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# $S$-Module and the New Massey-Product

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In this paper, we give the definition of $S$-Module and with this module theory prove the properties of the new Massey-product defined in [6]. As an application, we find new relations of $h_i$ and $b_j$ in the $E_2$-term of the classical Adams spectral sequence. © 1997 Academic Press

This is a paper that succeeds [6]. Throughout this paper, Hopf algebras mean graded commutative co-associative Hopf algebras over a field $K$, $(R,d)$ is the cobar complex for a Hopf algebra $H$, that is, the elements of $R$ have the form $[a_1|a_2|\cdots|a_n]$ with $a_i \in \overline{H}$, and the differential is defined by

$$d[a_1|a_2|\cdots|a_n] = \sum\left(-[a_1'|a_1''|\cdots|a_n]\cdots+(-1)^n[a_1|\cdots|a_{n-1}|a_n'|a_n'']\right),$$

where $\Delta(a_i) = 1 \otimes a_i + a_i \otimes 1 + \sum a_i' \otimes a_i''$. Then $(R, d)$ is a DGA (see [4]) with the concatenation product. We use $|\cdot|$ to denote the cohomological degree of $R$ and $\|\cdot\|$ to denote the resolution degree. To simplify the notion, we omit the product of $R$ and all module actions.

Note that the Massey-product defined in [6] may be defined on any DGA-module over a given DGA with unit that satisfies 1 and 2 of Lemma 2.1 in [6], but such a Massey-product will not satisfy 2 of Theorem 2.1 in [6]. For the same reason, although the $S$-module may be defined when $(R, d)$ is a DGA as above and has a higher homotopy $S_2$ (needed in the proof of Lemma 2.1), we do not generalize here.

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The purpose of this paper is to study some new properties of the Massey-product defined in [6]. These properties are Theorems 3.1 and 3.2. The interesting point here is that although these properties totally depends on the properties of the complex $(R, d)$ itself, if we focus on the definition of the Massey-product only as we did in [6], the analogous proof is too complicated to realize. Therefore, we must regard this Massey-product as an obstruction for a class of elements to be a cohomology class in a chain complex that has $(R, d)$ as a subcomplex. This "larger" chain complex is just the $S$-module.

## 1. $S$-MODULES

In this paper, all modules are bigraded bimodules over an algebra over the field $K$; that is, if $M$ is a module over $A$, then there is a left action $A \otimes_K M \to M$ and a right action $M \otimes_K A \to M$ such that for all $a, b \in A$, $m \in M$,

$$(am)b = a(mb), \qquad (ab)m = a(bm), \qquad m(ab) = (ma)b.$$

If for all $a \in A$, $m \in M$, $am = (-1)^{|a|\,|m| + \|a\|\,\|m\|} ma$, then the module is called symmetric.

DEFINITION 1.1. $(M, d)$ is called an $S$-module over $(R, d)$ if $R$ is a submodule of $M$ generated by 1 such that $1m = m1 = m$ for all $m \in M$ and $(M, d)$ is a DGA-module over $(R, d)$ (see [4]) and there is a homotopy $S$: $R \otimes_K M \to M$ such that

1. $S(1, m) = 0$, $S(a, 1) = 0$ for all $m \in M$, $a \in R$.
2. For all $a \in R$, $m \in M$,

$$dS(a, m) + S(da, m) + (-1)^{|a|} S(a, dm) = am - (-1)^{|a|\,|m| + \|a\|\,\|m\|} ma.$$

3. For all $a, b \in R$, $m \in M$,

$$S(a, bm) = S(a, b)m + (-1)^{(|a|-1)|b| + \|a\|\,\|b\|} bS(a, m),$$

$$S(a, mb) = S(a, m)b + (-1)^{(|a|-1)|m| + \|a\|\,\|m\|} mS(a, b).$$

Notice that this definition implies that $(R, d)$ itself has a homotopy $S$ that satisfies 1 and 2 of Lemma 1.2 in [6], so $H^{**}(R)$ is a commutative algebra. It is obvious that if $(M, d)$ is an $S$-module over $(R, d)$, then $H^{**}(M)$ is a symmetric module over $H^{**}(R)$. If there is no confusion, we write $M$ instead of $(M, d)$.

