WHAT WE KNOW FROM LAST TIME

Wednesday, November 4, 2020 1:47 PM

① FOR ANY SPACE X AND ANY ABELIAN GROUP A, THERE IS A SPLIT SES

$$0 \to H_n(X)\otimes A \longrightarrow H_n(X;A) \longrightarrow \mathrm{Tor}_1(H_{n-1}(X), A) \to 0$$

$$H_n(X;A) \cong H_n(X)\otimes A \oplus \mathrm{Tor}_1(H_{n-1}(X), A)$$

BUT THE ISOMORPHISM IS NOT NATURAL.

② SIMILARLY THERE IS A SPLIT SES

$$0 \leftarrow \mathrm{Hom}(H_n(X), A) \longleftarrow H^n(X;A) \longleftarrow \mathrm{Ext}^1(H_{n-1}(X), A) \leftarrow 0$$

③ LET C' AND C'' BE CHAIN COMPLEXES OF FREE ABELIAN GROUPS WITH $C = C'\otimes C''$. THERE IS A SPLIT SES

$$0 \to \bigoplus_{0\le i\le n} H_i(C')\otimes H_{n-i}(C'') \to H_n(C) \to \bigoplus_{0\le i\le n-1} \mathrm{Tor}_1(H_i(C'), H_{n-i-1}(C'')) \to 0$$

THE PROOF IS SIMILAR TO THAT OF ①.

WE WOULD LIKE A SIMILAR STATEMENT ABOUT $H_*(X\times Y)$ FOR SPACES X AND Y.

WE HAVE CHAIN COMPLEXES $C(X)$, $C(Y)$ AND $C(X \times Y)$, BUT $C(X\times Y) \not\cong C(X) \otimes C(Y)$. THERE IS A MAP

$$C(X) \otimes C(Y) \longrightarrow C(X \times Y)$$

WHICH IS KNOWN TO BE A CHAIN HOMOTOPY EQUIVALENCE. THE PROOF IS EXTREMELY BORING.

"METHOD OF ACYCLIC MODELS"

WILL CW-COMPEXES AND CELLULAR CHAIN COMPLEXES

CHANGE IN NOTATION

THE SINGULAR CHAIN COMPLEX OF A SPACE X, FORMERLY DENOTED BY $C(X)$, WILL NOW BE DENOTED BY $S(X)$.

<u>DEF</u> A <u>CW COMPLEX</u> IS A SPACE X CONSTRUCTED AS THE UNION OF SUBSPACES X^n (THE n-SKELETON)

FOR $n \geq 0$ AS FOLLOWS

$K_0 = X^0$ = DISCRETE SET

X^n IS OBTAINED FROM X^{n-1} AS FOLLOWS. THERE IS SET K_n AND A MAP

$i_n : S^{n-1} \hookrightarrow D^n$

$$\begin{tikzcd}
K_n \times S^{n-1} \arrow[r, "f_n"] \arrow[d, "K_n \times i_n"'] & X^{n-1} \arrow[d] \\
K_n \times D^n \arrow[r] & X^n = \text{PUSHOUT}
\end{tikzcd}$$

f_n IS THE ATTACHING MAP OF X^{n-1}

WE ARE ATTACHING SOME n-CELLS (INDEXED BY K_n) TO X^{n-1} TO GET X^n.

MANY FAMILIAR SPACES CAN B BUILT THIS WAY.

EXAMPLE

$S^0 = \{\pm 1\}$

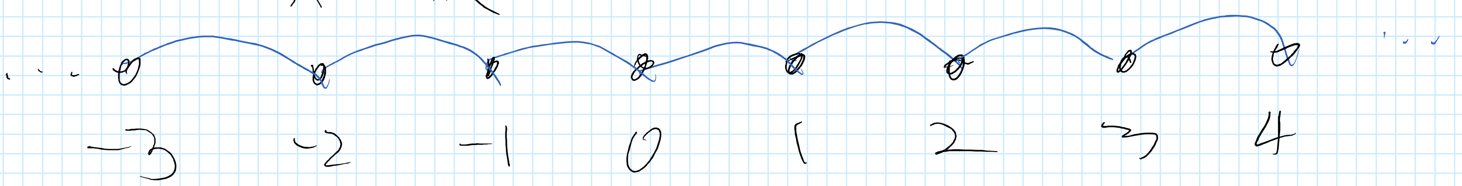
(1) $X = \mathbb{R}$ $\quad X^0 = \mathbb{Z}$

$K_1 = \mathbb{Z}$ $\quad K_1 \times S^0 \xrightarrow{f_1} X^0 = \mathbb{Z}$

$(n, 1) \longmapsto n$

$$(n,-1) \longmapsto n-1$$

$X^1 \approx \mathbb{R}$



CAN CONSTRUCT $\mathbb{R}^d$ IN A SIMILAR WAY.

