WHY ASK THESE ALGEBRAIC QUESTIONS?

<u>DEF</u> LET A BE AN ABELIAN GROUP. THEN THE <u>HOMOLOGY OF A SPACE X WITH COEFFICIENTS IN A</u> IS

$$H_*(C(X)\otimes A) =: H_*(X;A)$$

THIS IS CONVENIENT WHEN A IS A FIELD k, E.G. $\mathbb{Q}$ OR $\mathbb{Z}/p$ FOR p A PRIME. IT TURNS OUT THAT $H_*(X\times Y;k) \cong H_*(X;k)\otimes_k H_*(Y;k)$

EVEN THOUGH

$$H_*(X) =: H_*(X\times Y;\mathbb{Z}) \not\cong H_*(X;\mathbb{Z})\otimes H_*(Y;\mathbb{Z})$$

<u>DEF</u> THE <u>COHOMOLOGY OF X WITH COEFFICIENTS IN A</u> IS

$$H^*(\mathrm{Hom}(C(X),A)) =: H^*(X;A)$$

BACKGROUND:

A <u>COCHAIN COMPLEX</u> C IS DIAGRAM COLLECTION OF ABELIAN GROUPS

$$C^0 \xrightarrow{\delta^0} C^1 \xrightarrow{\delta^1} C^2 \xrightarrow{\delta^2} C^3 \xrightarrow{\delta^3} \cdots$$

WITH $\delta^{i+1}\delta^i = 0$, FOR $i \geq 0$

$H^i(C) = \ker \delta^i / \operatorname{im} \delta^{i-1}$

GIVEN A CHAIN COMPLEX C AND AN ABELIAN GROUP A, WE DEFINE A COCHAIN COMPLEX $\hat{C}$ BY $\hat{C}^i = \operatorname{Hom}(C_i, A) \ni \alpha$

$$\begin{tikzcd} C_{i-1} & C_i \arrow[l, "d_i"'] \arrow[d, "\alpha"'] & C_{i+1} \arrow[l, "d_{i+1}"'] \arrow[dl] & \cdots \arrow[l] \\ & A & & \end{tikzcd}$$

$\alpha d_{i+1} : C_{i+1} \to A$, SO $\alpha d_{i+1} \in \hat{C}^{i+1}$

AND $\delta^i(\alpha) := \alpha d_{i+1}$.

NOTE : A MAP $X \xrightarrow{f} Y$

$$\begin{tikzcd} C_n(X) \arrow[r, "C_n(f)"] \arrow[dr, "\alpha C_n(f)"'] & C_n(Y) \arrow[d, "\alpha"] \\ & A \end{tikzcd}$$

THIS LEADS TO A MAP

$$H^*(X;A) \longleftarrow H^*(Y;A)$$

NOTE REVERSAL OF ARROW.

$H^*(-;A)$ IS A CONTRAVARIANT FUNCTOR

A FEATURE OF $H^*(-;\mathbb{Z})$

$$X \xrightarrow{\ \Delta\ } X \times X$$
$$x \longmapsto (x,x)$$

$$\begin{tikzcd}
H^*(X;\mathbb{Z}) & H^*(X\times X;\mathbb{Z}) \arrow[l, "\Delta^*"'] \\
\alpha \cup \beta & \\
& H^*(X;\mathbb{Z}) \otimes H^*(X;\mathbb{Z}) \arrow[uul] \arrow[uu, "\exists"'] \\
& \alpha \otimes \beta \arrow[uul, mapsto]
\end{tikzcd}$$

\cup

LET $\alpha \in H^i(X;\mathbb{Z})$ $\beta \in H^j(X;\mathbb{Z})$

THEN THE <u>CUP PRODUCT</u>

$\alpha\beta = \alpha \cup \beta \in H^{i+j}(X;\mathbb{Z})$

$H^*(X;\mathbb{Z})$ IS A <u>GRADED RING</u>

THIS ADDITIONAL STRUCTURE IS VERY USEFUL.