Recall that for two modules $M_1$ and $M_2$ over $R$, $M_1 \otimes_R M_2$ is the quotient module of $M_1 \otimes_K M_2$ modulo the submodule generated by elements of the form $m_1 a \otimes m_2 - m_1 \otimes a m_2$ with $a \in R$, $m_i \in M_i$. Now if $M_1$ and $M_2$ are two $S$-modules over $R$, then this submodule of $M_1 \otimes_K M_2$ is also a subchain complex of it, so $M_1 \otimes_R M_2$ inherits a differential from $M_1 \otimes_K M_2$. We call this the tensor product of $M_1$ and $M_2$ over $R$.

THEOREM 1.1. *If $M_1$ and $M_2$ are two $S$-modules over $R$, then $M_1 \otimes_R M_2$ is also an $S$-module.*

*Proof.* Define $S\colon R \otimes_K M_1 \otimes_K M_2 \to M_1 \otimes_K M_2$ by

$$S(a, m_1 \otimes m_2) = S(a, m_1) \otimes m_2 + (-1)^{(|a|-1)|m_1| + \|a\| \|m_1\|} m_1 \otimes S(a, m_2)$$

for all $a \in R$, $m_i \in M_i$, $i = 1,2$, where we still use $S$ to denote the homotopy for $M_i$. Then it is easy to check that this $S$ makes $M_1 \otimes_K M_2$ an $S$-module. It is also easy to check that the submodule $I$ of $M_1 \otimes_K M_2$ generated by elements of the form $m_1 a \otimes m_2 - m_1 \otimes a m_2$ satisfies $S(R, I) \subset I$, so the induced homotopy makes $M_1 \otimes_R M_2$ an $S$-module over $R$. ∎

## 2. EXAMPLES OF $S$-MODULES

DEFINITION 2.1. For $b \in R^{s,t}$, $db = 0$, $E_R(b)$ is a DGA-module over $R$ such that as a left module over $R$ it is freely generated by 1 and $x$, $dx = b$, and for all $a \in R$,

$$xa - (-1)^{(s-1)|a| + t\|a\|} ax = S(a, b),$$

where $S$ is the homotopy for $R$ itself as defined in [6].

LEMMA 2.1. *$E_R(b)$ is an $S$-module.*

*Proof.* Define $\tilde{S}\colon R \otimes_K E_R(b) \to E_R(b)$ by

$$\tilde{S}(a, x) = (-1)^{|a|-1} S_2(a, b) \qquad \text{for all } a \in R,$$

$$\tilde{S}(a, c) = S(a, c) \qquad \text{for all } a, c \in R,$$

where $S$ is as above and $S_2$ is the homotopy for $R$ that satisfies for all $x, y \in R$,

$$dS_2(x, y) - S_2(dx, y) - (-1)^{|x|} S_2(x, dy)$$
$$= S(x, y) - (-1)^{|x||y| + \|x\| \|y\|} S(y, x).$$

Extend $\tilde{S}$ to $E_R(b)$ by 3 of Definition 1.1 and it is easy to check that this makes $E_R(b)$ an $S$-module. ∎

We call $E_R(b)$ the exterior $S$-module (we call it $S$-algebra if $x^2 = 0$ does not cause contradiction) over $R$ generated from $b$ and we define the exterior $S$-module over $R$ generated from $b_1, b_2, \ldots, b_n$ to be

$$E_R(b_1, b_2, \ldots, b_n) = \bigotimes_{i=1}^{n} E_R(b_i).$$

where $\otimes$ means $\otimes_R$.

Now let

$$T_n = \underbrace{E_R(b) \otimes_R E_R(b) \otimes_R \cdots \otimes_R E_R(b)}_{n}, \qquad n > 0,$$

and let $\tilde{P}(b) = \oplus_{i=1}^{\infty} T_n$. Then $\otimes_R$ makes $\tilde{P}$ an algebra over $R$, so we have

DEFINITION 2.2. For $b \in R$, $db = 0$, $P_R(b)$ is the quotient DGA-algebra of $\tilde{P}(b)$ modulo the ideal generated by $1 \otimes x - x$, $x \otimes 1 - 1$, and $1 \otimes 1 - 1$.

We call this algebra the polynomial $S$-algebra over $R$ generated from $b$ and define the polynomial $S$-algebra generated from $b_1, b_2, \ldots, b_n$ to be

$$P_R(b_1, b_2, \ldots, b_n) = \bigotimes_{i=1}^{n} P_R(b_i).$$

where $\otimes$ means $\otimes_R$.

