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A New Massey Product on **Ext** Groups*

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Let H be a commutative coassociative Hopf algebra over a field K; then $\mathbf{Ext}_H(K, K)$ is a DGA. Suppose M is a right DGA-module over this DGA. We give a new Massey product on this module and prove the basic properties of this Massey product. We have also discussed the case when M is a left DGA-module. © 1996 Academic Press, Inc.

This paper is a refinement of [5]. Throughout this paper, H is a graded commutative coassociative Hopf algebra over a field K, and (R, d) is the cobar complex for H; that is, the elements of R have the form $[a_1|a_2|\cdots|a_n]$ with $a_i \in \overline{H}$ and the differential is defined by

$$d[a_1|a_2|\cdots|a_n] = \sum\left(-[a_1'|a_1''|\cdots|a_n]\cdots+(-1)^n[a_1|\cdots|a_{n-1}|a_n'|a_n'']\right),$$

where $\Delta(a_i) = 1 \otimes a_i + a_i \otimes 1 + \sum a_i' \otimes a_i''$. Then (R, d) is a DGA with the bar product.

Let (M, d) be a right DGA-module over (R, d) (see [2]); that is, for any $a \in R$, $m \in M$ (we continue to use the bar to denote the module action)

$$d(m|a) = dm|a + (-1)^{|m|}m|da,$$

where we always use $|\cdot|$ to denote the cohomological degree and $\|\cdot\|$ to denote the second degree.

The purpose of this paper is to define a new Massey product on (M, d) and to prove some basic properties of this product.

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1. STANDARD HOMOTOPY

In this section, $\mathbf{Z}$ ($\mathbf{Z}_p$) is the set of integers (modular p) and when we compute sign functions, "$\equiv$" implies "$\equiv \pmod 2$."

We first give the definitions of some sign functions. For integers $0 < i_1 < i_2 < \cdots < i_n$, $n = 2, 3, \ldots,$ let

$$\mathscr{S}(i_1, i_2, \ldots, i_n) = \left\{(i_{s_1}, i_{s_2}, \ldots, i_{s_n}) \middle| (s_1, s_2, \ldots, s_n) \text{ is a permutation of } (1, 2, \ldots, n)\right\}.$$

If $f\colon \{i_1, i_2, \ldots, i_n\} \to \mathbf{Z}$ is a map, then $\mu_f\colon \mathscr{S}(i_1, i_2, \ldots, i_n) \to \mathbf{Z}_2$ is defined according to the following rules.

1. $\mu_f(i_1, i_2, \ldots, i_n) = 0.$
2. If $\mu_f(i_{s_1}, i_{s_2}, \ldots, i_{s_n})$ is defined, then for any $1 \le k \le n$,

$$\mu_f\left(i_{s_1}, \ldots, i_{s_{k-1}}, i_{s_k}, i_{s_{k+1}}, \ldots, i_{s_n}\right) \equiv \mu_f\left(i_{s_1}, \ldots, i_{s_{k-1}}, i_{s_{k+1}}, i_{s_k}, \ldots, i_{s_n}\right) + f\left(i_{s_k}\right) f\left(i_{s_{k+1}}\right).$$

It is easy to prove that μ_f is well-defined.

For a finite set $S = \{b_1, b_2, \ldots, b_n\}$ and a map $f\colon S \to \mathbf{Z}$, if $(i_1, i_2, \ldots, i_n)$ is a permutation of $(1, 2, \ldots, n)$, we regard it as an element of $\mathscr{S}(1, 2, \ldots, n)$ and define the sign function of $(b_{i_1}, b_{i_2}, \ldots, b_{i_n})$ relative to $(b_1, b_2, \ldots, b_n)$ by

$$\mu_f\left(b_{i_1}, b_{i_2}, \ldots, b_{i_n}\right) = \mu_{f'}(i_1, i_2, \ldots, i_n),$$

where $f'(k) = f(b_k)$, $k = 1, 2, \ldots, n$.

For $0 < i_1 < i_2 < \cdots < i_n$, $n > 1$, let

$$\mathscr{S}'(i_1, i_2, \ldots, i_n) = \{(p_1, p_2, \ldots, p_s | q_1, q_2, \ldots, q_{n-s}) | i_1 \le p_1 < p_2 < \cdots < p_s \le i_n, i_1 \le q_1 < q_2 < \cdots < q_{n-s} \le i_n, (p_1, \ldots, p_s, q_1, \ldots, q_{n-s}) \in \mathscr{S}(i_1, \ldots, i_n), s = 1, 2, \ldots, n-1\}.$$

Notice that $(1|2, 3, \ldots, n) \ne (1, 2|3, \ldots, n)$.

Let $f\colon \{i_1, i_2, \ldots, i_n\} \to \mathbf{Z}$ be a map. We still denote its restrictions by f, and we inductively define $\sigma_f\colon \mathscr{S}'(i_1, i_2, \ldots, i_n) \to \mathbf{Z}_2$ by

1. $$\sigma_f(i_1 | i_2, \ldots, i_n) \equiv 0$$
$$\sigma_f(i_2, \ldots, i_n | i_1) \equiv f(i_1) + f(i_2) + \cdots + f(i_n).$$

2. For $(p_1, \ldots, p_s | q_1, \ldots, q_{n-s-1}) \in \mathscr{S}(i_2, i_3, \ldots, i_n)$, $s \geq 1$, $n - s > 1$,

$$\sigma_f(i_1, p_1, \ldots, p_s | q_1, \ldots, q_{n-s}) \equiv f(i_1) + \sigma_f(p_1, \ldots, p_s | q_1, \ldots, q_{n-s-1})$$
$$\sigma_f(p_1, \ldots, p_s | i_1, q_1, \ldots, q_{n-s-1}) \equiv \sigma_f(p_1, \ldots, p_s | q_1, \ldots, q_{n-s-1}).$$

LEMMA 1.1. *Let*

$$(i_1, \ldots, i_s | j_1, \ldots, j_t) \in \mathscr{S}'(p_1, \ldots, p_{s+t}),$$
$$(j_1, \ldots, j_t | k_1, \ldots, k_u) \in \mathscr{S}'(q_1, \ldots, q_{t+u}),$$
$$(i_1, \ldots, i_s, j_1, \ldots, j_t, k_1, \ldots, k_u) \in \mathscr{S}(r_1, \ldots, r_{s+t+u}),$$

and let $f\colon \{r_1, \ldots, r_{s+t+u}\} \to \mathbf{Z}$ *be a map* (*we continue to denote its restrictions by* f). *Then we have*

$$\begin{aligned}\mu_f(i_1, \ldots, i_s, q_1, \ldots, q_{t+u}) &+ \mu_f(j_1, \ldots, j_t, k_1, \ldots, k_u)\\ &\equiv \mu_f(p_1, \ldots, p_{s+t}, k_1, \ldots, k_u) + \mu_f(i_1, \ldots, i_s, j_1, \ldots, j_t)\\ &\equiv \mu_f(i_1, \ldots, i_s, j_1, \ldots, j_t, k_1, \ldots, k_u)\end{aligned}$$

$$\begin{aligned}\sigma_f(p_1, \ldots, p_{s+t} | k_1, \ldots, k_u) &+ \sigma_f(i_1, \ldots, i_s | j_1, \ldots, j_t)\\ &\equiv f(i_1) + f(i_2) + \cdots + f(i_s)\\ &\quad + \sigma_f(i_1, \ldots, i_s | q_1, \ldots, q_{t+u}) + \sigma_f(j_1, \ldots, j_t | k_1, \ldots, k_u).\end{aligned}$$

Proof. Direct checkings. ∎

DEFINITION 1.1. For a Hopf algebra H, the standard homotopy on its cobar complex $S\colon R \otimes_K R \to R$ is defined by

1. $S(u, 1) = S(1, u) = 0$ for any $u \in R$.
2. For $u_i \in \overline{H}$, $v_j \in \overline{H}$, $i = 1, \ldots, m$, $j = 1, \ldots, n$, $m, n = 1, 2, \ldots$, and

$$\Delta^{(m-1)}(v_j) = \sum v_j^{(1)} \otimes v_j^{(2)} \otimes \cdots \otimes v_j^{(m)}, \qquad v_j^{(k)} \in H$$

$$\begin{aligned}S([u_1 | u_2 | \cdots | u_m], [v_1 | v_2 | \cdots | v_n])\\ = \sum_{k=1}^{n} \sum (-1)^{k(m-1)+\tau+1} [v_1 | \cdots | v_{k-1} | u_1 v_k^{(1)} | \cdots | u_m v_k^{(m)} | v_{k+1} | \cdots | v_n],\end{aligned}$$

where τ is the sign function of

$$\left(v_1, \ldots, v_{k-1}, u_1, v_k^{(1)}, \ldots, u_m, v_k^{(m)}, v_{k+1}, \ldots, v_n\right)$$

relative to

$$\left(u_1, \ldots, u_m, v_1, \ldots, v_{k-1}, v_k^{(1)}, \ldots, v_k^{(m)}, v_{k+1}, \ldots, v_n\right)$$

and f is taken to be the second degree $\|\cdot\|$.

