RECALL EVERY R-MODULE M HAS A PROJECTIVE (FREE) RESOLUTION, I.E A LONG EXACT SEQUENCE

$$0 \leftarrow M \leftarrow P_0 \leftarrow P_1 \leftarrow P_2 \leftarrow \cdots$$

WITH P_i PROJECTIVE

REMARKS

1) IF R IS A FIELD, EVERY MODULE IS FREE, SO

$$0 \leftarrow M = P_0 \leftarrow 0$$

LENGTH 0 RESOLUTION

2) IF $R = \mathbb{Z}$, IT IS KNOWN THAT WE ALWAYS HAVE

$$0 \leftarrow M \leftarrow P_0 \leftarrow P_1 \leftarrow 0$$

FREE ABELIAN GROUP

LENGTH 1 RESOLUTION

3) FOR EACH $n \geq 0$ $\exists$ RING R SUCH THAT EACH MODULE HAS A PROJECTIVE RESOLUTION OF LENGTH, AND SOME MODULES DO NOT HAVE SHORTER ONE.

4) $\exists$ R AND MODULE M FOR WHICH $P_i \neq 0$ FOR ALL $i \geq 0$

EXAMPLE k = FIELD

$$R = k[x]/(x^2)$$

AN R-MODULE A VECTOR SPACE V/k WITH AN ENDOMORPHISM $V \xrightarrow{x} V$ WITH $x^2 = 0$. LET $M = k$ WITH $x = 0$. THEN A PROJECTIVE RESOLUTION

$$0 \leftarrow M \leftarrow R \xleftarrow{x} R \xleftarrow{x} R \xleftarrow{x} R \leftarrow \cdots$$

$0 \leftarrow\!\shortmid x$

$0 \leftarrow\!\shortmid x$

$P_i = R \quad \forall i \geq 0.$

DEF A <u>PRINCIPAL IDEAL DOMAIN</u> (PID) IS AN INTEGRAL DOMAIN IN WHICH EACH IDEAL IS PRINCIPAL I.E. GENERATED BY A SINGLE ELEMENT.

<u>EXAMPLES</u>

① $\mathbb{Z}$ IS A PID

FOR $m, n \in \mathbb{Z}$, $(m,n) \subset \mathbb{Z}$ IS $(\gcd(m,n))$.

② LET k BE A FIELD. THEN $k[x]$ IS A PID. USE EUCLIDEAN ALGORITHM.

<u>THEOREM</u> LET M BE A FINITELY GENERATED MODULE OVER A PID R. THEN

$$M \cong \text{FREE MODULE} \oplus \bigoplus_i R/(x_i)$$

EXAMPLE $R = \mathbb{Z}$, $M = \mathbb{Q}$ (NOT FINITELY GENERATED). IT DOES NOT HAVE THE STATED FORM.

<u>COR</u> EVERY SUCH M HAS A LENGTH ONE PROJECTIVE

RESOLUTION:

$$0 \leftarrow M \leftarrow P_0 \leftarrow P_1 \leftarrow 0$$

$$0 \leftarrow R/(x_i) \leftarrow R \xleftarrow{x_i} R \leftarrow 0$$

PROJECTIVE RESOLUTIONS ARE NOT UNIQUE.

<u>LEMMA</u> LET $P_\bullet$ AND $P_\bullet'$ BE TWO PROJ. RESOLUTIONS OF M. THEY ARE CHAIN HOMOTOPY EQUIVALENT.

$$P_\bullet: 0 \leftarrow M \xleftarrow{d_0} P_0 \xleftarrow{d_1} P_1 \xleftarrow{d_2} P_2 \leftarrow \cdots$$

$$P_\bullet': 0 \leftarrow M \xleftarrow{d_0'} P_0' \xleftarrow{d_1'} P_1' \xleftarrow{d_2'} P_2' \leftarrow \cdots$$

$\exists$ MAPS $P_i \xrightarrow{f_i} P_i'$ AND $g_i: P_i' \to P_i$ WITH SUITABLE PROPERTIES.

PROOF: WILL CONSTRUCT $f_i: P_i \to P_i'$ BY INDUCTION ON i.

$$\begin{array}{ccccc} 0 & \leftarrow & M & \xleftarrow{d_0} & P_0 \\ & & \| & & \downarrow f_0 \\ 0 & \leftarrow & M & \underset{d_0'}{\xleftarrow{\text{ONTO}}} & P_0' \end{array}$$

$\exists f_0$ BECAUSE d_0' IS ONTO AND P_0 IS PROJECTIVE

STARTS INDUCTION

RECALL A MODULE P IS PROJECTIVE

$$\begin{array}{ccccc} & & P & & \\ & \tilde{f} \swarrow & \downarrow f & & \\ M' & \underset{\text{ONTO}}{\xrightarrow{j}} & M & \to & 0 \end{array}$$

$\exists \tilde{f}$ WITH $j\tilde{f} = f$

$$\begin{array}{ccccccccc} 0 & \leftarrow & M & \xleftarrow{d_0} & P_0 & \leftarrow & K_0 & \leftarrow & P_1 \\ & & \| & & \downarrow f_0 & & \downarrow f_0 & & \downarrow f_1 \\ 0 & \leftarrow & M & \xleftarrow[d_0']{} & P_0' & \leftarrow & K_0' & \underset{\text{ONTO}}{\xleftarrow{d_1'}} & P_1' \end{array}$$

$\exists f_1$ BECAUSE d_1' IS ONTO AND P_1 IS PROJECTIVE

WE CAN DO THE SAME FOR ALL $i \geq 0$,
WE GET MAPS g_i THE SAME WAY.
HENCE WE HAVE MAPS

$$P_\bullet \xrightarrow{f_\bullet} P'_\bullet \xrightarrow{g_\bullet} P_\bullet$$