We have a reason to give them such names. Suppose $H_1$ is a sub-Hopf algebra of $H_2$, a Hopf algebra over the field of integers modulo a prime $p$. Then there is an ideal of $H_2$ generated by $H_1$. Modulo this ideal we obtain a quotient Hopf algebra $H_2/H_1$. If $H_2/H_1$ is generated by one primitive element $\xi$, then its cohomology is known. Suppose $p$ is an odd number and $H_2/H_1 = P(\xi)/(\xi^{p^{n+1}})$ with $\|\xi\|$ an even number (other cases are simpler), where $P$ is the polynomial algebra. Then by [7] we have

$$H^{**}(H_2/H_1) = E\left(\tilde{h}_0, \tilde{h}_1, \ldots, \tilde{h}_n\right) \otimes P\left(\tilde{b}_0, \tilde{b}_1, \ldots, \tilde{b}_n\right),$$

where $E$ is the exterior algebra, $P$ is the polynomial algebra, and $\tilde{h}_i$ and $\tilde{b}_i$ are represented, respectively, by $[\xi_1^{p^i}]$ and $\sum_{k=1}^{p-1} \binom{p}{k}/p[\xi^{kp^i} \mid \xi^{(p-k)p^i}]$ in the cobar complex. Let $(R, d)$ be the cobar complex of $H_1$. Then in [5] we prove that there are $h_i, b_i \in R$ such that if we define

$$C = E_R(h_1, h_2, \ldots, h_n) \otimes_R P_R(b_1, b_2, \ldots, b_n)$$

with $d\tilde{h}_i = h_i$, $d\tilde{b}_i = b_i$, then $H^{**}(H_2) = H^{**}(C)$. The pity is that the cohomology of $C$ is still too hard to compute. We will not discuss this complex in this paper.

In [6], we give a new Massey-product $\langle a; b_1, b_2, \ldots, b_n\rangle$. Now we will prove some of its properties when $a \in R$. All the corresponding definitions are defined as in [6]. Particularly, we use $\bar{u}$ to denote a given element such that $d\bar{u} = u$.

For $a, b_i \in R$, $da = 0$, $db_i = 0$, $i = 1, 2, \ldots, n$, we define a class of elements in the $S$-module $E_R(b_1, \ldots, b_n)$ $(dx_i = b_i)$ by

$$\begin{aligned}\{a; b_1, b_2, \ldots, b_n\} &= (-1)^{|a|} a x_1 x_2 \cdots x_n \\ &\quad - \sum_{\alpha \in \mathscr{S}'(1, \ldots, n)} (-1)^{\mu(\alpha)} a(i_1, \ldots, i_s) x_{j_1} \cdots x_{j_{n-s}},\end{aligned}$$

where $\alpha = (i_1, \ldots, i_s \mid j_1, \ldots, j_{n-s})$, $\mu$ and $\mathscr{S}'(1, \ldots, n)$ are defined as in [6], and $a(i_1, \ldots, i_s) \in R$. This class of elements seems too arbitrary, but if we require $d\{a; b_1, b_2, \ldots, b_n\} \in R$, then every $a(i_1, \ldots, i_s)$ must satisfy $da(i_1, \ldots, i_s) = \langle a; b_{i_1}, \ldots, b_{i_s}\rangle$.

LEMMA 2.2. *Suppose* $a, b_i \in R$, $da = 0$, $db_i = 0$, $i = 1, 2, \ldots, n$, $M$ *is DGA-module over R such that* $E_R(b_1, \ldots, b_n)$ *is a submodule of it. If a given* $\langle a; b_1, b_2, \ldots, b_n\rangle$ *is defined, then a class* $\{a; b_1, b_2, \ldots, b_n\}$ *is defined such that* $d\{a; b_1, b_2, \ldots, b_n\} = \langle a; b_1, b_2, \ldots, b_n\rangle$. *Conversely, if a given* $\{a; b_1, b_2, \ldots, b_n\} \in M$ *satisfies* $d\{a; b_1, b_2, \ldots, b_n\} \in R$, *then a given* $\langle a; b_1, b_2, \ldots, b_n\rangle$ *is defined such that* $d\{a; b_1, b_2, \ldots, b_n\} = \langle a; b_1, b_2, \ldots, b_n\rangle$.