LEMMA 1.2. *For any $u, v, w \in R$, we have*

1. $dS(u,v) + S(du,v) + (-1)^{|u|}S(u,dv) = u|v - (-1)^{|u\|v| + \|u\|\,\|v\|}v|u$
2. $S(u, v|w) = S(u,v)|w + (-1)^{(|u|-1)|v| + \|u\|\,\|v\|}v|S(u,w)$
3. $S(u, S(v,w)) - S(S(u,v),w) = (-1)^{(|u|-1)(|v|-1) + \|u\|\,\|v\|}$
$(S(v, S(u,w)) - S(S(v,u),w))$ *and if* $|w| = 1$, *then* $S(u, S(v,w)) = S(S(u,v),w)$.

Proof. Part 1 is proved by a definition of [3]. Since (R, d) is the cobar complex of a Hopf algebra H, there is a group module morphism (see [3]),

$$\theta\colon W \otimes_K R \otimes_K R \to R,$$

where W is the free resolution of the group $\mathbf{Z}_2$; in fact for any $x, y \in R$, $S(x,y) = \theta(e_1 \otimes x \otimes y)$, so part 1 is proved.

Part 2 is proved by induction on the first degree.

For any $u_i, v_j, w_k \in \overline{H}$, $i = 1, \ldots, m$, $j = 1, \ldots, n$, $k = 1, \ldots, l$, and

$$\Delta^{(r-1)} w_k = \sum w_k^{(1)} \otimes \cdots \otimes w_k^{(r)}$$

we have by definition ($l > 1$, if $l = 1$, the sum is 0)

$$\begin{aligned}
S\big([u_1|\cdots|u_m], S([v_1|\cdots|v_n],[w_1|\cdots|w_l])\big)&\\
- S\big(S([u_1|\cdots|u_m],[v_1|\cdots|v_n]),[w_1|\cdots|w_l]\big)&\\
= \sum_{1\le s<t\le l} (-1)^{s(m-1)+t(n-1)+\tau}&\\
\times\big[w_1|\cdots|w_{s-1}|u_1w_s^{(1)}|\cdots|u_mw_s^{(m)}|w_{s+1}|\cdots&\\
|w_{t-1}|v_1w_t^{(1)}|\cdots|v_nw_t^{(n)}|w_{t+1}|\cdots|w_l\big]&\\
+ \sum_{1\le s<t\le l} (-1)^{s(n-1)+(t+n-1)(m-1)+\tau'}&\\
\times\big[w_1|\cdots|w_{s-1}|v_1w_s^{(1)}|\cdots|v_nw_s^{(n)}|w_{s+1}|\cdots&\\
|w_{t-1}|u_1w_t^{(1)}|\cdots|u_mw_t^{(m)}|w_{t+1}|\cdots|w_l\big],&
\end{aligned}$$

where τ, τ' are respectively the sign functions relative to the sequence

$$\big(u_1, \dots, u_m, v_1, \dots, v_n, w_1, \dots, w_{s-1}, w_s^{(1)}, \dots, w_s^{(m)}, w_{s+1}, \dots, w_{t-1}, w_t^{(1)}, \dots, w_t^{(n)}, w_{t+1}, \dots, w_l\big)$$

$$\big(u_1, \dots, u_m, v_1, \dots, v_n, w_1, \dots, w_{s-1}, w_s^{(1)}, \dots, w_s^{(n)}, w_{s+1}, \dots, w_{t-1}, w_t^{(1)}, \dots, w_t^{(m)}, w_{t+1}, \dots, w_l\big)$$

and the map f is taken to be $\|\cdot\|$. ∎

For $b_i \in C(H)$, $i = 1, \dots, n$, we inductively define

$$S(b_1) = b_1, S(b_1, b_2, \dots, b_n) = S\big(b_1, S(b_2, \dots, b_n)\big).$$

For $1 \le i_1 < i_2 < \cdots < i_n$, $\beta = (p_1, \dots, p_s | q_1, \dots, q_{n-s}) \in \mathscr{S}'(i_1, i_2, \dots, i_n)$, we denote

$$S(b_\beta) = S\big(b_{p_1}, \dots, b_{p_s}\big)\big|S\big(b_{q_1}, \dots, b_{q_{n-s}}\big)$$

$$\mu\big(b_\beta\big) = \mu_f(p_1, \dots, p_s, q_1, \dots, q_{n-s}) + \mu_{f'}(p_1, \dots, p_s, q_1, \dots, q_{n-s}) + \sigma_f(p_1, \dots, p_s | q_1, \dots, q_{n-s}),$$

$$f(i_k) = |b_{i_k}| - 1, f'(i_k) = \|b_{i_k}\|, k = 1, \dots, n.$$

Then by definition, if $(p_1, \dots, p_s | q_1, \dots, q_{n-s-1}) \in \mathscr{S}'(2, 3, \dots, n)$, we have

$$\mu\big(b_{(1, p_1, \dots, p_s | q_1, \dots, q_{n-s-1})}\big) \equiv \mu\big(b_{(p_1, \dots, p_s | q_1, \dots, q_{n-s-1})}\big) + |b_1| - 1$$

$$\mu\big(b_{(p_1, \dots, p_s | 1, q_1, \dots, q_{n-s-1})}\big) \equiv \mu\big(b_{(p_1, \dots, p_s | q_1, \dots, q_{n-s-1})}\big) + \big(|b_{p_1}| + \cdots + |b_{p_s}| - s\big)\big(|b_1| - 1\big) + \big(\|b_{p_1}\| + \cdots + \|b_{p_s}\|\big)\|b_1\|.$$

THEOREM 1.1. *For $b_i \in R$, $i = 1, 2, \dots, n$, $n > 1$, we have*

1. *If $db_i = 0$, $i = 1, 2, \dots, n$, then*

$$dS(b_1, b_2, \dots, b_n) = \sum_{\alpha \in \mathscr{S}'(1, 2, \dots, n)} (-1)^{\mu(b_\alpha)} S(b_\alpha).$$

2. *If* $db_i = 0$, $i = 1, 2, \ldots, n$, $1 \leq k \leq n$, $d\bar{b}_k = b_k$, *then*

$$dS\big(b_1, \ldots, b_{k-1}, \bar{b}_k, b_{k+1}, \ldots, b_n\big)$$
$$= (-1)^{|b_1|+|b_2|+\cdots+|b_{k-1}|-k+\epsilon} S(b_1, \ldots, b_{k-1}, b_k, b_{k+1}, \ldots, b_n)$$
$$+ \sum_{\alpha \in \mathscr{S}'(1,2,\ldots,n)} (-1)^{\mu(c_\alpha)} S(c_\alpha),$$

where $c_i = b_i$ *if* $i \neq k$, $c_k = \bar{b}_k$, $\epsilon = 0$, *if* $k < n$, $\epsilon = 1$ *if* $k = n$.

3. *For* $b_{n+1} \in R$,

$$S\big(b_1, b_2, \ldots, b_{n-1}, b_n | b_{n+1}\big) = \sum_{\alpha \in \mathscr{S}'} (-1)^{|b_1|+\cdots+|b_{n-1}|-n+1+\mu(b_\alpha)} S(b_\alpha),$$

where $\mathscr{S}'$ *is a subset of* $\mathscr{S}'(1, 2, \ldots, n, n+1)$ *of such elements* $\alpha = (i_1, \ldots, i_s, n | j_1, \ldots, j_{n-s-1}, n+1)$, $s = 0, 1, \ldots, n-1$.

