Monday, August 31, 2020 2:03 PM

Recall $\pi_1(X, x_0)$ = SET OF HOMOTOPY CLASSES OF CLOSED PATHS IN X STARTED/ENDING AT x_0

I.E. OF MAP

$$(I, \partial I) \longrightarrow (X, x_0)$$

s = HOMOTOPY PARAMETER

t = PATH PARAMETER

WE GET A BINARY OPERATION $*$ ON THIS SET BY

$$(p_1 * p_2)(t) = \begin{cases} p_1(2t) & 0 \le t \le 1/2 \\ p_2(2t-1) & 1/2 \le t \le 1 \end{cases}$$

<u>ASSOCIATIVITY.</u>

NEED TO SHOW $(p_1 * p_2) * p_3 \simeq p_1 * (p_2 * p_3)$

$$((p_1 * p_2) * p_3)(t) = \begin{cases} p_1(4t) & 0 \le t \le 1/4 \\ p_2(4t-1) & 1/4 \le t \le 1/2 \\ p_3(2t-1) & 1/2 \le t \le 1 \end{cases}$$

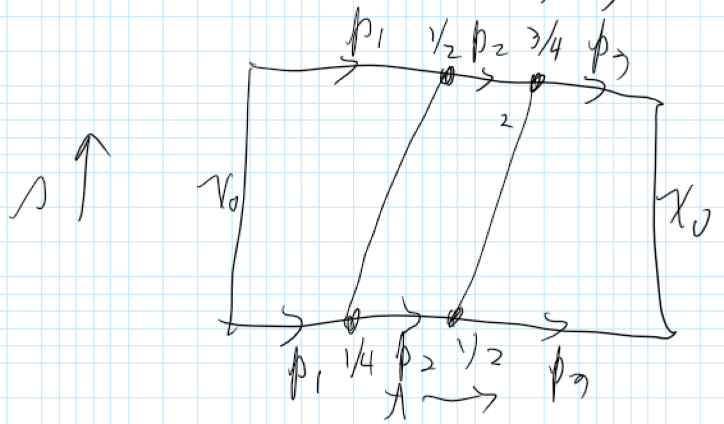
THE HOMOTOPY WE WANT IS A MAP

$I \times I \xrightarrow{h} X$ WITH

$$\left.\begin{aligned} h(0, t) &= ((p_1 * p_2) * p_3)(t) \\ h(1, t) &= (p_1 * (p_2 * p_3))(t) \end{aligned}\right\} \text{FOR ALL } t$$

$h(s, 0) = h(s, 1) = x_0$ FOR ALL s

$h(s,0) = h(s,1) = x_0$ FOR ALL s



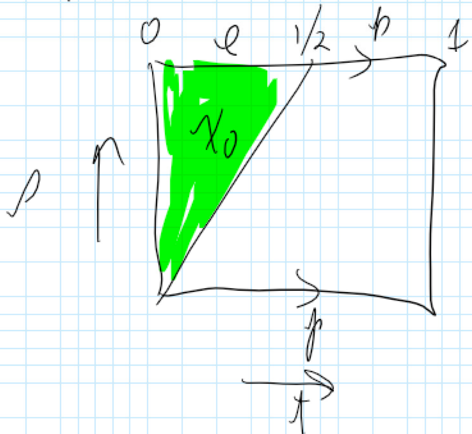
$h(s,t)$ = LINEAR INTERPOLATION $h(0,t)$ AND $h(1,t)$

EXERCISE

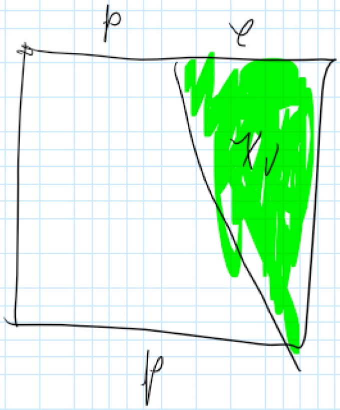
IDENTITY ELEMENT

e = CONSTANT PATH $I \longrightarrow x_0$

WE NEED TO SHOW $e * p \simeq p \simeq p * e$ FOR ANY CLOSED PATH p.