*Proof.* The first part of the lemma is obvious if we define $(a; b_{i_1}, \ldots, b_{i_s}) = \overline{\langle a; b_{i_1}, \ldots, b_{i_s}\rangle}$ for all $1 \le i_1 < \cdots < i_s \le n$, $s < n$. The second part depends on the fact that $E_R(b_1, \ldots, b_n)$ is a submodule of $M$. We call an element in $E_R(b_1, \ldots, b_n)$ in its standard form if it is the sum of monomials $cx_{i_1}, \ldots, x_{i_s}$ with $c \in R$, $1 \le i_1 < \cdots < i_s \le n$. Write $d\{a; b_1, b_2, \ldots, b_n\}$ in its standard form and we will see by induction that to cancel the term $cx_{j_1} \cdots x_{j_{n-s}}$ with $1 \le j_1 < \cdots < j_{n-s} \le n$, $s > 0$, all $\mathrm{In}\langle a; b_{i_1}, b_{i_2}, \ldots, b_{i_s}\rangle$ with $0 \le i_1 < \cdots < i_s \le n$, $s < n$ are trivial and $d(a; b_{i_1}, \ldots, b_{i_s}) = \langle a; b_{i_1}, \ldots, b_{i_s}\rangle$, so naturally $d\{a; b_1, b_2, \ldots, b_n\} = \langle a; b_1, b_2, \ldots, b_n\rangle$. ∎

## 3. MASSEY-PRODUCT

THEOREM 3.1. *Suppose* $a_1, a_2, b_i \in R$, $db_i = 0$, $i = 1, 2, \ldots, m+n$. *If* $\mathrm{In}\langle a_1; b_1, b_2, \ldots, b_n\rangle \sim 0$ *and* $\mathrm{In}\langle a_2; b_{m+1}, b_{m+2}, \ldots, b_{m+n}\rangle \sim 0$, *then*

$$\mathrm{In}\langle a_1 a_2; b_1, b_2, \ldots, b_m, b_{m+1}, b_{m+2}, \ldots, b_{m+n}\rangle \sim 0.$$

*Proof.* We define $S$-module $M = E_R(b_1, \ldots, b_{m+n})$ with $dx_i = b_i$, $i = 1, \ldots, m+n$. Let

$$z_1 = (-1)^{|a_1|} a_1 x_1 \cdots x_m - \sum_{\alpha \in \mathscr{S}'(1,\ldots,m)} (-1)^{\mu(\alpha)} \overline{\langle a_1; b_{i_1}, \ldots, b_{i_s} \rangle} x_{j_1} \cdots x_{j_{m-s}},$$

$$z_2 = (-1)^{|a_2|} a_2 x_{m+1} \cdots x_{m+n} - \sum_{\beta \in \mathscr{S}'(m+1,\ldots,m+n)} (-1)^{\mu(\beta)} \overline{\langle a_2; b_{k_1}, \ldots, b_{k_t} \rangle} x_{l_1} \cdots x_{l_{n-t}},$$

where $\alpha = (i_1, \ldots, i_s \,|\, j_1, \ldots, j_{m-s})$, $\beta = (k_1, \ldots, k_t \,|\, l_1, \ldots, l_{n-t})$ and $\overline{\langle \cdots \rangle} \in R$. Then by Lemma 2.2,

$$dz_1 = \langle a_1; b_1, \ldots, b_n \rangle, \qquad dz_2 = \langle a_2; b_{m+1}, \ldots, b_{m+n} \rangle,$$

so we have ($u_1 = \langle a_1; b_1, \ldots, b_n \rangle$, $u_2 = \langle a_2; b_{m+1}, \ldots, b_{m+n} \rangle$)

$$d\left( z_1 z_2 - \bar{u}_1 z_2 - z_1 \bar{u}_2 + \bar{u}_1 \bar{u}_2 \right) = 0,$$

where $\bar{u}_1, \bar{u}_2 \in R$. Now writing $z_1 z_2 - \bar{u}_1 z_2 - z_1 \bar{u}_2 + \bar{u}_1 \bar{u}_2$ in its standard form as in the proof of Lemma 2.2, we get

$$d\left( (-1)^{|a_1 + a_2|} a_1 a_2 x_1 \cdots x_{m+n} \right) - d\left( \sum_{\alpha \in \mathscr{S}'(1,\ldots,m+n)} (-1)^{\mu(\alpha)} \eta(i_1, \ldots, i_s) x_{j_1} \cdots x_{j_{m+n-s}} - \eta \right) = 0,$$

where $\eta(\cdots), \eta \in R$, $\alpha = (i_1, \ldots, i_s \,|\, j_1, \ldots, j_{m+n-s})$. By Lemma 2.2, this equality implies $d\{a_1 a_2; b_1, \ldots, b_{m+n}\} = d\eta \in R$, so the theorem is proved. ∎