DIFFERENTIAL FORMS ON A SMOOTH MANIFOLD M LEAD TO A COCHAIN COMPLEX OF REAL VECTOR SPACES. A THEOREM OF DE RHAM SAYS ITS COHOMOLOGY

IS $H^*(M;\mathbb{R})$

BASIC ALGEBRAIC QUESTION:

GIVEN A CHAIN COMPLEX C

AND AN ABELIAN GROUP A,

HOW CAN WE DESCRIBE

$H^*(\mathrm{Hom}(C,A))$ IN TERMS OF

A AND $H_*(C)$

NAIVE GUESS:

$$H^n(-) = \mathrm{Hom}(H_n(C), A)$$

EXAMPLE

$$C: \quad \begin{array}{ccccccc} \mathbb{Z} & \xleftarrow{2} & \mathbb{Z} & \leftarrow & 0 & \leftarrow & \cdots \\ \| & & \| & & \| & & \\ C_0 & & C_1 & & C_2 & & \end{array}$$

$$H_i(C) = \begin{cases} \mathbb{Z}/2 & \text{FOR } i=0 \\ 0 & i>0 \end{cases}$$

$A = \mathbb{Z}/4 \qquad \mathrm{Hom}(\mathbb{Z}, \mathbb{Z}/4) \cong \mathbb{Z}/4$

$\mathrm{Hom}(\mathbb{Z}, A) \cong A$ FOR ANY A

$$\begin{array}{ccccc} \mathrm{Hom}(C_0, \mathbb{Z}/4) & \longrightarrow & \mathrm{Hom}(C_1, \mathbb{Z}/4) & \longrightarrow & 0 \cdots \\ \| & & \| & & \\ \mathbb{Z}/4 & \xrightarrow[\delta^0]{2} & \mathbb{Z}/4 & \xrightarrow{0} & \end{array}$$

THIS MEANS

$$H^i(-) = \begin{cases} \mathbb{Z}/2 \cong \ker \delta^0 & \text{FOR } i=0 \\ \mathbb{Z}/2 \cong \operatorname{coker} \delta^1 & \text{FOR } i=1 \\ 0 & i>1 \end{cases}$$

BASIC DIFFICULTY:

SUPPOSE WE HAVE A SHORT EXACT SEQUENCE OF R-MODULES

$$0 \longrightarrow M' \xrightarrow{1-1} M \xrightarrow{\text{ONTO}} M'' \longrightarrow 0$$

AND ANOTHER R-MODULE A.

$$M' \otimes_R A \xrightarrow{\text{NOT } 1-1} M \otimes_R A \xrightarrow{\text{ONTO}} M'' \otimes_R A \to 0$$

THE FUNCTOR $(- \otimes_R A)$ DOES NOT PRESERVE EXACTNESS.

$$0 \longrightarrow M' \xrightarrow{1-1} M \xrightarrow{\text{ONTO}} M'' \longrightarrow 0$$

$$\mathrm{Hom}_R(M', A) \xleftarrow[\text{ONTO}]{\text{NOT}} \mathrm{Hom}_R(M, A) \xleftarrow{1-1} \mathrm{Hom}_R(M'', A)$$

$$\mathrm{Hom}_R(M',A) \xleftarrow[\text{ONTO}]{\text{NOT}} \mathrm{Hom}_R(M,A) \xleftarrow{1-1} \mathrm{Hom}_R(M'',A)$$

THE FUNCTOR $\mathrm{Hom}_R(-,A)$ ALSO DOES NOT PRESERVE EXACTNESS.

WHAT TO DO ???

DEF AN R-MODULE P IS **PROJECTIVE** IF g IS ONTO

$$\begin{tikzcd} & P \arrow[dl, dashed, "\hat{f}"'] \arrow[d, "f"] & \\ m \in M \arrow[r, "g"'] & N \arrow[r] & 0 \end{tikzcd}$$

$\exists\, \hat{f} : P \to M$ WITH $g\hat{f} = f$

EXAMPLE $P = R \ni 1$. f IS DETERMINED BY $f(1) \in N$, WHICH COULD BE ANYTHING. SINCE g IS ONTO, $\exists\, m \in M$ WITH $g(m) = f(1)$. THEN DEFINE $\hat{f}$ BY $\hat{f}(1) = m$.