Proof. All is proved by induction on n. ∎

2. THE NEW MASSEY PRODUCT ON **Ext** GROUPS

For $a \in M$, $da = 0$, $b_i \in R$, $db_i = 0$, $i = 1, 2, \ldots, n$, the product $\langle a; b_1, b_2, \ldots, b_n \rangle$ is defined modular a class of elements, and we always use $\langle a; b_1, b_2, \ldots, b_n \rangle$ to denote a concrete element and use $In\langle a; b_1, b_2, \ldots, b_n \rangle$ to denote the whole class. We say $In\langle a; b_1, b_2, \ldots, b_n \rangle$ is trivial if there is $c \in In\langle a; b_1, b_2, \ldots, b_n \rangle$ such that $c \sim 0$ in (M, d).

Definition 2.1. Let (M, d) be a DGA-module over (R, d), $a \in M$, $da = 0$, $b_i \in R$, $db_i = 0$, $i = 1, \ldots, n$. We inductively define $\langle a; b_1, b_2, \ldots, b_n \rangle$ by

1. $\langle a; b_i \rangle = a | b_i$, $i = 1, 2, \ldots, n$.

2. If for any $1 \leq i_1 < i_2 < \cdots < i_s \leq n$, $s \langle n, n \rangle 1$, $In\ \langle a; b_{i_1}, b_{i_2}, \ldots, b_{i_s} \rangle$ is trivial and we let $d\overline{\langle a; b_{i_1}, b_{i_2}, \ldots, b_{i_s} \rangle} = \langle a; b_{i_1}, b_{i_2}, \ldots, b_{i_s} \rangle$, then a given element in $In\langle a; b_1, b_2, \ldots, b_n \rangle$ is defined by

$$\langle a; b_1, b_2, \ldots, b_n \rangle$$
$$= a | S(b_1, b_2, \ldots, b_n)$$
$$- \sum_{\alpha \in \mathscr{S}'(1,2,\ldots,n)} (-1)^{|a|+\mu(b_\alpha)} \overline{\langle a; b_{i_1}, \ldots, b_{i_s} \rangle} | S\big(b_{j_1}, \ldots, b_{j_{n-s}}\big),$$

where $\alpha = (i_1, \ldots, i_s | j_1, \ldots, j_{n-s})$.

Notice that if we suppose

$$(i_1, \ldots, i_s | j_1, \ldots, j_t) \in \mathscr{S}'(p_1, \ldots, p_{s+t}),$$

$$(j_1, \ldots, j_t | k_1, \ldots, k_u) \in \mathscr{S}'(q_1, \ldots, q_{t+u}),$$

$$(i_1, \ldots, i_s, j_1, \ldots, j_t, k_1, \ldots, k_u) \in \mathscr{S}(r_1, \ldots, r_{s+t+u}),$$

then by Lemma 1.1 and by definition we have

$$\begin{aligned} &\mu\big(b_{(i_1, \ldots, i_s | q_1, \ldots, q_{i+u})}\big) + \mu\big(b_{(j_1, \ldots, j_t | k_1, \ldots, k_u)}\big) \\ &\quad \equiv \mu\big(b_{(p_1, \ldots, p_{s+t} | k_1, \ldots, k_u)}\big) + \mu\big(b_{(i_1, \ldots, i_s | j_1, \ldots, j_t)}\big) \\ &\quad\quad + |b_{i_1}| + |b_{i_2}| + \cdots + |b_{i_s}| - s \end{aligned}$$

so we have

$$\begin{aligned} &d\langle a; b_1, b_2, \ldots, b_n \rangle \\ &\quad = \sum_{\alpha \in \mathscr{S}'(1, 2, \ldots, n)} (-1)^{|a| + \mu(b_\alpha)} a | S(b_\alpha) \\ &\quad\quad - \sum_{\alpha \in \mathscr{S}'(1, 2, \ldots, n)} (-1)^{|a| + \mu(b_\alpha)} \langle a; b_{i_1}, \ldots, b_{i_s} \rangle | S\big(b_{j_1}, \ldots, b_{j_{n-s}}\big) \\ &\quad\quad - \sum_{\alpha \in \mathscr{S}'(1, 2, \ldots, n)} \sum_{\beta \in \mathscr{S}'(j_1, \ldots, j_{n-1})} (-1)^{\mu(b_\alpha) + \mu(b_\beta) + \mu} \\ &\quad\quad\quad \overline{\langle a; b_{i_1}, \ldots, b_{i_s} \rangle} | S\big(b_\beta\big) \\ &\quad = \sum_{\alpha \in \mathscr{S}'(1, \ldots, n)} \sum_{\gamma \in \mathscr{S}'(i_1, \ldots, i_s)} (-1)^{\mu(b_\alpha) + \mu(b_\gamma)} \overline{\langle a; b_{p_1}, \ldots, b_{p_i} \rangle} \\ &\quad\quad \big| S\big(b_{q_1}, \ldots, b_{q_{s-i}}\big) \big| S\big(b_{j_1}, \ldots, b_{j_{n-s}}\big) \\ &\quad\quad - \sum_{\alpha \in \mathscr{S}'(1, \ldots, n)} \sum_{\beta \in \mathscr{S}'(j_1, \ldots, j_{n-s})} (-1)^{\mu(b_\alpha) + \mu(b_\beta) + \mu} \\ &\quad\quad \times \overline{\langle a; b_{i_1}, \ldots, b_{i_s} \rangle} | S\big(b_\beta\big) = 0, \end{aligned}$$

where $\alpha = (i_1, \ldots, i_s | j_1, \ldots, j_{n-s})$, $\gamma = (p_1, \ldots, p_t | q_1, \ldots, q_{s-t})$, $\mu = |b_{i_1}| + \cdots + |b_{i_s}| - s$, so $\langle a; b_1, b_2, \ldots, b_n \rangle \in \ker d$.

LEMMA 2.1. *Suppose* $\langle a; b_1, b_2, \ldots, b_n \rangle$ *is defined, if* $a \sim 0$ *in* (M, d), *or for some* $1 \le k \le n$, $b_k \sim 0$ *in* (R, d); *then* $In\langle a; b_1, \ldots, b_n \rangle$ *is trivial.*

Proof. If $a \sim 0$ in (M, d), let $d\bar{a} = a$; then for $1 \le i_1 < \cdots < i_s \le n$, $s = 1, 2, \ldots, n$, we define

$$\overline{\langle a; b_{i_1}, \ldots, b_{i_s} \rangle} = \bar{a} | S\big(b_{i_1}, \ldots, b_{i_s}\big).$$

Then it is easy to check that this is well-defined.

If for $1 \leq k \leq n$, $d\bar{b}_k = b_k$, then for $1 \leq i_1 < i_2 < \cdots < i_s \leq n$, $s = 1, 2, \ldots, n$, we define

$$\begin{aligned}
&(-1)^{\epsilon + \sum_{i_j < k}(|b_{i_j}| - 1)} \overline{\langle a; b_{i_1}, b_{i_2}, \ldots, b_{i_s} \rangle} \\
&\quad = (-1)^{|a|} a | S(c_{i_1}, c_{i_2}, \ldots, c_{i_s}) \\
&\qquad - \sum_{\alpha \in \mathscr{S}_k{}'(i_1, \ldots, i_s)} (-1)^{\mu(c_\alpha)} \overline{\langle a; b_{p_1}, \ldots, b_{p_i} \rangle} | S(c_{q_1}, \ldots, c_{q_{s-i}}),
\end{aligned}$$

where $\mathscr{S}_k{}'(i_1, i_2, \ldots, i_s)$ is the subset of $\mathscr{S}'(i_1, i_2, \ldots, i_s)$ of element $\alpha = (p_1, \ldots, p_t | q_1, \ldots, q_{s-t})$ such that $k \in \{q_1, q_2, \ldots, q_{s-t}\}$, $c_{i_j} = b_{i_j}$ if $i_j \neq k$, $c_k = \bar{b}_k$, $\epsilon = 0$ if $k = n$, $\epsilon = 1$ if $k < n$, then we have

