FINAL FOR 2019

1. LET $A, B \subset X$ BE FINITE CW-COMPLEXES WITH $A \cup B = X$. $C = A \cap B$

$$\begin{tikzcd} & A \arrow[dr] & \\ C \arrow[ur] \arrow[dr] & & X \\ & B \arrow[ur] & \end{tikzcd}$$

EACH MAP IS THE INCLUSION OF A SUB-CW-COMPLX

$$L = \chi(X) = \chi(A) + \chi(B) - \chi(A \cap B).$$

LET c_i, a_i, b_i AND x_i BE THE # OF i-CELLS IN C, A, B AND X.

$$c_i \le a_i, b_i \le x_i$$

(1) $x_i = a_i + b_i - c_i$

c_i = # OF i-CELLS LYING IN BOTH A AND B.

$$\chi(X) = \sum_{i \ge 0} (-1)^i x_i$$

$$= \sum_{i \ge 0} (-1)^i (a_i + b_i - c_i)$$

$$= \sum (-1)^i a_i + \sum (-1)^i b_i - \sum (-1)^i c_i$$

$$= \chi(A) + \chi(B) - \chi(A \cap B)$$

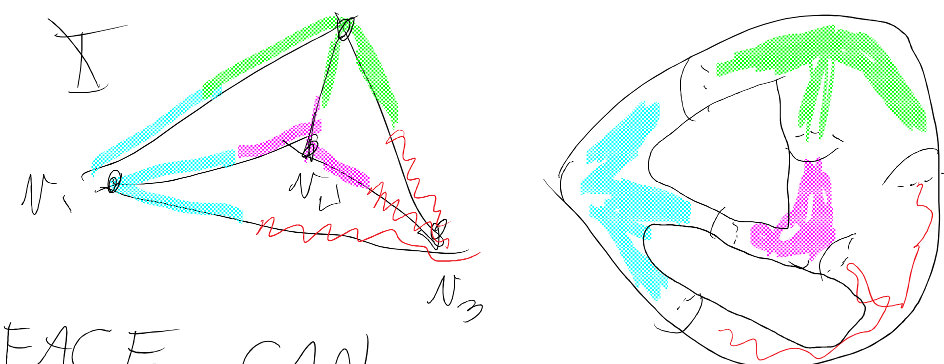
QED

(2) X = GRAPH WITH v VERTICES AND e EDGES

Y = THICKENED IMAGE OF X IN $\mathbb{R}^3$

FIND $\chi(\partial Y)$

EXAMPLE

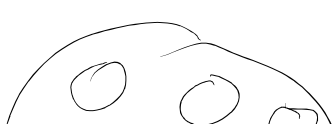


THE SURFACE CAN BE DIVIDED INTO v REGIONS EACH IS HOMEO TO $S^2 - $(SOME DISKS)

THE TOTAL # OF SUCH DISKS IN $2e$.

CLAIM

$$\chi(S^2 - k \text{ DISKS}) = 2 - k$$



$S^2 = (S^2 - k$ DISKS$)$

$S^2 = (S^2 - k \text{ DISKS})$

$\quad \cup \; k \text{ DISKS}$

$= A \cup B$

$A \cap B = k$ CIRCLES

$\chi(A \cap B) = k\chi(S^1) = 0$

$$\chi(S^2) = \chi(A) + \chi(B)$$

$$= \chi(A) + k\chi(D^2)$$

$$\chi(A) = \chi(S^2) - k\chi(D^2)$$

$$= 2 - k$$

IN OUR SURFACE, THE SUM OF THESE EULER CHARACTERISTICS IS

$(2 - k_1) + (2 - k_2) \cdots (2 - k_v)$

$\boxed{k_i = \# \text{ EDGES MEET } i\text{TH VERTEX}}$

$= 2v - \sum k_i = 2v - 2e$

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3. WE HAVE AN INFINITE

$K \subset \mathbb{R}^3$ WITH CERTAIN VERTICES. EACH HAS 8 NEAREST NEIGHBORS

EG THOSE OF $(0,0,0)$ ARE $(\pm 1/2, \pm 1/2, \pm 1/2)$ WITH ALL POSSIBLE SIGN COMBINATIONS.

