THE HOMOTOPY AXIOM

Monday, October 19, 2020 8:38 AM

THEOREM SUPPOSE WE HAVE TWO MAPS

ABSOLUTE $f, g : X \to Y$ RELATIVE $[f, g : (X, A) \to (Y, B)]$.

IF $f \simeq g$, THEN $H_*(f) = H_*(g)$.

WE NEED SOME ALGEBRA

FIVE LEMMA: SUPPOSE WE HAVE A COMMUTATIVE DIAGRAM OF ABELIAN GROUPS WITH EXACT ROWS.

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "\alpha"'] & B \arrow[r] \arrow[d, "\beta"'] & C \arrow[r] \arrow[d, "\gamma"'] & D \arrow[r] \arrow[d, "\delta"'] & E \arrow[d, "\varepsilon"'] \\
A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & D' \arrow[r] & E'
\end{tikzcd}$$

IF α, β, δ AND ε ARE ISOMORPHISMS, SO IS γ.

PROOF IS A DIAGRAM CHASE.

REMARK. IF α, β, δ AND ε ARE TRIVIAL, γ NEED NOT BE

EXAMPLE

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & \mathbb{Z}/2 \arrow[r, "2"] \arrow[d, "2"', "0"] & \mathbb{Z}/4 \arrow[r] \arrow[d, "2"', "\neq 0"] & \mathbb{Z}/2 \arrow[r] \arrow[d, "2"', "0"] & 0 \arrow[d] \\
0 \arrow[r] & \mathbb{Z}/2 \arrow[r, "2"] & \mathbb{Z}/4 \arrow[r] & \mathbb{Z}/2 \arrow[r] & 0
\end{tikzcd}$$

COROLLARY TO 5-LEMMA

SUPPOSE $f : (X, A) \to (Y, B)$, WHERE

$f': X \to Y$ AND $f'': A \to B$ BOTH INDUCE ISOMORPHISMS IN H_* THEN, SO DOES f.

PROOF:

$$\begin{tikzcd}
H_i(A) \arrow[r] \arrow[d, "H_i(f'')"'] & H_i(X) \arrow[r] \arrow[d, "H_i(f')"'] & H_i(X,A) \arrow[r] \arrow[d, "H_i(f)"'] & H_{i-1}(A) \arrow[r] \arrow[d, "H_{i-1}(f'')"'] & H_{i-1}(X) \arrow[d, "H_{i-1}(f')"'] \\
H_i(B) \arrow[r] & H_i(Y) \arrow[r] & H_i(Y,B) \arrow[r] & H_{i-1}(B) \arrow[r] & H_{i-1}(Y)
\end{tikzcd}$$

$H_i(f)$ IS AN ISOMORPHISM BY THE FIVE LEMMA. QED

TO PROVE THE THEOREM WE WILL SHOW HOMOTOPIC MAPS OF [PAIRS OF] SPACE INDUCE CHAIN HOMOTOPIC MAPS OF SINGULAR CHAIN COMPLEXES, SO THEY INDUCE THE SAME MAP IN HOMOLOGY. LET $h: X \times I \to Y$ BE A HOMOTOPY BETWEEN f AND g.

$$\begin{tikzcd}
X \times \{0\} \arrow[d] \arrow[dr, "f"] & \\
X \times [0,1] \arrow[r, "h"] & Y \\
X \times \{1\} \arrow[u] \arrow[ur, "g"'] &
\end{tikzcd}$$

A CHAIN HOMOTOPY BETWEEN $C(f)$ AND $C(g)$ IS A COLLECTION OF HOMS $D_n : C_n(X) \longrightarrow C_{n+1}(Y)$ SUCH THAT

$$\begin{tikzcd}
C_{n-1}(X) \arrow[dr, "D_{n-1}"'] & C_n(X) \arrow[l, "d_n"'] \arrow[d, "C_n(f)-C_n(g)"] \arrow[dr, "D_n"] & \\
& C_n(Y) & C_{n+1}(Y) \arrow[l, "d'_{n+1}"]
\end{tikzcd}$$

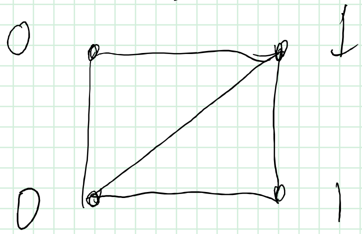
$$D_{n-1}\, d_n + d'_{n+1}\, D_n = C_n(f) - C_n(g)$$

