<u>DEF</u> A SMOOTH n-MANIFOLD IS A SPACE M IN WHICH EACH $x \in M$ HAS A NBD HOMEOMORPHIC TO $\mathbb{R}^n$ SUCH WHEN TWO SUCH NEIGHORHOODS U_1 AND U_2 OVERLAP WITH $U_1 \cap U_2 = U_{12}$, THEN

$$\begin{tikzcd}
\mathbb{R}^n \supset W_2 & U_1 \cap U_2 \arrow[l, "h_{U_2}"', "\cong"] \arrow[r, "h_{U_1}", "\cong"'] & W_1 \subset \mathbb{R}^n
\arrow[from=1-1, to=1-3, bend right, "h_{U_1} \circ h_{U_2}^{-1} = f_{12}"']
\end{tikzcd}$$

f_{12} AS ABOVE IS A SMOOTH HOMEO (DIFFEOMORPHISM) BETWEEN TWO SUBSETS OF $\mathbb{R}^n$. AT EACH $x \in U_1 \cap U_2$ WE HAVE PARTIAL DERIVATIVES $\frac{\partial f_i}{\partial x_j}$ FOR $1 \le i, j \le n$

WHERE f_i = iTH COORDINATE OF f THESE FORM AN INVERTIBLE $n \times n$ MATRIX, SO WE HAVE

$$U_1 \cap U_2 \longrightarrow GL_n(\mathbb{R}) = \text{INVERTIBLE } n \times n \text{ MATRICES}/\mathbb{R}$$

WE CAN USE THIS DATA TO

DEFINE THE TANGENT BUNDLE OF M. RECALL AN $\mathbb{R}^n$-BUNDLE/X IS A MAP $\xi : E \to X$ S.T.

① EACH $x \in X$ HAS A NBD U WITH $\xi^{-1}(U) \approx \mathbb{R}^n \times U$.

② WHEN TWO SUCH NBDS U_1, U_2 OVERLAP THESE HOMEOMORPHISMS ARE RELATED BY A MAP

$$U_1 \cap U_2 \to GL_n(\mathbb{R}).$$

THESE MAPS DETERMINE ξ.

WHEN $X = M^n$ AS ABOVE, WE HAVE SUCH MAPS AND HENCE AN $\mathbb{R}^n$-BUNDLE $\tau(M)$, THE <u>TANGENT BUNDLE</u> OF M.

WE DO NOT NEED AN EMBEDDING OF M INTO $\mathbb{R}^{n+k}$ TO DEFINE THIS.

USE CONSTRUCTION

LET $G^{\mathbb{R}}_{n,k}$ DENOTE THE SET OF n-DIMENSIONAL SUBSPACES IN $\mathbb{R}^{n+k}$. IT CAN BE TOPOLOGIZED AS FOLLOWS: FOR EVERY OPEN SUBSET $U \subset \mathbb{R}^{n+k}$, THE SET OF n-PLANES INTERSECTING IT IS OPEN IN $G_{n,k}(\mathbb{R})$ = GRASSMANNIAN OF...

G IS GRASSMANN

THEOREM: $G_{n,k}(\mathbb{R})$ IS A COMPACT SMOOTH MANIFOLD OF DIMENSION nk.

EXAMPLE $G_{1,k} = \mathbb{RP}^k$

LET

$$E^{\mathbb{R}}_{n,k} = \{(x,g) \in \mathbb{R}^{n+k} \times G^{\mathbb{R}}_{n,k} : x \in g\}$$

WE HAVE A MAP

$$E^{\mathbb{R}}_{n,k} \longrightarrow G^{\mathbb{R}}_{n,k}$$

$$(x,g) \longmapsto g$$

IT IS AN $\mathbb{R}^n$-BUNDLE / $G^{\mathbb{R}}_{n,k}$

$\gamma^{\mathbb{R}}_{n,k}$ TAUTOLOGICAL OR CANONICAL BUNDLE

DEF SUPPOSE WE HAVE AN $\mathbb{R}^n$-BUNDLE ξ OVER A SPACE Y AND A MAP $f: X \to Y$. THEN WE DEFINE AN $\mathbb{R}^n$-BUNDLE OVER X, $f^*(\xi)$ = BUNDLE INDUCED BY f.