NEIGHBORS OF (x, y, z) ARE $(x \pm 1/2, y \pm 1/2, z \pm 1/2)$

THE CENTERS OF THOSE EDGES ARE $(x \pm 1/4, y \pm 1/4, z \pm 1/4)$

$G = \mathbb{Z}^3$ ACTS FREELY ON $\mathbb{R}^3$ AND ON K. THICKENING K IN $\mathbb{R}^3$ LEADS TO A BOUNDING SURFACE M ALSO ACTED ON FREELY BY G.

PROBLEM: DESCRIBE K/G AND M/G: BOTH ARE IN $\mathbb{R}^3/G = S^1 \times S^1 \times S^1 = 3\text{-TORUS}$.

ANSWER: THINK OF $\mathbb{R}^3/G$ AS

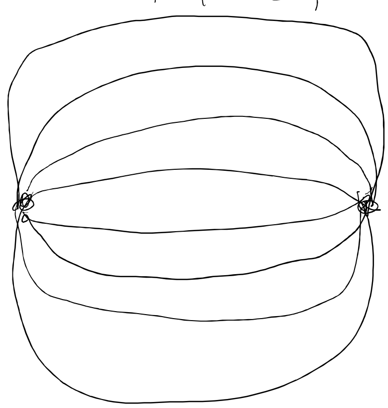
QUOTIENT OF $[0,1]^3$

K/G HAS 2 VERTICES,

$(0,0,0)$ AND $(1/2, 1/2, 1/2)$

CORNER CENTER OF CUBE

THERE ARE 8 EDGES CONNECTING THESE POINTS



$v = 2$

$e = 8$

$\chi(M) = 2v - 2e$

$= 4 - 16 = -12$

$= 2 - 2g$

g = GENUS OF M

$2 - 2g = -12$ SO $g = 7$.

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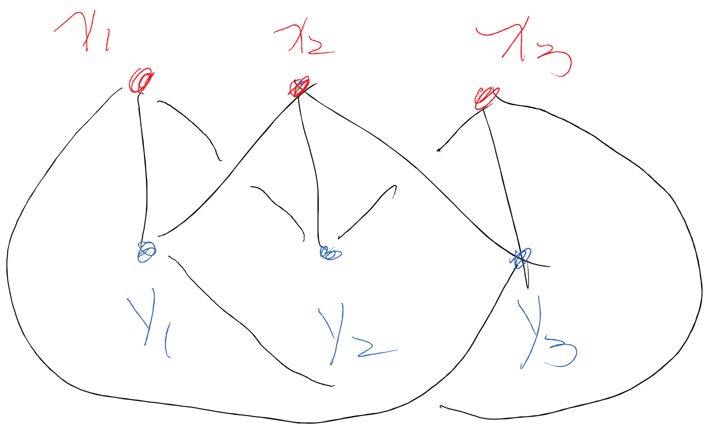
4. SHOW THE H-U GRAPH K IS NON PLANAR.