$e * p \simeq p$



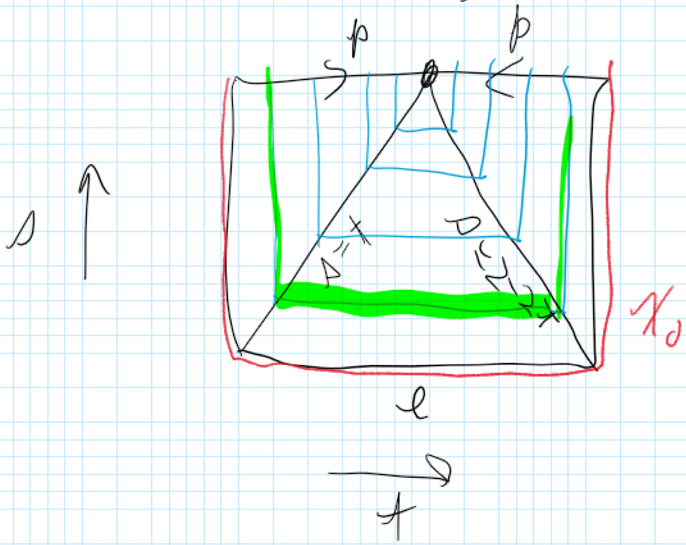
$p * e \simeq p$

INVERSES. GIVEN A CLOSED PATH p, DEFINE $\bar{p}$ BY $\bar{p}(t) = p(1-t)$.

NOTE $\bar{\bar{p}} = p$. NEED TO SHOW $p * \bar{p} \simeq e$.

$\bar{p}$ REPRESENTS THE INVERSE OF THE

CLASS OF p.



h IS CONSTANT ALONG EACH BLUE LINE

THUS WE HAVE A GROUP STRUCTURE ON $\pi_1(X, x_0)$

EXAMPLES

① $(X, x_0) = (\mathbb{R}^n, 0)$ THEN $\pi_1 = e =$ TRIVIAL GROUP

GIVEN $p : (I, \partial I) \to (\mathbb{R}^n, 0)$

DEFINE $h(s,t) = s\,p(t) \in \mathbb{R}^n$

$h(0,t) = 0$ FOR ALL t

$h(1,t) = p(t)$

② LET $S^n = \left\{ (x_0, \cdots x_n) \in \mathbb{R}^{n+1} : \sum_{0 \le i \le n} x_i^2 = 1 \right\}$

= UNIT SPHERE IN $\mathbb{R}^{n+1}$

CAN USE $(1, 0, \cdots 0)$ AS BASE POINT.

CAN USE $(1,0,\dots,0)$ AS BASE POINT.

$\pi_1(S^1) \cong \mathbb{Z}$ = INTEGERS

THEOREM TO BE PROVED LATER.

FOR $n>1$, $\pi_1(S^n) = 0$ = TRIVIAL GROUP

PROOF SKETCH: ASSUME THE

$p:(I,\partial I) \to (S^n, x_0)$ IS NOT ONTO

LET $x_1 \in S^n$ BE A POINT NOT IN THE IMAGE.

$$\begin{tikzcd}
(I,\partial I) \arrow[r, "p"] \arrow[dr, dashed, "p"'] & (S^n, x_0) \\
& (S^n - x_1, x_0) \arrow[u] & \approx (\mathbb{R}^n, 0)
\end{tikzcd}$$

HENCE p IS <u>NULL HOMOTOPIC</u>, I.E. HOMOTOPIC TO CONSTANT MAP.

HATCHER SHOW EVERY IS HOMOTOPIC TO ONE THAT IS NOT ONTO.

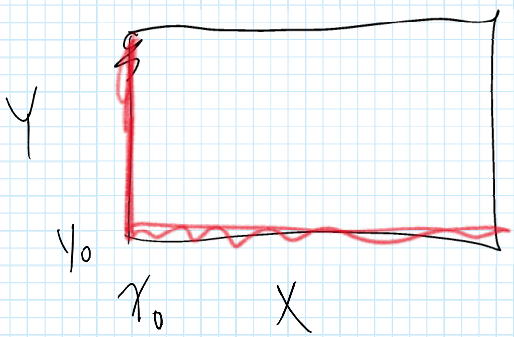
<u>DEF</u> LET (X, x_0) AND (Y, y_0) BE POINTED SPACES. THEN THEIR <u>ONE POINT UNION</u> OR <u>WEDGE</u>

$$X \vee Y := X \amalg Y / x_0 = y_0$$

\vee   NOTE \wedge $\wedge$

THE SMASH PRODUCT

$$X \wedge Y := X \times Y / (x_0 \times Y \cup X \times y_0)$$

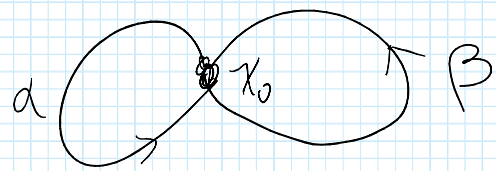


EXERCISE

$$S^m \wedge S^n \approx S^{m+n}$$

EXAMPLE 3

$$X = S^1 \vee S^1$$



$\pi_1(X)$ = FREE GROUP ON TWO GENERATORS. α AND β

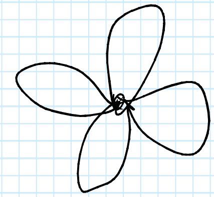
$\alpha * \beta \neq \beta * \alpha$. ONE CAN FORM "WORDS" FROM $\alpha, \beta, \alpha^{-1}, \beta^{-1}$