$$\begin{aligned}
&d\overline{\langle a; b_1, b_2, \ldots, b_n \rangle} \\
&\quad = a | S(b_1, \ldots, b_k, \ldots, b_n) + \sum_{\alpha \in \mathscr{S}'(1, \ldots, n)} (-1)^{\mu + \mu(c_\alpha) + \epsilon} a | S(c_\alpha) \\
&\qquad - \sum_{\alpha \in \mathscr{S}_k{}'(1, \ldots, n)} (-1)^{\mu + \mu(c_\alpha) + \epsilon} \langle a; b_{i_1}, \ldots, b_{i_s} \rangle | S(c_{j_1}, \ldots, c_{j_{n-s}}) \\
&\qquad - \sum_{\alpha \in \mathscr{S}_k{}'(1, \ldots, n)} (-1)^{\mu + \mu(c_\alpha) + |a| + \mu_1 + \mu_2} \overline{\langle a; b_{i_1}, \ldots, b_{i_s} \rangle} | S \\
&\qquad \big(b_{j_1}, \ldots, b_k, \ldots, b_{j_{n-s}}\big) \\
&\qquad - \sum_{\alpha \in \mathscr{S}_k{}'(1, \ldots, n)} \sum_{\gamma \in \mathscr{S}'(j_1, \ldots, j_{n-s})} (-1)^{\mu + \mu(c_\alpha) + \epsilon + |a| + \mu_1 + \mu(c_\gamma)} \\
&\qquad \overline{\langle a; b_{i_1}, \ldots, b_{i_s} \rangle} | S(c_\gamma) \\
&\quad = a | S(b_1, \ldots, b_k, \ldots, b_n) \\
&\qquad + \sum_{\alpha \in (\mathscr{S}_k{}'(1, \ldots, n))^c} (-1)^{\mu + \mu(c_\alpha) + \epsilon} a | S(c_\alpha) \\
&\qquad + \sum_{\alpha \in \mathscr{S}_k{}'(1, \ldots, n)} \sum_{\beta \in \mathscr{S}'(i_1, \ldots, i_s)} (-1)^{|a| + \mu + \mu(c_\alpha) + g3 + \mu(b_\beta)} \\
&\qquad \overline{\langle a; b_{p_1}, \ldots, b_{p_i} \rangle} | S\big(b_{q_1}, \ldots, b_{q_{s-i}}\big) | S(c_{j_1}, \ldots, c_{j_{n-s}}) \\
&\qquad - \sum_{\alpha \in \mathscr{S}_k{}'(1, \ldots, n)} (-1)^{|a| + \mu(b_\alpha)} \\
&\qquad \overline{\langle a; b_{i_1}, \ldots, b_{i_s} \rangle} | S\big(b_{j_1}, \ldots, b_k, \ldots, b_{j_{n-s}}\big)
\end{aligned}$$

$$
\begin{aligned}
&- \sum_{\alpha \in \mathscr{S}_k'(1,\dots,n)} \sum_{\gamma \in \mathscr{S}'(j_1,\dots,j_{n-a})} (-1)^{\mu+\mu(c_\alpha)+\epsilon+\mu_1+\mu(c_\gamma)+|a|} \\
&\overline{\langle a; b_{i_1},\dots,b_{i_s}\rangle}|S(c_\gamma) \\
= {}& a|S(b_1,\dots,b_k,\dots,b_n) \\
&- \sum_{\alpha \in (\mathscr{S}_k'(1,\dots,n))^c} (-1)^{\mu_1'+\mu(b_\alpha)+\epsilon} a|S(c_{i_1},\dots,c_{i_s})|S\big(b_{j_1},\dots,b_{j_{n-s}}\big) \\
&+ \sum_{\alpha \in \mathscr{S}_k'(1,\dots,n)} \sum_{\beta \in \mathscr{S}_k'(i_1,\dots,i_s)} (-1)^{|a|+\mu_1'+\mu(b_\alpha)+\epsilon+\mu(c_\beta)} \\
&\overline{\langle a; b_{p_1},\dots,b_{p_i}\rangle}|S(c_{q_1},\dots,c_{q_{s-i}})|S\big(b_{j_1},\dots,b_{j_{n-s}}\big) \\
&- \sum_{\alpha \in \mathscr{S}_k'(1,\dots,n)} (-1)^{|a|+\mu(b_\alpha)} \\
&\overline{\langle a; b_{i_1},\dots,b_{i_s}\rangle}|S\big(b_{j_1},\dots,b_k,\dots,b_{j_{n-s}}\big) \\
= {}& \langle a; b_1, b_2,\dots,b_k,\dots,b_n\rangle,
\end{aligned}
$$

where $\alpha = (i_1,\dots,i_s|j_1,\dots,j_{n-s})$, $\beta = (p_1,\dots,p_i|q_1,\dots,q_{s-i})$, $\mu = \sum_{i=1}^{k-1}(|b_i|-1)$, $\mu_1 = \sum_{j=1}^{s}(|b_{i_j}|-1)$, $\mu_1' = \sum_{i_i<k}(|b_{i_i}|-1)$, and by definition $\mu+\mu(c_\alpha)+\mu_1+\mu_2 \equiv \mu(b_\alpha)$ and by differentiating part 2 of Theorem 1.1, we have $\mu+\mu(c_\alpha)+\mu_1'+\mu(b_\alpha) \equiv 0$, so the lemma is proved. ∎

THEOREM 2.1. *Suppose all the following Massey products are defined. Then*

1.
$$
\begin{aligned}
&In\langle a; b_1,\dots,b_n\rangle + In\langle a'; b_1,\dots,b_n\rangle \\
&\subset In\langle a+a'; b_1,\dots,b_n\rangle \\
&In\langle a; b_1,\dots,b_k,\dots,b_n\rangle + In\langle a; b_1,\dots,b_k',\dots,b_n\rangle \\
&\subset In\langle a; b_1,\dots,b_k+b_k',\dots,b_n\rangle.
\end{aligned}
$$

2. *For* $1 \le k \le n-1$, $n > 1$,

$$
\begin{aligned}
&In\langle a; b_1,\dots,b_k,b_{k+1},\dots,b_n\rangle \\
&\sim (-1)^{(|b_k|-1)(|b_{k+1}|-1)+\|b_k\|\,\|b_{k+1}\|} In\langle a; b_1,\dots,b_{k+1},b_k,\dots,b_n\rangle.
\end{aligned}
$$

3.
$$
In\langle a; b_1,\dots,b_n\rangle|b_{n+1} \prec In\langle a; b_1,\dots,b_n|b_{n+1}\rangle.
$$

Here $A+B\subset C$ *implies that for any* $a\in A$, *there exists* $b\in B$ *such that* $a+b\in C$; $A\prec B$ *implies that for any* $a\in A$, *there exists* $b\in B$ *such that* $a\sim b$; $A\sim B$ *implies that* $A\prec B$ *and* $B\prec A$, $b|A = \{b|a|a\in A\}$.

Proof. Part 1 is obvious. We only prove parts 2 and 3.

Since (R, d) is the cobar complex of a Hopf algebra H, there is a group module morphism (see [3])

$$\theta\colon W \otimes_K R \otimes_K R \to R,$$

where W is the free resolution of the group $\mathbf{Z}_2$. In fact for any $x, y \in R$, $S(x, y) = \theta(e_1 \otimes x \otimes y)$, now define $S_2(x, y) = \theta(e_2 \otimes x \otimes y)$; then we have

$$dS_2(x, y) - S_2(dx, y) - (-1)^{|x|} S_2(x, dy)$$
$$= S(x, y) + (-1)^{|x||y| + \|x\|\,\|y\|} S(y, x).$$

To prove part 2, we define $c_i = b_i$, $i \neq k, k+1$, $c_k = S_2(b_k, b_{k+1})$, since by part 3 of Lemma 1.2, we have

$$S(b_1, \ldots, b_k, b_{k+1}, \ldots, b_n) - (-1)^{\mu} S(b_1, \ldots, b_{k+1}, b_k, \ldots, b_n)$$
$$= S(b_1, \ldots, b_{k-1}, S(b_k, b_{k+1}), b_{k+2}, \ldots, b_n)$$
$$+ (-1)^{\mu} S(b_1, \ldots, b_{k-1}, S(b_{k+1}, b_k), b_{k+2}, \ldots, b_n),$$

where $\mu = (|b_k| - 1)(|b_{k+1} - 1) + \|b_k\|\,\|b_{k+1}\|$, so by part 2 of Theorem 1.1 we have

$$dS(c_1, \ldots, c_{k-1}, c_k, c_{k+2}, \ldots, c_n)$$
$$= (-1)^{|b_1| + \cdots + |b_{k-1}| - k + \epsilon} \big(S(b_1, \ldots, b_k, b_{k+1}, \ldots, b_n)$$
$$- (-1)^{\mu} S(b_1, \ldots, b_{k+1}, b_k, \ldots, b_n) \big)$$
$$+ \sum_{\alpha \in \mathscr{S}'(1, \ldots, k, k+2, \ldots, n)} (-1)^{\mu(c_\alpha)} S(c_\alpha),$$

where $\mu = (|b_k| - 1)(|b_{k+1}| - 1) + \|b_k\|\,\|b_{k+1}\|$, $\epsilon = 0$ if $k < n$, $\epsilon = 1$ if $k = n$.

