CUP PRODUCTS

GIVEN $\alpha \in H^i(X)$, $\beta \in H^j(X)$, THERE IS A CLASS $\alpha\beta \in H^{i+j}(X)$.

FORMAL PROPERTIES

1) LINEAR IN EACH FACTOR, IE

$$(\alpha_1 + \alpha_2)\beta = \alpha_1\beta + \alpha_2\beta$$
$$\alpha(\beta_1 + \beta_2) = \alpha\beta_1 + \alpha\beta_2$$

2) ASSOCIATIVE $(\alpha\beta)\gamma = \alpha(\beta\gamma)$

3) COMMUTATIVE UP TO SIGN

$$\beta\alpha = (-1)^{ij}\alpha\beta$$

POSITIVE UNLESS i AND j ARE BOTH ODD

4) NATURAL, GIVEN $X \xrightarrow{f} Y$

$$\begin{array}{ccc} H^i(X) \otimes H^j(X) & \xleftarrow{f^* \otimes f^*} & H^i(Y) \otimes H^j(Y) \\ \downarrow & & \downarrow \\ H^{i+j}(X) & \xleftarrow{f^*} & H^{i+j}(Y) \end{array}$$

COMMUTES.

HOW IS IT CONSTRUCTED?

RECALL FOR SPACES X AND Y, THERE IS A SHORT EXACT SEQUENCE

$$0 \to \bigoplus_{0 \le i \le n} H^i X \otimes H^{n-i} Y \xrightarrow{\alpha} H^n(X \times Y) \to \bigoplus \mathrm{Tor}(\ ) \to 0$$

WE HAVE THE DIAGONAL MAP

$$X \xrightarrow{\Delta} X \times X$$
$$x \longmapsto (x, x)$$

$$\begin{tikzcd} H^{i+j} X & H^{i+j}(X \times X) \arrow[l, "\Delta^*"'] & H^i(X) \otimes H^j(X) \arrow[l, "\alpha"'] \arrow[ll, bend left, "\text{CUP PRODUCT}"] \end{tikzcd}$$

THE HOLDS FOR $H^*(X; R)$

FOR ANY COMM RING R

EXAMPLES

1) RECALL $H^i(\mathbb{RP}^n; \mathbb{Z}/2) = \begin{cases} \mathbb{Z}/2 & \text{FOR } 0 \le i \le n \\ 0 & \text{FOR } i > n \end{cases}$

LET $x \in H^1$ BE THE GENERATOR

THEN $x^i \in H^i$ IS THE GENERATOR

FOR $0 \le i \le n$.

THE GRADED RING

$$H^*(\mathbb{RP}^n; \mathbb{Z}/2) := \bigoplus_{i \ge 0} H^i(\mathbb{RP}^n; \mathbb{Z}/2)$$

$$= \mathbb{Z}/2[x] / (x^{n+1})$$

TRUNCATED POLYNOMIAL RING

OF HEIGHT n

OF HEIGHT n.

2) SIMILAR STATEMENT FOR $\mathbb{C}P^n$

RECALL

$$H^i(\mathbb{C}P^n;\mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{FOR } i \text{ EVEN AND } 0 \le i \le 2n \\ 0 & \text{ELSE} \end{cases}$$

LET x BE A GENERATOR OF H^2 THEN x^i GENERATES H^{2i} FOR $0 \le i \le n$.

3) LET k BE A FIELD. THEN

$$H^*(X \times Y; k) = H^*(X;k) \otimes_k H^*(Y;k)$$

AS GRADED RINGS

<u>COROLLARY</u> $\mathbb{C}P^2$ AND $S^2 \vee S^4$ ARE <u>NOT</u> HOMOTOPY EQUIVALENT

<u>PROOF</u> NOTE

$$H^i(\mathbb{C}P^2) = H^i(S^2 \vee S^4) = \begin{cases} \mathbb{Z} & \text{FOR } i = 0,2,4 \\ 0 & \text{ELSE} \end{cases}$$

WE KNOW THERE IS A GENERATOR $x \in H^2\mathbb{C}P^2 = \mathbb{Z}$ SUCH THAT x^2 GENERATES $H^4\mathbb{C}P^2$.