LET $\sigma : \Delta^n \longrightarrow X$ BE ANY MAP. CONSIDER

$$\Delta^n \times I \xrightarrow{\sigma \times I} X \times I \xrightarrow{h} Y$$

WILL SHOW $\Delta^n \times I$ IS THE UNION OF $(n+1)$ COPIES OF Δ^{n+1}

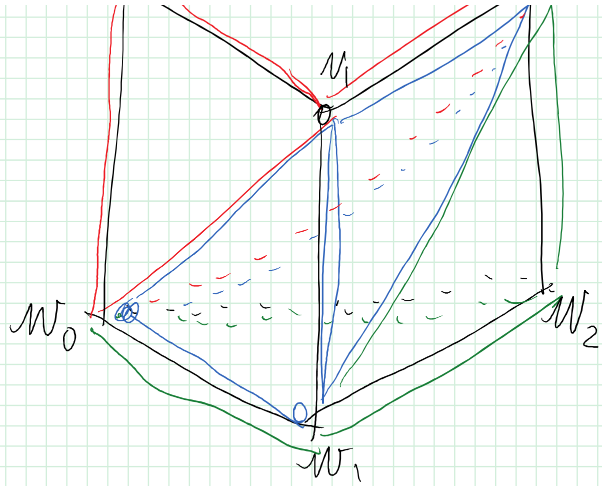
PICTURE FOR $n=1$



$\Delta^1 \times I \cong$ UNION OF TWO COPIES OF Δ^2

PICTURE FOR $n=2$

3 COPIES OF Δ^3 (TETRAHEDRA) WITH THE FOLLOWING



(TETRAHEDRA) WITH THE FOLLOWING SETS OF VERTICES

$\{w_0, v_0, v_1, v_2\}$

$\{w_0, w_1, v_1, v_2\}$

$\{w_0, w_1, w_2, v_2\}$

IN GENERAL OUR Δ^{n+1}s HAVE VERTICES

$$\{w_0, \dots w_i, v_i, v_{i+1}, \dots v_n\} \qquad 0 \le i \le n$$

$n+1$ VALUES

CALL THIS Δ^{n+1}_i

$$\Delta^n \times I = \bigcup_{0 \le i \le n} \Delta^{n+1}_i$$

FOR $0 \le i \le n$ WE HAVE

$$\Delta^{n+1}_i \xrightarrow{g_i} \Delta^n \times I \xrightarrow{\sigma \times I} X \times I \xrightarrow{h} Y \qquad (1)$$

PRISM

WE NEED A MAP

$$D_n : C_n(X) \longrightarrow C_{n+1}(Y)$$

$$[\sigma] \longmapsto \sum_{0 \le i \le n} [h(\sigma \times I) g_i]$$

CAN SHOW THIS IS THE DESIRED

CHAIN HOMOTOPY BETWEEN $C(f)$ AND $C(g)$. DETAILS ARE IN HATCHER. THIS IS THE ABSOLUTE CASE.

FOR THE RELATIVE CASE

$$C_n(X,A) := C_n(X)/C_n(A)$$

$$C_n(Y,B) := C_n(Y)/C_n(B)$$

$\exists!\ \beta$ $\qquad \tilde{\alpha}', \tilde{\alpha} \longmapsto \alpha$ $\qquad \exists!\ \beta$ WITH $i(\beta) = \tilde{\alpha} - \tilde{\alpha}'$

$$\begin{array}{ccccccccc}
0 & \to & C_n(A) & \xrightarrow{i} & C_n(X) & \xrightarrow{j} & C_n(X,A) & \to & 0 \\
 & & \downarrow D_n' & \text{COMMUTES} & \downarrow D_n & & \vdots\ D_n'' & & \\
0 & \to & C_{n+1}(B) & \xrightarrow{i} & C_{n+1}(Y) & \xrightarrow{j} & C_{n+1}(Y,B) & \to & 0
\end{array}$$

$iD_n'(\beta) = D_n(\tilde{\alpha} - \tilde{\alpha}')$ $\qquad D_n\tilde{\alpha} \longrightarrow jD_n\tilde{\alpha}$

WANT TO "DEFINE" $D_n''\alpha$ TO BE $jD_n\tilde{\alpha}$

IS IT WELL DEFINED?

$D_n(\tilde{\alpha} - \tilde{\alpha}') = iD_n'(\beta)$ IN $C_{n+1}(Y)$

$jD_n(\tilde{\alpha} - \tilde{\alpha}') = jiD_n'(\beta) = 0$ IN $C_{n+1}(Y,B)$

$\qquad \| $

$jD_n\tilde{\alpha} - jD_n\tilde{\alpha}'$. HENCE $D_n''(\alpha) = jD_n\tilde{\alpha}$

IS INDEPENDENT OF THE CHOICE

OF $\tilde{\alpha}$. THIS IS THE DESIRED CHAIN HOMOTOPY. (SEE HATCHER)

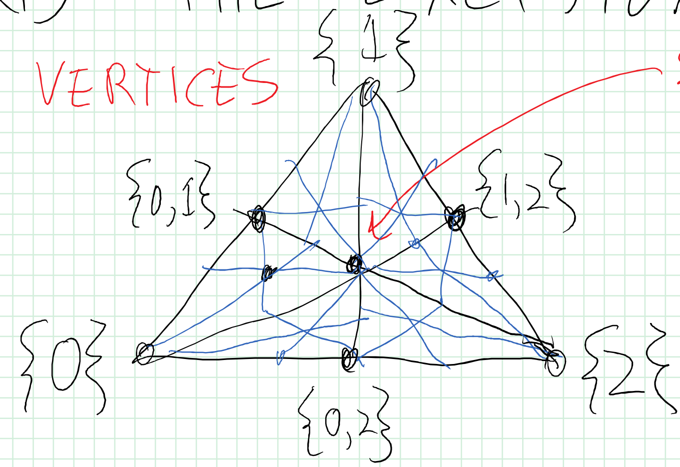
QED

BARYCENTRIC SUBDIVISION

TOWARD THE EXCISION AXIOM

7 VERTICES

2



Δ^2 GETS DIVIDED INTO 6 SMALLER Δ^2S.

SIMILARY Δ^n CAN BE DIVIDED INTO $(n+1)!$ SMALLER ONES.

LET $\underline{n} = \{0, 1, 2, \cdots n\}$

IN THE GENERALIZATON OF 2, THERE ARE $2^{n+1}-1$ VERTICES, ONE FOR NONEMPTY SUBSET OF $\underline{n}$.

THERE IS A SMALL Δ^n FOR EACH CHAIN OF SUBSETS OF THE FORM

$$\{i_0\} \subset \{i_0, i_1\} \subset \{i_0, i_1, i_2\} \subset \cdots \subset \underline{n}$$

WHERE $i_0, i_1, \ldots i_n$ ARE DISTINCT.

THERE $n+1$ CHOICES FOR i_0

n " FOR i_1

ETC.

THERE ARE $(n+1)!$ SUCH Δ_n'S.

THIS PROCESS CAN BE ITERATED.