

# CUP PRODUCTS

GIVEN  $\alpha \in H^i(X)$ ,  $\beta \in H^j(X)$ , THERE IS A CLASS  $\alpha\beta \in H^{i+j}(X)$ .

## FORMAL PROPERTIES

1) LINEAR IN EACH FACTOR, I.E.

$$(\alpha_1 + \alpha_2)\beta = \alpha_1\beta + \alpha_2\beta$$

$$\alpha(\beta_1 + \beta_2) = \alpha\beta_1 + \alpha\beta_2$$

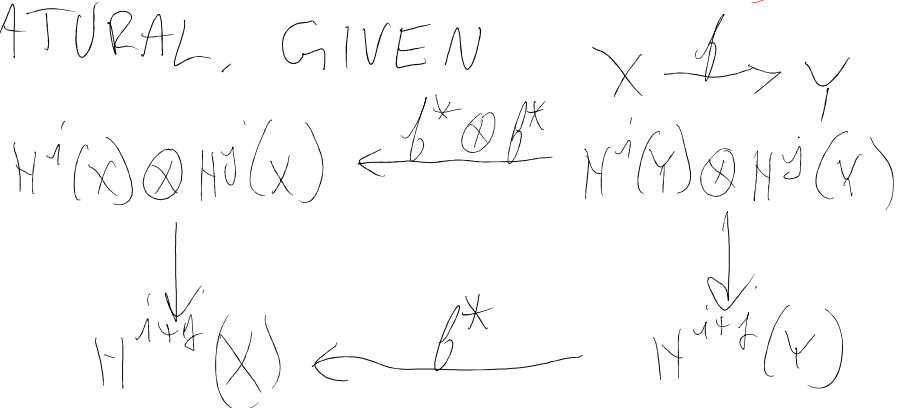
2) ASSOCIATIVE  $(\alpha\beta)\gamma = \alpha(\beta\gamma)$

3) COMMUTATIVE UP TO SIGN

$$\beta\alpha = (-1)^{ij} \alpha\beta$$

POSITIVE UNLESS  $i$  AND  $j$  ARE BOTH ODD

4) NATURAL. GIVEN



COMMUTES.

HOW IS IT CONSTRUCTED?

RECALL FOR SPACES  $X$  AND  $Y$ , THERE IS A SHORT EXACT SEQUENCE

$$0 \rightarrow \bigoplus_{0 \leq i \leq n} H^i(X) \otimes H^{n-i}(Y) \xrightarrow{\alpha} H^n(X \times Y) \rightarrow \bigoplus \text{Tor}(\ ) \rightarrow 0$$

WE HAVE THE DIAGONAL MAP

$$\begin{array}{ccc} X & \xrightarrow{\Delta} & X \times X \\ x & \longmapsto & (x, x) \end{array}$$

$$\begin{array}{ccc} H^{i+j}(X) & \xleftarrow{\Delta^*} & H^{i+j}(X \times X) \xleftarrow{\alpha} H^i(X) \otimes H^j(X) \\ \uparrow & \text{CUP PRODUCT} & \uparrow \end{array}$$

THIS HOLDS FOR  $H^*(X; R)$   
FOR ANY COMM RING  $R$

EXAMPLE  $\rightarrow$

$$1) \text{ RECALL } H^i(\mathbb{R}P^n; \mathbb{Z}/2) = \begin{cases} \mathbb{Z}/2 & \text{FOR } 0 \leq i \leq n \\ 0 & \text{FOR } i > n \end{cases}$$

LET  $x \in H^1$  BE THE GENERATOR  
THEN  $x^i \in H^i$  IS THE GENERATOR  
FOR  $0 \leq i \leq n$ .

