WILL PROVE THEOREM ABOUT

Monday, October 5, 2020 5:18 AM

THE UNIVERSAL COVERING

BEFORE THAT WE NEED

DEFINITION A GROUPOID $\mathcal{C}$ IS A SMALL CATEGORY IN WHICH EACH MORPHISM IS INVERTIBLE.

"SMALL" MEANS THE COLLECTION OF OBJECTS IS A SET

A MORPHISM $A \xrightarrow{f} B$ IS INVERTIBLE IF $\exists$ $B \xrightarrow{g} A$ SUCH THAT

$$gf = 1_A$$
$$fg = 1_B.$$

EXAMPLE ① A GROUPOID WITH ONE OBJECT IS A GROUP.

ITS MORPHISMS FORM A GROUP.

② SUPPOSE A GROUP G ACTS ON A SET X. THERE IS GROUPOID $\mathcal{B}_X G$

WITH OBJECT SET X

AND $\mathcal{B}_X G(x,y)$ (THE SET OF MORPHISMS $x \to y$ FOR $x,y \in X$.)

IS $\{\gamma \in G : \gamma(x) = y\}$. (IT COULD BE EMPTY.)

\Pi

EXAMPLE. LET X BE A TOPOLOGICAL SPACE. LET $\Pi(X)$, THE FUNDAMENTAL GROUPOID OF X,

~~IS THE GROUPOID WHOSE~~

OBJECTS ARE THE POINTS OF X AND WHOSE MORPHISMS ARE HOMOTOPY CLASSES OF PATHS IN X. FOR $x_1, x_2 \in X$ THE $\Pi(X)(x_1, x_2)$, THE SET OF MORPHISMS $x_1 \to x_2$, IS THE SET OF HOMOTOPY CLASSES OF PATHS $p : (I, 0, 1) \to (X, x_1, x_2)$

(HOMOTOPIES DO NOT MOVE END POINTS)

NOTE: ① ANY PATH p FROM x_1 TO x_2 HAS AN INVERSE $\bar{p}$ FROM x_2 TO x_1. $\bar{p} * p \simeq$ CONSTANT PATH AT x_1

$p * \bar{p} \simeq$ " x_2.

② IF x_1 AND x_2 LIE IN DIFFERENT PATH CONNECTED COMPONENTS OF X THEN THE MORPHISM SET IS EMPTY.

③ THE MORPHISM FROM x_0 to x_0 IS $\pi_1(X, x_0)$ AS PREVIOUSLY DEFINED.

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THEOREM. LET X BE A SPACE WHICH IS

1) PATH CONNECTED
2) LOCALLY PATH CONNECTED (LPC)
3) SLSC (AS DEFINED BEFORE).

THEN THERE A PATH CONNECTED, LPC SIMPLY CONNECTED (SC) COVERING $\tilde{X} \xrightarrow{p} X$.

PROOF: CHOOSE A BASE POINT $x_0 \in X$. SAY TWO PATHS ω AND ω' STARTING AT x_0 ARE <u>EQUIVALENT</u> IF

1) $\omega(1) = \omega'(1) =: x_1$ (SAME END POINTS)

2) THERE IS AND END POINT PRESERVING HOMOTOPY

BETWEEN THEM.

LET $\tilde{X}$ BE THE SET OF ALL SUCH EQUIVALENCE CLASSES $[\omega]$

DEFINE $p : \tilde{X} \to X$ BY $p([\omega]) = \omega(1)$.

WE WILL DEFINE A TOPOLOGY ON $\tilde{X}$, SHOW p IS CONTINUOUS AND A COVERING, AND $\tilde{X}$ IS SC, PC AND LPC.

$\tilde{X}$ IS THE SET OF MORPHISMS IN $\Pi(X)$ WITH SOURCE (DOMAIN) x_0. NOTE THAT $p^{-1}(x_0)$ IS THE SET UNDERLYING $\pi_1(X, x_0)$

ONE ELEMENT OF THIS SET IS THE CLASS OF CONSTANT PATH AT x_0, $[e]$.

WILL DESCRIBE A BASIS FOR THE TOPOLOGY ON $\tilde{X}$.