$$\begin{tikzcd} E' \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d, "p"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

PULLBACK

$$E' = \{(x,e) \in X \times E : f(x) = p(e)\}$$

EXAMPLE M^n = n-MANIFOLD WITH EMBEDDING $M^n \xrightarrow{i} \mathbb{R}^{n+k}$

WE GET A MAP

$$\begin{tikzcd} M^n \arrow[r, "f"] & G^{\mathbb{R}}_{n,k} \end{tikzcd}$$

$$x \longmapsto \text{TANGENT } n\text{-PLANE OF } i(x)$$

THEN $\tau(M) \cong f^*\left(\gamma^{\mathbb{R}}_{n,k}\right)$
$\quad\ \ \|$
TANGENT BUNDLE

REMARK $G^{\mathbb{R}}_{n,k} \cong G^{\mathbb{R}}_{k,n}$

CONSIDER

$$\mathbb{R}^n \hookrightarrow \mathbb{R}^{n+1} \hookrightarrow \mathbb{R}^{n+2} \hookrightarrow \cdots \quad \mathbb{R}^\infty$$

$$G^{\mathbb{R}}_{n,0} \xhookrightarrow{f_0} G^{\mathbb{R}}_{n,1} \xhookrightarrow{f_1} G^{\mathbb{R}}_{n,2} \xhookrightarrow{f_2} \cdots \quad G^{\mathbb{R}}_{n,\infty}$$

THE BUNDLES $\gamma^{\mathbb{R}}_{n,k}$ OVER $G^{\mathbb{R}}_{n,k}$ ARE COMPATIBLE UNDER THESE MAPS.

$G^{\mathbb{R}}_{n,\infty}$ = SPACE OF n-PLANES THRU 0 IN $\mathbb{R}^\infty$ AND CONTAINED IN SOME $\mathbb{R}^{n+k}$

= UNION OF ALL $G^{\mathbb{R}}_{n,k}$ FOR $k \geq 0$

IT HAS AN $\mathbb{R}^n$-BUNDLE

$$\gamma_n^{\mathbb{R}} = \gamma_{n,\infty}^{\mathbb{R}}$$

CLASSIFICATION THEOREM FOR $\mathbb{R}^n$-BUNDLES. LET X BE A PARACOMPACT SPACE. LET ξ BE AN $\mathbb{R}^n$-BUNDLE OVER IT. THEN THERE IS A MAP

$$X \xrightarrow{f} G_{n,\infty}^{\mathbb{R}}$$

SUCH THAT $\xi \cong f^*(\gamma_n^{\mathbb{R}})$. TWO SUCH MAPS INDUCE ISOMORPHIC BUNDLES / X IFF THEY ARE HOMOTOPIC.

FOR THIS REASON $G_{n,\infty}^{\mathbb{R}}$ IS CALLED THE CLASSIFYING SPACE FOR $\mathbb{R}^n$-BUNDLES

ALTERNATIVE DESCRIPTION OF $G_{n,\infty}^{\mathbb{R}}$

LET Γ BE A TOPOLOGICAL GROUP, $E_n\Gamma = \underbrace{\Gamma * \Gamma * \cdots \Gamma}_{(n+1)\text{ FACTORS}}$

Γ ACTS FREELY BY MULTIPLICATION

LET $B_n\Gamma$ = ORBIT SPACE OF $E_n\Gamma$.

WE HAVE MAPS

$$\Gamma = E_0\Gamma \longrightarrow E_1\Gamma \longrightarrow E_2\Gamma \longrightarrow \cdots$$

$$* = B_0\Gamma \longrightarrow B_1\Gamma \longrightarrow B_2\Gamma \longrightarrow \cdots$$

LET $B\Gamma = \operatorname{colim} B_n\Gamma$, THE CLASSIFYING SPACE OF Γ.

THEOREM $G^{\mathbb{R}}_{n,\infty}$ IS HOMOTOPY EQUIVALENT TO $BGL_n(\mathbb{R})$ AND TO $BO(n)$ WHERE $O(n) \subset GL_n(\mathbb{R})$ IS THE nTH ORTHOGONAL MATRICES.

QUESTIONS

① WHAT WE SAY ABOUT $BO(n)$?