It is easy to check that

$$\langle a; b_1, \ldots, b_k, b_{k+1}, \ldots, b_n \rangle - (-1)^{\mu} \langle a; b_1, \ldots, b_{k+1}, b_k, \ldots, b_n \rangle$$
$$= a | S(b_1, \ldots, b_k, b_{k+1}, \ldots, b_n) - (-1)^{\mu} a | S(b_1, \ldots, b_{k+1}, b_k, \ldots, b_n)$$
$$- \sum_{\alpha \in \mathscr{S}_1'} (-1)^{\mu(b_\alpha) + |a|} \Big(\overline{\langle a; b_{i_1}, \ldots, b_k, b_{k+1}, \ldots, b_{i_s} \rangle}$$
$$- (-1)^{\mu} \overline{\langle a; b_{i_1}, \ldots, b_{k+1}, b_k, \ldots, b_{i_s} \rangle} \Big) | S(b_{j_1}, \ldots, b_{j_{n-s}})$$
$$- \sum_{\alpha \in \mathscr{S}_2'} (-1)^{\mu(b_\alpha) + |a|} \overline{\langle a; b_{i_1}, \ldots, b_{i_s} \rangle} \big| \Big(S(b_{j_1}, \ldots, b_k, b_{k+1}, \ldots, b_{j_{n-s}})$$
$$- (-1)^{\mu} S(b_{j_1}, \ldots, b_{k+1}, b_k, \ldots, b_{j_{n-s}}) \Big),$$

where $\alpha = (i_1, \dots, i_s | j_1, \dots, j_{n-s})$, $\mathscr{S}_1'$ is the subset of $\mathscr{S}'(1, \dots, n)$ of α such that $k, k+1 \in \{i_1, \dots, i_s\}$, $\mathscr{S}_2'$ is the subset of $\mathscr{S}'(1, \dots, n)$ of α such that $k, k+1 \in \{j_1, \dots, j_{n-s}\}$, $\mu = (|b_k| - 1)(|b_{k+1}| - 1) + \|b_k\| \, \|b_{k+1}\|$.

For $1 \le i_1 < \cdots < i_s \le n$, $s = 1, \dots, n-1$, $k \in \{i_1, \dots, i_s\}$, $k+1 \notin \{i_1, \dots, i_s\}$, we define

$$(-1)^{\mu_1}\left(\overline{\langle a; b_{i_1}, \dots, b_k, b_{k+1}, \dots, b_{i_s}\rangle - (-1)^{\mu}\langle a; b_{i_1}, \dots, b_{k+1}, b_k, \dots, b_{i_s}\rangle}\right)$$

$$= (-1)^{|a|} a | S(c_{i_1}, \dots, c_{i_s})$$

$$- \sum_{\alpha \in \mathscr{S}_k'(i_1, \dots, i_n)} (-1)^{\mu(c_\alpha)} \overline{\langle a; b_{p_1}, \dots, b_{p_i}\rangle} | S(c_{q_1}, \dots, c_{q_{s-i}}),$$

where $\alpha = (p_1, \dots, p_t | q_1, \dots, q_{s-t})$, $\mathscr{S}_k'(i_1, \dots, i_s)$ is a subset of $\mathscr{S}'(i_1, \dots, i_s)$ of α such that $k \in \{q_1, \dots, q_{s-t}\}$, $\mu = (|b_k| - 1)(|b_{k+1}| - 1) + \|b_k\| \, \|b_{k+1}\|$.

$$\mu_1 = \begin{cases} \sum_{i_j < k} \left(|b_{i_j}| - 1\right) + 1, & \text{if } k < i_s \\ \sum_{i_j < k} \left(|b_{i_j}| - 1\right), & \text{if } k = i_s. \end{cases}$$

It is obvious that the above definition is well-defined for $n = 2$. Suppose it is well-defined for $n < N$; then (we denote $(1, \dots, k, k+2, \dots, N)$ simply by $(1, \dots, N)$),

$$d\Bigg((-1)^{|a| + \mu_1} a | S(c_1, \dots, c_N)$$

$$- \sum_{\alpha \in \mathscr{S}_k'(1, \dots, N)} (-1)^{\mu_1 + \mu(c_\alpha)} \overline{\langle a; b_{i_1}, \dots, b_{i_s}\rangle} | S(c_{j_1}, \dots, c_{j_{N-s-1}})\Bigg)$$

$$= a | S(b_1, \dots, b_k, b_{k+1}, \dots, b_N) - (-1)^{\mu} a | S(b_1, \dots, b_{k+1}, b_k, \dots, b_N)$$

$$+ \sum_{\alpha \in \mathscr{S}'(1, \dots, N)} (-1)^{\mu_1 + \mu(c_\alpha)} a | S(c_\alpha)$$

$$- \sum_{\alpha \in \mathscr{S}_k'(1, \dots, N)} (-1)^{\mu_1 + \mu(c_\alpha)} \langle a; b_{i_1}, \dots, b_{i_s}\rangle | S(c_{j_1}, \dots, c_{j_{N-s-1}})$$

$$- \sum_{\alpha \in \mathscr{S}_k'(1, \dots, N)} (-1)^{\mu_1 + \mu(c_\alpha) + |a| + \mu(i) + \mu_2} \overline{\langle a; b_{i_1}, \dots, b_{i_s}\rangle} |$$

$$\left(S(b_{j_1}, \dots, b_k, b_{k+1}, \dots, b_{j_{N-s-1}}) - (-1)^{\mu}\right.$$

$$S\big(b_{j_1},\ldots,b_{k+1},b_k,\ldots,b_{j_{N-s-1}})\big)$$

$$-\sum_{\alpha\in\mathscr{S}_k'(1,\ldots,N)}\ \sum_{\gamma\in\mathscr{S}'(j_1,\ldots,j_{N-s-1})}(-1)^{\mu_1+\mu(c_\alpha)+|a|+\mu(c_\gamma)+\mu(i)}$$

$$\overline{\langle a;b_{i_1},\ldots,b_{i_s}\rangle}|S(c_\gamma)$$

$$= a|S(b_1,\ldots,b_k,b_{k+1},\ldots,b_N) - (-1)^{\mu}a|S(b_1,\ldots,b_{k+1},b_k,\ldots,b_N)$$

$$+\sum_{\alpha\in(\mathscr{S}_k'(1,\ldots,N))^c}(-1)^{\mu_1+\mu(c_\alpha)}a|S(c_{i_1},\ldots,c_{i_s})|S\big(b_{j_1},\ldots,b_{j_{N-s-1}}\big)$$

$$+\sum_{\alpha\in\mathscr{S}_k'(1,\ldots,N)}\ \sum_{\beta\in\mathscr{S}'(i_1,\ldots,i_s)}(-1)^{|a|+\mu(b_\beta)+\mu_1+\mu(c_\alpha)}\overline{\langle a;b_{p_1},\ldots,b_{p_i}\rangle}|$$

$$S\big(b_{q_1},\ldots,b_{q_{s-i}}\big)|S(c_{j_1},\ldots,c_{j_{N-s-1}})$$

$$-\sum_{\alpha\in\mathscr{S}_k'(1,\ldots,N)}(-1)^{|a|+\mu(b_\alpha)}\overline{\langle a;b_{i_1},\ldots,b_{i_s}\rangle}|$$