THEOREM 3.2. *Suppose* $a_i, b_j \in R$, $da_i = 0$, $db_j = 0$, $i = 1, 2$, $j = 1, \ldots, m+n$. *If* $\mathrm{In}\langle a_1; b_1, b_2, \ldots, b_m \rangle$ *and* $\mathrm{In}\langle a_2; b_{m+1}, b_{m+2}, \ldots, b_{m+n} \rangle$ *are defined, then*

$$(-1)^{\mu} \,\mathrm{In}\langle a_1; b_1, b_2, \ldots, b_m \rangle \mathrm{In}\langle a_2; b_{m+1}, b_{m+2}, \ldots, b_{m+n} \rangle \subset \mathrm{In}\langle a_1 a_2; b_1, \ldots, b_{m-1}, b_m b_{m+1}, b_{m-2}, \ldots, b_{m+n} \rangle,$$

*where* $\subset$ *is defined as in Theorem* 2.1 *of* [6] *and* $\mu = |a_2|(|b_1| + \cdots + |b_m| - m) + \|a_2\|(\|b_1\| + \cdots + \|b_m\|)$.

*Proof.* We define the $S$-module

$$M = E_R(b_1, \dots, b_m, b_m b_{m+1}, b_{m+2}, \dots, b_{m+n})$$

such that $dx_i = b_i$, $i \neq m+1$, $dx_{m+1}a = b_m b_{m+1}$, and define elements in $M$ by

$$\begin{aligned} u_1 = {} & (-1)^{|a_1|} a_1 x_1 \cdots x_{m-1} \\ & - \sum_{\alpha \in \mathscr{S}'(1,\dots,m-1)} (-1)^{\mu(\alpha)} \overline{\langle a_1; b_{i_1}, \dots, b_{i_s} \rangle} x_{j_1} \cdots x_{j_{m-s-1}} \\ & - \overline{\langle a_1; b_1, \dots, b_{m-1} \rangle}, \\ u_2 = {} & (-1)^{|a_2|} a_2 x_{m+2} \cdots x_{m+n} \\ & - \sum_{\beta \in \mathscr{S}'(m+2,\dots,m+n)} (-1)^{\mu(\beta)} \overline{\langle a_2; b_{k_1}, \dots, b_{k_i} \rangle} x_{l_1} \cdots x_{l_{n-i-1}} \\ & - \overline{\langle a_2; b_{m+2}, \dots, b_{m+n} \rangle}, \\ u_3 = {} & (-1)^{|a_1|} a_1 x_1 \cdots x_m \\ & - \sum_{\gamma \in \mathscr{S}'(1,\dots,m)} (-1)^{\mu(\gamma)} \overline{\langle a_1; b_{u_1}, \dots, b_{u_q} \rangle} x_{v_1} \cdots x_{v_{m-q}}, \\ u = {} & x_{m+1} - x_m b_{m+1}, \end{aligned}$$

where $\alpha = (i_1, \dots, i_s \mid j_1, \dots, j_{m-s-1})$, $\beta = (k_1, \dots, k_i \mid l_1, \dots, l_{n-i-1})$, $\gamma = (u_1, \dots, u_q \mid v_1, \dots, v_{m-q})$, and $\mu, \mathscr{S}'$ are as defined in [6].