WE KNOW $H^*(BO(n);\mathbb{Z}/2)$

② GIVEN AN n-MANIFOLD M, THE TANGENT BUNDLE $\tau(M)$ CORRESPONDS TO A MAP

$$M \longrightarrow BO(n)$$

WHAT CAN WE SAY ABOUT IT?

REFERENCE:

CHARACTERISTIC CLASSES BY MILNOR + STASHEFF.

ANOTHER CONSTRUCTION:

LET α AND β BE TWO VECTOR BUNDLES $/X$, OF DIMENSIONS m AND n.

WE HAVE MAPS $E_\alpha, E_\beta \longrightarrow X$.

AND HENCE

$$E_\alpha \times E_\beta \longrightarrow X \times X$$

IT IS AN $\mathbb{R}^{m+n}$-BUNDLE OVER $X \times X$, THE EXTERNAL SUM OF α AND β, $\alpha \times \beta$

$$\begin{tikzcd}
E_{\alpha\oplus\beta} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & E_\alpha \times E_\beta \arrow[d] \\
X \arrow[r, "\Delta"] & X \times X \\
x \arrow[r, mapsto] & (x, x)
\end{tikzcd}$$

DEF THE WHITNEY SUM $\alpha \oplus \beta = \Delta^*(\alpha \times \beta)$. IT IS AN $\mathbb{R}^{m+n}$-BUNDLE / X.

EXAMPLE LET $M^n \xhookrightarrow{i} \mathbb{R}^{n+k}$

$\tau(M)$ = TANGENT $\mathbb{R}^n$- BUNDLE/M.

$\nu_i(M)$ = NORMAL $\mathbb{R}^k$-BUNDLE / M

THEN $\tau(M) \oplus \nu_i(M)$ IS THE TRIVIAL $\mathbb{R}^{n+k}$-BUNDLE / M.

THEOREM

$$H^*(BO(n); \mathbb{Z}/2) \cong \mathbb{Z}/2[w_1, w_2 \dots w_n]$$

$H^*(BO(n);\mathbb{Z}/2) \cong \mathbb{Z}/2[w_1, w_2 \cdots w_n]$

WHERE $w_i \in H^i(-)$.

DEFINITION. LET M BE A SMOOTH n-MANIFOLD. THEN

$H^i(M;\mathbb{Z}/2) \ni w_i(M) = f^*(w_i)$

WHERE $f: M \longrightarrow BO(n)$ CLASSIFIES THE TANGENT BUNDLE.

w_i = iTH STIEFEL-WHITNEY CLASS.

THESE GIVE GOOD INFORMATION ABOUT M.

ABOUT COVERINGS OF

$\mathbb{RP}^2 \vee \mathbb{RP}^2 \quad \pi_1 = C_2 * C_2 = G$

= INFINITE DIHEDRAL

GROUP

G HAS VARIOUS SUBGROUPS

SOME NORMAL, SOME NOT

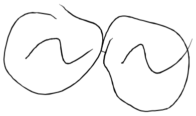
$$0 \longrightarrow \mathbb{Z} \longrightarrow G \longrightarrow C_2 \longrightarrow 0$$

x y

$$yxy = x^{-1}$$

$$G = \langle a, b : a^2 = b^2 = e \rangle$$

$$X = \mathbb{RP}^2 \vee \mathbb{RP}^2 = A \vee B$$

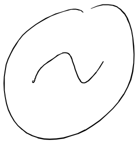


IT HAS A SINGULAR POINT x_0

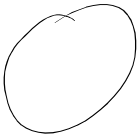
AND IS A "2-MANIFOLD"