IF $K \subset S^2$, IT DIVIDES IT INTO FACES, EACH WITH ≥ 4 SIDES



LET f = NUMBER OF FACES

$6 = v = \#$ OF VERTICES

$9 = e =$ " EDGES

$18 = 2e =$ SUM OF # OF EDGES ON EACH FACE

$\geq 4f$

$f \leq \frac{18}{4} = \frac{9}{2}$ SO $f \leq 4$

$2 = \chi = v - e + f = 6 - 9 + f$

$\leq 6 - 9 + 4 = 1$

CONTRADICTION

K IS NOT PLANAR

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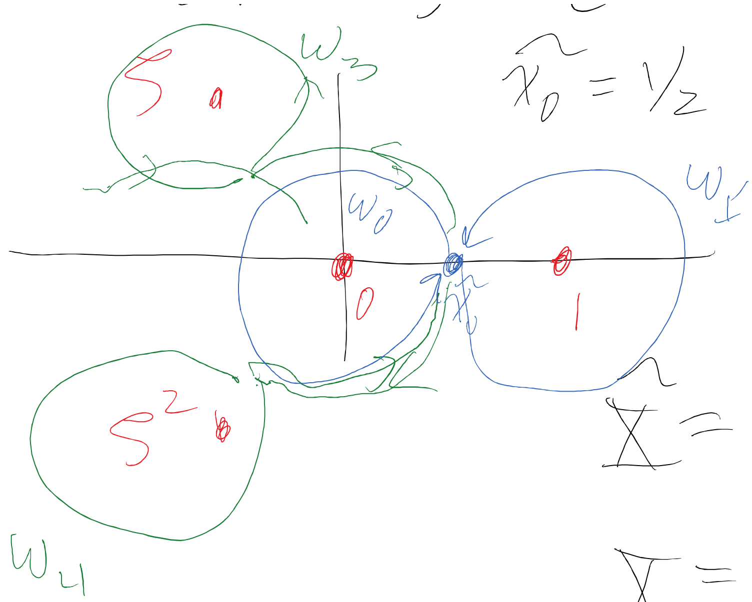
5 LET $\zeta = e^{2\pi i/3} = \frac{-1 + i\sqrt{3}}{2}$



$\tilde{x}_0 = 1/2$

$p(\tilde{x}_0) = 1/8 = x_0 \in X$

$$\tilde{X} = \mathbb{C} - \{0, 1, \zeta, \zeta^2\}$$

$$X = \mathbb{C} - \{0, 1\}$$

THE MAP $\mathbb{C} \longrightarrow \mathbb{C}$ NOT A COVERING

$x \longmapsto x^3$

$\tilde{X} \xrightarrow{\ p\ } X$ TRIPLE COVERING

WE HAVE $\tilde{X} \xrightarrow{\ p\ } X = \mathbb{C} - \{0,1\}$

$\tilde{X} = \mathbb{C} - \{0, 1, \zeta, \zeta^2\}$

$p(x) = x^3$

$$\pi_1(\tilde{X}) = \langle \overset{0}{a_0}, \overset{1}{a_1}, \overset{\zeta}{a_2}, \overset{\zeta^2}{a_3} \rangle$$

$$\pi_1(X) = \langle \overset{0}{x}, \overset{1}{y} \rangle$$

$\tilde{x}_0 = 1/2$ $\quad \pi_1(\tilde{X}, \tilde{x}_0) \xrightarrow{\ p_*\ } \pi_1(X, x_0)$ $\quad x_0 = 1/8$

$a_0 \longmapsto x^3$ $\quad \varphi$

$a_0 \longmapsto x^3$

$a_1 \longmapsto y$

$a_2 \longmapsto xyx^{-1}$

$a_3 \longmapsto x^{-1}yx$

$\longrightarrow C_3 \ni \gamma$

$x \longmapsto \gamma$

$y \longmapsto e$

SO $\quad x^3 \longmapsto \gamma^3 = e$

$x^{-1}yx,\ xyx^{-1} \longmapsto e$

\# 4 FROM 2016

D^2 = UNIT DISK IN $\mathbb{C}$

IDENTIFY $x \in \partial D$ WITH ζx

WHERE $\zeta = e^{2\pi i/5}$

TO GET X. FIND $\pi_1 X$

ALTERNATE DESCRIPTION

SPACES
$$\begin{tikzcd}
S^1 \arrow[r, "\text{5TH POWER}"] \arrow[d] & S^1 \arrow[d] \\
D^2 \arrow[r] & X
\end{tikzcd}$$
S^1 = UNIT CIRCLE IN $\mathbb{C}$

GROUPS
$$\mathbb{Z} \xrightarrow{5} \mathbb{Z}$$

GROUPS

$$
\begin{tikzcd}
\mathbb{Z} \arrow[r, "v"] \arrow[d] \arrow[dr, phantom, "\ulcorner" very near end] & \mathbb{Z} \arrow[d] \\
0 \arrow[r] & \pi_1 X = \mathbb{Z}/5
\end{tikzcd}
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OFFICE HOURS 2-3
FRIDAY

FIND ZOOM LINK ON
MATH 443 PAGE.

FINAL 4-7 PM MONDAY.