$$D^n \cong D^i \times D^{n-i}$$

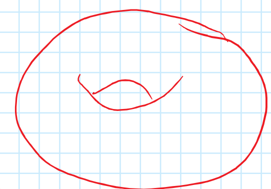
$$S^{n-1} = \partial D^n \cong (\partial D^i \times D^{n-i}) \cup (D^i \times \partial D^{n-i})$$

$S^{i-1} \times S^{n-i-1}$

$n = 4 \quad i = 2$

$$S^3 \cong (\partial D^2 \times D^2) \cup (D^2 \times \partial D^2)$$
$$= (S^1 \times D^2) \cup (D^2 \times S^1)$$

SOLID TORI



THE COMPLEMENT OF $S^1 \times D^2 \subset S^3$ (STANDARD EMBEDDING) IS ALSO $D^2 \times S^1$. THEIR INTERSECTION IS $S^1 \times S^1$

RECALL WE HAVE CW COMPLEXES X AND Y. WE WANT A CW-STRUCTURE ON $X \times Y$ SUCH

$$C(X \times Y) \cong C(X) \otimes C(Y)$$

LET $\{K_i, f_i\}$ AND $\{L_i, g_i\}$ BE CW DATA FOR X AND Y.

K_i, L_i = SETS OF i-CELLS

$f_i : K_i \times S^{i-1} \longrightarrow X^{i-1}$

$g_j : L_j \times S^{j-1} \longrightarrow Y^{j-1}$

WILL DENOTE SUCH DATA FOR $X \times Y$ BY (M_i, h_i)

$$M_n = \coprod_{0 \le i \le n} K_i \times L_{n-i} \qquad h_n : M_n \times S^{n-1} \longrightarrow (X \times Y)^{n-1}$$

$$M_n \times S^{n-1} = \coprod_{0 \le i \le n} K_i \times L_{n-i} \times S^{n-1}$$

$$= \coprod_{0 \le i \le n} K_i \times L_{n-i} \times \left(S^{i-1} \times D^{n-i} \cup D^i \times S^{n-i-1} \right)$$

NOTE $$(X \times Y)^{n-1} = \bigcup_{0 \le i \le n-1} X^i \times Y^{n-i-1}$$

WE NEED A MAP TO IT FROM

$$(K_i \times S^{i-1}) \times L_{n-i} \times D^{n-i} \xrightarrow{f_i \times inc} X^{i-1} \times Y^{n-i}$$

AND

$$(K_i \times D^i) \times L_{n-i} \times S^{n-i-1} \xrightarrow{inc \times g_{n-i}}$$

THE DEFINITIONS IMPLY THAT

$$C(X\times Y)\simeq C(X)\otimes C(Y).$$

QED

SINCE $C(X)$ IS CHAIN HOMOTOPY EQUIVALENT TO $S(X)$, $C(X)\otimes C(Y)\simeq C(X\times Y)$ IS CHE TO $S(X\times Y)$.

HENCE $H_* S(X\times Y)\simeq H_*(S(X)\otimes S(Y))$

WE NEED TO ASSUME X AND Y ARE CW-COMPEXES

LET Y BE ANY SPACE.

WE WILL CONSTRUCT A CW-CX $\hat{Y}$ WITH A MAP $\hat{Y}\xrightarrow{j} Y$

SUCH $H_*(j)$ AND $\pi_*(j)$ ARE ISOMORPHISMS.

WILL DEFINE SIMPLICIAL SETS

LET $[n]=\{0,1,\dots n\}$

Δ = CATEGORY WITH OBJECTS

$[n]$ FOR $n \geq 0$
AND MORPHISMS CONSISTING
OF ORDER PRESERVING MAP.
(SEE ESHT BOOK (ON MY WEB PAGE) PAGE 242

<u>DEF</u> A SIMPICIAL SET X
IS A CONTRAVARIANT FUNCTOR

$$\Delta \longrightarrow Set$$

$$[n] \longmapsto X_n$$

$$\begin{array}{ccc} [n-1] & & X_{n-1} \\ d_i \downarrow & & \uparrow \\ [n] & & X_n \end{array}$$

