 5$, $2\leqslant s\leqslant p-1$, $k\geqslant 2$, $\tilde{\beta}_sh_0h_{k+1}$ *survives to* E_∞.

(2) *For* $p\geqslant 7$, $3\leqslant s\leqslant p-1$, $k\geqslant 3$, $\tilde{\gamma}_sh_0h_{k+1}$ *survives to* E_∞.

Proof. From above, we know that, for $n = 2, 3$, $\tilde{j}\bar{\Lambda}\tilde{v}_n\bar{\Lambda}\cdots\bar{\Lambda}\tilde{v}_n\bar{\Lambda}\tilde{\phi}_1\bar{\Lambda}\pi^*(h_0h_{k+1}) = \tilde{\alpha}_s^{(n)}h_0h_{k+1}$. And then the theorem follows if we have proved that $\tilde{\alpha}_s^{(n)}h_0h_{k+1}\neq$ and it is not the differential of some $\tilde{h}\in E_2$.

For (1), see the proof of Theorem 2 in reference [6].

For (2), suppose that the r-th differential of $\tilde{h} = \alpha[x_1 \mid x_2 \mid \cdots \mid x_m] + \cdots$in $C^{s-r+2}(Z_p, Z_p)/\bar{B}(\infty)$ is $\tilde{\gamma}_sh_0h_{k+1}$, where x_i is one of τ_{k_i}, $\xi_i^{p^j}$, or $b(k_i, j)$, and $r=1$ means that its coboundary is $\tilde{\gamma}_sh_0h_{k+1}$. Then $1\leqslant m\leqslant s-r+1<p$. Soppose that $\deg x_i = 2(p-1)(a_{ik+1}p^{k+1} + \cdots + a_{i3}p^3 + a_{i2}p^2 + \cdots + a_{i0}) + b_i$, where $a_{ij} = 0$ or 1 and for $x_i = \tau_{k_i}$, $b_i = 1$, otherwise $b_i = 0$. So we have

$$\begin{aligned}\deg\tilde{h} &= 2(p-1)\left(\left(\sum_{i=1}^m a_{ik+1}\right)p^{k+1} + \cdots + \left(\sum_{i=1}^m a_{i2}\right)p^2 + \left(\sum_{i=1}^m a_{i1}\right)p + \left(\sum_{i=1}^m a_{i0}\right)\right) + \sum_{i=1}^m b_i \\ &= 2(p-1)(p^{k+1} + sp^2 + (s-1)p + (s-1)) + s - n - r + 1.\end{aligned}$$

First of all, we know that $s-n-r+1\geqslant 0$, otherwise $p > \sum_{i=1}^m b_i = 2(p-1) + s - n - r + 1 \geqslant p$. And then

$$\sum_{i=1}^m b_i = s - n - r + 1 \geqslant 0.$$

Meanwhile, from $m<p$ and $a_{ij}=0$ or 1, we know that $\sum a_{ik+1} = 1$, $\sum ai_3 = 0$, and $\sum a_{i2} = s$. So in x_1, x_2, $\cdots$, x_m, there should be a $\xi_1^{p^{k+1}}$ or $b(1,k)$. Suppose that x_m is so. Then $\sum_{i=1}^{m-1} a_{i2} = s\leqslant m-1$, and from $m\leqslant s-r+2$, we know that, for $r\geqslant 2$, these $x_1,\cdots,x_{m-1},x_m$ do not exist.

As for $r=1$ and $x_{s+1} = \xi_1^{p^{k+1}}$, if the coboundary of $\tilde{h} = [x_1 \mid \cdots \mid x_s \mid \xi_1^{p^{k+1}}]$ contains the first term

$$[\tau_3 \mid \cdots \mid \tau_3 \mid \xi_3 \mid \xi_2^p \mid \xi_1^{p^{k+1}} \mid \xi_1^{p^2} \mid \xi_1]$$

of $\tilde{\gamma}_sh_0h_{k+1}$, then, from $\sum a_{i3} = 0$, we know that there are

$$s-3\ \tau_3\text{'s in } x_1, \cdots, x_s, \text{ i.e. } \tilde{h} = [\tau_3 \mid \cdots \mid \tau_3 \mid \xi_1^{p^{k+1}} \mid x_1 \mid x_2 \mid x_3].$$

So $\deg[x_1 \mid x_2 \mid x_3] = 2(p-1)(3p^2+2p+2)$ and $x_i = \xi_{m_i}^{p^j}$. That is impossible unless two ξ_3's are allowed to appear.

From the Thom map $\Phi:\mathrm{Ext}_{BP_*BP}^{s,*}(BP_*, BP_*) \longrightarrow \mathrm{Ext}_A^{s,*}(Z_p, Z_p)$, we know that $\tilde{\Phi}(\beta_{p^k/p^k-1}) = h_0h_{k+1}$, $\Phi(\beta_2) = [\xi_2 \mid \xi_1^p] = \tilde{\beta}_2$ (see ref. [4]) while $\tilde{\beta}_sh_0h_{k+1} \neq 0$. So from $\Phi(\beta_2\beta_{p^k/p^k-1}) = \tilde{\beta}_2h_0h_{k+1}$, we know that $\beta_2\beta_{p^k/p^k-1}\neq 0$.

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