THE GRADED RING

$$H^*(\mathbb{R}P^n; \mathbb{Z}/2) \cong \bigoplus_{i \geq 0} H^i(\mathbb{R}P^n; \mathbb{Z}/2)$$

$$= \mathbb{Z}/2[x] / (x^{n+1})$$

TRUNCATED POLYNOMIAL RING  
OF HEIGHT  $n$

## OF HEIGHT $n$ .

2) SIMILAR STATEMENT FOR  $\mathbb{C}P^n$

RECALL

$$H^i(\mathbb{C}P^n; \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{FOR } i \text{ EVEN} \\ & \text{AND } 0 \leq i \leq 2n \\ 0 & \text{ELSE} \end{cases}$$

LET  $x$  BE A GENERATOR OF  $H^2$   
THEN  $x^i$  GENERATES  $H^{2i}$  FOR  
 $0 \leq i \leq n$ .

3) LET  $k$  BE A FIELD. THEN

$$H^*(X \times Y; k) = H^*(X; k) \otimes_{\mathbb{Z}} H^*(Y; k)$$

AS GRADED RINGS

COROLLARY  $\mathbb{C}P^2$  AND  $S^2 \vee S^4$

ARE NOT HOMOTOPY EQUIVALENT

PROOF NOTE

$$H^i(\mathbb{C}P^2) = H^i(S^2 \vee S^4) = \begin{cases} \mathbb{Z} & \text{FOR } i=0,2,4 \\ 0 & \text{ELSE} \end{cases}$$

WE KNOW THERE IS A GENERATOR  
 $x \in H^2 \mathbb{C}P^2 \Rightarrow$  SUCH THAT  
 $x^2$  GENERATES  $H^4 \mathbb{C}P^2$ .

LET  $\gamma_2 \in H^2(S^2 \vee S^4)$  AND  $\gamma_4 \in H^4(S^2 \vee S^4)$   
 BE GENERATORS. CLAIM  $\gamma_2^2 = 0$   
 SO  $\gamma_4 \neq \gamma_2^2$ .

EACH SPACE HAS A CW-STRUCTURE  
 WITH A SINGLE CELL IN  
 DIMENSIONS 0, 2 AND 4.

THERE ARE MAPS

$$\begin{array}{ccccc}
 S^2 & \longrightarrow & S^2 \vee S^4 & \longrightarrow & S^2 \\
 & & \longleftarrow & & \uparrow \\
 & & \text{IDENTITY} & & 
 \end{array}$$

$$H^2 S^2 \ni \gamma_2 \longleftarrow \gamma_2 \longleftarrow \gamma_2 \in H^2 S^2$$

$$0 = \gamma_2^2 \longleftarrow 0 = \gamma_2^2 \in H^4 S^2 = 0$$

MORE INFO ABOUT THESE SPACES  
 BOTH ARE OBTAINED FROM  $S^2$   
 BY ATTACHING A 4-CELL VIA  
 A MAP  $S^3 \rightarrow S^2$

FOR  $S^2 \vee S^4$ , THIS MAP IS CONSTANT

FOR  $\mathbb{C}P^2$  IT IS THE HOPF MAP

$$\mathbb{C}^2 \supset S^3 \xrightarrow{\eta} S^2 = \mathbb{C} \cup \{\infty\}$$

$$(z_1, z_2) \mapsto \begin{cases} \infty \\ \dots \end{cases} \quad \text{FOR } z_2 = 0$$

$\mathbb{Z}, \mathbb{Z}_2$  FOR  $\mathbb{Z}_2 \neq 0$

THE HOMOTOPICAL DISTINCTION  
BETWEEN  $\mathbb{C}P^2$  AND  $S^2 \vee S^4$   
PROVES THAT  $\eta$  IS NOT NULL  
HOMOTOPIC.