$$\Big(S\big(b_{j_1},\ldots,b_k,b_{k+1},\ldots,b_{j_{N-s-1}}\big)$$

$$-(-1)^{\mu}S\big(b_{j_1},\ldots,b_{k+1},b_k,\ldots,b_{j_{N-s-1}})\big)\Big)$$

$$-\sum_{\alpha\in\mathscr{S}_k'(1,\ldots,N)}\ \sum_{\gamma\in\mathscr{S}'(j_1,\ldots,j_{N-s-1})}(-1)^{\mu_1+\mu(c_\alpha)+|a|+\mu(c_\gamma)+\mu(i)}$$

$$\overline{\langle a;b_{i_1},\ldots,b_{i_s}\rangle}|S(c_\gamma)$$

$$= a|S(b_1,\ldots,b_k,b_{k+1},\ldots,b_N) - (-1)^{\mu}a|S(b_1,\ldots,b_{k+1},b_k,\ldots,b_N)$$

$$+\sum_{\alpha\in(\mathscr{S}_k'(1,\ldots,N))^c}(-1)^{\mu_1+\mu(c_\alpha)}a|S(c_{i_1},\ldots,c_{i_s})|S\big(b_{j_1},\ldots,b_{j_{N-s-1}}\big)$$

$$-\sum_{\alpha\in(\mathscr{S}_k'(1,\ldots,N))^c}\ \sum_{\beta\in\mathscr{S}_k'(i_1,\ldots,i_s)}(-1)^{|a|+\mu_1+\mu(c_\alpha)+\mu(c_\beta)}\overline{\langle a;b_{p_1},\ldots,b_{p_i}\rangle}|$$

$$S(c_{q_1},\ldots,c_{q_{s-i}})|S\big(b_{j_1},\ldots,b_{j_{N-s-1}}\big)$$

$$-\sum_{\alpha\in\mathscr{S}_k'(1,\ldots,N)}(-1)^{|a|+\mu(b_\alpha)}\overline{\langle a;b_{i_1},\ldots,b_{i_s}\rangle}|$$

$$\Big(S\big(b_{j_1},\ldots,b_k,b_{k+1},\ldots,b_{j_{N-s-1}}\big)$$

$$-(-1)^{\mu}S\big(b_{j_1},\ldots,b_{k+1},b_k,\ldots,b_{j_{N-s-1}})\big)\Big)$$

$$= a|S(b_1,\ldots,b_k,b_{k+1},\ldots,b_N) - (-1)^{\mu}a|S(b_1,\ldots,b_{k+1},b_k,\ldots,b_N)$$

$$- \sum_{\alpha \in (\mathscr{S}_k'(1,\ldots,N))^c} (-1)^{\mu_3 + \mu(b_\beta)} a|S(c_{i_1},\ldots,c_{i_s})|S\big(b_{j_1},\ldots,b_{j_{N-s-1}}\big)$$

$$+ \sum_{\alpha \in (\mathscr{S}_k'(1,\ldots,N))^c} \sum_{\beta \in \mathscr{S}_k'(i_1,\ldots,i_s)} (-1)^{|a| + \mu(b_\alpha) + \mu_3 + \mu(c_\beta)} \overline{\langle a; b_{p_1},\ldots,b_{p_i}\rangle}|$$

$$S(c_{q_1},\ldots,c_{q_{s-i}})|S\big(b_{j_1},\ldots,b_{j_{N-s-1}}\big)$$

$$- \sum_{\alpha \in \mathscr{S}_k'(1,\ldots,N)} (-1)^{|a| + \mu(b_\alpha)} \overline{\langle a; b_{i_1},\ldots,b_{i_s}\rangle}|$$

$$\Big(S\big(b_{j_1},\ldots,b_k,b_{k+1},\ldots,b_{j_{N-s-1}}\big)$$

$$-(-1)^{\mu} S\big(b_{j_1},\ldots,b_{k+1},b_k,\ldots,b_{j_{N-s-1}}\big)\Big)$$

$$= \langle a; b_1,\ldots,b_k,b_{k+1},\ldots,b_N\rangle - (-1)^{\mu}\langle a; b_1,\ldots,b_{k+1},b_k,\ldots,b_N\rangle,$$

where $\alpha = (i_1,\ldots,i_s|j_1,\ldots,j_{N-s-1})$, $\beta = (p_1,\ldots,p_t|q_1,\ldots,q_{s-t})$, and

$$\mu_1 = \begin{cases} \sum_{i<k} \big(|b_i| - 1\big) + 1 & \text{if } k < N-1 \\ \sum_{i<k} \big(|b_i| - 1\big) & \text{if } k = N-1 \end{cases}$$

$$\mu_2 = \begin{cases} \sum_{j_m<k} \big(|b_{j_m}| - 1\big) + 1 & \text{if } k < j_{N-s-1} \\ \sum_{j_m<k} \big(|b_{j_m}| - 1\big) & \text{if } k = j_{N-s-1} \end{cases}$$

$$\mu_3 = \begin{cases} \sum_{i_m<k} \big(|b_{i_m}| - 1\big) + 1 & \text{if } k < i_s \\ \sum_{i_m<k} \big(|b_{i_m}| - 1\big) & \text{if } k = i_s \end{cases}$$

$$\mu = \big(|b_k| - 1\big)\big(|b_{k+1}| - 1\big) + \|b_k\|\,\|b_{k+1}\|$$

$$\mu(i) = |b_{i_1}| + |b_{i_2}| + \cdots + |b_{i_s}| - s.$$

Thus it is well-defined for $n = N$, and part 2 is proved.

To prove part 3, for $1 \le i_1 < i_2 < \cdots < i_s < n$, $s = 1,\ldots,n-1$, we define

$$\overline{\langle a; b_{i_1},\ldots,b_{i_s},b_n|b_{n+1}\rangle - \langle a; b_{i_1},\ldots,b_{i_s},b_n\rangle|b_{n+1}}$$

$$= \sum_{\alpha \in (\mathscr{S}_n'(i_1,\ldots,i_s,n))^c} (-1)^{\mu(b_{\alpha'})} \overline{\langle a; b_{p_1},\ldots,b_{p_i},b_n\rangle}|S\big(b_{q_1},\ldots,b_{q_{s-i}},b_{n+1}\big),$$

where $\alpha = (p_1, \dots, p_t, \ \ n|q_1, \dots, q_{s-t})$, $\alpha' = (p_1, \dots, p_t, \ \ n|q_1, \dots, q_{s-t}, n+1)$, $\tilde{\mu}(b_{\alpha'}) + |b_{i_1}| + \cdots + |b_{i_s}| - s$, and $\mathscr{S}_n'(i_1, \dots, i_s, n)$ is the subset of $\mathscr{S}'(i_1, \dots, i_s, n)$ of such $\beta = (p_1, \dots, p_t|q_1, \dots, q_{s-t}, n)$.

It is easy to check that if $s = 1$, the above definition is well-defined. Now suppose it is well-defined for $s < n$; then