LET $y_2 \in H^2(S^2 \vee S^4)$ AND $y_4 \in H^4(S^2 \vee S^4)$ BE GENERATORS. CLAIM $y_2^2 = 0$ SO $y_4 \neq y_2^2$.

EACH SPACE HAS A CW-STRUCTURE WITH A SINGLE CELL IN DIMENSIONS 0, 2 AND 4.

THERE ARE MAPS

$$\begin{tikzcd} S^2 \arrow[r] \arrow[rr, bend right, "\text{IDENTITY}"'] & S^2 \vee S^4 \arrow[r] & S^2 \end{tikzcd}$$

$$H^2S^2 \ni y_2 \longleftarrow\!\shortmid\; y_2 \longleftarrow\!\shortmid\; y_2 \in H^2 S^2$$

$$0 = y_2^2 \longleftarrow\!\shortmid\; 0 = y_2^2 \in H^4 S^2 = 0$$

MORE INFO ABOUT THESE SPACES

BOTH ARE OBTAINED FROM S^2 BY ATTACHING A 4-CELL VIA A MAP $S^3 \longrightarrow S^2$

FOR $S^2 \vee S^4$, THIS MAP IS CONSTANT

FOR $\mathbb{C}P^2$ IT IS THE HOPF MAP

$$\mathbb{C}^2 \supset S^3 \xrightarrow{\eta} S^2 = \mathbb{C} \cup \{\infty\}$$

$$(z_1, z_2) \longmapsto \begin{cases} \infty, & \text{FOR } z_2 = 0 \end{cases}$$

$\ldots$ $(z_1/z_2$ FOR $z_2 \neq 0$

THE HOMOTOPICAL DISTINCTION BETWEEN $\mathbb{C}P^2$ AND $S^2 \vee S^4$ PROVES THAT η IS NOT NULL HOMOTOPIC.

MOREOVER, LET $S^3 \xrightarrow{f} S^2$ BE ANY MAP. USE IT TO ATTACH A 4-CELL TO S^2 TO GET A SPACE X WITH

$$H^i X = \begin{cases} \mathbb{Z} & \text{FOR } i = 0, 2, 4 \\ 0 & \text{ELSE} \end{cases}$$

WHAT IS ITS CUP PRODUCT STRUCTURE? LET x_2 AND x_4 GENERATE H^2 AND H^4. THEN $x_2^2 = k x_4$ FOR SOME $k \in \mathbb{Z}$. (k IS WELL DEFINED UP TO SIGN)

k IS CALLED THE <u>HOPF INVARIANT OF f</u>, $HI(f)$

INVARIANT OF f, $HI(f)$

$HI(\eta) = 1$ AND $HI(\text{constant}) = 0$

WE CAN MAKE THE SAME DEFINITION FOR A MAP

$S^{4m-1} \xrightarrow{f} S^{2m}$ $HI(f)$ IS DEFINED IN TERMS OF THE PRODUCT IN $H^* X$, WHERE IS THE MAPPING CONE OF f.

SUPPOSE WE HAVE A MAP

$S^{4m+1} \xrightarrow{f} S^{2m+1}$ FOR $m > 0$

WITH MAPPING CONE X. THEN

$$H^i(X) = \begin{cases} \mathbb{Z} & \text{FOR } i = 0, 2m+1, 4m+2 \\ 0 & \text{ELSE} \end{cases}$$

LET x_{2m+1} AND x_{4m+2} GENERATE H^{2m+1} AND H^{4m+2}, SO

$$x_{2m+1}^2 = k\, x_{4m+2}$$

CLAIM $k = 0$ IN ALL CASES.

BY THE SIGN RULE

$$x_{2m+1}^2 = (-1)^{(2m+1)^2} x_{2m+1}^2$$

$$= -x_{2m+1}^2$$

SO $2x_{2m+1}^2 = 0$. SINCE $x_{2m+1}^2 \in H^{4m+2}X = \mathbb{Z}$, IT MUST BE ZERO.