EXAMPLE

W = ICOSAHEDRON

X_0 = SET OF ITS 12 VERTICES
X_1 = " 30 EDGES
X_2 = " 20 FACES

EACH FACE HAS 3 EDGES
SO WE HAVE 3 MAPS $X_2 \longrightarrow X_1$

CORRESPONDING TO $[1] \xrightarrow{d_i} [2]$

FOR $0 \le i \le 2$

OUR SPACE X IS A QUOTIENT OF

$$\underbrace{\left(X_0 \times \Delta^0\right)}_{\text{VERTICES}} \amalg \underbrace{\left(X_1 \times \Delta^1\right)}_{\text{EDGES}} \amalg \underbrace{\left(X_2 \times \Delta^2\right)}_{\text{FACES}}$$

THE FUNCTOR CONTAINS "GLUING INSTRUCTIONS"

Δ^n = STANDARD n-SIMPLEX

THE RESULTING SPACE $|X| = W$ IS THE GEOMETRIC REALIZATION OF THE SIMPLICIAL SET $X_\bullet$

IT IS A CW-COMPLEX WHERE EACH SIMPLEX IS A CELL.

LET Y BE ANY SPACE

$\hat{Y}$ WILL BE THE GEOMETRIC REALIZATION OF A SIMPLICIAL

SET W, WHERE W_n IS THE SET OF ALL CONTINUOUS MAPS
$\Delta^n \xrightarrow{\sigma} Y$

CAN CHECK THAT THIS LEADS TO A MAP $\hat{Y} = |W| \xrightarrow{\hat{j}} Y$ INDUCED BY THE σ'S ABOVE.
NOTE $\hat{Y} = |W|$ IS A CW-COMPLEX AND $C(\hat{Y}) \cong S(Y)$. THIS FOLLOWS FROM THE DEFINITION.
THIS IMPLIES $H_*(j)$ IS AN ISO AS CLAIMED.

GOOGLE "SIMPLICIAL SETS"

ANOTHER USEFUL SIMPLICIAL SET
LET J BE A SMALL CATEGORY (THE COLLECTION OF OBJECTS IS A SET.)
<u>EXAMPLE</u> LET G BE A GROUP.

<u>EXAMPLE</u> LET G BE A GROUP.
AND LET $\mathcal{B}G$ BE THE CATEGORY WITH ONE OBJECT, AND ONE MORPHISM FOR EACH $\gamma \in G$.

<u>EXAMPLE</u> (WALKING ARROW CATEGORY)
W HAS TWO OBJECTS a AND b AND ONE MORPHISM $a \to b$.

WE ASSOCIATE TO J A SIMPLICIAL SET NJ (THE <u>NERVE</u> OF J)

$(NJ)_n$ = SET OF DIAGRAMS IN J

$$j_0 \to j_1 \to j_1 \to j_2 \cdots \to j_n \in NJ_{n+1}$$

$$j_0 \to j_1 \to \cdots \to j_n \in NJ_n$$

$$\boxed{j_0 \to j_1 \to j_3 \to \cdots j_n} \in NJ_{n-1}$$

$$j_2$$

THE SPACE $|NJ| =: BJ$ IS THE CLASSIFYING SPACE OF J.

WHEN $J = BG$, THE SPACE $|N\,BG| =: BBG = BG$, THE CLASSIFYING SPACE OF G.

IT HAS A LOT OF INTERESTING INFORMATION ABOUT G

EXAMPLE

HOMOTOPY EQUIVALENT TO

① $G = C_2$ $\qquad BG \simeq \mathbb{RP}^\infty = \underset{\overrightarrow{n}}{\mathrm{colim}}\ \mathbb{RP}^n$

② $G = S^1$ $\qquad BG \simeq \mathbb{CP}^\infty = \underset{\overrightarrow{n}}{\mathrm{colim}}\ \mathbb{CP}^n$

ALTERNATE DESCRIPTION OF BG

<u>DEF</u> THE JOIN $X * Y$ IS THE QUOTIENT OF $X \times I \times Y$ $\quad I = [0,1]$

SUBJECT TO

$$(x, 0, y) \sim (x', 0, y) \qquad x, x' \in X$$

$$(x, 1, y) \sim (x, 1, y') \qquad y, y' \in Y$$

EXERCISE: SHOW

$$C_2 * C_2 * \cdots * C_2 \simeq S^n$$

$$\underbrace{S^1 * S^1 * S^1 \cdots S^1}_{n+1 \text{ FACTORS}} \simeq S^{2n+1}$$

TO BE CONTINUED.