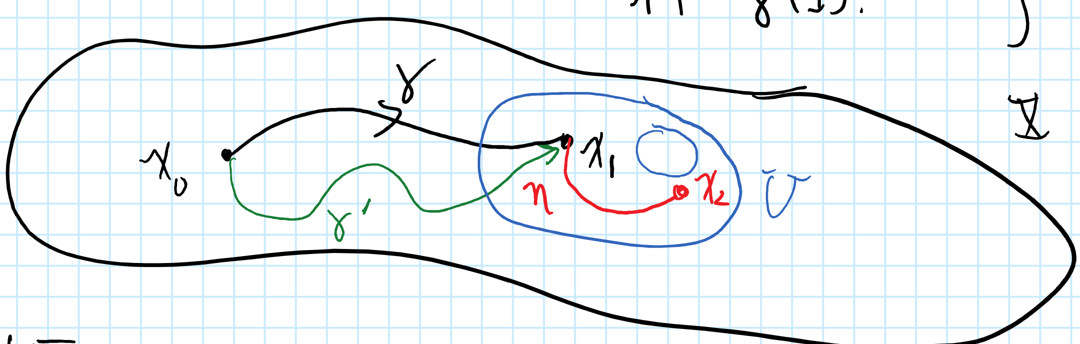
LET $\mathcal{U} = \left\{ U \subset X : \begin{array}{l} U \text{ IS OPEN} \\ U \text{ IS PC} \\ \pi_1(U) \to \pi_1(X) \\ \text{IS TRIVIAL} \end{array} \right\}$

BECAUSE X IS LPC AND SLSC,

$\mathcal{U}$ IS A BASIS FOR THE TOPOLOGY ON X.

DEFINE SUBSETS OF $\tilde{X}$ AS FOLLOWS. CHOOSE $[\gamma] \in \tilde{X}$ AND A NEIGHBORHOOD U OF $\gamma(1)$ THAT IS IN $\mathcal{U}$.

LET $U_{[\gamma]} = \{ [\gamma * \eta]$ FOR η A PATH IN U STARTING AT $\gamma(1)$. $\}$



THE COLLECTION OF ALL SUCH $U_{[\gamma]} \subset \tilde{X}$ IS THE BASIS OF A TOPOLOGY. SEE HATCHER ~~pp~~ 64–65.

UNDER THIS TOPOLOGY, $\tilde{X} \xrightarrow{p} X$ IS CONTINUOUS.

REMARKS

1) $\tilde{X} \xrightarrow{p} X$ IS ONTO SINCE X IS P.C.

2) p SENDS $U_{[\gamma]}$ TO U

BECAUSE U IS PATH CONNECTED

3) THE MAP $U_{[\gamma]} \xrightarrow{p} U$ IS 1-1

BECAUSE $\pi_1(U)$ MAPS TRIVIALLY TO $\pi_1(X)$.

WHY p IS A COVERING:

FOR U AS ABOVE

$$p^{-1}(U) = \bigcup_{\gamma} U_{[\gamma]}, \text{ WHERE } \gamma$$

CAN ANY PATH FROM x_0 TO SOME POINT IN U.

WE WANT TO SAY THIS IS $U \times D$ FOR DISCRETE.

THIS D IS $\pi_1(X)$ BECAUSE GIVEN ANY OTHER PATH γ' FROM x_0 TO x_1, $[\gamma * \bar{\gamma}^*] \in \pi_1(X, x_0)$

SO THE SET OF SUCH $[\gamma]$ IS ISO TO $\pi_1(X, x_0)$.

WHY $\tilde{X}$ IS SIMPLY CONNECTED:

A CLOSED PATH $\tilde{\gamma}$ IN $\tilde{X}$ IS

A CLOSED PATH $\tilde{\gamma} : I \longrightarrow (\tilde{X}, [\tilde{x}_0])$

$$I^2 \xrightarrow{\hat{\gamma}} X$$

$$(s, t) \longmapsto \gamma(t)(s) \in X$$

RED $\longrightarrow x_0$

$\hat{h}(s, t, u) = \hat{\gamma}(s, ut)$

$s, t, u \in [0, 1]$

$\hat{h}$ DEFINES A HOMOTOPY BETWEEN $\hat{\gamma}$ AND CONSTANT

WHERE $\tilde{x}_0$ IS THE CONSTANT PATH x_0

WANT TO SHOW γ IS HOMOTOPIC TO CONSTANT PATH IN $\tilde{X}$.

$\therefore \exists$ HOMOTOPY $h: I \times I \xrightarrow{h} \tilde{X}$ CORRESPONDS TO $I^3 \xrightarrow{\hat{h}} X$ DEFINED ABOVE.

QED