MOREOVER, LET  $S^3 \xrightarrow{f} S^2$   
BE ANY MAP. USE IT TO  
ATTACH A 4-CELL TO  $S^2$   
TO GET A SPACE  $X$  WITH

$$H^i X = \begin{cases} \mathbb{Z} & \text{FOR } i=0, 2, 4 \\ 0 & \text{ELSE} \end{cases}$$

WHAT IS ITS CUP PRODUCT  
STRUCTURE? LET  $\chi_2$  AND  
 $\chi_4$  GENERATE  $H^2$  AND  $H^4$   
THEN  $\chi_2^2 = k \chi_4$  FOR SOME  
 $k \in \mathbb{Z}$ . ( $k$  IS WELL DEFINED  
UP TO SIGN)

$k$  IS CALLED THE HOPF  
INVARIANT OF  $f$ ,  $HI(f)$

INVARIANTS OF  $f$ ,  $HI(f)$   
 $HI(\eta) = 1$  AND  $HI(\text{constant}) = 0$

WE CAN MAKE THE SAME  
DEFINITION FOR A MAP

$S^{4m-1} \xrightarrow{f} S^{2m}$   $HI(f)$  IS  
DEFINED IN TERMS OF  
THE PRODUCT IN  $H^*X$ , WHERE  
IS THE MAPPING CONE OF  $f$ .

SUPPOSE WE HAVE A MAP

$S^{4m+1} \xrightarrow{f} S^{2m+1}$  FOR  $m > 0$

WITH MAPPING CONE  $X$ . THEN

$$H^i(X) = \begin{cases} \mathbb{Z} & \text{FOR } i = 0, 2m+1, 4m+2 \\ 0 & \text{ELSE} \end{cases}$$

LET  $\chi_{2m+1}$  AND  $\chi_{4m+2}$  GENERATE  
 $H^{2m+1}$  AND  $H^{4m+2}$ , SO

$$\chi_{2m+1}^2 = R \chi_{4m+2}$$

CLAIM  $R = 0$  IN ALL CASES

BY THE SIGN RULE

$$\begin{aligned}\chi_{2m+1}^2 &= (-1)^{(2m+1)^2} \chi_{2m+1}^2 \\ &= -\chi_{2m+1}^2\end{aligned}$$

SO  $2\chi_{2m+1}^2 = 0$ . SINCE

$\chi_{2m+1}^2 \in \mathbb{H}^{4m+2} X = \mathbb{Z}$ , IT MUST  
BE ZERO.

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MORE ABOUT THE HOPF INVARIANT

RECALL:  $\eta: S^3 \rightarrow S^2$ ,  $\forall x \in S^2$ ,

$\eta^{-1}(x) \cong S^1 \subset S^3$ . ANY 2 SUCH

CIRCLES ARE LINKED.

HOW IS LINKING DEFINED?

GIVEN TWO CIRCLES IN

$S^3$  OR  $\mathbb{R}^3$ , SUPPOSE ONE

BOUNDS A DISK IN  $S^3$

THE SECOND CIRCLE WILL

INTERSECT THAT DISK IN

SOME # OF POINTS. THESE

CAN BE COUNTED ALGEBRAICALLY  
THE # IS THE LINKING  
NUMBER OF THE TWO CIRCLES.

FOR A SMOOTH MAP

$$S^3 \xrightarrow{f} S^2 \supset W$$

THEN WE KNOW THAT  $S^2$   
HAS AN OPEN DENSE SUBSET  
SUCH THAT  $\forall w \in W$ ,  $f^{-1}(w)$   
IS A FINITE UNION OF CIRCLES IN  
 $S^3$ , SO WE GET A LINKING  
# FOR TWO POINTS IN  $W$ .

IT IS THE SAME FOR ANY  
TWO POINTS IN  $W$ .

THEOREM (HOFF ~1930?)

THIS LINKING # IS  $HI(f)$ .