Write $u_1 x_{m+1} u_2$ in its standard form as in the proof of Lemma 2.2. Then we get

$$\begin{aligned} & u_1 x_{m+1} u_2 \\ & \quad = (-1)^{|a_1|+|a_2|+\mu} a_1 a_2 x_1 \cdots x_{m-1} x_{m+1} x_{m+2} \cdots x_{m+n} \\ & \qquad - \sum_{\alpha \in \mathscr{S}_1} (-1)^{\mu(\alpha)+\mu'} \eta(i_1, \dots, i_s) x_{j_1} \cdots x_{j_{m+n-s-1}} + \eta, \end{aligned}$$

where $\mathscr{S}_1$ is a subset of $\mathscr{S}'(1, \dots, m-1, m+1, \dots, m+n)$ of elements $\alpha = (i_1, \dots, i_s \mid j_1, \dots, j_{m+n-s-1})$ such that $m+1 \in \{j_1, \dots, j_{m+n-s-1}\}$ and $\eta, \eta(\cdots) \in R$ and $\mu' = \mu + |a_2|\,|b_{m+1}| + \|a_2\|\,\|b_{m+1}\|$, $\mu$ is as stated in the theorem.

We have

$$\begin{aligned} & u_1 x_m b_{m+1} u_2 \\ & \quad = u_3 b_{m+1} u_2 + \sum_{\alpha \in \mathscr{S}_2} (-1)^{\mu(\alpha)+\mu'} \eta(i_1, \dots, i_s) x_{j_1} \cdots x_{j_{m+n-s-1}} + \eta', \end{aligned}$$

where $\alpha = (i_1, \dots, i_s \,|\, j_{m+n-s-1})$, $\mathscr{S}_2 = \mathscr{S}'(1, \dots, m-1, m+1, \dots, m+n) - \mathscr{S}_1$, $\mu$, $\mu'$, $\mathscr{S}_1$ are as above, and $\eta', \eta(\cdots) \in R$. Thus

$$\begin{aligned} u_1 u u_2 &\\ &= -u_3 b_{m+1} u_2 + (-1)^{|a_1|+|a_2|+\mu} a_1 a_2 x_1 \cdots x_{m-1} x_{m+1} x_{m+2} \cdots x_{m+n} \\ &\quad - \sum_{\alpha \in \mathscr{S}'(1, \dots, m-1, m+1, \dots, m+n)} (-1)^{\mu(\alpha)+\mu'} \eta(i_1, \dots, i_s) \\ &\quad \times x_{j_1} \cdots x_{j_{m+n-s-1}} + \eta - \eta', \end{aligned}$$

where $\alpha = (i_1, \dots, i_s \,|\, j_1, \dots, j_{m+n-s-1})$. Apply the differential to this equality and we get

$$-\langle a_1; b_1, \dots, b_m \rangle b_{m+1} u_2 + (-1)^{\mu'} d\{a_1 a_2; b_1, \dots, b_{m+n}\} + d(\eta - \eta') = 0, \tag{$*$}$$

where $\{a_1 a_2; b_1, \dots, b_{m+n}\}$ is as defined in Lemma 2.2.

It is easy to check that there is an element $z$ in its standard form

$$z = \sum_{\alpha \in \mathscr{S}'(m+2, \dots, m+n)} (-1)^{\mu(\alpha)+\mu'} c(i_1, \dots, i_s) x_{j_1} \cdots x_{j_{n-s-1}}$$

with $c(\cdots) \in R$ such that $dz = b_{m+1} u_2 - (-1)^{\mu-\mu'} \langle a_2; b_{m+1}, \dots, b_{m+n} \rangle$, so

$$\begin{aligned} d\big((-1)^{\tau} &\langle a_1; b_1, b_2, \dots, b_m \rangle z\big) \\ &= \langle a_1; b_1, b_2, \dots, b_m \rangle b_{m+1} u_2 \\ &\quad - (-1)^{\mu-\mu'} \langle a_1; b_1, \dots, b_m \rangle \langle a_2; b_{m+1}, \dots, b_{m+n} \rangle, \end{aligned}$$

where $\tau = |\langle a_1; b_1, b_2, \dots, b_m \rangle|$, so we have from $(*)$,

$$\begin{aligned} -(-1)^{\mu} &\langle a_1; b_1, b_2, \dots, b_m \rangle \langle a_2; b_{m+1}, b_2, \dots, b_{m+n} \rangle \\ &- d\big((-1)^{\tau} \langle a_1; b_1, b_2, \dots, b_m \rangle z + (-1)^{\mu'} \{a_1 a_2; b_1, \dots, b_{m+n}\}\big) \\ &+ d(\eta - \eta') = 0. \end{aligned}$$