AWAY FROM IT

SOME COVERINGS OF X

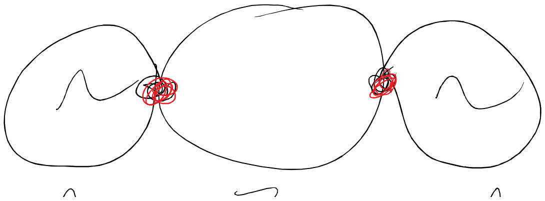


$= \mathbb{RP}^2$



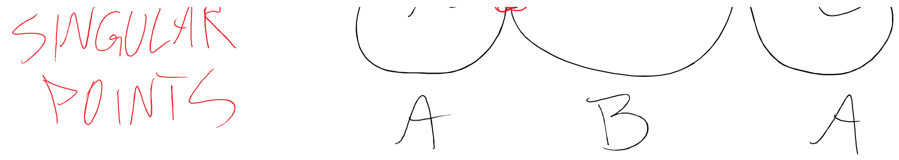
$= S^2$

TWO SINGULAR POINTS



$= \mathbb{RP}^2 \vee S^2 \vee \mathbb{RP}^2$

DOUBLE



$$\Big\downarrow \text{DOUBLE COVERING}$$

$$\mathbb{RP}^2 \vee \mathbb{RP}^2$$

$$\underset{A}{S^2} \vee \underset{B}{S^2} \longrightarrow \mathbb{RP}^2 \vee \mathbb{RP}^2$$

NOT A COVERING

MORE COVERINGS

$$RP^2 \vee \underbrace{S^2 \vee \cdots \vee S^2}_{k \text{ COPIES}} \vee RP^2$$

DEGREE $(k+1)$

$$\underset{A}{RP^2} \vee \underset{B}{S^2} \vee \underset{A}{S^2} \vee \underset{B}{RP^2}$$

DEGREE 3

UNIVERSAL COVER $\tilde{X}$

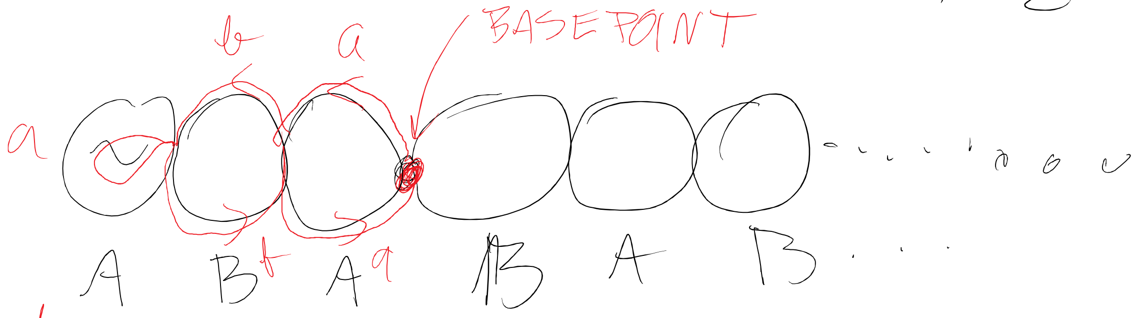
$$\cdots \vee \underset{B}{S^2} \overset{a}{\curvearrowright} \underset{A}{S^2} \underset{\uparrow \text{BASEPOINT}}{\vee} \underset{B}{S^2} \vee \underset{A}{S^2} \vee \underset{B}{S^2} \vee \cdots$$

a ACTS ANTIPODALLY ON ONE S^2 OVER A AND PAIRWISE PERMUTES

OTHER SUMS.

SIMILARLY FOR b.

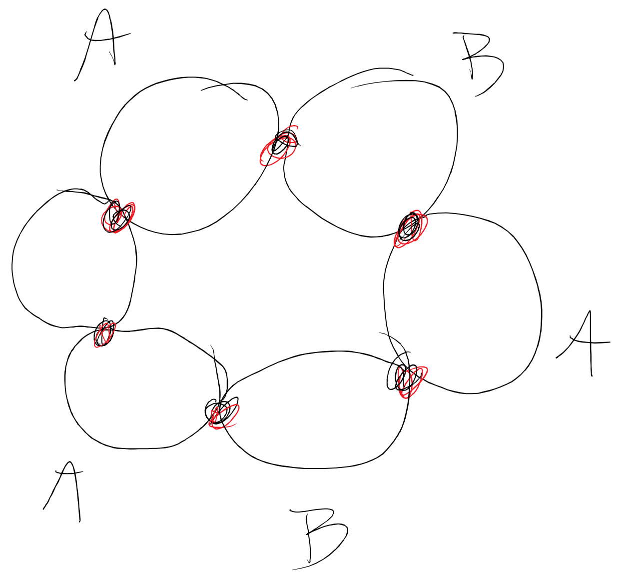
THE ORBIT SPACE $\tilde{X}/C_2$ IS



ababa EG ORDER TWO

FOR $m \geq 1$ WE HAVE

$m=3$



DEGREE 6