NEED THIS TO BE CHAIN HOMOTOPIC
TO THE IDENTITY. WE NEED D_{i-1}, D_i
WITH

$$\begin{tikzcd}
P_{i-1} \arrow[dr, "D_{i-1}"'] & P_i \arrow[l, "d_i"'] \arrow[dr, "D_i"] & \\
& P_i & P_{i+1} \arrow[l, "d_{i+1}"]
\end{tikzcd}$$

$$d_{i+1} D_i + D_{i-1} d_i = 1 - g_i f_i$$

THESE MAPS CAN ALSO BE
CONSTRUCTED INDUCTIVELY.
DETAILS OMITTED. QED

RECALL THE FUNDAMENTAL PROBLEM.
GIVEN A SHORT EXACT SEQUENCE

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

OF R-MODULES AND ANOTHER
R-MODULE N, THE SEQUENCES

$$M' \otimes_R N \longrightarrow M \otimes_R N \longrightarrow M'' \otimes_R N$$

$$\mathrm{Hom}(M', N) \longleftarrow \mathrm{Hom}_R(M, N) \longleftarrow \mathrm{Hom}_R(M'', N)$$

MAY NOT BE SHORT EXACT.

$\boxed{\text{HOW TO DEAL WITH THIS}}$

LET $P_\bullet$ BE A PROJECTIVE RESOLUTION

OF M

$$0 \leftarrow M \leftarrow P_0 \leftarrow P_1 \leftarrow P_2 \cdots\cdots$$

APPLY THE FUNCTORS $(-)\otimes_R N$ AND $Hom_R(-,N)$ WE GET A CHAIN COMPLEX

$$(1) \qquad P_0 \otimes_R N \leftarrow P_1 \otimes_R N \leftarrow P_2 \otimes N \leftarrow \cdots\cdots$$

AND A COCHAIN COMPLEX

$$(2) \qquad Hom_R(P_0,N) \to Hom_R(P_1,N) \to Hom_R(P_2,N) \to \cdots$$

DEF $Tor_i^R(M,N) = H_i$ OF (1)

$Ext_R^i(M,N) = H^i$ OF (2)

THE LEMMA IMPLIES THAT REPLACING $P_\bullet$ BY $P'_\bullet$ DOES NOT CHANGE $H_*(- \otimes_R N)$ OR $H^*(Hom_R(-,N))$

HENCE Tor AND Ext AS DEFINED ABOVE ONLY DEPEND ON M AND N.

WE WILL SHOW THAT A SHORT EXACT SEQUENCE

$$0 \to M' \to M \to M'' \to 0$$

LEADS TO LONG EXACT SEQUENCES OF Tor AND Ext GROUPS.

$$\cdots\cdots \to Tor_i^R(M',N) \to Tor_i^R(M,N) \to Tor_i^R(M'',N) \to$$

$$\cdots \to \mathrm{Tor}_i^R(M',N) \to \mathrm{Tor}_i^R(M,N) \to \mathrm{Tor}_i^R(M'',N) \to \mathrm{Tor}_{i-1}(M',N) \to \cdots$$

$$\cdots \leftarrow \mathrm{Ext}_R^i(M',N) \leftarrow \mathrm{Ext}_R^i(M,N) \leftarrow \mathrm{Ext}_R^i(M'',N) \leftarrow \mathrm{Ext}^{i-1}(M',N) \leftarrow \cdots$$

CAN SHOW

$$\mathrm{Tor}_0^R(M,N) = M \otimes_R N$$

AND $\mathrm{Ext}_R^0(M,N) = \mathrm{Hom}_R(M,N)$

LEMMA GIVEN A SHORT EXACT SEQUENCE

(3) $$0 \to M' \to M \to M'' \to 0$$

$\exists$ PROJECTIVE RESOLUTIONS $P_\bullet'$ OF M', $P_\bullet$ OF M AND $P_\bullet''$ OF M'' SUCH THAT THERE A SHORT EXACT SEQUENCE OF CHAIN COMPLEXES

(4) $$0 \to P_\bullet' \to P_\bullet \to P_\bullet'' \to 0$$

WILL SEE THAT APPLYING $-\otimes_R N$ OR $\mathrm{Hom}_R(-,N)$ TO (4), WE GET A SHORT EXACT

SEQUENCE OF (CO)CHAIN COMPLEXES. THIS LEADS TO THE DESIRED LONG EXACT SEQUENCES OF Tor AND Ext GROUPS.

PROOF OF LEMMA: CHOOSE $P_\bullet'$ AND $P_\bullet''$. WILL CONSTRUCT A RESOLUTION $P_\bullet$ OF M WITH $P_i \cong P_i' \oplus P_i''$. THIS DOES <u>NOT</u> MEAN $P_\bullet \cong P_\bullet' \oplus P_\bullet''$ AS CHAIN COMPEXES.

USE INDUCTION ON i.

$$\begin{tikzcd}
0 \arrow[r] & M' \arrow[r, "\alpha"] & M \arrow[r, "\beta", "\text{ONTO}"'] & M'' \arrow[r] & 0 \\
0 \arrow[r] & P_0' \arrow[u, "d_0'"] \arrow[r] \arrow[ur, "f"] & P_0' \oplus P_0'' \arrow[u, "d_0"] \arrow[r] & P_0'' \arrow[u, "d_0''"'] \arrow[ul, dashed, "g"'] \arrow[r] & 0
\end{tikzcd}$$

$\exists g$ BECAUSE P_0'' IS PROJECTIVE AND β IS ONTO, AND $\beta g = d_0''$

$f = \alpha d_0'$. $d_0 = f \oplus g$.

TO BE CONTINUED.