Notice that the second term in the parentheses is another given $(-1)^{\mu'} \{a_1 a_2; b_1, \dots, b_{m+n}\}$ defined in Lemma 2.2, so by Lemma 2.2

$$\begin{aligned} (-1)^{\mu} &\langle a_1; b_1, b_2, \dots, b_m \rangle \langle a_2; b_{m+1}, b_{m+2}, \dots, b_{m+n} \rangle \\ &\sim \langle a_1 a_2; b_1, \dots, b_{m-1}, b_m, b_{m+1}, b_{m+2}, \dots, b_{m+n} \rangle. \end{aligned}$$ ∎

## 4. APPLICATIONS

Let $H_1$ be a sub-Hopf algebra of $H_2$ and $(R_i, d)$ be the cobar complex for $H_i$, $i = 1, 2$. Then there is a natural inclusion i: $R_1 \to R_2$ that induces $\mathrm{i}_*: H^{**}(R_1) \to H^{**}(R_2)$ and we have

THEOREM 4.1. *Let $H_i$ be as above, $a, b_i \in H^{**}(R_1)$ and $b_i \in \ker \mathrm{i}_*$, $i = 1, \dots, n$. If $\mathrm{In}\langle a; b_1, b_2, \dots, b_n\rangle$ is defined and nontrivial in $(R_1, d)$, then $\mathrm{In}\langle a; b_1, b_2, \dots, b_n\rangle \in \ker \mathrm{i}_*$.*

*Proof.* We define an $S$-module over $(R_2, d)$ by

$$M_2 = E_{R_2}(b_1, b_2, \dots, b_n)$$

with $dx_i = b_i$. Since $b_i \in \ker \mathrm{i}_*$, there is $m_i \in R_2$ such that $dm_i = b_i$. For a monomial $cx_{i_1}x_{i_2} \cdots x_{i_s}$, we define its filtration degree to be $s$ and define $F_s$ to be the submodule of $M_2$ consisting of all monomials of filtration degree not more than $s$. Then these $\{F_s\}$ give a spectral sequence $E^{r,s,t}$ such that

$$E^{1,**} = H^{**}(R_2) \otimes E(x_1, x_2, \dots, x_n),$$

where $E$ is the exterior algebra and we continue to use $x_i$ to denote its quotient class in $F_1/F_0$. Since $d(x_i - m_i) = 0$ in $M_2$, the spectral sequence collapses and we have that $H^{**}(M_2)$ is a symmetric module over $H^{**}(R_2)$ freely generated by a base of the exterior algebra $E(u_1, u_2, \dots, u_n)$, where $u_i$ is represented by $x_i - m_i$. Define an $S$-module over $(R_1, d)$ by

$$M_1 = E_{R_1}(b_1, b_2, \dots, b_n)$$

and choose the same $x_i$ to make $M_1$ a subchain complex of $M_2$, so we have the following commutative diagram of chain complexes:

$$\begin{array}{ccc} R_1 & \xrightarrow{\mathrm{i}} & R_2 \\ \downarrow \mathrm{j}' & & \downarrow \mathrm{j} \\ M_1 & \xrightarrow{\mathrm{i}'} & M_2 \end{array}$$

By the above computation we know that $\ker \mathrm{j}_* = 0$, so $\ker \mathrm{i}_* = \ker(\mathrm{ji})_* = \ker(\mathrm{i}'\mathrm{j}')_*$ and, by Lemma 2.2, $\mathrm{In}\langle a; b_1, b_2, \dots, b_n\rangle \in \ker \mathrm{j}'_*$, so $\mathrm{In}\langle a; b_1, b_2, \dots, b_n\rangle \in \ker(\mathrm{i}'\mathrm{j}')_* = \ker \mathrm{i}_*$. ∎

Now let $H_2$ be $A^*$, the dual Steenrod algebra over an odd prime $p$. Let $H_1$ be $P(\xi_1)$ the sub-Hopf algebra of $A^*$ generated by $\xi_1$. It is known (see [7]) that

$$H^{**}(R_1) = E(h_0, h_1 \dots) \otimes P(b_0, b_1, \dots),$$

where $E$ is the exterior algebra, $P$ is the polynomial algebra, and $h_i$ and $b_i$ are represented, respectively, by $[\xi_1^{p^i}]$ and $\sum_{k=1}^{p-1}\binom{p}{k}/p[\xi_1^{kp^i} \mid \xi_1^{(p-k)p^i}]$ in the cobar complex. By [1] we know that $h_i$ and $b_j$ are nontrivial cohomology classes in $H^{**}(R_2)$, which is just the $E_2$-term of the Adams spectral sequence, so applying Theorem 4.1 we have