(2) $X = S^n$ $\qquad X^0 = *$ ONE POINT

$K_i = \emptyset$ FOR $0 < i < n$

$K_n = *$

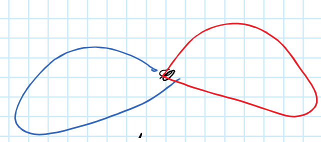
$$\begin{tikzcd} S^{n-1} \arrow[r] \arrow[d, "i"'] & X^0 = pt \arrow[d] \\ D^n \arrow[r] & X^n \approx S^n \end{tikzcd}$$

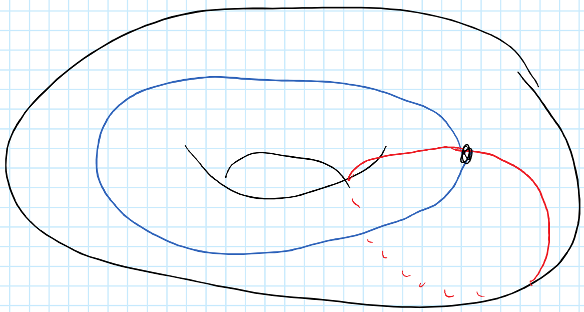
(3) $X = S^1 \times S^1$

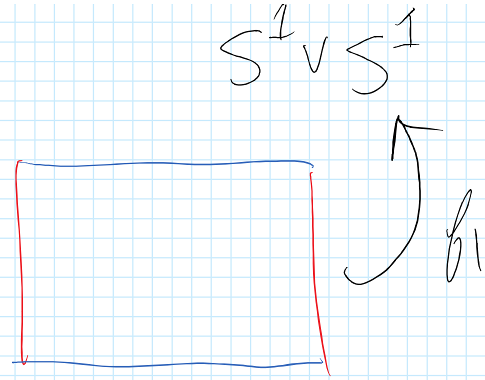
$K_0 = *$

$K_1 = 2$ POINTS

$$\begin{tikzcd} K_1 \times S^0 \arrow[r] \arrow[d] & X^0 = \text{POINT} \arrow[d] \\ K_1 \times D^1 \arrow[r] & X^1 \end{tikzcd}$$





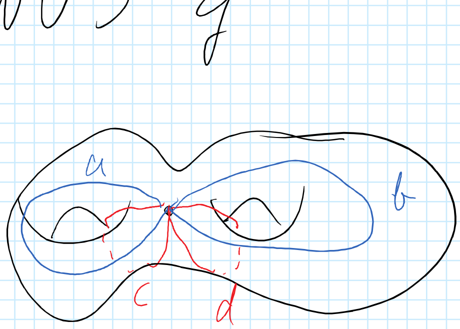


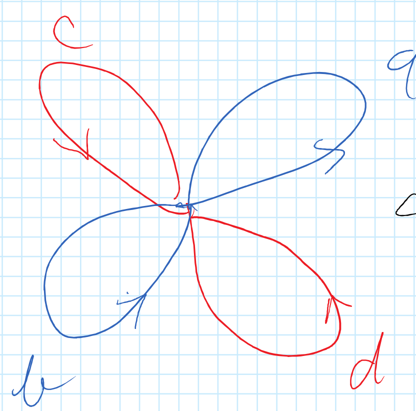
$\partial(I \times I) \approx S^1 = \partial D^2$

1 0-CELL
2 1-CELLS
1 2-CELL

$$\begin{tikzcd} S^1 \arrow[r, "f_1"] \arrow[d] & X^1 = S^1 \vee S^1 \arrow[d] \\ D^2 \arrow[r] & X^2 = \text{TORUS} \end{tikzcd}$$