SIMILARLY, ANY FREE R-MODULE IS PROJECTIVE.

REMARK FOR SOME RINGS,

e.g. $R = \mathbb{Z}$, EVERY PROJECTIVE MODULE IS FREE. FOR OTHER RINGS THIS MAY NOT BE TRUE

SUBTLE QUESTION. $\mathbb{Z}[\sqrt{-5}] = R$

<u>DEF</u> A <u>PROJECTIVE (FREE) RESOLUTION</u> OF AN R-MODULE M IS A LONG EXACT SEQUENCE

$$0 \leftarrow M \xleftarrow{\text{ONTO}} P_0 \leftarrow P_1 \leftarrow P_2 \leftarrow \cdots$$

WHERE P_i IS PROJECTIVE (FREE) FOR ALL $i \geq 0$.

<u>EXAMPLE</u> $R = \mathbb{Z}$, $M = \mathbb{Z}/7$

$$\begin{array}{ccccccccc} 0 \leftarrow & \mathbb{Z}/7 & \leftarrow & \mathbb{Z} & \xleftarrow{7} & \mathbb{Z} & \leftarrow 0 \\ & \| & & \| & & \| & \\ & M & & P_0 & & P_1 & \end{array}$$

EXERCISE: FIND A PROJECTIVE (FREE) RESOLUTION OF $\mathbb{Q}$

(NOTE $\mathbb{Q}$ IS NOT PROJECTIVE)

<u>PROP</u> EVERY R-MODULE M HAS A

PROJECTIVE RESOLUTION.

PROOF: CLAIM ~~IT~~ SUFFICES THAT $\exists$ PROJECTIVE R-MODULE P_0 WITH A MAP ONTO M.

$$
\begin{tikzcd}
0 & M \arrow[l] \arrow[d, equal] & P_0 \arrow[l, "d_0"'] & M_1 \arrow[l, "i_1"'] \arrow[d, equal] & 0 \arrow[l] \\
 & M_0 & & \ker d_0 &
\end{tikzcd}
$$

$$
\begin{tikzcd}
0 & M_1 \arrow[l] & P_1 \arrow[l, "d_1"'] & M_2 \arrow[l, "i_2"'] \arrow[d, equal] & 0 \arrow[l] \\
 & & & \ker d_1 &
\end{tikzcd}
$$

ETC.

$$
\begin{tikzcd}
0 & M \arrow[l] & P_0 \arrow[l, "d_0"'] & & P_1 \arrow[ll, "e_1"'] \arrow[dl, "d_1"] & & P_2 \arrow[ll, "e_2"'] \arrow[dl] & \cdots \arrow[l] \\
 & & & M_1 \arrow[ul, "i_1"] & & M_2 \arrow[ul, "i_2"] & &
\end{tikzcd}
$$

NOTE $\ker d_0 = \operatorname{im} i_1 = \operatorname{im} i_1 d_1 = \operatorname{im} e_1$

CAN SHOW IT IS EXACT AT EACH STAGE, i.e.

$\ker e_i = \operatorname{im} e_{i+1}$ QED FOR CLAIM.

HOW TO FIND P_0 ?

CONSIDER M AS A SET AND LET $R(M)$ BE THE FREE R-MODULE GENERATED BY IT.

THERE IS A MAP

$$M \xleftarrow{\text{ONTO}} R(M)$$

$$M \xleftarrow{\text{ONTO}} R(M)$$

$$\parallel$$

PROJECTIVE

MORE SUBTLE APPROACH

LET $K \subset M$ IS A SUBSET THAT GENERATES M AS AN R-MODULE.

$$M \xleftarrow{\text{ONTO}} R(K) = \text{PROJECTIVE}$$

QED