COROLLARY 4.1. *In the $E_2$-term of the classical Adams spectral sequence, the set $K$ of zero relations of $h_i$ and $b_j$ satisfies*:

1. *$K$ is an ideal of $H^{**}(R_1)$ and $h_0h_1 \in K$.*
2. *If $b \in K$, then $\mathscr{P}^i(b) \in K$, where $\mathscr{P}^i$ is the Steenrod cohomological operation.*
3. *If $a \in H^{**}(R_1)$, $c_i \in K$, $i = 1, \ldots, n$, and $\operatorname{In}\langle a; c_1, c_2, \ldots, c_n\rangle$ is defined in $(R_1, d)$, then $\operatorname{In}\langle a; c_1, c_2, \ldots, c_n\rangle \in K$.*

It is still not known yet whether $K$ is the set defined just by the above three conditions. Even so, the set that is defined just by the above three conditions is very complicated. For example, an easy computation shows that in $H^{**}(R_1)$,

$$\langle h_i; \underbrace{h_i, \ldots, h_i}_{p-1}\rangle = b_i$$

and by Theorems 3.1 and 3.2, we have

$$\operatorname{In}\langle h_{i_1} \cdots h_{i_s}; \underbrace{h_{i_1}, \ldots, h_{i_1}}_{j_1}, \ldots, \underbrace{h_{i_s}, \ldots, h_{i_s}}_{j_s}\rangle \sim 0,$$

$$\operatorname{In}\langle h_{i_1} \cdots h_{i_s}; \underbrace{h_{i_1}, \ldots, h_{i_1}}_{p-2}, h_{i_1}h_{i_2}, \underbrace{h_{i_2}, \ldots, h_{i_2}}_{p-3}, h_{i_2}h_{i_3},$$
$$\cdots, h_{i_{s-1}}h_{i_s}, \underbrace{h_{i_s}, \ldots, h_{i_s}}_{p-2}\rangle = b_{i_1}b_{i_2} \cdots b_{i_s},$$

where $i_1 < i_2 < \cdots < i_s$, $j_r < p-1$, $r = 1, \ldots, s$.

Notice that $h_nh_{n+1} \in K$, $h_{n+2}b_n - h_{n+1}b_{n+1} \in K$, so we have inductively

$$\langle h_1h_2; h_0h_1, \underbrace{h_2b_0 - h_1b_1, \ldots, h_2b_0 - h_1b_1}_{p-2}\rangle = h_0h_2b_1^{p-1} \in K,$$

$$\langle h_2h_3; h_0h_2b_1^{p-1}, \underbrace{h_3b_1 - h_2b_2, \ldots, h_3b_1 - h_2b_2}_{p-2}\rangle = h_0h_3b_1^{p-1}b_2^{p-1} \in K,$$

$$\vdots \qquad\qquad\qquad \vdots$$

$$\langle h_{n+1}h_{n+2}; h_0h_{n+1}b_1^{p-1} \cdots b_n^{p-1}, \underbrace{h_{n+2}b_n - h_{n+1}b_{n+1}, \ldots, h_{n+2}b_n - h_{n+1}b_{n+1}}_{p-2}\rangle$$

$$= h_0 h_{n+2} b_1^{p-1} b_2^{p-1} \cdots b_n^{p-1} b_{n+1}^{p-1}$$
$$\sim h_0 h_{n+1} b_1^{p-1} b_2^{p-1} \cdots b_n^{p-2} b_{n+1}^{p} \quad (\text{modulo } h_{n+2} b_n - h_{n+1} b_{n+1})$$
$$\sim h_0 h_n b_1^{p-1} \cdots b_{n-1}^{p-2} b_n^{p-1} b_{n+1}^{p} \quad (\text{modulo } h_{n+1} b_{n-1} - h_n b_n)$$
$$\vdots$$
$$\sim h_0 h_2 b_1^{p-2} b_2^{p-1} \cdots b_n^{p-1} b_{n+1}^{p} \in K.$$

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