$$\begin{aligned}
&\langle a; b_1, \dots, b_{n-1}, b_n|b_{n+1}\rangle - \langle a; b_1, \dots, b_{n-1}, b_n\rangle|b_{n+1} \\
&\quad = a|S(b_1, \dots, b_{n-1}, b_n|b_{n+1}) - a|S(b_1, \dots, b_{n-1}, b_n)|b_{n+1} \\
&\qquad - \sum_{\alpha \in \mathscr{S}'(1, \dots, n)} (-1)^{|a| + \mu(c_\alpha)} \overline{\langle a; c_{i_1}, \dots, c_{i_s}\rangle}|S(c_{j_1}, \dots, c_{j_{s-i}}) \\
&\qquad + \sum_{\alpha \in \mathscr{S}'(1, \dots, n)} (-1)^{|a| + \mu(b_\alpha)} \overline{\langle a; b_{i_1}, \dots, b_{i_s}\rangle}|S(b_{j_1}, \dots, b_{j_{s-i}})|b_{n+1} \\
&\quad = \sum_{\alpha \in (\mathscr{S}_n'(1, \dots, n))^c} (-1)^{\tilde{\mu}(b_{\alpha'})} a|S(b_{i_1}, \dots, b_{i_s}, b_n)|S(b_{j_1}, \dots, b_{j_{n-s}}, b_{n+1}) \\
&\qquad - \sum_{\alpha \in \mathscr{S}_n'(1, \dots, n)} (-1)^{|a| + \mu(b_\alpha)} \overline{\langle a; b_{i_1}, \dots, b_{i_s}\rangle}| \\
&\qquad \left(S(b_{j_1}, \dots, b_{j_{n-1}}, b_n|b_{n+1}) - S(b_{j_1}, \dots, b_{j_{n-1}}, b_n)|b_{n+1}\right) \\
&\qquad - \sum_{\alpha \in (\mathscr{S}_n'(1, \dots, n))^c} (-1)^{|a| + \mu(c_\alpha)} \overline{\langle a; b_{i_1}, \dots, b_{i_{s-1}}, b_n|b_{n+1}\rangle}| \\
&\qquad S(b_{j_1}, \dots, b_{j_{n-s}}) \qquad (1) \\
&\qquad + \sum_{\alpha \in (\mathscr{S}_n'(1, \dots, n))^c} (-1)^{|a| + \mu(b_\alpha)} \overline{\langle a; b_{i_1}, \dots, b_{i_{s-1}}, b_n\rangle}| \\
&\qquad S(b_{j_1}, \dots, b_{j_{n-s}})|b_{n+1} \\
&\quad = \sum_{\alpha \in (\mathscr{S}_n'(1, \dots, n))^c} (-1)^{\tilde{\mu}(b_{\alpha'})} a|S(b_{\alpha'}) \\
&\qquad - \sum_{\alpha \in \mathscr{S}_n'(1, \dots, n)} \ \sum_{\beta \in (\mathscr{S}_n'(j_1, \dots, j_{n-s-1}, n))^c} (-1)^{|a| + \mu(b_\alpha) + \tilde{\mu}(b_{\beta'})} \\
&\qquad \overline{\langle a; b_{i_1}, \dots, b_{i_s}\rangle}|S(b_{\beta'}) \\
&\qquad + \sum_{\alpha \in (\mathscr{S}_n'(1, \dots, n))^c} (-1)^{|a| + \mu(i) + \tilde{\mu}(b_{\alpha'}) + \mu(b_{(n+1|j_1, \dots, j_{n-s})})} \\
&\qquad \overline{\langle a; b_{i_1}, \dots, b_{i_{s-1}}, b_n\rangle}|b_{n+1}|S(b_{j_1}, \dots, b_{j_{n-s}}) \qquad (2)
\end{aligned}$$

$$- \sum_{\alpha \in (\mathscr{S}_n{}'(1,\dots,n))^c} \sum_{\gamma \in (\mathscr{S}_n{}'(i_1,\dots,i_{s-1},n))^c} (-1)^{|a| + \mu(c_\alpha) + \tilde{\mu}(b_{\gamma'})}$$

$$\overline{\langle a; b_{p_1}, \dots, b_{p_{i-1}}, b_n \rangle} | S(b_{q_1}, \dots, b_{q_{s-i}}, b_{n+1}) | S(b_{j_1}, \dots, b_{j_{n-s}}) \tag{3}$$

$$+ \sum_{\alpha \in (\mathscr{S}_n{}'(1,\dots,n))^c} (-1)^{|a| + \mu(b_\alpha)}$$

$$\overline{\langle a; b_{i_1}, \dots, b_{i_{s-1}}, b_n \rangle} | S(b_{j_1}, \dots, b_{j_{n-s}}) | b_{n+1}$$

$$= \sum_{\alpha \in (\mathscr{S}_n{}'(1,\dots,n))^c} (-1)^{\tilde{\mu}(b_{\alpha'})} a | S(b_{\alpha'})$$

$$- \sum_{\alpha \in (\mathscr{S}_n{}'(1,\dots,n))^c} \sum_{\gamma \in \mathscr{S}_n{}'(i_1,\dots,i_{s-1},n)} (-1)^{|a| + \tilde{\mu}(b_{\alpha'}) + \mu(b_\gamma)}$$

$$\overline{\langle a; b_{p_1}, \dots, b_{p_i} \rangle} \Big| S(b_{q_1}, \dots, b_{q_{s-i-1}}, b_n) \Big| S(b_{j_1}, \dots, b_{j_{n-s}}, b_{n+1})$$

$$- \sum_{\alpha \in (\mathscr{S}_n{}'(1,\dots,n))^c} \sum_{\gamma \in (\mathscr{S}_n{}'(i_1,\dots,i_{s-1},n))^c} (-1)^{|a| + \tilde{\mu}(b_{\alpha'}) + \mu(b_\gamma)}$$

$$\overline{\langle a; b_{p_1}, \dots, b_{p_{i-1}}, b_n \rangle} \Big| S(b_{q_1}, \dots, b_{q_{s-i}}) \Big| S(b_{j_1}, \dots, b_{j_{n-s}}, b_{n+1})$$

$$+ \sum_{\alpha \in (\mathscr{S}_n{}'(1,\dots,n))^c} \sum_{\beta \in \mathscr{S}'(j_1,\dots,j_{n-s})} (-1)^{|a| + \tilde{\mu}(b_{\alpha'}) + \mu(b_{\beta'}) + \mu(i)}$$

$$\overline{\langle a; b_{i_1}, \dots, b_{i_{s-1}}, b_n \rangle} | S(b_{\beta'})$$

$$+ \sum_{\alpha \in (\mathscr{S}_n{}'(1,\dots,n))^c} (-1)^{|a| + \mu(i) + \tilde{\mu}(b_{\alpha'}) + \mu(b_{(n+1|j_1,\dots,j_{n-s})})}$$

$$\overline{\langle a; b_{i_1}, \dots, b_{i_{s-1}}, b_n \rangle} | b_{n+1} | S(b_{j_1}, \dots, b_{j_{n-s}})$$

$$+ \sum_{\alpha \in (\mathscr{S}_n{}'(1,\dots,n))^c} \sum_{\beta \in \mathscr{S}'(j_1,\dots,j_{n-s})} (-1)^{|a| + \tilde{\mu}(b_{\alpha'}) + \mu(b_{\beta''}) + \mu(i)}$$

$$\overline{\langle a; b_{i_1}, \dots, b_{i_{s-1}}, b_n \rangle} | S(b_{\beta''}) \tag{4}$$

$$+ \sum_{\alpha \in (\mathscr{S}_n{}'(1,\dots,n))^c} (-1)^{|a| + \mu(i) + \tilde{\mu}(b_{\alpha'}) + \mu(b_{(j_1,\dots,j_{n-s}|n+1)})}$$

$$\overline{\langle a; b_{i_1}, \dots, b_{i_{s-1}}, b_n \rangle} \Big| S(b_{j_1}, \dots, b_{j_{n-s}}) \Big| b_{n+1}$$

$$= \sum_{\alpha \in (\mathscr{S}_n{}'(1,\dots,n))^c} (-1)^{\tilde{\mu}(b_{\alpha'})} \langle a; b_{i_1}, \dots, b_{i_{s-1}}, b_n \rangle |$$

$$S(b_{j_1}, \dots, b_{j_{n-s}}, b_{n+1})$$

$$+ \sum_{\alpha \in (\mathscr{S}_n'(1,\dots,n))^c} (-1)^{|a| + \mu(i) + \tilde{\mu}(b_{\alpha'})}$$

$$\overline{\langle a; b_{i_1}, \dots, b_{i_{s-1}}, b_n \rangle} | dS\big(b_{j_1}, \dots, b_{j_{n-s}}, b_{n+1}\big),$$

where $c_i = b_i$ if $i < n$, $c_n = b_n | b_{n+1}$, $\alpha = (i_1, \dots, i_s | j_1, \dots, j_{n-s})$, $\gamma = (p_1, \dots, p_t | q_1, \dots, q_{s-t})$. If $\beta = (k_1, \dots, k_u | l_1, \dots, l_{n-s-u})$, then

$$\beta' = \big(k_1, \dots, k_u | l_1, \dots, l_{n-s-u}, n+1\big)$$

$$\beta'' = \big(k_1, \dots, k_u, n+1 | l_1, \dots, l_{n-s-u}\big).$$

$\mu(i) = |b_{i_1}| + \cdots + |b_{i_s}| - s$, but the sign function is very complicated, and so we only give the computations of some of them:

$$\begin{aligned}
\mu(c_\alpha) &= \mu(c_{(i_1,\dots,i_{s-1},n|j_1,\dots,j_{n-s})}) \qquad \big(\alpha \in (\mathscr{S}_n(1,\dots,n))^c\big) \\
&\equiv (\mu_f + \mu_{f'})(c_{(i_1,\dots,i_{s-1},n,j_1,\dots,j_{n-s})}) + \sigma_f(c_{(i_1,\dots,i_{s-1},n|j_1,\dots,j_{n-s})}) \\
&\equiv (\mu_f + \mu_{f'})\big(b_{(i_1,\dots,i_{s-1},n,j_1,\dots,j_{n-s})}\big) + (\mu_f + \mu_{f'})\big(b_{(n+1,j_1,\dots,j_{n-s})}\big) \\
&\qquad + |b_{j_1}| + \cdots + |b_{j_{n-s}}| - n + s + \sigma_f\big(b_{(i_1,\dots,i_{s-1},n|j_1,\dots,j_{n-s})}\big) + |b_{n+1}| \\
&\equiv (\mu_f + \mu_{f'})\big(b_{(i_1,\dots,i_{s-1},n,j_1,\dots,j_{n-s})}\big) + (\mu_f + \mu_{f'})\big(b_{(n+1,j_1,\dots,j_{n-s})}\big) \\
&\qquad + |b_{j_1}| + \cdots + |b_{j_{n-s}}| - n + s \\
&\qquad + \sigma_f\big(b_{(i_1,\dots,j_{s-1},n|j_1,\dots,j_{n-s},n+1)}\big) + |b_n| + |b_{j_{n-s}}| + |b_{n+1}| - 2 \\
&\equiv |b_{i_1}| + \cdots + |b_{i_s}| + |b_n| - s + \tilde{\mu}(b_{\alpha'}) + \mu\big(b_{(n+1|j_1,\dots,j_{n-s})}\big) + 1 \\
&\equiv 1 + \mu(i) + \tilde{\mu}(b_{\alpha'}) + \mu\big(b_{(n+1|j_1,\dots,j_{n-s})}\big).
\end{aligned}$$

So (1) = (2) + (3), and

$$\begin{aligned}
&\mu(c_\alpha) + \tilde{\mu}\big(b_{\gamma'}\big) \qquad \big(\gamma \in (\mathscr{S}_n(i_1,\dots,i_{s-1},n))^c\big) \\
&\quad = \mu(c_{(i_1,\dots,i_{s-1},n|j_1,\dots,j_{n-s})}) + \tilde{\mu}\big(b_{(p_1,\dots,p_{i-1},n|q_1,\dots,q_{s-i},n+1)}\big) \\
&\quad \equiv (\mu_f + \mu_{f'})(c_{(i_1,\dots,i_{s-1},n|j_1,\dots,j_{n-s})}) + \sigma_f(c_{(i_1,\dots,i_{s-1},n|j_1,\dots,j_{n-s})}) \\
&\qquad + \mu\big(b_{(p_1,\dots,p_{i-1},n|q_1,\dots,q_{s-i},n+1)}\big) + |b_{p_1}| + \cdots \\
&\qquad + |b_{p_{i-s}}| + |b_{q_1}| + \cdots + |b_{q_{s-i}}| - s + 1 \\
&\quad \equiv (\mu_f + \mu_{f'})\big(b_{(i_1,\dots,i_{s-1},n,j_1,\dots,j_{n-s})}\big) \\
&\qquad + (\mu_f + \mu_{f'})\big(b_{(n+1,j_1,\dots,j_{n-s})}\big)
\end{aligned}$$

$$
\begin{aligned}
&+|b_{j_1}| + \cdots + |b_{j_{n-s}}| - n + s + \sigma_f\big(b_{(i_1,\ldots,i_{s-1},n|j_1,\ldots,j_{n-s})}\big)\\
&+|b_{n+1}| + (\mu_f + \mu_{f'})\big(b_{(p_1,\ldots,p_{i-1},n,q_1,\ldots,q_{s-i},n+1)}\big)\\
&+\sigma_f\big(b_{(p_1,\ldots,p_{i-1},n|q_1,\ldots,q_{s-i},n+1)}\big) + |b_{p_1}| + \cdots\\
&+|b_{p_{i-1}}| + |b_{q_1}| + \cdots + |b_{q_{s-i}}| - s + 1\\
\equiv{}& (\mu_f + \mu_{f'})\big(b_{(p_1,\ldots,p_{i-1},n,q_1,\ldots,q_{s-i},n+1,j_1,\ldots,j_{n-s})}\big)\\
&+\sigma_f\big(b_{(i_1,\ldots,i_{s-1},n|j_1,\ldots,j_{n-s})}\big) + \sigma_f\big(b_{(p_1,\ldots,p_{i-1},n|q_1,\ldots,q_{s-i},n+1)}\big)\\
&+|b_{p_1}| + \cdots + |b_{p_{i-1}}| + |b_{q_1}| + \cdots + |b_{q_{s-i}}| + |b_{n+1}| + |b_{j_1}| + \cdots\\
&+|b_{j_{n-s}}| - n + 1\\
\equiv{}& (\mu_f + \mu_{f'})\big(b_{(p_1,\ldots,p_{i-1},n,w_1,\ldots,w_{n-i},n+1)}\big)\\
&+(\mu_f + \mu_{f'})\big(b_{(q_1,\ldots,q_{s-i},n+1,j_1,\ldots,j_{n-s})}\big)\\
&+\sigma_f\big(b_{(p_1,\ldots,p_{i-1},n|w_1,\ldots,w_{n-i},n+1)}\big)\\
&+\sigma\big(b_{(q_1,\ldots,q_{s-i},n+1|j_1,\ldots,j_{n-s})}\big)\\
&+|b_{q_1}| + \cdots + |b_{q_{s-i}}| + |b_n| - s + t + |b_{j_1}|\\
&+ \cdots + |b_{j_{n-s}}| - n + s\\
\equiv{}& \tilde{\mu}\big(b_{(p_1,\ldots,p_{i-1},n|w_1,\ldots,w_{n-i},n+1)}\big) + \mu\big(b_{(q_1,\ldots,q_{s-i},n+1|j_1,\ldots,j_{n-s})}\big)\\
&+|b_{p_1}| + \cdots + |b_{p_{i-1}}| + |b_n| - t + 1,
\end{aligned}
$$

where $(q_1,\ldots,q_{s-t},n+1,j_1,\ldots,j_{n-s}) \in \mathscr{S}(w_1,\ldots,w_{n-t},n+1)$, so (3) = (4). Other signs are direct checkings, so part 3 is proved. ∎

When (M,d) is a left DGA-module over (R,d), we have all the corresponding definitions for (M,d). We only give the corresponding result.

THEOREM 2.2. *Suppose all the following Massey products are defined; Then*

1.
$$
\begin{aligned}
&In\langle b_n,\ldots,b_1;a\rangle + In\langle b_n,\ldots,b_1;a'\rangle\\
&\subset In\langle b_n,\ldots,b_1;a+a'\rangle\\
&In\langle b_n,\ldots,b_k,\ldots,b_1;a\rangle + In\langle b_n,\ldots,b_k',\ldots,b_1;a\rangle\\
&\subset In\langle b_n,\ldots,b_k+b_k',\ldots,b_1;a\rangle.
\end{aligned}
$$

2. *For* $1 \leq k \leq n-1$, $n > 1$,

$$In\langle b_n, \ldots, b_{k+1}, b_k, \ldots, b_1; a\rangle \sim (-1)^{(b|b_k|-1)(|b_{k+1}|-1)+\|b_k\|\|b_{k+1}\|} In\langle b_n, \ldots, b_k, b_{k+1}, \ldots, b_1; a\rangle.$$

3. $b_{n+1}|In\langle b_n, \ldots, b_1; a\rangle \prec In\langle b_{n+1}|b_n, \ldots, b_1; a\rangle$. *Here* $A + B \subset C$ *implies that for any* $a \in A$, *there exists* $b \in B$ *such that* $a + b \in C$; $A \prec B$ *implies that for any* $a \in A$, *there exists* $b \in B$ *such that* $a \sim b$; $A \sim B$ *implies that* $A \prec B$ *and* $B \prec A$, $A|b = \{a|b|a \in A\}$.

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