HIGHER DIMENSIONAL  
GENERALIZATION

$$S^{4m-1} \xrightarrow{f} S^{2m} \quad m \geq 0$$



$\bigcup$   
 $W$  OPEN DENSE

FOR ANY  $w \in W$ ,  $f^{-1}(w)$  IS  
 A FINITE DISJOINT UNION OF  
 $S^{m-1}$  OR  $(m-1)$ -MANIFOLDS  
 A LINKING # CAN BE  
 DEFINED AS IN THE CASE  
 $m=1$ . IT EQUALS THE  
 HOPF INVARIANT AS DEFINED  
 IN TERMS OF CUP PRODUCT  
 ABOVE.

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DEEP QUESTION:

WHAT VALUES CAN  $HI(\beta)$   
 HAVE?

EXAMPLE OF A MAP

$$S^{4m-1} \xrightarrow{\beta} S^{2m} \quad \text{WITH } HI(\beta) = 2.$$

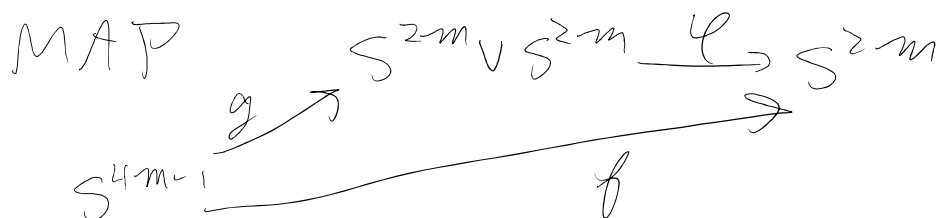
CONSIDER THE SPACE

$S^{2m} \times S^{2m}$ . IT HAS A  
 CW-STRUCTURE WITH  
 TWO  $2m$ -CELLS AND

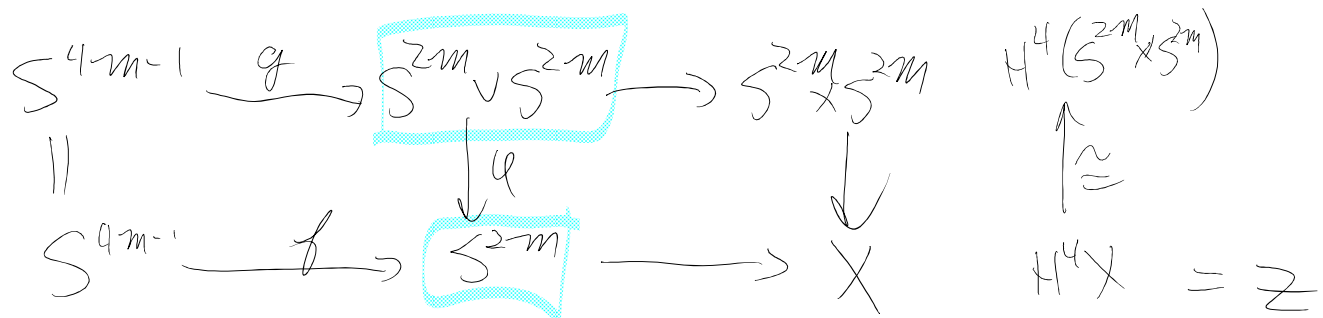
ONE  $4m$ -CELL  
 WITH AN ATTACHING MAP  
 $S^{4m-1} \xrightarrow{g} S^{2m} \vee S^{2m}$

EXERCISE: DESCRIBE  $g$   
 EXPLICITLY.

COMPOSE  $g$  WITH THE FOLD



CLAIM  $H_1(f) = \mathbb{Z}$



LET  $x \in H^{2m}(S^{2m})$

$x', x'' \in H^{2m}(S^{2m} \times S^{2m})$

BE GENERATORS WITH

$\varphi^*(x) = x' + x''$

$x' x'' \in H^{4m} S^{2m} \times S^{2m}$

GENERATOR

$(x')^2 = (x'')^2 = 0$

$$(x' + x'')^2 = (x')^2 + 2x'x'' + (x'')^2 = 2x'x''$$

THE CLAIM FOLLOWS.