SIMILARLY A SURFACE OF GENUS g

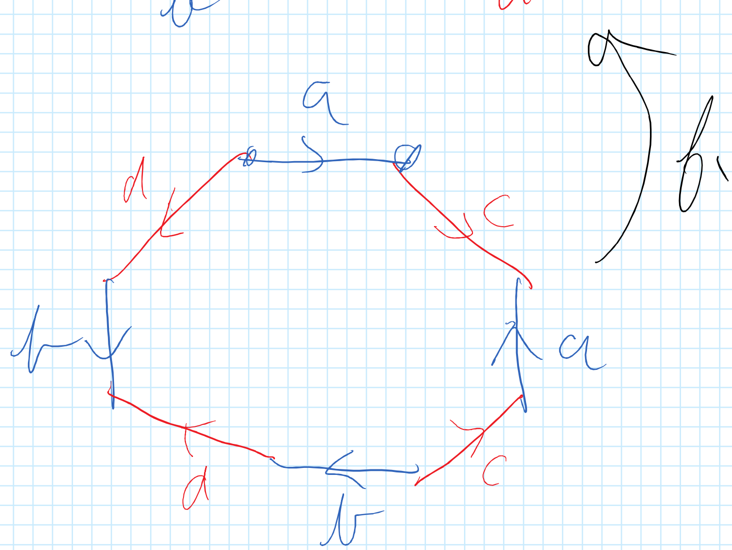




$S^1 \vee S^1 \vee S^1 \vee S^1$

WE GET A CW-COMPLEX WITH

1 0-CELL
2g 1-CELLS
1 2-CELL



DEF LET X BE A CW-COMPLEX AS ABOVE. ITS CELLULAR CHAIN COMPLEX $C(X)$ IS GIVEN

$$C_n(X) = \text{FREE ABELIAN GROUP GENERATED BY THE SET } K_n$$

TO DEFINE $d_n : C_n(X) \to C_{n-1}(X)$,

NOTE $X^n/X^{n-1} \cong \bigvee_{K_n} S^n$ SINGLE 0-CELL $|K_n|$ n-CELLS

CONSIDER

$$\text{(1)} \quad \begin{array}{ccccc} X^{n-1}/X^{n-2} & \longrightarrow & X^n/X^{n-2} & \longrightarrow & X^n/X^{n-1} \\ \cong & & \shortparallel & & \cong \\ \bigvee_{K_{n-1}} S^{n-1} & & ? & & \bigvee_{K_n} S^n \end{array}$$

CELLS IN DIMENSION 0, $n-1$ AND n

LET $\bar{C}(X)$ BE THE KERNEL OF

$$C(X) \longrightarrow C(\text{point})$$

THERE IS A SES C' $\bar{C}$ $\bar{C}''$

$$0 \to \bar{C}(X^{n-1}/X^{n-2}) \to \underbrace{\bar{C}(X^{n}/X^{n-2})}_{} \to \bar{C}(X^{n}/X^{n-1}) \to 0$$

CONCENTRATED IN DIM $n-1$

CONCENTRATED IN DIM n

WE GET A LES IN H_*

$$0 \to H_n \bar{C} \to H_n \bar{C}'' \xrightarrow{\partial} H_{n-1} \bar{C}' \to H_{n-1} \bar{C} \to 0$$

$$H_n \bar{C}'' = C_n(X) \xrightarrow{d_n} C_{n-1}(X) = H_{n-1} \bar{C}'$$

OUR BOUNDARY OPERATOR IS THE CONNECTED

CAN SHOW $d_{n-1} d_n = 0$.

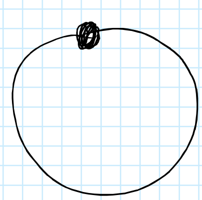
<u>THEOREM</u> FOR ANY CW-COMPLEX,

$$H_* C(X) \cong H_* S(X) =: H_*(X)$$

$C(X)$ IS A MUCH SMALLER SUBSTITUTE FOR $S(X)$.

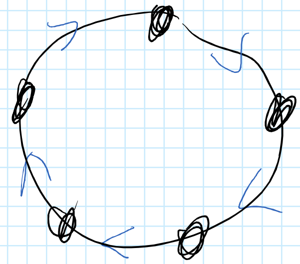
EXAMPLE: A SPACE CAN HAVE MANY CW-STRUCTURES

$X = S^1$



STRUCTURE DESCRIBED ABOVE

5 0-CELLS
5 1-CELLS



THEOREM CAN BE PROVED BY INDUCTION ON SKELETA.

THEOREM LET X AND Y BE CW-COMPLEXES. THEN THERE IS A CW-STRUCTURE ON $X \times Y$ SUCH THAT

$$C(X \times Y) \cong C(X) \otimes C(Y)$$

PROOF. LET $\{K_n, f_n : n \geq 0\}$ AND $\{L_m, g_m : m \geq 0\}$ BE THE CW-DATA FOR X AND Y. THE DATA FOR $X \times Y$ IS $\{M_n, h_n ; n \geq 0\}$

$M_n = \coprod_{0 \leq i \leq n} K_i \times L_{n-i}$. WE NEED AN ATTACHING MAP

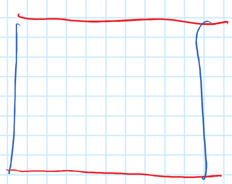
$$M_n \times S^{n-1} \xrightarrow{h_n} (X \times Y)^{n-1}$$

NOTE $D^n \cong D^i \times D^{n-i}$ $\forall$ $0 \leq i < n$

$$\partial D^n \cong (\partial D^i \times D^{n-i}) \cup (D^i \times \partial D^{n-i})$$
$$= (S^{i-1} \times D^{n-i}) \cup (D^i \times S^{n-i-1})$$

e.g. $n=2$ $i=1$

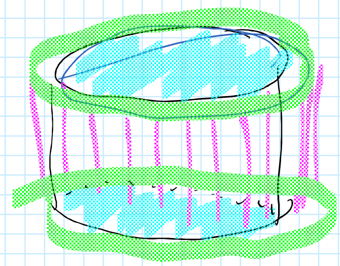
$$\partial D^2 = (S^0 \times D^1) \cup (D^1 \times S^0)$$



$D^1 \times S^0$

$S^0 \times D^1$

$n=3$, $i=1$ $\quad D^3 \cong D^1 \times D^2$



$$\partial D^3 = (\partial D^1 \times D^2) \cup (D^1 \times \partial D^2)$$
$$= (S^0 \times D^2) \cup (D^1 \times S^1)$$

$S^0 \times S^1$

$S^0 \times D^2$

$D^1 \times S^1$

THINK ABOUT THIS
FOR $n=4$, $i=2$