MORE ABOUT THE HOPF INVARIANT

RECALL: $\eta: S^3 \longrightarrow S^2$, $\forall\, x \in S^2$, $\eta^{-1}(x) \approx S^1 \subset S^3$. ANY 2 SUCH CIRCLES ARE LINKED.

HOW IS LINKING DEFINED? GIVEN TWO CIRCLES IN S^3 OR $\mathbb{R}^3$, SUPPOSE ONE BOUNDS A DISK IN S^3. THE SECOND CIRCLE WILL INTERSECT THAT DISK IN SOME # OF POINTS. THESE

CAN BE COUNTED ALGEBRAICALLY THE # IS THE LINKING NUMBER OF THE TWO CIRCLES.

FOR A SMOOTH MAP

$$S^3 \xrightarrow{f} S^2 \supset W$$

THEN WE KNOW THAT S^2 HAS AN OPEN DENSE SUBSET SUCH THAT $\forall\, w \in W$, $f^{-1}(w)$ IS A FINITE UNION OF CIRCLES IN S^3, SO WE GET A LINKING # FOR TWO POINTS IN W. IT IS THE SAME FOR ANY TWO POINTS IN W.

THEOREM (HOPF ~1930?)

THIS LINKING # IS HI(f).

HIGHER DIMENSIONAL GENERALIZATION

$$S^{4m-1} \xrightarrow{f} S^{2m} \qquad m > 0$$

$\cup$
W OPEN DENSE

FOR ANY $w \in W$, $f^{-1}(w)$ IS A FINITE DISJOINT UNION OF S^{m-1} OR $(m-1)$-MANIFOLDS

A LINKING # CAN BE DEFINED AS IN THE CASE $m=1$. IT EQUALS THE HOPF INVARIANT AS DEFINED IN TERMS OF CUP PRODUCT ABOVE.

<u>DEEP QUESTION:</u>

WHAT VALUES CAN $HI(f)$ HAVE?

EXAMPLE OF A MAP

$S^{4m-1} \xrightarrow{f} S^{2m}$ WITH $HI(f)=2$.

CONSIDER THE SPACE $S^{2m} \times S^{2m}$. IT HAS A CW-STRUCTURE WITH TWO $2m$-CELLS AND

ONE 4m-CELL
WITH AN ATTACHING MAP
$S^{4m-1} \xrightarrow{g} S^{2m} \vee S^{2m}$

EXERCISE: DESCRIBE g EXPLICITLY.

COMPOSE g WITH THE FOLD MAP $S^{2m} \vee S^{2m} \xrightarrow{\varphi} S^{2m}$

$$\begin{tikzcd} & S^{2m} \vee S^{2m} \arrow[r, "\varphi"] & S^{2m} \\ S^{4m-1} \arrow[ur, "g"] \arrow[urr, "f"'] & & \end{tikzcd}$$

CLAIM $HI(f) = 2$

$$\begin{tikzcd} S^{4m-1} \arrow[r, "g"] \arrow[d, equal] & \boxed{S^{2m} \vee S^{2m}} \arrow[r] \arrow[d, "\varphi"] & S^{2m} \times S^{2m} \arrow[d] & H^4(S^{2m} \times S^{2m}) \\ S^{4m-1} \arrow[r, "f"] & \boxed{S^{2m}} \arrow[r] & X & H^4 X = \mathbb{Z} \arrow[u, "\cong"'] \end{tikzcd}$$

LET $x \in H^{2m}(S^{2m})$
$x', x'' \in H^{2m}(S^{2m} \times S^{2m})$
BE GENERATORS WITH
$\varphi^*(x) = x' + x''$

$x' x'' \in H^{4m} S^{2m} \times S^{2m}$ GENERATOR
$(x')^2 = (x'')^2 = 0$

$$(x' + x'')^2 = (x')^2 + 2x'x'' + (x'')^2 = 2x'x''$$

THE CLAIM FOLLOWS.