3A $\quad X = \underbrace{S^1 \vee S^1 \vee S^1 \vee \cdots \vee S^1}_{n \text{ FACTORS}}$

$n = 4$

n FACTORS

$n=4$

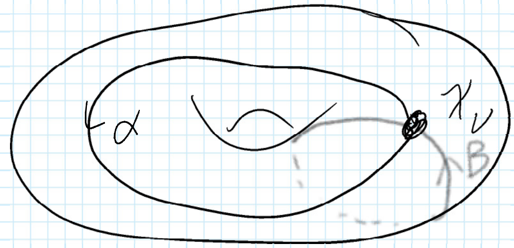


$\pi_1 X$ = FREE GROUP ON n GENERATORS $=: F_n$

<u>WEIRD PROPERTY</u>

FOR EACH $n > 0$, F_2 HAS A SUBGROUP ISOMORPHIC TO F_n. WILL PROVE THIS LATER.

EXAMPLE 4 $X = S^1 \times S^1 =:$ TORUS $= T^2$



$\pi_1(T^2) \approx \mathbb{Z} \oplus \mathbb{Z}$

IT CONTAINS A SUBSPACE HOMEO TO $S^1 \vee S^1$. THE INCLUSION MAP INDUCES

$$\begin{array}{ccc}\pi_1(S^1 \vee S^1) & \longrightarrow & \pi(S^1 \times S^1) \\ \| & & \| \\ F & \xrightarrow{\text{ONTO}} & \mathbb{Z} \oplus \mathbb{Z}\end{array}$$

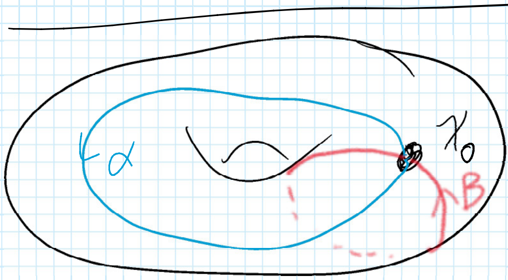
$$F_2 \xrightarrow{\text{ONTO}} \mathbb{Z} \oplus \mathbb{Z}$$
$$\alpha \longmapsto \alpha$$
$$\beta \longmapsto \beta$$

THEOREM (EASY) FOR SPACES (X, x_0) AND (Y, y_0),

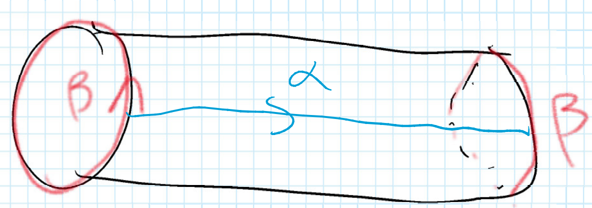
$$\pi_1(X \times Y, x_0 \times y_0) \approx \pi_1(X, x_0) \times \pi_1(Y, y_0)$$

DEF: GIVEN GROUPS G_1 AND G_2, THEIR PRODUCT $G_1 \times G_2$

WITH $(\gamma_1, \gamma_2) \times (\gamma_1', \gamma_2') = (\gamma_1\gamma_1', \gamma_2\gamma_2')$ $\quad \gamma_1, \gamma_1' \in G_1$, $\gamma_2, \gamma_2' \in G_2$

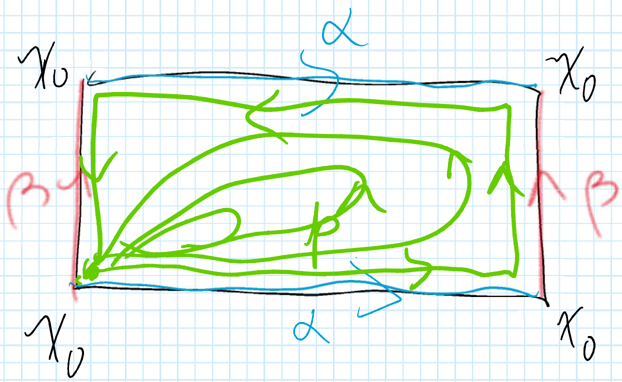


IF WE CUT ALONG β, WE GET A CYLINDER



CUT AGAIN ALONG α AND GET

CUT AGAIN ALONG α AND GET



THE CLOSED p MAPS TO A CLOSED PATH IN THE TORUS. IT IS NULL HOMOTOPIC IN THE RECTANGLE AND HENCE IN THE TORUS

LET $a, b \in \pi_1(S^1 \times S^1)$ BE REPRESENTED BY α AND β. THEN p REPRESENTS $aba^{-1}b^{-1} \in \pi_1(S^1 \times S^1)$

HENCE $$aba^{-1}b^{-1} = e$$
$$aba^{-1} = b$$
$$ab = ba$$

AND π_1 IS ABELIAN.